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ABSTRACT. Bateman and Duquette have initiated the study of Salem elements in positive characteristic. This work extends their results to 2-Salem elements whose minimal polynomials are of the type $Y^n + \lambda_{n-1}Y^{n-1} + \ldots + \lambda_1 Y + \lambda_0 \in \mathbb{F}_q[X][Y]$ where $n \geq 2, \lambda_0 \neq 0$ and $\deg \lambda_{n-1} < \deg \lambda_{n-2} = \max_{i \neq n-2} \deg(\lambda_i)$. This work provides an analogue of their results for 2-Salem elements whose minimal polynomials meet certain requirements.

Keywords: Finite field, Laurent series, 2-Salem series, 2-Salem element, irreducible polynomial, Newton polygon, Salem element.

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CONTENTS

1. INTRODUCTION

A *Salem number* is a real algebraic integer $\theta > 1$ of even degree at least 4, having θ^{-1} as a conjugate over $\mathbb{Q}$, having all its conjugates θ_i excluding θ and θ^{-1}, of modulus exactly 1 [8]. The monic minimal polynomial, over $\mathbb{Q}$, $\Lambda(z)$ of a Salem number θ is reciprocal: it satisfies the equation $z^{\deg\Lambda(z)}\Lambda(\frac{1}{z}) = \Lambda(z)$. To put is simply, this means that its coefficients form a palindromic sequence: they read the same backwards as forwards. Therefore $\theta+\theta^{-1}$ is a real algebraic integer $\theta > 2$ such that its conjugates $\neq \theta+\theta^{-1}$ lie in the real interval $[-2,2]$. The Mahler measure $M(\theta) := \prod_{i=1}^{\deg\theta} \max\{1,|\theta_i|\}$ of θ satisfies $M(\theta) = \theta$. A Salem number is the Mahler measure of itself.

The *set of Salem numbers* is traditionally denoted by T [2]. The smallest known element of T is Lehmer's number $\beta_0 = 1.1762\ldots$ of degree 10 , as *dominant* root ("i.e. if β is another root, then $|\beta| < \beta_0$") of Lehmer's polynomial:

$$P(X) = X^{10}+X^9-X^7-X^6-X^5-X^4-X^3+X+1. \tag{1.0.1}$$

The Surveys [8] [10] take stock of various problems on Salem numbers and more generally Mahler measures in all their forms.

Kerada [5] defined and studied, as a generalization of a *Salem number*, j-*Salems*, $j \geq 2$ (also called *j-Salem numbers* in the literature, e.g. in [9]). In particular, a *2-Salem* is a pair (β_1,β_2) of conjugate algebraic integers of modulus > 1 whose remaining conjugates have modulus at most 1, with at least one having modulus exactly 1. The *set of 2-Salems* is denoted by T_2. It is partitioned as $T_2 = T_2' \cup T_2''$ where T_2' is the set of 2-Salems with $\beta_1,\beta_2 \in \mathbb{R}$ and T_2'' the set of 2-Salems for which β_1 and β_2 are complex non-real (and so complex conjugates of one another, $\beta_1 = \bar{\beta_2}$).

In 1962 Bateman and Duquette [1] introduced and characterized the Salem and Pisot (PV) elements in the field of Laurent series. We first start recalling their theorem prior to stating an analogue theorem for 2-Salem elements, extending Kerada's study.

Theorem 1.1 (Bateman - Duquette)**.** *An element ω in $\mathbb{F}_q((X^{-1}))$ is a Salem (resp. Pisot) element if and only if its minimal polynomial can be written*

$$\Lambda(Y) = Y^s + \lambda_{s-1}Y^{s-1} + \ldots + \lambda_0, \qquad \lambda_i \in \mathbb{F}_q[X] \quad \text{for } i = 0,\ldots,s-1,$$

with $|\lambda_{s-1}| = |\omega| > 1$ and $|\lambda_{s-1}| = \max_{0\leq i\leq s-2}|\lambda_i|$ (resp. $|\lambda_{s-1}| > \max_{0\leq i\leq s-2}|\lambda_i|$).

In this work, instead of the classical setting of the real numbers, the analogues of Kerada's 2-*Salems* over the ring of formal Laurent series over finite fields are investigated. In the context of the original study of *Salem elements* in positive characteristic by Bateman and Duquette [1], 2-*Salem elements* in positive characteristic will also be called *2-Salem series*. The objectives of the present note consist in extending some of the results of Bateman and Duquette to 2-Salem series over $\mathbb{F}_q[X]$, $q \neq 2^r$, and to study the analogues of the above-mentioned properties of 2-Salem series. More precisely, let $\mathbb{F}_q$ denote the finite field having q elements, $q \geq 3$, and let p be the characteristic of $\mathbb{F}_q$; q is a power of p. Let X be an indeterminate over $\mathbb{F}_q$ and denote $k := \mathbb{F}_q(X)$. Let ∞ be the unique place of k which is a pole of X, and denote $k_\infty := \mathbb{F}_q((\frac{1}{X}))$. Let C_∞ be a completion of an algebraic closure of k_∞. Then C_∞ is algebraically closed and complete, and we denote by υ_∞ the valuation on

C_∞ normalized by $\upsilon_\infty(X) = -1$. We fix an embedding of an algebraic closure of k in C_∞ so that all the finite extensions of k mentioned in this work will be contained in C_∞. An explicit description of υ_∞ is done in section 2. For simplicity's sake the algebraic closure of k_∞ will be often denoted by $\mathbb{F}_q((X^{-1}))$.

2-Salem series over $\mathbb{F}_q[X]$ may belong to k_∞ or to finite extensions of k_∞. By analogy with Kerada's notations we denote by T_2^* the set of 2-Salem series. It can be partitioned as $T_2^* = T_2'^* \cup T_2''^*$ where $T_2'^*$ is by definition those 2-Salem series (ω_1, ω_2) over $\mathbb{F}_q[X]$ which (both) belong to $\mathbb{F}_q((X^{-1}))$, and $T_2''^*$, by definition, those 2-Salem series, not in $\mathbb{F}_q((X^{-1}))$, such that $(\omega_1^n, \omega_2^n) \in T_2'^*$ for some integer $n \geq 2$.

Theorem 1.2. *Suppose $q \neq 2^r$ for any integer $r \geq 1$, and $n \geq 3$. Let Λ be the polynomial defined by*

$$\Lambda(Y) = Y^n + \lambda_{n-1}Y^{n-1} + \lambda_{n-2}Y^{n-2} + \ldots + \lambda_1 Y + \lambda_0 \in \mathbb{F}_q[X][Y], \tag{1.0.2}$$

assumed irreducible and such that $\lambda_0 \neq 0$. Let us assume

$$\deg \lambda_{n-1} < \deg \lambda_{n-2} = \max_{i \neq n-2} \deg(\lambda_i).$$

Denote by ω_1 and ω_2 the dominant roots of Λ. Then

- *(i) for $n \geq 4$: if $\deg \lambda_{n-2} > 2\deg \lambda_{n-1}$, then $(\omega_1, \omega_2) \in T_2'^*$ if and only if $\deg \lambda_{n-2}$ is even, the dominant coefficient α_{2s} of $\lambda_{n-2} = \alpha_{2s}X^{2s} + \ldots + \alpha_0$ is equal to $-a^2$ for some $a \in \mathbb{F}_q^*$, and $\deg \lambda_{n-3} < \deg \lambda_{n-2}$,*
- *(ii) for $n = 3$: if $\deg \lambda_1 > 2\deg \lambda_2$, then $(\omega_1, \omega_2) \in T_2'^*$ if and only if $\deg \lambda_1$ is even, the dominant coefficient α_{2s} of $\lambda_1 = \alpha_{2s}X^{2s} + \ldots + \alpha_0$ is equal to $-a^2$ for some $a \in \mathbb{F}_q^*$,*
- *(iii) for $n \geq 3$: if $\deg \lambda_{n-2} < 2\deg \lambda_{n-1}$, then $(\omega_1, \omega_2) \in T_2'^*$.*

The paper is organized as follows. In section 2 the fields of formal power series and the valuations used in this study are recalled. The main Theorem 1.1 of Bateman and Duquette, characterizing Salem elements, is stated above with these notations. Section 3 is devoted to the arithmetical and topological properties of 2-Salem series in $\overline{\mathbb{F}_q((X^{-1}))}$. In section 4 Weiss's method of the upper Newton polygon is explicited to characterize 2-Salem series in $\overline{\mathbb{F}_q((X^{-1}))}$. In section 5 attention is focused on those 2-Salem series which lie in the field $\mathbb{F}_q((X^{-1}))$, by establishing criteria discriminating whether they belong to $\mathbb{F}_q((X^{-1}))$ or to $\overline{\mathbb{F}_q((X^{-1}))} \setminus \mathbb{F}_q((X^{-1}))$. The proof of Theorem 1.2 is given in section 6. In Theorem 1.2 the polynomial given by (1.0.2) is assumed irreducible. More generally, the question of irreducibility of a polynomial Λ of the general form (1.0.2) is discussed in section 7 under the hypothesis that Λ has no root in $\mathbb{F}_q$.

2. Salem series in $\mathbb{F}_q((X^{-1}))$

For p a prime and q a power of p, let $\mathbb{F}_q((X^{-1}))$ be the set of Laurent series over $\mathbb{F}_q$ which is defined as follows

$$\mathbb{F}_q((X^{-1})) = \{\omega = \sum_{i \geq n_0} \omega_i X^{-i} : n_0 \in \mathbb{Z} \text{ and } \omega_i \in \mathbb{F}_q\}.$$

We know that every algebraic element over $\mathbb{F}_q[X]$ can be written explicitly as a formal series because $\mathbb{F}_q[X] \subseteq \mathbb{F}_q((X^{-1}))$. However, as $\mathbb{F}_q((X^{-1}))$ is not algebraically closed, such an

element is not necessarily expressed as a power series. We refer to Kedlaya [4] for a full characterization of the algebraic closure of $\mathbb{F}_q[X]$. We denote by $\overline{\mathbb{F}_q((X^{-1}))}$ an algebraic closure of $\mathbb{F}_q((X^{-1}))$. Indifferently we will speak of 2-Salem elements or 2-Salem series in the present context.

Let ω be an element of $\mathbb{F}_q((X^{-1}))$, its polynomial part is denoted by $[\omega] \in \mathbb{F}_q[X]$ and $\{\omega\}$ its fractional part. We can remark that $\omega = [\omega] + \{\omega\}$. If $\omega \neq 0$, then the *polynomial degree* $\deg \omega$ of ω is $\gamma(\omega) = \sup\{-i : \ \omega_i \neq 0\}$, the degree of the highest-degree nonzero monomial in ω, with the convention $\gamma(0) = -\infty$. The generic form of ω, with $n_0 \in \mathbb{Z}$ and $\omega_i \in \mathbb{F}_q$, $n_0 = -\gamma(\omega)$, is

$$\omega = \sum_{i \geq n_0} \omega_i X^{-i}.$$

Note that if $[\omega] \neq 0$ then $\gamma(\omega)$ is the degree of the polynomial $[\omega]$. Thus, we define the absolute value

$$|\omega| = \begin{cases} q^{\gamma(\omega)} & \text{for } \omega \neq 0; \\ 0 & \text{for } \omega = 0. \end{cases}$$

Since $|.|$ is not archimedean, $|.|$ fulfills the strict triangle inequality

$$\begin{aligned} |\omega + \nu| &\leq \max(|\omega|, |\nu|) \quad \textit{and} \\ |\omega + \nu| &= \max(|\omega|, |\nu|) \quad \textit{if } |\omega| \neq |\nu|. \end{aligned}$$

Definition 2.1. A *Salem element* ω in $\mathbb{F}_q((X^{-1}))$ is an algebraic integer over $\mathbb{F}_q[X]$ such that $|\omega| > 1$, whose remaining conjugates in $\overline{\mathbb{F}_q((X^{-1}))}$ have an absolute value no greater than 1, and at least one has absolute value exactly 1. A *Pisot element* ω in $\mathbb{F}_q((X^{-1}))$ is an algebraic integer over $\mathbb{F}_q[X]$ such that $|\omega| > 1$, whose remaining conjugates in $\overline{\mathbb{F}_q((X^{-1}))}$ have an absolute value strictly less than 1. The set of Salem elements, resp. Pisot elements, is denoted T^*, resp. S^*.

In the following we will focus on 2-Salem series in k_∞: a *2-Salem element* is a pair of series (ω_1, ω_2) in $\mathbb{F}_q((X^{-1})) \times \mathbb{F}_q((X^{-1}))$, which has an absolute value greater than 1, in the sense that it is such that ω_1 is an algebraic integer over $\mathbb{F}_q[X]$, with the property that all of its conjugates ω_i, $i \neq 1, 2$, lie on or within the unit circle, and at least one conjugate lies on the unit circle. This implies that all 2-Salem elements are necessarily separable over $\mathbb{F}_q(X)$. Note that the pair (ω_1, ω_2) is not ordered.

Let us remark that it is easy to construct a 2-Salem element over $\mathbb{F}_q$ with $q = 2$ and then to show that 2-Salem elements do exist without the assumption $q \neq 2^r, r \geq 1$, taken in Theorem 1.2. The exclusion case $q \neq 2^r$ of Theorem 1.2 will arise in a general setting from Lemma 5.2 and its consequences.

3. Multiplicative properties of 2-Salem series

Proposition 3.1. *Let $(\omega_1, \omega_2) \in T_2'^*$, then $(\omega_1^n, \omega_2^n) \in T_2'^*$, for all $n \geq 1$.*

Proof. Let $M \in \mathbb{F}_q[X][Y]$ the minimal polynomial of the algebraic integer $\omega = \omega_1$ of degree d and $\omega_2, \ldots, \omega_d$ the conjugates of ω. We consider that the conjugate ω_2 of ω_1 is the only conjugate which lies outside the unit disk. Evidently, since ω_1 is an algebraic integer over $\mathbb{F}_q[X]$, ω_1^n, for $n \geq 1$, is also an algebraic integer over $\mathbb{F}_q[X]$.

Let $n \geq 1$ and $\Lambda \in \mathbb{F}_q[X][Y]$ be the minimal polynomial of ω_1^n. We consider the embedding σ_i of $\mathbb{F}_q(X)(\omega_1)$ into $\overline{\mathbb{F}_q((X^{-1}))}$, which fixes $\mathbb{F}_q(X)$ and maps ω_1 to ω_i. Obviously, for $i = 1, 2, \ldots, d$, ω_i^n is a root of the equation $\Lambda(Y) = 0$, and $\omega_1^n, \omega_2^n, \ldots, \omega_d^n$ are all the roots of Λ, since

$$\Lambda(\omega_i^n) = \Lambda((\sigma_i(\omega_1))^n)) = \Lambda(\sigma_i(\omega_1^n)) = \sigma_i(\Lambda(\omega_1^n)) = \sigma_i(0) = 0.$$

We deduce $\deg(\Lambda) \leq \deg(M)$ since

$$[\mathbb{F}_q(X)(\omega_1^n) : \mathbb{F}_q(X)] \leq [\mathbb{F}_q(X)(\omega_1) : \mathbb{F}_q(X)].$$

If $3 \leq i \leq d$, then $|\omega_i^n| = |\omega_i|^n \leq 1$ and there exists at least one j, $3 \leq j \leq n$, such that $|\omega_j^n| = |\omega_j|^n = 1$. Therefore $(\omega_1^n, \omega_2^n) \in T_2'^*$, for all $n \geq 1$. □

Note that the converse is false in general. For instance, take $q = 3, d = 4$ and $n = 2$. Then, the polynomial

$$Y^4 - 2X^2Y^2 + 2X^2$$

over $\mathbb{F}_3$ is irreducible and its two roots of absolute value > 1 defined by

$$(\omega_1, \omega_2) = ((\sqrt{2}(X - \frac{1}{X^3} + \ldots), -(\sqrt{2}(X - \frac{1}{X^3} + \ldots)),$$

not lie in $\mathbb{F}_3((X^{-1}))$. The other conjugates defined by

$$(\omega_3, \omega_4) = (1 - \frac{1}{X^2} + \ldots, -(1 + \frac{1}{X^2} + \ldots)).$$

We can see that (ω_1^2, ω_2^2) lie in $\mathbb{F}_3((X^{-1}))$.

For a 2-Salem series θ, let us define its trace by $\mathrm{Tr}(\theta) := \sum_{i=1}^{\deg\theta} \theta_i$. The 2-Salem series have the following basic property, as it can easily be seen by considering its trace. Recall that in the real case the trace of a Salem number is an integer ($\in \mathbb{Z}$) which is not bounded and can take arbitrary negative values [6].

Proposition 3.2. *Let $(\omega_1, \omega_2) \in T_2'^*$, then the sequence $(\{\omega_1^n + \omega_2^n\})_{n \geq 1}$ is bounded.*

Proof. Let (ω_1, ω_2) be a 2-Salem series and $\omega_3, \ldots, \omega_d$ the other conjugates of ω_1 and ω_2. From Proposition 3.1, for all $n \geq 1$, ω_1^n and ω_2^n are the roots of the same irreducible polynomial, say Λ_n in $\mathbb{F}_q[X]$, of degree d. We have

$$\mathrm{Tr}(\Lambda_n) = \sum_{i=1}^{d} \omega_i^n \in \mathbb{F}_q[X].$$

Thus $\{\mathrm{Tr}(\Lambda_n)\} = 0$, which can be rewritten

$$0 = \{\mathrm{Tr}(\Lambda_n) = \sum_{i=1}^{d} \omega_i^n\} = \{\omega_1^n + \omega_2^n + \sum_{i=3}^{d} \omega_i^n\}.$$

But $|\omega_i| \leq 1$, for $3 \leq i \leq d$, and there exists at least one j, $3 \leq j \leq n$ such that $|\omega_j^n| = |\omega_j|^n = 1$. Therefore, taking the absolute values, we deduce $|\{\omega_1^n + \omega_2^n\}| = |\{\omega_3^n\} + \{\omega_4^n\} + \ldots + \{\omega_d^n\}|$ and

$$\lim_{n \mapsto +\infty} \Big| \{\sum_{i=3}^{d} \omega_i^n\} \Big| \leq \max_{i=3,\ldots,d} \big\{ \big| \{\omega_i^n\} \big| \big\} \leq C \in \mathbb{F}_q,$$

and then $\{\omega_1^n+\omega_2^n\}$ is bounded. □

Remark 3.3. If the 2-Salem series $(\omega_1,\omega_2)\in T_2'^*$ of Proposition 3.2 admits only one root ω_3 having absolute value equal to 1 and for which the other conjugates have an absolute value strictly less than 1, then $\lim_{n\to+\infty}\{\omega_1^n+\omega_2^n\}=0$.

Proof. It is a consequence of the definition of the upper Newton polygon of the polynomial Λ_n, recalled in Proposition 4.1 below. From Proposition 4.1 we can see that $\omega_3\in\mathbb{F}_q((X^{-1}))$. Thus

$$\lim_{n\mapsto+\infty}\{\omega_3^n\}=0. \tag{3.0.1}$$

From the proof of Proposition 3.2, we have $\omega_1^n+\omega_2^n=\mathrm{Tr}(\omega_1^n)-\omega_3^n-\omega_4^n-\ldots-\omega_d^n$, $n\geq 1$, what implies

$$\begin{aligned}|\{\omega_1^n+\omega_2^n\}| &= |\{\omega_3^n\}+\{\omega_4^n\}+\ldots+\{\omega_d^n\}|\\ &\leq |\{\omega_3^n\}+\omega_4^n+\ldots+\omega_d^n|\\ &\leq \max_{i=4,\ldots,d}\{|\{\omega_3^n\}|,|\omega_i^n|\}.\end{aligned}$$

Since $|\omega_i|<1$ for $i=4,\ldots,d$ and by (3.0.1), the assertion of the Remark follows. □

Proposition 3.4. *Let $(\omega_1,\omega_2)\in T_2'^*$ be a 2-Salem series. Assume that $\Lambda\in\mathbb{F}_q[X][Y]$ is its minimal polynomial, that the degree of Λ is equal to 4 and $\omega_1,\omega_2,\omega_3,\omega_4$ are its four roots, the root ω_3 satisfying $\deg\omega_3=0$. If $\Lambda(0)\in\mathbb{F}_q^*$, then $\omega_1\omega_2\omega_3\in T^*$.*

Proof. Let $(\omega_1,\omega_2)\in T_2'^*$ and

$$\Lambda(Y)=Y^4+\lambda_3Y^3+\lambda_2Y^2+\lambda_1Y+\lambda_0,\qquad \lambda_0\in\mathbb{F}_q^*,$$

the minimal polynomial of ω_1 and ω_2. We have $\lambda_0=\omega_1\omega_2\omega_3\omega_4$. Consider the reciprocal polynomial of Λ

$$Q(Y):=Y^4\Lambda(\frac{1}{Y}).$$

Clearly Q is an irreducible polynomial over $\mathbb{F}_q[X]$, and admits the four roots

$$\frac{1}{\omega_1},\ \frac{1}{\omega_2},\ \frac{1}{\omega_3}=\lambda_0^{-1}\omega_1\omega_2\omega_4\quad\text{ and }\quad\frac{1}{\omega_4}=\lambda_0^{-1}\omega_1\omega_2\omega_3.$$

We have

$$|\frac{1}{\omega_3}|=|\omega_1\omega_2\omega_4|=1,\qquad |\frac{1}{\omega_4}|=|\omega_1\omega_2\omega_3|>1$$

and $|\frac{1}{\omega_i}|<1$, for $i=1,2$. Therefore $\omega_1\omega_2\omega_3=\frac{\lambda_0}{\omega_4}$ is a Salem series. □

4. A first characterization of 2-Salem series

The theory of the Newton polygon of a bivariate polynomial is used in the present study. The following Proposition of Weiss in [11] is the main tool for our purposes. Let us recall it. Let

$$\Lambda(X,Y)=\lambda_nY^n+\lambda_{n-1}Y^{n-1}+\ldots+\lambda_1Y+\lambda_0\quad\in\mathbb{F}_q[X,Y]=\mathbb{F}_q[X][Y] \tag{4.0.1}$$

be a nonzero polynomial. To each monomial $\lambda_i Y^i \neq 0$, we assign the point $(i, \deg(\lambda_i)) \in \mathbb{Z}^2$. For $\lambda_i = 0$, we ignore the corresponding point $(i, -\infty)$. If we consider the upper convex hull of the set of points

$$\{(0, deg(\lambda_0)), \ldots, (n, deg(\lambda_n))\},$$

we obtain the upper Newton polygon of $\Lambda(X,Y)$ with respect to Y. The polygon is a sequence of line segments $E_1, E_2, \ldots E_t$, with monotonous decreasing slopes.

The slope of a segment of the Newton polygon of $\Lambda(X,Y)$ joins, for instance, the point $(r, \deg(A_r))$ to $(r+s, \deg(A_{r+s}))$ for some $0 \leq r < r+s \leq m$. The corresponding slope is

$$k = \frac{\deg(A_{r+s}) - \deg(A_r)}{s}.$$

Denote by K_Λ the set of the slopes. For any slope $k \in K_\Lambda$, denote by s the length of the facet of slope k.

Proposition 4.1 (Weiss)**.** *Let*

$$\Lambda(X,Y) = Y^n + \lambda_{n-1}Y^{n-1} + \ldots + \lambda_1 Y + \lambda_0 \in \mathbb{F}_q[X,Y]$$

and K_Λ the set of the slopes of its upper Newton polygon. Then, for every $k \in K_\Lambda$,

i) $\Lambda(X,Y)$, as a polynomial in Y, has s roots $\alpha_1, \ldots, \alpha_s$ with the same degree $-k$ and

$$|\alpha_1| = \ldots = |\alpha_s| = q^{-k},$$

ii) the polynomial

$$\Lambda_k(X,Y) = \prod_{i=1}^{s}(Y - \alpha_i) \quad \in \mathbb{F}_q((X^{-1}))[Y]$$

divides $\Lambda(X,Y)$, with

$$\Lambda(X,Y) = \prod_{k \in K_\Lambda} \Lambda_k(X,Y).$$

Corollary 4.2 is an application of Proposition 4.1 obtained by Ben Nasr and Kthiri in [7] in the context of 2-Pisot elements. In the case of 2-Salem elements, Proposition 4.1 has several direct consequences: the following Corollary 4.3 and Theorem 4.4.

Corollary 4.2. *Let*

$$\Lambda(X,Y) = \lambda_n Y^n + \lambda_{n-1}Y^{n-1} + \ldots + \lambda_1 Y + \lambda_0 \in \mathbb{F}_q[X][Y]. \tag{4.0.2}$$

and ω a root of Λ. If $|\lambda_n| = \max_{0 \leq k \leq n} |\lambda_k|$, then $|\omega| \leq 1$.

Corollary 4.3. *Let $n \geq 3$. Let*

$$\Lambda(X,Y) = Y^n + \lambda_{n-1}Y^{n-1} + \ldots + \lambda_1 Y + \lambda_0 \in \mathbb{F}_q[X][Y] \tag{4.0.3}$$

with $\lambda_0 \neq 0$. Let us assume

$$\deg \lambda_{n-1} < \max_{0 \leq k < n-2} \deg \lambda_k = \deg \lambda_{n-2} < 2 \deg \lambda_{n-1}.$$

Then, Λ has only two roots $\omega_1, \omega_2 \in \mathbb{F}_q((X^{-1}))$ satisfying $|\omega_1| > 1$ and $|\omega_2| > 1$ and at least one conjugate which lies on the unit circle.

Proof. First let us notice that the stated condition implies that $\deg\lambda_{n-1} > 0$. Moreover the upper Newton polygon of Λ contains the line with a slope k_1 joining $(n-1, \deg\lambda_{n-1})$ and $(n,0)$, the line with a slope k_2 joining $(n-2, \deg\lambda_{n-2})$ and $(n-1, \deg\lambda_{n-1})$ and the line with a slope $k_3 = 0$ joining $(n-2, \deg\lambda_{n-2})$ and $(n-k, \deg\lambda_{n-k} = \deg\lambda_{n-2})$ for some $0 \leq k < n-2$. We have: $\deg\lambda_{n-2} - \deg\lambda_{n-1} < \deg\lambda_{n-1}$. By Proposition 4.1 (i), Λ has exactly two dominant roots ω_1, ω_2

$$\begin{cases} |\omega_1| = & q^{\deg\lambda_{n-1}} = q^{-k_1} > 1, \\ |\omega_2| = & q^{\deg\lambda_{n-2} - \deg\lambda_{n-1}} = q^{-k_2} > 1. \end{cases}$$

There exists $0 \leq k < n-2$ such that $\deg\lambda_k = \deg\lambda_{n-2}$; hence Λ has one root, say ω_3, such that

$$|\omega_3| = q^{\frac{-\deg\lambda_{n-2} + \deg\lambda_k}{n-2-k}} = q^{-k_3} = 1.$$

By Proposition 4.1 (ii), Λ admits the two factors $\Lambda_{k_1}(X,Y) = (Y - \omega_1) \in \mathbb{F}_q((X^{-1}))[Y]$ and $\Lambda_{k_2}(X,Y) = (Y - \omega_2) \in \mathbb{F}_q((X^{-1}))[Y]$. Hence ω_1 and $\omega_2 \in \mathbb{F}_q((X^{-1}))$. □

Theorem 4.4. *Let Λ be the polynomial of degree $n \geq 3$ defined by*

$$\Lambda(Y) = Y^n + \lambda_{n-1}Y^{n-1} + \lambda_{n-2}Y^{n-2} + \ldots + \lambda_1 Y + \lambda_0 \in \mathbb{F}_q[X][Y]$$

with $\lambda_0 \neq 0$. Then, Λ has exactly 2 roots in $\overline{\mathbb{F}_q((X^{-1}))}$ which have an absolute value strictly greater than 1 and the remaining roots in $\overline{\mathbb{F}_q((X^{-1}))}$ which have an absolute value less or equal to 1, with at least one conjugate lying on the unit circle, if and only if the following conditions are satisfied: $|\lambda_{n-1}| < |\lambda_{n-2}| = \max_{0 \leq i < n-2} |\lambda_i|$.

Proof. Let $\omega_1, \omega_2, \ldots, \omega_n$ be the roots of Λ. The conditions are necessary. Suppose $|\omega_1| \geq |\omega_2| > 1 \geq |\omega_3| \geq \ldots \geq |\omega_n|$ and that there exists at least one j, $3 \leq j \leq n$, such that $|\omega_j| = 1$. We have $|\lambda_{n-2}| > |\lambda_{n-1}|$. For $k \in \{1, \ldots, n\}$, $k \neq 2$,

$$|\lambda_{n-k}| = \left| \sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq n} \omega_{i_1}\omega_{i_2}\ldots\omega_{i_k} \right| \leq |\omega_1\omega_2\ldots\omega_k| \leq |\omega_1\omega_2| = |\lambda_{n-2}|$$

and

$$|\lambda_{n-j}| = \left| \sum_{1 \leq i_1 < i_2 < \ldots < i_j \leq n} \omega_{i_1}\omega_{i_2}\ldots\omega_{i_j} \right| = |\omega_1\omega_2\ldots\omega_j| = |\omega_1\omega_2| = |\lambda_{n-2}|.$$

Then

$$|\lambda_{n-2}| = \max_{i \neq n-2} |\lambda_i|.$$

The conditions are sufficient. The converse easily follows from Proposition 4.1. □

Example 4.5.

Let

$$\Lambda(Y) = Y^3 + (X+1)Y^2 + (X^4 + X^3)Y + X^4 + X^3 + X^2 + X + 1 \in \mathbb{F}_2[X][Y].$$

By Theorem 4.4, $\Lambda(Y)$ has two roots ω_1 and ω_2 having absolute value strictly greater than 1 and one root ω_3 which has an absolute value exactly equal to 1. Using the facts that

- $[\omega_1 + \omega_2 + \omega_3] = X + 1$,
- $[\omega_1\omega_2 + \omega_1\omega_3 + \omega_2\omega_3] = X^4 + X^3$,
- $[\omega_1\omega_2\omega_3] = X^4 + X^3 + X^2 + X + 1$,

then $\omega_1,\ \omega_2$ and ω_3 are defined by:

$$\begin{cases} \omega_1 = & X^2+1+\dfrac{1}{Z_1} \quad \text{such that } |Z_1|>1, \\ \omega_2 = & X^2+X+\dfrac{1}{Z_2} \quad \text{such that } |Z_2|>1, \end{cases}$$

and $\omega_3 = 1+\dfrac{1}{Z_3}$ such that $|Z_3|>1$. For $j=1$, resp. $j=2$, the fact that $\Lambda(\omega_j)=0$ implies that Z_1, resp. Z_2, is a root of the polynomial H_1, resp. H_2, defined by

$$H_1 = Z^3+(X^3+1)Z^2+(X^2+X)Z+1, \tag{4.0.4}$$

resp.

$$H_2 = (X^2+X+1)Z^3+(X^3+X^2)Z^2+(X^2+1)Z+1. \tag{4.0.5}$$

Applying Proposition 4.1 to the equations (4.0.4) and (4.0.5), we obtain $Z_1, Z_2 \in \mathbb{F}_2((X^{-1}))$. Therefore $\omega_1, \omega_2 \in \mathbb{F}_2((X^{-1}))$. Since Λ is monic and irreducible over $\mathbb{F}_2[X]$, we deduce that (ω_1,ω_2) is a 2-Salem series and Λ is the minimal polynomial of ω_1.

5. Criteria of existence of roots and conjugates in $\mathbb{F}_q((X^{-1}))$

Before giving the proof of our results, we establish some lemmas that will be needed.

Lemma 5.1. *Let $n \geq 3$. Let Λ be defined by*

$$\Lambda(Y) = Y^n+\lambda_{n-1}Y^{n-1}+\lambda_{n-2}Y^{n-2}+\ldots+\lambda_1 Y+\lambda_0 \in \mathbb{F}_q[X][Y],$$

with $\lambda_0 \neq 0$. Suppose $\max\limits_{i\neq n-2} \deg\lambda_i = \deg\lambda_{n-2} \geq 2\deg\lambda_{n-1}$. If $\deg(\lambda_{n-2})$ is odd, then Λ has no root in $\mathbb{F}_q((X^{-1}))$ with absolute value >1.

Proof. By Theorem 4.4, Λ has two roots ω_1 and ω_2 such that $|\omega_1|>1$ and $|\omega_2|>1$. The remaining roots $\omega_3,\ldots,\omega_n$ have an absolute value less or equal to 1 and at least one conjugate ω_j lies on the unit circle for $3\leq j\leq n$. As $\deg\lambda_{n-2}\geq 2\deg\lambda_{n-1}$, then the upper Newton polygon of Λ contains the line connecting the points $(n-2,\deg\lambda_{n-2})$ and $(n,0)$. The slope of this line is $k=-\dfrac{\deg\lambda_{n-2}}{2}$. By Proposition 4.1 (i), Λ has $n-(n-2)=2$ roots ω_1 and ω_2 having the absolute value $q^{-k}>1$. Since they have the same absolute value q^{-k}, we would have

$$\deg\omega_1 = \deg\omega_2 = -k = \frac{\deg\lambda_{n-2}}{2} \notin \mathbb{Z}. \tag{5.0.1}$$

Therefore $\omega_1,\omega_2 \notin \mathbb{F}_q((X^{-1}))$. □

Lemma 5.2. *Let $q\neq 2^r$ for any $r\geq 1$ and $n\geq 3$. Let Λ be the polynomial defined by*

$$\Lambda(Y) = Y^n+\lambda_{n-1}Y^{n-1}+\lambda_{n-2}Y^{n-2}+\ldots+\lambda_1 Y+\lambda_0 \in \mathbb{F}_q[X][Y]$$

with $\lambda_0\neq 0$. Suppose $\deg\lambda_{n-2}\geq \max\limits_{i\neq n-2}\deg(\lambda_i)$ and $\deg\lambda_{n-2}>2\deg\lambda_{n-1}$. Let ω_1 be a root of Λ such that $|\omega_1|>1$. If $\deg\lambda_{n-3}=\deg\lambda_{n-2}$, then $\omega_1 \in \overline{\mathbb{F}_q((X^{-1}))}\setminus\mathbb{F}_q((X^{-1}))$.

Proof. By Theorem 4.4 the polynomial Λ has exactly two roots ω_1, ω_2, such that $|\omega_1| > 1, |\omega_2| > 1$, and at least one, ω_3, such that $|\omega_3| = 1$. According to Lemma 5.1, we conclude that $\deg \lambda_{n-2}$ is even. Set $\deg \lambda_{n-2} = 2s > 0$, then $\deg \omega_1 = \deg \omega_2 = s$. Let us assume $\omega_1 \in \mathbb{F}_q((X^{-1}))$. Consider

$$\omega_1 = \sum_{i=0}^{s} a_i X^i + \frac{1}{Z_1}, \qquad \text{resp.} \qquad \omega_2 = \sum_{i=0}^{s} b_i X^i + \frac{1}{Z_2} \tag{5.0.2}$$

such that $a_s \neq 0, b_s \neq 0$ and $|Z_1| > 1, |Z_2| > 1$. Let $\lambda_n = 1$,

$$\lambda_i = \sum_{k_i=0}^{m_i} \alpha_{(k_i,i)} X^{k_i}$$

with $m_i \leq 2s$ for $i = 0, \ldots, n-4$, $m_{n-3} = 2s$, and

$$\lambda_{n-2} = \sum_{j=0}^{2s} \alpha_{(j,n-2)} X^j$$

such that $\alpha_{(2s,n-2)} \neq 0$. We now prove that necessarily $|Z_1| \leq 1$, in contradiction with $|Z_1| > 1$.

Indeed, the identity $\Lambda(\omega_1) = 0$ implies $0 =$

$$\Big([\omega_1] + \frac{1}{Z_1}\Big)^n + \lambda_{n-1}\Big([\omega_1] + \frac{1}{Z_1}\Big)^{n-1} + \lambda_{n-2}\Big([\omega_1] + \frac{1}{Z_1}\Big)^{n-2} + \ldots + \lambda_1\Big([\omega_1] + \frac{1}{Z_1}\Big) + \lambda_0.$$

Multiplying it by Z_1^n, we obtain

$$Z_1^n\Big(\sum_{k=0}^{n} \lambda_k [\omega_1]^k\Big) + Z_1^{n-1}\Big(\sum_{k=1}^{n} k\lambda_k [\omega_1]^{k-1}\Big) + Z_1^{n-2}\Big(\sum_{k=2}^{n} \frac{k(k-1)}{2}\lambda_k [\omega_1]^{k-2}\Big)$$

$$+ \ldots + Z_1^{n-j}\Big(\sum_{k=j}^{n} \frac{k(k-1)\ldots(k-j+1)}{j!}\lambda_k [\omega_1]^{k-j}\Big) + \ldots + 1 = 0.$$

Whence Z_1 is the root of the polynomial H defined by

$$H(Z) = A_n Z^n + A_{n-1} Z^{n-1} + \ldots + 1 \in \mathbb{F}_q[X][Z]$$

where

$$A_i = \sum_{k=0}^{i} \binom{n-k}{i-k} \lambda_{n-k} [\omega_1]^{i-k}, \quad 0 \leq i \leq n. \tag{5.0.3}$$

Moreover

$$-\lambda_{n-1} = [\omega_1] + [\omega_2] + [\omega_3] \tag{5.0.4}$$

and

$$\lambda_{n-2} = \omega_1\omega_2 + \omega_1\omega_3 + \ldots + \omega_{n-1}\omega_n \tag{5.0.5}$$

$$= [\omega_1][\omega_2] + Q \tag{5.0.6}$$

with $Q \in \mathbb{F}_q[X]$ and $\deg Q \leq s-1$. Notice that $\deg \lambda_{n-2} > 2\deg \lambda_{n-1}$ implies

$$\deg \lambda_{n-1} = \deg([\omega_1] + [\omega_2]) < s. \tag{5.0.7}$$

then $a_s + b_s = 0$. Hence $[\omega_1] - [\omega_2] = 2a_sX^s + (a_{s-1} - b_{s-1})X^{s-1} + \ldots + (a_0 - b_0)$. Since $q \neq 2^r$, for any $r \geq 1$, then $\deg([\omega_1] - [\omega_2]) = s$. It follows from (5.0.4) and (5.0.5) that, for $0 \leq i \leq n$, $0 \leq k \leq i$,

$$\begin{cases} \deg(\lambda_{n-k}[\omega_1]^{i-k}) = is & \text{for } k = 0, 2, \\ \deg(\lambda_{n-k}[\omega_1]^{i-k}) < is & \text{for } k \neq 0, 2. \end{cases}$$

Then

$$\deg A_i \leq is, \quad 0 \leq i \leq n.$$

In view of (5.0.3), (5.0.4) and (5.0.5), we can write

$$\begin{aligned} A_n &= [\omega_1]^n + \lambda_{n-1}[\omega_1]^{n-1} + \lambda_{n-2}[\omega_1]^{n-2} + \ldots + \lambda_0 \\ &= -[\omega_3][\omega_1]^{n-1} + [\omega_1]^{n-2}Q + \lambda_{n-3}[\omega_1]^{n-3} + \ldots + \lambda_0. \end{aligned}$$

Thus

$$\deg A_n = (n-1)s.$$

Again, by (5.0.3), it is easy to show

$$\deg A_i \leq (n-1)s, \quad \text{for } 0 \leq i \leq n-1.$$

As a result, by applying Corollary 4.2, we obtain $|Z_1| \leq 1$, a contradiction. □

6. Proof of Theorem 1.2

For establishing the proof of Theorem 1.2 the cases $n = 3$ and $n \geq 4$ are dissociated. Proposition 6.1 and Theorem 6.2, interesting in their own rights, play an important role in the characterization of the 2-Salem elements.

Proposition 6.1. *Let Λ be the polynomial defined by*

$$\Lambda(Y) = Y^3 + \lambda_2 Y^2 + \lambda_1 Y + \lambda_0 \in \mathbb{F}_q[X][Y] \tag{6.0.1}$$

where $2\deg\lambda_2 < \deg\lambda_1 = \deg\lambda_0$. Suppose $q \neq 2^r$ for any $r \geq 1$. Let ω_1 be a root of Λ such that $|\omega_1| > 1$. Then $\omega_1 \in \mathbb{F}_q((X^{-1}))$ if and only if $[\omega_1] \in \mathbb{F}_q[X]$ and $\deg\lambda_1$ is even ($\neq 0$).

Proof. The condition is necessary. Indeed, from Theorem 4.4, the root ω_1 belongs to $\overline{\mathbb{F}_q((X^{-1}))}$. Imposing $\omega_1 \in \mathbb{F}_q((X^{-1}))$ implies $[\omega_1] \in \mathbb{F}_q[X]$, and, from Lemma 5.1, $\deg\lambda_1$ is even. For sufficiency, we consider that the decomposition $\omega_1 = [\omega_1] + 1/Z_1$, with $|Z_1| > 1$, holds, and we keep the same notations for $[\omega_1]$ as in (5.0.2). Then the steps of the proof are those of the proof of Lemma 5.2 until the equality (5.0.7).

In view of (5.0.3), with $\deg\lambda_1 = 2s > 0$, we can write

$$\begin{aligned} A_3 &= [\omega_1]^3 + \lambda_2[\omega_1]^2 + \lambda_1[\omega_1] + \lambda_0 \\ &= [\omega_1]^3 - ([\omega_1] + [\omega_2] + [\omega_3])[\omega_1]^2 + ([\omega_1][\omega_2] + [\omega_1][\omega_3] + [\omega_2][\omega_3] + Q)[\omega_1] \\ &\quad - [\omega_1][\omega_2][\omega_3] + Q' \\ &= Q" \end{aligned}$$

where $\deg Q \leq s-1$, and Q' and $Q"$ are two polynomials with degree less than or equal to $2s-1$. Thus

$$\deg A_3 \leq 2s - 1.$$

Notice that $\deg\lambda_1 > 2\deg\lambda_2$ implies

$$\deg\lambda_2 = \deg([\omega_1]+[\omega_2]+[\omega_3]) < s. \tag{6.0.2}$$

then $a_s + b_s = 0$. Hence

$$[\omega_1]-[\omega_2] = 2a_sX^s + (a_{s-1}-b_{s-1})X^{s-1} + \ldots + (a_0-b_0).$$

Since $q \neq 2^r$, then $\deg([\omega_1]-[\omega_2]) = s$. Since

$$A_2 = 3[\omega_1]^2 + 2\lambda_2[\omega_1] + \lambda_1 = ([\omega_1]-[\omega_2]+[\omega_3])[\omega_1] + [\omega_2][\omega_3] + Q$$

we have

$$\deg A_2 = \deg([\omega_1]-[\omega_2]) + s = 2s.$$

We have

$$\deg A_1 = s.$$

Notice that $A_3 \neq 0$; if not, by Corollary 4.2, we would have $|Z_1| \leq 1$, a contradiction. We conclude that

$$\deg A_2 > \max_{i\neq 2} \deg A_i.$$

Finally, by Proposition 4.1, the only root of H with an absolute value > 1 is Z_1 and H admits the factor $(Z-Z_1) \in \mathbb{F}_q((X^{-1}))[Z]$. Then $Z_1 \in \mathbb{F}_q((X^{-1}))$ and $\omega_1 = [\omega_1] + \dfrac{1}{Z_1} \in \mathbb{F}_q((X^{-1}))$, completing the proof. □

Theorem 6.2. *Let $n \geq 4$ and suppose $q \neq 2^r$ for any $r \geq 1$. Let Λ be the polynomial*

$$\Lambda(Y) := Y^n + \lambda_{n-1}Y^{n-1} + \lambda_{n-2}Y^{n-2} + \ldots + \lambda_1 Y + \lambda_0 \quad \in \mathbb{F}_q[X][Y] \tag{6.0.3}$$

with $\lambda_0 \neq 0$. Suppose $\deg\lambda_{n-2} = \max\limits_{i\neq n-2} \deg(\lambda_i)$ and $\deg\lambda_{n-2} > 2\deg\lambda_{n-1}$. Let ω_1 be a root of Λ such that $|\omega_1| > 1$. Then $\omega_1 \in \mathbb{F}_q((X^{-1}))$ if and only if $[\omega_1] \in \mathbb{F}_q[X]$, $\deg\lambda_{n-2}$ is even ($\neq 0$) and $\deg\lambda_{n-3} < \deg\lambda_{n-2}$.

Proof. Let us show that the condition is necessary. From Theorem 4.4 the root ω_1 belongs to $\overline{\mathbb{F}_q((X^{-1}))}$. Assuming $\omega_1 \in \mathbb{F}_q((X^{-1}))$ implies $[\omega_1] \in \mathbb{F}_q[X]$; from Lemma 5.1, $\deg\lambda_{n-2}$ is even, and, from Lemma 5.2, $\deg\lambda_{n-3} < \deg\lambda_{n-2}$.

For sufficiency, we consider that the root $\omega_1 \in \overline{\mathbb{F}_q((X^{-1}))}$ can be decomposed as $\omega_1 = [\omega_1] + 1/Z_1$, with $|Z_1| > 1$ and $[\omega_1] \in \mathbb{F}_q[X]$. We keep the same notations for $[\omega_1]$ as in (5.0.2). The steps of the proof are now those of the proof of Lemma 5.2 until the equality (5.0.7). Denote $2s := \deg\lambda_{n-2} > 0$. We have $\deg\lambda_{n-3} \leq 2s-1$.

Since $n \geq 4$, the assumption $\deg\lambda_{n-2} = \max\limits_{i\neq n-2} \deg(\lambda_i)$ means that the upper Newton polygon of Λ has an horizontal facet of length ≥ 2. Then there exists at least one root of Λ, say ω_3, such that $|\omega_3| = 1$. Using the expressions of the symmetric functions λ_js of the roots $\omega_1, \omega_2, \omega_3, \ldots$ as functions of $[\omega_1], [\omega_2], [\omega_3], \ldots$, as above, in (5.0.3), i.e. in

$$A_n = [\omega_1]^n + \lambda_{n-1}[\omega_1]^{n-1} + \lambda_{n-2}[\omega_1]^{n-2} + \ldots + \lambda_0,$$

we deduce $\deg A_n \leq (n-1)s - 1$.

From the assumption $2s = \deg\lambda_{n-2} > 2\deg\lambda_{n-1}$ we deduce

$$\deg\lambda_{n-1} = \deg([\omega_1]+[\omega_2]+[\omega_3]+\sum_{j=4}^{n}[\omega_j]) < s. \tag{6.0.4}$$

Hence $a_s + b_s = 0$. The condition $q \neq 2^r$, $r \geq 1$, implies $a_s \neq b_s$ and $[\omega_1] \neq [\omega_2]$. Hence the degree of

$$[\omega_1] - [\omega_2] = 2a_s X^s + (a_{s-1} - b_{s-1})X^{s-1} + \ldots + (a_0 - b_0)$$

is exactly $\deg([\omega_1] - [\omega_2]) = s$.

Now the expressions of the coefficients A_{n-1} and A_{n-2} are respectively:

$$\begin{aligned} A_{n-1} &= [\omega_1]^{n-2}([\omega_1] - [\omega_2]) + (n-2)(Q[\omega_1]^{n-3} + \lambda_{n-3}[\omega_1]^{n-4}) \\ &\quad -\lambda_{n-3}[\omega_1]^{n-4} + (n-4)\lambda_{n-4}[\omega_1]^{n-5} + \ldots + \lambda_1 \end{aligned}$$

and

$$\begin{aligned} A_{n-2} &= (n-1)[\omega_1]^{n-3}([\omega_1] - [\omega_2]) + [\omega_1]^{n-3}[\omega_2] + \\ &+ \frac{(n-2)(n-3)}{2}[\omega_1]^{n-4}Q + \frac{(n-3)(n-4)}{2}\lambda_{n-3}[\omega_1]^{n-5} + \ldots + \lambda_1. \end{aligned}$$

Therefore

$$\deg A_{n-1} = (n-2)s + \deg([\omega_1] - [\omega_2]) = (n-1)s$$

and

$$\deg A_{n-2} = (n-2)s.$$

We have: $\deg A_n < \deg A_{n-1}, \deg A_{n-2} < \deg A_{n-1}$ and it is easy to show

$$\max_{i \neq n-1} \deg A_i < \deg A_{n-1}.$$

Now $A_n \neq 0$; if not, by Corollary 4.2, we would have $|Z_1| \leq 1$, a contradiction. Finally, by Proposition 4.1, the only root of H which has an absolute value > 1 is Z_1 and H admits the factor $(Z - Z_1) \in \mathbb{F}_q((X^{-1}))[Z]$. Then $Z_1 \in \mathbb{F}_q((X^{-1}))$ and $\omega_1 = [\omega_1] + \dfrac{1}{Z_1} \in \mathbb{F}_q((X^{-1}))$, completing the proof. □

Remark 6.3.

(i) We mention that Theorem 6.2 is not always true in characteristic 3 in the case $\deg \lambda_{n-2} = 2 \deg \lambda_{n-1}$ (see Example 6.4).
(ii) We note also that this theorem is not always true for any field of characteristic $p = 2$ (see Example 4.5).

Example 6.4.

Let

$$\Lambda(Y) = Y^3 + (X+1)Y^2 + X^2 Y - X^2 + 2 \in \mathbb{F}_3[X][Y]. \tag{6.0.5}$$

By Theorem 4.4, $\Lambda(Y)$ has two roots ω_1 and ω_2 having an absolute value strictly greater than 1 and one root ω_3 having an absolute value equal to 1. Set $\omega_1 = X + \dfrac{1}{Z_1} \in \mathbb{F}_3((X^{-1}))$ such that $|Z_1| > 1$. Z_1 is the root of the polynomial defined by

$$2Z^3 + 2XZ^2 + (X+1)Z + 1 = 0. \tag{6.0.6}$$

By Proposition 4.1, we deduce that $Z_1 \in \mathbb{F}_4((X^{-1}))$ and $\omega_1 \in \mathbb{F}_3((X^{-1}))$.
Now set $\omega_2 = X+1+\dfrac{1}{Z_2} \in \mathbb{F}_3((X^{-1}))$ with $|Z_2|>1$. We obtain Z_2 as a root of the polynomial defined by

$$Z^3+(X^2+X+1)Z^2+(2X^2+X+2)Z+1=0. \tag{6.0.7}$$

Again by Proposition 4.1, we deduce that $Z_2 \in \mathbb{F}_3((X^{-1}))$ and $\omega_2 \in \mathbb{F}_3((X^{-1}))$.
Since Λ is monic and irreducible over $\mathbb{F}_3[X]$, it follows that (ω_1,ω_2) is a 2-Salem series and Λ is the minimal polynomial of ω_1.

Proof of Theorem 1.2. Let us prove the necessary condition for (i) and (ii). Assume that $\omega_1 \in \mathbb{F}_q((X^{-1}))$ and $n \geq 3$. By Proposition 6.1 or Theorem 6.2, and the notations in their respective proofs, we deduce that $\deg \lambda_{n-2}$ is even and $\neq 0$. Still with these notations, set

$$\lambda_{n-2} = \alpha_{2s}X^{2s}+\alpha_{2s-1}X^{2s-1}+\ldots+\alpha_0 = [\omega_1][\omega_2]+Q \tag{6.0.8}$$

$$= (a_sX^s+a_{s-1}X^{s-1}+\ldots+a_0)(b_sX^s+b_{s-1}X^{s-1}+\ldots+b_0)+Q. \tag{6.0.9}$$

From (6.0.2) or (6.0.4), we have $\deg \lambda_{n-1} < s$. Hence $a_s = -b_s \in \mathbb{F}_q$, what implies the claim

$$-\alpha_{2s} = -a_sb_s = a_s^2 \neq 0.$$

In addition, for $n \geq 4$, Theorem 6.2 implies that $\deg \lambda_{n-3} < \deg \lambda_{n-2}$ holds.

Let us prove the sufficient condition for (i). By Theorem 4.4 the polynomial Λ has two roots ω_1 and ω_2 such that $|\omega_1|>1, |\omega_2|>1$, with at least one conjugate ω_j, $3 \leq j \leq n$, on the unit circle. Let k denote the length of the horizontal facet of the upper Newton polygon. Since $\deg \lambda_{n-3} < \deg \lambda_{n-2}$, we have $k \geq 2$. There are k conjugates ω_j, $j=3,\ldots,3+k-1$, on the unit circle, by Proposition 4.1. Let

$$\omega_j = c_0^{(j)}+c_{-1}^{(j)}X^{-1}+\ldots \in \overline{\mathbb{F}_q((X^{-1}))}, \qquad j=3,\ldots,3+k-1.$$

From Proposition 4.1 (ii), we can see $\displaystyle\sum_{j=3}^{3+k-1}\omega_j \in \mathbb{F}_q((X^{-1}))$ and therefore $\displaystyle\sum_{i=3}^{3+k-1}c_0^{(i)} \in \mathbb{F}_q$.
Now

$$\lambda_{n-1} = \beta_sX^s+\beta_{s-1}X^{s-1}+\ldots+\beta_0$$
$$= -\big([\omega_1]+[\omega_2]+\sum_{i=3}^{3+k-1}c_0^{(i)}\big).$$

Thus

$$-\beta_i = a_i+b_i, \qquad 1 \leq i \leq s. \tag{6.0.10}$$

and

$$-\beta_0 = a_0+b_0+\sum_{i=3}^{3+k-1}c_0^{(i)}. \tag{6.0.11}$$

Suppose $\alpha_{2s} = -a^2$ where $s \geq 1$ and $a \in \mathbb{F}_q$ is nonzero. Let us put $a_s = a$. Then $b_s = -a$ and $\beta_s = 0$. We deduce

$$\alpha_{2s-1} = a_sb_{s-1}+a_{s-1}b_s = a(b_{s-1}-a_{s-1}),$$

then $b_{s-1}-a_{s-1}\in\mathbb{F}_q$. Since $q\neq 2^r$, for any $r\geq 1$, and that $b_{s-1}+a_{s-1}=-\beta_{s-1}\in\mathbb{F}_q$, we have

$$a_{s-1},b_{s-1}\in\mathbb{F}_q.$$

Let us show recursively that

$$a_{s-i},b_{s-i}\in\mathbb{F}_q,\qquad i=2,3,\ldots,s.$$

Let us assume that $a_{s-j},b_{s-j}\in\mathbb{F}_q$ holds for $j=0,1,\ldots,i-1$. From (6.0.9), we deduce

$$\begin{aligned}\alpha_{2s-i}&=a_sb_{s-i}+a_{s-1}b_{s-i+1}+\ldots+a_{s-i}b_s\\&=a(b_{s-i}-a_{s-i})+d_{s-i}\end{aligned}$$

where

$$d_{s-i}:=a_{s-1}b_{s-i+1}+\ldots+b_{s-1}a_{s-i+1}\in\mathbb{F}_q,\qquad i=2,\ldots,s.$$

Hence

$$b_{s-i}-a_{s-i}=a^{-1}(\alpha_{2s-i}-d_{s-i})\in\mathbb{F}_q. \tag{6.0.12}$$

Since $b_{s-i}+a_{s-i}=-\beta_{s-i}\in\mathbb{F}_q$, we have

$$a_{s-i},b_{s-i}\in\mathbb{F}_q.$$

Let us note $d_{s-1}=0$. Combining (6.0.10) (6.0.11) and (6.0.12), we obtain

$$a_i=-2^{-1}(\beta_i+a^{-1}(\alpha_{s+i}-d_i)),\qquad 0\leq i\leq s-1. \tag{6.0.13}$$

Therefore, $[\omega_1]\in\mathbb{F}_q[X]$ and from Theorem 6.2, we obtain $\omega_1\in\mathbb{F}_q((X^{-1}))$. In the same way, we can show that $\omega_2\in\mathbb{F}_q((X^{-1}))$. As Λ is monic and irreducible over $\mathbb{F}_q[X]$, then ω_1 is an algebraic integer. Therefore (ω_1,ω_2) is a 2-Salem element in $T_2'^*$.

Let us give the proof of the sufficency condition for (ii), in the same way. By Theorem 4.4 the polynomial Λ has two roots ω_1 and ω_2 such that $|\omega_1|>1,|\omega_2|>1$, and the third one ω_3 is on the unit circle. For $n=3$, the assumptions $\deg\lambda_1=\deg(\lambda_0)$ and $\deg\lambda_1>2\deg(\lambda_2)$ hold. Then Proposition 6.1 can be applied to obtain the result. We have just to show that $[\omega_1]\in\mathbb{F}_q[X]$. For proving $[\omega_1]\in\mathbb{F}_q[X]$ we proceed as above, from (6.0.8) to (6.0.13), except that $-\beta_0$ is now equal to $a_0+b_0+c_0$ with

$$\omega_3=c_0+c_1X^{-1}+\ldots\quad\in\overline{\mathbb{F}_q((X^{-1}))}.$$

(iii) This assertion follows immediately from Corollary 4.3.

□

Remark 6.5.

Note that Theorem 1.2 (i) is not always true in the case $\deg\lambda_{n-2}=2\deg\lambda_{n-1}$. To show this, we construct two counter-examples.

Example 6.6.

Let Λ the polynomial over $\mathbb{F}_3[X]$ which is defined by (6.0.5). Then, in view of the above, Λ satisfies the conditions $\deg\lambda_{n-2}=2\deg\lambda_{n-1}$ and -1 is not a square in $\mathbb{F}_3$. In contrast, Λ has two dominant roots $\omega_1,\omega_2\in\mathbb{F}_3((X^{-1}))$.

Example 6.7.

The polynomial

$$\Lambda_2 = Y^4 - XY^3 + X^2Y^2 + XY + X^2 + 1 \in \mathbb{F}_5[X][Y]$$

satisfies the conditions $\deg \lambda_{n-2} = 2\deg \lambda_{n-1}$ and -1 is a square in $\mathbb{F}_5$. By Proposition 4.1 (i), Λ_2 has exactly two dominant roots ω_1 and ω_2 with

$$\deg \omega_1 = \deg \omega_2 = 1.$$

The other conjugated roots ω_3 and ω_4 have the same degree equal to 0. Suppose $[\omega_1] \in \mathbb{F}_5[X]$, using the fact that

$$[\omega_1] + [\omega_2] + [\omega_3] + [\omega_4] = X,$$

this yields that $[\omega_2] \in \mathbb{F}_5[X]$. Let

$$[\omega_1] = a_1X + a_0 \ , \qquad [\omega_2] = b_1X + b_0$$

and

$$[\omega_3] = c_0 \ , \qquad [\omega_2] = d_0$$

where a_1, b_1, c_0 and d_0 are four integers in $\mathbb{F}_5\backslash\{0\}$. It follows that

$$a_1 + b_1 = a_1b_1 = 1.$$

These equations have no solutions in $\mathbb{F}_5$.

7. A CRITERIUM OF IRREDUCIBILITY

In the following the assumption "$\lambda_0 \neq 0$" is replaced by the stronger hypothesis "Λ has no root in $\mathbb{F}_q$" in order to reach the property of being irreducible.

Lemma 7.1. *Let $n \geq 3$. Let Λ be defined by*

$$\Lambda(Y) = Y^n + \lambda_{n-1}Y^{n-1} + \lambda_{n-2}Y^{n-2} + \ldots + \lambda_1 Y + \lambda_0 \quad \in \mathbb{F}_q[X][Y].$$

Suppose that Λ has no root in $\mathbb{F}_q$ and $\max_{i<n-3} \deg \lambda_i < \deg \lambda_{n-3} = \deg \lambda_{n-2} \geq 2\deg \lambda_{n-1}$. If $\deg(\lambda_{n-2})$ is odd, then Λ is irreducible over $\mathbb{F}_q[X]$.

Proof. By considering the upper Newton polygon of Λ, the polynomial Λ has exactly two roots ω_1 and ω_2 such that $|\omega_1| > 1$ and $|\omega_2| > 1$, one root ω_3 such that $|\omega_3| = 1$ and the remaining roots $\omega_4, \ldots, \omega_n$ have an absolute value strictly less than 1. Suppose that $\Lambda(Y)$ admits the decomposition

$$\Lambda(Y) \ = \ \Lambda_1(Y).\Lambda_2(Y)$$

$$(7.0.1) \qquad = \ (Y^s + A_{s-1}Y^{s-1} + \ldots + A_1Y + A_0)(Y^m + B_{m-1}Y^{m-1} + \ldots + B_1Y + B_0)$$

with $\Lambda_1, \Lambda_2 \in \mathbb{F}_q[X][Y]$ and $s > 0, \ m > 0$.

There are several cases to show the contradiction. If we had $\Lambda_1(\omega_i) = 0$ for $i = 1, 2, 3$, all the roots of Λ_2 would have an absolute value strictly less 1, which is a contradiction, because $|B_0| > 1$. If we had $\Lambda_1(\omega_i) = 0$ for $i = 1, 2$, and $\Lambda_2(\omega_3) = 0$, with $m = \deg \Lambda_2 > 1$, then one of the roots of Λ_2 would have an absolute value equal to 1 and the other roots of Λ_2 have an absolute value strictly less 1, which is a contradiction, since $|B_0| > 1$. Now, if $\Lambda_1(\omega_1) = \Lambda_1(\omega_2) = 0$, and $\Lambda_2(\omega_3) = 0$ with $\deg \Lambda_2 = 1$, all the other conjugates of ω_1 are roots of Λ_1, then, from (7.0.1),

$$(7.0.2) \qquad \Lambda(Y) = (Y^{n-1} + A_{n-2}Y^{n-2} + \ldots + A_1Y + A_0)(Y + B_0) \quad \in \mathbb{F}_q[X][Y]$$

with $\deg B_0 = \deg(\omega_3) = 0$, and then $B_0 = b_0 \in \mathbb{F}_q \backslash \{0\}$. This is in contradiction with the assumption.

Then we can conclude that $\Lambda_1(\omega_1) = 0$ and $\Lambda_2(\omega_2) = 0$. The remaining roots of Λ_1 and Λ_2 have an absolute value ≤ 1. $(***)$

Let us continue the generic case, assuming $\Lambda_1(\omega_1) = \Lambda_1(\omega_3) = 0$ and $\Lambda_2(\omega_2) = 0$. Since $-A_{s-1}$ (resp. $-B_{m-1}$) is the sum of the roots of Λ_1 (resp. Λ_2) and by the symmetric functions of the roots, it follows that

$$\deg A_{s-1} = \deg \omega_1 = \max_{i \neq s-1} \deg A_i \quad \text{and} \quad \deg B_{m-1} = \deg \omega_2 > \max_{j \neq m-1} \deg B_j.$$

In particular we have: $|A_{s-2}| \leq |\omega_1|$ and $|B_{m-2}| < |\omega_2|$. Then

$$\deg \lambda_{n-2} = \deg(A_{s-2} + A_{s-1}B_{m-1} + B_{m-2}) = \deg A_{s-1} + \deg B_{m-1}.$$

But the assumption $\deg \lambda_{n-2} \geq 2 \deg \lambda_{n-1}$ means that

$$\deg A_{s-1} + \deg B_{m-1} \geq 2 \max\{\deg A_{s-1}, \deg B_{m-1}\},$$

from which we deduce

$$\deg A_{s-1} = \deg B_{m-1},$$

and then $\deg \lambda_{n-2} = 2 \deg A_{s-1}$. By Lemma 5.1, Λ would have no root in $\mathbb{F}_q((X^{-1}))$ with absolute value > 1, a contradiction. We deduce the irreducibility of Λ over $\mathbb{F}_q[X]$. □

Theorem 7.2. *Let $n \geq 4$ and suppose $q \neq 2^r$ for any $r \geq 1$. Let Λ be the polynomial*

$$\Lambda(Y) := Y^n + \lambda_{n-1}Y^{n-1} + \lambda_{n-2}Y^{n-2} + \ldots + \lambda_1 Y + \lambda_0 \quad \in \mathbb{F}_q[X][Y]. \tag{7.0.3}$$

Suppose that Λ has no root in $\mathbb{F}_q$, and assume that the coefficients λ_i satisfy

(i) $\max_{i \in \{1,2,\ldots,n-4\} \cup \{n-1\}} \deg \lambda_i < \deg \lambda_{n-3} = \deg \lambda_{n-2} < 2 \deg \lambda_{n-1}$,

(ii) $\dfrac{\deg \lambda_{i+1} + \deg \lambda_{i-1}}{2} < \deg \lambda_i \qquad \text{for } 1 \leq i \leq n-4$,

(iii) $\deg \lambda_{n-2} - \deg \lambda_{n-1} < \deg \lambda_{n-4} < \deg \lambda_{n-1}$.

Then (ω_1, ω_2) is a 2-Salem element and Λ is its minimal polynomial.

Proof. By Corollary 4.3, $\Lambda(Y)$ has two roots ω_1 and ω_2 in $\mathbb{F}_q((X^{-1}))$, such that $|\omega_1| > 1$ and $|\omega_2| > 1$, and there is exactly one conjugate ω_3 which lies on the unit circle. Denote $s := \deg(\omega_2)$ and $m := \deg(\omega_1)$ respectively. They satisfy

$$1 < |\omega_2| = q^{\deg \lambda_{n-2} - \deg \lambda_{n-1}} = q^m < |\omega_1| = q^{\deg \lambda_{n-1}} = q^s.$$

The other conjugates $\omega_4, \ldots, \omega_n \in \overline{\mathbb{F}_q((X^{-1}))}$ have an absolute value strictly less than 1. Since $\dfrac{\deg \lambda_{i+1} + \deg \lambda_{i-1}}{2} < \deg \lambda_i$, for $1 \leq i \leq n-4$, then $\deg \lambda_{i+1} - \deg \lambda_i < \deg \lambda_i - \deg \lambda_{i-1}$; all the facets of the upper Newton polygon of Λ are of length 1. We have

$$|\omega_j| = q^{-k_j} < 1, \qquad 4 \leq j \leq n,$$

with

$$-k_j = \deg \omega_j = \deg \lambda_{n-j} - \deg \lambda_{n-j+1}. \tag{7.0.4}$$

We now assume that Λ is reducible and show the contradiction. With the same notations as in the proof of Lemma 7.1, let us suppose that $\Lambda(Y)$ admits the decomposition

$$\Lambda(Y) \;=\; \Lambda_1(Y).\Lambda_2(Y)$$

$$(7.0.5) \qquad =\; (Y^s + A_{s-1}Y^{s-1} + \ldots + A_1Y + A_0)(Y^m + B_{m-1}Y^{m-1} + \ldots + B_1Y + B_0)$$

as in (7.0.1). Then we discard the impossible cases as in the proof of Lemma 7.1, i.e. from (7.0.1) until $(***)$ in the same steps. We conclude that $\Lambda_1(\omega_1) = 0$ and $\Lambda_2(\omega_2) = 0$.

Now suppose that $\Lambda_2(\omega_3) = 0$, without loss of generality; so we obtain $\omega_1 \in S^*$ and $\omega_2 \in T^*$. Applying Theorem 1.1, we get

$$(7.0.6) \qquad s = \deg A_{s-1} = \deg \omega_1 > \max_{i \le s-2} \deg A_i$$

and

$$(7.0.7) \qquad m = \deg B_{m-1} = \deg B_{m-2} = \deg \omega_2 > \max_{j \le m-3} \deg B_j.$$

The contradiction will come from the coefficient λ_{n-4}. From (7.0.5),

$$(7.0.8) \qquad \lambda_{n-4} = A_{s-4} + A_{s-3}B_{m-1} + A_{s-2}B_{m-2} + A_{s-1}B_{m-3} + B_{m-4}.$$

Let us examine the degrees of the terms of the sum. First we can see that $\Lambda_1(\omega_4) = 0$. Indeed, if we assume $\Lambda_1(\omega_4) \neq 0$, by the symmetric functions of the roots of Λ_2 we would obtain, using (i) and (7.0.4),

$$\deg B_{m-3} = \deg(\omega_2\omega_3\omega_4) = \deg \lambda_{n-4} - \deg \lambda_{n-1} < 0,$$

a contradiction. In the list $\{\omega_1, \omega_2, \omega_3, \omega_4, \ldots, \omega_n\}$ the roots ω_1 and ω_4 are roots of Λ_1, the roots ω_2 and ω_3 are roots of Λ_2, and the other roots are distributed as roots of Λ_1 or Λ_2. Then $\deg B_{m-3} > 1$. From (7.0.6) we deduce

(7.0.9)

$$\max\{\deg A_{s-3}, \deg A_{s-4}\} < s = \deg A_{s-1} < \deg A_{s-1} + \deg B_{m-3} = \deg(A_{s-1}B_{m-3}).$$

On the other hand, $\deg A_{s-2} > 0$. From (7.0.7) we deduce

$$\max\{\deg B_{m-3}, \deg B_{m-4}\} < \deg B_{m-2} = m = \deg B_{m-1} < \deg A_{s-2} + \deg B_{m-2}.$$

Let us show that $\deg(A_{s-2}B_{m-2}) < s$.

Indeed, from (iii), $\lambda_{n-4} < \deg \lambda_{n-1} = s$; then

$$\deg \lambda_{n-4} = \deg(\omega_1\omega_2\omega_3\omega_4) = \deg(\omega_1) + \deg(\omega_2) + \deg(\omega_4) = s + m + \deg(\omega_4) < s.$$

Thus

$$\deg(\omega_4) < -\deg(\omega_2) = -m,$$

what means

$$\deg(A_{s-2}B_{m-2}) = \deg(\omega_1) + \deg(\omega_4) + \deg B_{m-1} < s - m + m = s.$$

In the same way, using (ii),

$$\deg(A_{s-3}B_{m-1}) = \deg(\omega_1) + \deg(\omega_4) + \deg(\omega_5) + \deg B_{m-1} < s - m - (m+1) + m < s.$$

We deduce

$$\deg(\lambda_{n-4}) = \deg(A_{s-1}B_{m-3}).$$

But, from (7.0.9), we have

$$\deg(\lambda_{n-4}) = \deg(A_{s-1}B_{m-3}) > s.$$

The contradiction comes from (iii) since $\deg(\lambda_{n-4})$ should be $< s = \deg \lambda_{n-1}$.

Therefore $\Lambda(Y)$ is irreducible over $\mathbb{F}_q[X]$. Finally, since $\Lambda(Y)$ is monic, then (ω_1, ω_2) is a 2-Salem element and Λ is its minimal polynomial. □

Example 7.3. 2-Salem series of degree 5 in $\mathbb{F}_3((X^{-1}))$.

Let

$$\Lambda(Y) = Y^5 + X^4Y^4 + X^5Y^3 + X^5Y^2 + X^3Y + 1 \in \mathbb{F}_3[X][Y].$$

We deduce from Theorem 7.2 that Λ is irreducible over $\mathbb{F}_3[X]$ and has 5 roots defined by

$$\begin{cases} \omega_1 = & X^5 + 2X + \dfrac{1}{X^2} + \ldots = X^5 + 2X + \dfrac{1}{Z_1} & \text{such that } |Z_1| > 1 \\ \omega_2 = & X + 1 + \dfrac{1}{Z_2} & \text{such that } |Z_2| > 1 \\ \omega_3 = & 2 + \dfrac{1}{Z_3} & \text{such that } |Z_3| > 1 \\ \omega_4 = & \dfrac{1}{X^2} + \ldots & \\ \omega_5 = & \dfrac{2}{X^3} + \ldots & \end{cases}$$

These roots correspond to the facets of the upper Newton polygon associated with the 2-Salem minimal polynomial Λ. Since Λ is monic then w_1 is an algebraic integer. Therefore (ω_1, ω_2) is a 2-Salem element.

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