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1 **VECTOR POTENTIALS WITH MIXED BOUNDARY CONDITIONS.**
2 **APPLICATION TO THE STOKES PROBLEM WITH PRESSURE**
3 **AND NAVIER-TYPE BOUNDARY CONDITIONS***

4 CHÉRIF AMROUCHE[†] AND IMANE BOUSSETOUAN[‡]

5 **Abstract.** In a three-dimensional bounded possibly multiply connected domain, we prove the
6 existence, uniqueness and regularity of some vector potentials, associated with a divergence-free
7 function and satisfying mixed boundary conditions. For such a construction, the fundamental tool
8 is the characterization of the kernel which is related to the topology of the domain. We also give
9 several estimates of vector fields via the operators div and \mathbf{curl} when mixing tangential and normal
10 components on the boundary. Furthermore, we establish some Inf-Sup conditions that are crucial
11 in the L^p -theory proofs. Finally, we apply the obtained results to solve the Stokes problem with
12 a pressure condition on some part of the boundary and Navier-type boundary condition on the
13 remaining part, where weak and strong solutions are considered.

14 **Key words.** Vector potentials, mixed boundary conditions, L^p theory, Stokes equations, Navier-
15 type boundary condition.

16 **AMS subject classifications.** 35J05, 35J20, 35J25, 76D03, 76D07

17 **1. Introduction.** A relevant problem in fluid mechanics is the appropriate choi-
18 ce of the boundary conditions type. Various physical phenomena, like lubrication or
19 air and blood flows, require suitable mixed boundary conditions to be prescribed on
20 the boundary [16, 20]. Problems involving such conditions have been widely discussed
21 in the literature, from theoretical and numerical point of views : let us mention here
22 only few selected references [10, 11, 13, 17, 18, 22]. Nevertheless, at our knowledge
23 the theory of elliptic problems with mixed boundary conditions has not been fully
24 investigated in complex 3D geometries.

25 Unless stated otherwise, we assume that Ω is a $\mathcal{C}^{1,1}$ domain in \mathbb{R}^3 , possibly multi-
26 ply connected. The boundary of the flow domain is decomposed of an inner and outer
27 wall as $\Gamma = \Gamma_D \cup \Gamma_N$. Furthermore, we suppose that Γ_D and Γ_N are not empty and
28 for the sake of simplification, $\overline{\Gamma_D} \cap \overline{\Gamma_N} = \emptyset$.

29 We do not assume that Γ_D and Γ_N are connected and we denote by Γ_D^ℓ , $0 \leq$
30 $\ell \leq L_D$, the connected components of Γ_D and similarly by Γ_N^ℓ , $0 \leq \ell \leq L_N$ the
31 connected components of Γ_N . Also, $\partial\Sigma$ stands for the union of the boundaries Σ_j
32 of an admissible set of cuts $1 \leq j \leq J$ such that each surface Σ_j is an open subset of a
33 smooth manifold \mathcal{M}_j . The boundary of each Σ_j is contained in Γ and the intersection
34 $\overline{\Sigma}_i \cap \overline{\Sigma}_j$ is empty for $i \neq j$. The open set $\Omega^o = \Omega \setminus \cup_{j=1}^J \Sigma_j$ is a simply-connected
35 domain. More details will be given in Section 2.

36 It is known that a divergence-free vector field is the \mathbf{curl} of another vector field
37 called vector potential when adequate boundary conditions are imposed at any given
38 part of the boundary. Furthermore, an amount that reflects the topological structure
39 of the domain needs to be added as it plays an important role in the uniqueness
40 results and in the well-posedness of the corresponding problems. The theory of vector
41 potentials is very useful in the Maxwell's theory, in other words in electromagnetism.

42 Vector potentials on arbitrary Lipschitz domains have been treated by Mitrea et
43 al [29]. Then, in the seminal work of Amrouche et al [2], the authors gave a fairly

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complete picture of the theory of vector potentials in non-smooth domains, in the Hilbert settings. These results were extended to the L^p -theory in [5]. In [33], an important estimate has been established via div and curl when $1 < p < \infty$ if and only if the first Betti number I vanishes, *i.e.* Ω is simply connected in the case of $\mathbf{u} \times \mathbf{n} = \mathbf{0}$ on Γ or if and only if the second Betti number J vanishes, *i.e.* Ω has only one connected component of the boundary in the case of $\mathbf{u} \cdot \mathbf{n} = 0$ on Γ , given by

$$(1.1) \quad \|\nabla \mathbf{u}\|_{\mathbf{L}^p(\Omega)} \leq C \left(\|\operatorname{div} \mathbf{u}\|_{L^p(\Omega)} + \|\operatorname{curl} \mathbf{u}\|_{\mathbf{L}^p(\Omega)} \right).$$

In [5], the authors generalized the inequality (1.1) to the case where Ω has arbitrary Betti numbers and for vector fields with vanishing tangential components or vanishing normal components on the boundary.

The main objective of this paper consists on a contribution to this topic that is focused on extending the previous results when mixing boundary conditions on the normal and tangential components of the vector potential where Ω has arbitrary Betti numbers, in the Hilbert and non-Hilbert cases. The methods of proofs are mainly based on the characterization of the kernel which is related to the geometrical properties of the domain. Since the boundary of the domain is decomposed into two parts, the dimension of the kernel depends on where the union of the boundaries of the admissible set of cuts $\partial\Sigma$ lies. Throughout this paper, we will deal separately with the case where $\partial\Sigma$ is included in Γ_N and the case where it is included in Γ_D because their treatments are entirely different in character. Our goal is also to improve the regularity of the obtained vector potentials to the L^p -theory for any $1 < p < +\infty$. For the general case $p \neq 2$, the standard arguments will not allow us to get the existence of the vector potentials. To overcome this obstacle, by use of the classical Helmholtz decomposition, we prove some important Inf-Sup conditions of the type :

$$(1.2) \quad \inf_{\substack{\boldsymbol{\varphi} \in \widetilde{\mathbf{V}}_0^p(\Omega) \\ \boldsymbol{\varphi} \neq \mathbf{0}}} \sup_{\substack{\boldsymbol{\xi} \in \widetilde{\mathbf{V}}_0^p(\Omega) \\ \boldsymbol{\xi} \neq \mathbf{0}}} \frac{\left| \int_{\Omega} \operatorname{curl} \boldsymbol{\xi} \cdot \operatorname{curl} \boldsymbol{\varphi} \right|}{\|\boldsymbol{\xi}\|_{\mathbf{W}^{1,p}(\Omega)} \|\boldsymbol{\varphi}\|_{\mathbf{W}^{1,p'}(\Omega)}} \geq \beta,$$

where $\beta > 0$ and the space $\widetilde{\mathbf{V}}_0^p(\Omega)$ will be defined later. It turns out that these conditions are the key point when solving various elliptic problems as the following one: find $\boldsymbol{\xi} \in \mathbf{W}^{1,p}(\Omega)$ such that

$$(1.3) \quad \begin{cases} -\Delta \boldsymbol{\xi} = \operatorname{curl} \mathbf{v} & \text{and } \operatorname{div} \boldsymbol{\xi} = 0 & \text{in } \Omega, \\ \boldsymbol{\xi} \cdot \mathbf{n} = 0, \quad (\operatorname{curl} \boldsymbol{\xi} - \mathbf{v}) \times \mathbf{n} = \mathbf{0} & \text{on } \Gamma_D & \text{and } \boldsymbol{\xi} \times \mathbf{n} = \mathbf{0} & \text{on } \Gamma_N, \\ \langle \boldsymbol{\xi} \cdot \mathbf{n}, 1 \rangle_{\Gamma_N^\ell} = 0, \quad 1 \leq \ell \leq L_N, \end{cases}$$

where $\partial\Sigma \subset \Gamma_N$ and $\mathbf{v} \in \mathbf{L}^p(\Omega)$.

As an application, we consider stationary motions of viscous incompressible fluid in Ω governed by the Stokes system

$$(1.4) \quad \begin{cases} -\Delta \mathbf{u} + \nabla \pi = \mathbf{f} & \text{in } \Omega, \\ \operatorname{div} \mathbf{u} = 0 & \text{in } \Omega, \end{cases}$$

where \mathbf{u} is the velocity field, π the pressure and \mathbf{f} denotes the external force. Here and in what follows, the unit outer normal to the boundary is denoted by \mathbf{n} and the unit tangent vector by $\boldsymbol{\tau}$. We respectively define the normal and the tangential velocities by $u_n = \mathbf{u} \cdot \mathbf{n}$ and $\mathbf{u}_\tau = \mathbf{u} - u_n \mathbf{n}$.

Stokes and Navier-Stokes systems are often studied with the no-slip Dirichlet condition. However, this idea, although successful for some kind of flows from a

83 mathematical point of view, is not well justified from a physical point of view. In
 84 fact, it has previously been shown that the conventional no-slip boundary condition
 85 predicts a singularity at a moving contact line and that forces us to take into account
 86 some form of slip [19]. In the last decades, several mathematical papers have been
 87 conducted in relation to the non standard boundary conditions involving some friction
 88 (see [15, 21, 31]). The L^p -theory for the Stokes problem with various types of boundary
 89 conditions can be found for instance in [28].

90 The Navier boundary conditions were proposed by Navier [30], these conditions
 91 assume that the tangential component of the strain tensor is proportional to the
 92 tangential component of the fluid velocity on the boundary, referred to as “stress-
 93 free” or “slip” boundary conditions

$$94 \quad (1.5) \quad \mathbf{u} \cdot \mathbf{n} = 0 \quad \text{and} \quad 2\mu[\mathbb{D}(\mathbf{u})\mathbf{n}]_\tau + \alpha\mathbf{u}_\tau = \mathbf{0},$$

95 where μ is the fluid viscosity, $\mathbb{D}(\mathbf{u}) = 1/2(\nabla\mathbf{u} + \nabla\mathbf{u}^T)$ is the strain rate tensor associ-
 96 ated to the velocity field and α is a friction coefficient, which measures the tendency
 97 of the fluid to slip on the boundary. These conditions appear in the study of climate
 98 modeling and oceanic dynamics [27]. They are particularly used in the large eddy sim-
 99 ulation for turbulent flows. Since the first work [32] treating the Stokes problem with
 100 Dirichlet boundary condition on some part of the boundary and (1.5) with ($\alpha = 0$)
 101 on the other part, where the authors proved an existence result of strong (local) solu-
 102 tions, the interest in this kind of conditions has been increasing over the years (see for
 103 instance [25, 26]). In [7], the author has established the existence and uniqueness of
 104 solutions to the Stokes problem involving Navier conditions in the L^2 -settings. This
 105 work was completed by Amrouche et al in [3] where the L^p -theory of such problems
 106 was developed. Recently in [1], the authors discussed the behavior of the weak and
 107 strong solutions with respect to the friction coefficient α assumed to be a function.

108 Let us consider any point P on Γ and choose any neighborhood W of P in Γ , small
 109 enough to allow the existence of C^2 curves on W . The lengths s_1, s_2 along each family
 110 of curves are a possible set of coordinates in W . The unit tangent vectors to each
 111 family of curves are denoted by $\boldsymbol{\tau}_1, \boldsymbol{\tau}_2$, with this notation we have $\mathbf{v} = \mathbf{v}_\tau + (\mathbf{v} \cdot \mathbf{n})\mathbf{n}$
 112 and $\mathbf{v}_\tau = \sum_{k=1}^2 v_k \boldsymbol{\tau}_k$, where $v_k = \mathbf{v} \cdot \boldsymbol{\tau}_k$. Then we can prove that

$$113 \quad 2\mu[\mathbb{D}(\mathbf{u})\mathbf{n}]_\tau = -\mathbf{curl} \mathbf{u} \times \mathbf{n} - 2\Lambda \mathbf{u},$$

114 where Λ is the operator $\Lambda \mathbf{u} = \sum_{j=1}^2 \left(\frac{\partial \mathbf{n}}{\partial s_j} \cdot \mathbf{u}_\tau \right) \boldsymbol{\tau}_j$.

115 One can observe that in the case of flat boundary and when $\alpha = 0$, the Navier
 116 boundary condition (1.5) with a right hand side equal to \mathbf{h} which is a given tangential
 117 vector field, may be replaced by the condition

$$118 \quad (1.6) \quad \mathbf{u} \cdot \mathbf{n} = 0 \quad \text{and} \quad \mathbf{curl} \mathbf{u} \times \mathbf{n} = \mathbf{h} \times \mathbf{n},$$

119 which is called Navier-type boundary condition. In [4], the authors have shown the
 120 existence and uniqueness of weak, strong and very weak solutions to the Stokes prob-
 121 lem subjected to Navier-type boundary conditions. We assume that (1.6) is imposed
 122 on Γ_D . Unfortunately, one cannot prescribe only the value of the pressure on the
 123 boundary, since such a problem is known to be ill-posed. We consider that the pres-
 124 sure values are prescribed, together with the condition of non-tangential flow on the
 125 remaining part of the boundary Γ_N

$$126 \quad (1.7) \quad \mathbf{u} \times \mathbf{n} = \mathbf{0} \quad \text{and} \quad \pi = \pi_0 \quad \text{on} \quad \Gamma_N.$$

127 These conditions are used in Poiseuille flows, blood vessels or pipelines [6]. Re-
 128 cently in [14], the authors have considered the Stokes problem with (1.7) on a part of
 129 the boundary with a numerical approach applied in hemodynamics modeling of the
 130 cerebral venous network. Numerical analysis of the discrete corresponding problem
 131 has been performed in [13]. Stokes and Navier-Stokes systems including both condi-
 132 tions (1.6) and (1.7) were firstly treated in [18] where the authors assume that the
 133 boundary is divided into three parts and Dirichlet boundary condition is imposed on
 134 the third part. They proved the existence and uniqueness of a variational solution
 135 and they showed that it is a solution of the original problem in the Hilbert setting.
 136 Better regularity properties have been successfully demonstrated by Bernard. Indeed,
 137 if the given pressure on a part of the boundary is more regular then the variational
 138 solution satisfies $\Delta \mathbf{u} \in \mathbf{L}^2(\Omega)$ and the corresponding boundary conditions [11], then
 139 a $\mathbf{W}^{m,r}(\Omega)$ regularity is obtained for any $m \in \mathbb{N}$, $m \geq 2$, $r \geq 2$ [12].

140 In this paper, we follow another strategy based on the fact that the pressure can
 141 independently be obtained of the velocity field and is solution of an elliptic problem
 142 with Dirichlet boundary condition on a part of the boundary and Neumann boundary
 143 condition on the remaining part. Indeed, by setting $\mathbf{F} = \mathbf{f} - \nabla \pi$ in the Stokes
 144 problem, we get a system of equations which only includes the velocity field.

$$145 \quad -\Delta \mathbf{u} = \mathbf{F} \quad \text{and} \quad \operatorname{div} \mathbf{u} = 0 \quad \text{in} \quad \Omega,$$

146 with the boundary condition (1.6) on Γ_D and $\mathbf{u} \times \mathbf{n} = \mathbf{0}$ on Γ_N . Note that variational
 147 formulations have solutions that can be given by vector potentials of the velocity field
 148 of the Stokes problem [9]. We use the obtained Inf-Sup condition (1.2) to prove the
 149 existence of the velocity field in $\mathbf{W}^{1,p}(\Omega)$.

150 Let us outline the structure of this paper. In Section 2, we introduce the math-
 151 ematical framework, we illustrate the geometry of the domain and we review some
 152 preliminary results.

153 In Section 3, we establish some estimates for vector fields dealing with mixed
 154 normal and tangential boundary conditions for any $1 < p < \infty$. Then, we characterize
 155 the kernels when $\partial \Sigma$ is included in Γ_D and then in Γ_N . Furthermore, we obtain in
 156 both cases some Friedrich's inequalities for any function $\mathbf{u} \in \mathbf{W}^{1,p}(\Omega)$ with $\mathbf{u} \times \mathbf{n} = \mathbf{0}$
 157 on Γ_D and $\mathbf{u} \cdot \mathbf{n} = 0$ on Γ_N by virtue of Peetre-Tartar Theorem.

158 Section 4 is devoted to the existence and uniqueness of vector potentials with
 159 divergence-free and satisfying vanishing tangential components on a part of the bound-
 160 ary and vanishing normal components on the other part, in the L^2 -theory. We also
 161 point out the case of less standard but useful vector potentials that have non van-
 162 ishing divergence and where Dirichlet boundary condition is imposed on a part of
 163 the boundary. In order to extend these results to the L^p -theory, we prove two Inf-
 164 Sup conditions when $\partial \Sigma$ is included either in Γ_N or in Γ_D that are necessary in the
 165 solvability of some elliptic problems as the system (1.3) and also in the last section.

166 Finally in Section 5, we focus the attention on the existence and uniqueness of
 167 the solution of Stokes problem with Navier-type boundary condition (1.6) on a part
 168 of the boundary and a pressure condition (1.7) on the other part and we give some
 169 regularity assertions to that solution. We restrict ourselves to the case where $\partial \Sigma$ lies
 170 in Γ_D in this section, the other case can be solved in a similar way.

171 The proofs of the Stokes problem are of great help in the analysis of the Navier-
 172 Stokes equations when mixing different boundary conditions, which is the main pur-
 173 pose of our forthcoming paper.

174 **2. Functional spaces and notations.** In this section, we give some basic nota-
 175 tions, we introduce the functional spaces that are used and we describe the geometry
 176 of the domain in which we are working.

177 We follow the convention that C is a constant that may vary from expression
 178 to expression. We denote by X' the dual space of the space X and by $\langle \cdot, \cdot \rangle_{X, X'}$ the
 179 duality product between X and X' . Vector fields are designated by bold letters and
 180 their corresponding spaces by bold capital characters.

181 We denote by $[\cdot]_j$ the jump of a function over Σ_j , *i.e.* the differences of the traces
 182 for any $1 \leq j \leq J$. For any function $q \in W^{1,p}(\Omega^o)$, ∇q is the gradient of q in the
 183 sense of distributions in $\mathcal{D}'(\Omega^o)$ which belongs to $\mathbf{L}^p(\Omega^o)$ and it can be extended to
 184 $\mathbf{L}^p(\Omega)$. Therefore, to distinguish this extension from the gradient of q , we denote it
 185 by $\mathbf{grad} q$.

186 Let us introduce for any $1 < p < \infty$ the following functional framework

$$187 \quad \mathbf{H}^p(\mathbf{curl}, \Omega) = \{\mathbf{v} \in \mathbf{L}^p(\Omega); \mathbf{curl} \mathbf{v} \in \mathbf{L}^p(\Omega)\}$$

$$188 \quad \mathbf{H}^p(\mathbf{div}, \Omega) = \{\mathbf{v} \in \mathbf{L}^p(\Omega); \mathbf{div} \mathbf{v} \in L^p(\Omega)\}$$

189 and we denote by $\mathbf{X}^p(\Omega)$ the space

$$190 \quad \mathbf{X}^p(\Omega) = \mathbf{H}^p(\mathbf{curl}, \Omega) \cap \mathbf{H}^p(\mathbf{div}, \Omega)$$

191 provided with the norm

$$192 \quad \|\mathbf{v}\|_{\mathbf{X}^p(\Omega)} = \left(\|\mathbf{v}\|_{\mathbf{L}^p(\Omega)}^p + \|\mathbf{curl} \mathbf{v}\|_{\mathbf{L}^p(\Omega)}^p + \|\mathbf{div} \mathbf{v}\|_{L^p(\Omega)}^p \right)^{1/p}.$$

193 We define also the following subspaces

$$194 \quad \mathbf{X}_0^p(\Omega) = \{\mathbf{v} \in \mathbf{X}^p(\Omega), \quad \mathbf{v} \times \mathbf{n} = \mathbf{0} \quad \text{on } \Gamma_D, \quad \mathbf{v} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_N\},$$

$$195 \quad \tilde{\mathbf{X}}_0^p(\Omega) = \{\mathbf{v} \in \mathbf{X}^p(\Omega), \quad \mathbf{v} \times \mathbf{n} = \mathbf{0} \quad \text{on } \Gamma_N, \quad \mathbf{v} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_D\}$$

196 and the kernels

$$197 \quad \mathbf{K}_0^p(\Omega) = \{\mathbf{v} \in \mathbf{X}_0^p(\Omega), \quad \mathbf{div} \mathbf{v} = 0, \quad \mathbf{curl} \mathbf{v} = \mathbf{0} \quad \text{in } \Omega\},$$

$$198 \quad \tilde{\mathbf{K}}_0^p(\Omega) = \{\mathbf{v} \in \tilde{\mathbf{X}}_0^p(\Omega), \quad \mathbf{div} \mathbf{v} = 0, \quad \mathbf{curl} \mathbf{v} = \mathbf{0} \quad \text{in } \Omega\}.$$

199 Let us shed some light on the geometry of the domain here, we emphasize that Ω
 200 contains simply-connected obstacles denoted by $\Omega_D^0, \dots, \Omega_D^{L_D}$ and $\Omega_N^0, \dots, \Omega_N^{L_N}$, the
 201 non simply-connected ones are denoted by $\Omega_\Sigma^1, \dots, \Omega_\Sigma^J$. It is important to identify
 202 the components of each part of the boundary in the following cases as we will be
 203 confronted with in the whole paper:

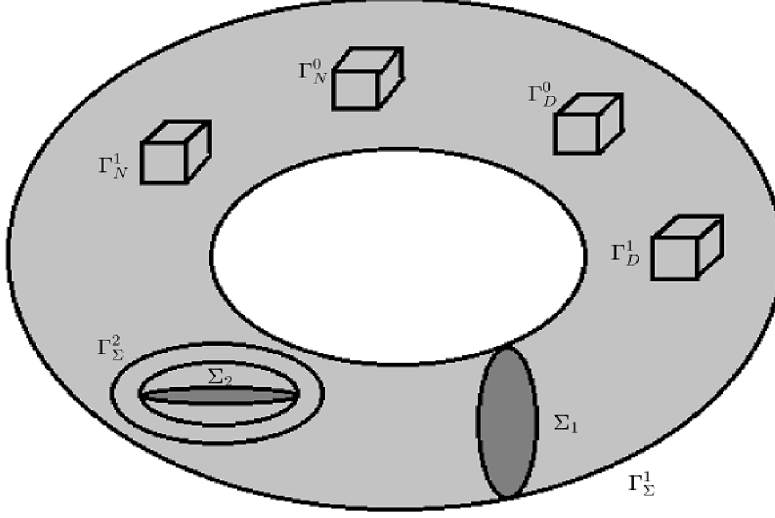


FIG. 1. Lipschitz flow domain

204 **Case 1.** When $\partial\Sigma \subset \Gamma_N$:

$$205 \quad \Gamma_D = \bigcup_{\ell=0}^{L_D} \Gamma_D^\ell \quad \text{and} \quad \Gamma_N = \left(\bigcup_{\ell=0}^{L_N} \Gamma_N^\ell \right) \cup \left(\bigcup_{j=1}^J \Gamma_\Sigma^j \right),$$

206 where Γ_D^ℓ is the boundary of Ω_D^ℓ , Γ_N^ℓ is the boundary of Ω_N^ℓ and Γ_Σ^j is the boundary
207 of Ω_Σ^j .

208 **Case 2.** When $\partial\Sigma \subset \Gamma_D$:

$$209 \quad \Gamma_N = \bigcup_{\ell=0}^{L_N} \Gamma_N^\ell, \quad \Gamma_D = \left(\bigcup_{\ell=0}^{L_D} \Gamma_D^\ell \right) \cup \left(\bigcup_{j=1}^J \Gamma_\Sigma^j \right).$$

210 As shown in figure 1, if $\partial\Sigma \subset \Gamma_N$, this means that $\Gamma_N = \Gamma_\Sigma^1 \cup \Gamma_\Sigma^2 \cup \Gamma_N^0 \cup \Gamma_N^1$ and
211 $\Gamma_D = \Gamma_D^0 \cup \Gamma_D^1$. In the other side, if $\partial\Sigma \subset \Gamma_D$, we interchange the notation in figure
212 1 such that $\Gamma_D = \Gamma_\Sigma^1 \cup \Gamma_\Sigma^2 \cup \Gamma_D^0 \cup \Gamma_D^1$ and $\Gamma_N = \Gamma_N^0 \cup \Gamma_N^1$.

213 *Remark 2.1.* We underline that in the case where $\overline{\Gamma_D} \cap \overline{\Gamma_N}$ form an edge, we lose
214 the H^2 regularity in some singularity points and for this reason, we avoid to work in
215 this case and we consider only the simplified one $\overline{\Gamma_D} \cap \overline{\Gamma_N} = \emptyset$.

216 It is worth recalling the obtained results in [5] where Γ_i , $1 \leq i \leq I$ represent
217 the connected components of the boundary Γ and Σ_j , $1 \leq j \leq J$, are the connected
218 open surfaces called ‘‘cuts’’. The authors have established the following Friedrich’s
219 inequality concerning tangential vector fields $\mathbf{u} \in \mathbf{W}^{1,p}(\Omega)$, $1 < p < \infty$ with $\mathbf{u} \times \mathbf{n} = \mathbf{0}$
220 on Γ

$$221 \quad (2.1) \quad \|\nabla \mathbf{u}\|_{\mathbf{L}^p(\Omega)} \leq C \left(\|\operatorname{div} \mathbf{u}\|_{L^p(\Omega)} + \|\operatorname{curl} \mathbf{u}\|_{\mathbf{L}^p(\Omega)} + \sum_{i=1}^I \left| \int_{\Gamma_i} \mathbf{u} \cdot \mathbf{n} \right| \right).$$

222 Similarly for normal vector fields, we have for any $\mathbf{u} \in \mathbf{W}^{1,p}(\Omega)$ with $\mathbf{u} \cdot \mathbf{n} = 0$ on Γ

$$223 \quad (2.2) \quad \|\nabla \mathbf{u}\|_{\mathbf{L}^p(\Omega)} \leq C \left(\|\operatorname{div} \mathbf{u}\|_{L^p(\Omega)} + \|\operatorname{curl} \mathbf{u}\|_{\mathbf{L}^p(\Omega)} + \sum_{j=1}^J \left| \int_{\Sigma_j} \mathbf{u} \cdot \mathbf{n} \right| \right).$$

224 The above estimates are proved by use of some integral representations, Calderón
 225 Zygmund inequalities and the traces properties [5]. Note that as soon as \mathbf{u} belongs to
 226 $\mathbf{H}^p(\operatorname{curl}, \Omega)$, the tangential boundary component $\mathbf{u} \times \mathbf{n}$ is defined in $\mathbf{W}^{-1/p,p}(\Gamma)$ and
 227 in the case where \mathbf{u} belongs to $\mathbf{H}^p(\operatorname{div}, \Omega)$, the normal boundary component $\mathbf{u} \cdot \mathbf{n}$ is
 228 also defined in $W^{-1/p,p}(\Gamma)$. Moreover, we have the Green's formulas

$$229 \quad (2.3) \quad \forall \varphi \in \mathbf{W}^{1,p'}(\Omega), \langle \mathbf{u} \times \mathbf{n}, \varphi \rangle_{\Gamma} = \int_{\Omega} \mathbf{u} \cdot \operatorname{curl} \varphi \, dx - \int_{\Omega} \operatorname{curl} \mathbf{u} \cdot \varphi \, dx,$$

230 where $\langle \cdot, \cdot \rangle_{\Gamma}$ denotes the duality product between $\mathbf{W}^{-1/p,p}(\Gamma)$ and $\mathbf{W}^{1/p,p'}(\Gamma)$ and

$$231 \quad (2.4) \quad \forall \varphi \in W^{1,p'}(\Omega), \langle \mathbf{u} \cdot \mathbf{n}, \varphi \rangle_{\Gamma} = \int_{\Omega} \mathbf{u} \cdot \nabla \varphi \, dx + \int_{\Omega} (\operatorname{div} \mathbf{u}) \varphi \, dx,$$

232 where $\langle \cdot, \cdot \rangle_{\Gamma}$ denotes the duality product between $W^{-1/p,p}(\Gamma)$ and $W^{1/p,p'}(\Gamma)$. In
 233 the case where the boundary conditions $\mathbf{u} \times \mathbf{n} = \mathbf{0}$ or $\mathbf{u} \cdot \mathbf{n} = 0$ on Γ are replaced by
 234 inhomogeneous ones, the authors have showed in [5] the following estimates

$$235 \quad \|\mathbf{u}\|_{\mathbf{W}^{1,p}(\Omega)} \leq C \left(\|\mathbf{u}\|_{\mathbf{L}^p(\Omega)} + \|\operatorname{div} \mathbf{u}\|_{L^p(\Omega)} + \|\operatorname{curl} \mathbf{u}\|_{\mathbf{L}^p(\Omega)} + \|\mathbf{u} \cdot \mathbf{n}\|_{W^{1-1/p,p}(\Gamma)} \right)$$

236

$$237 \quad \|\mathbf{u}\|_{\mathbf{W}^{1,p}(\Omega)} \leq C \left(\|\mathbf{u}\|_{\mathbf{L}^p(\Omega)} + \|\operatorname{div} \mathbf{u}\|_{L^p(\Omega)} + \|\operatorname{curl} \mathbf{u}\|_{\mathbf{L}^p(\Omega)} + \|\mathbf{u} \times \mathbf{n}\|_{\mathbf{W}^{1-1/p,p}(\Gamma)} \right).$$

238 **3. Harmonic vector fields and Friedrich's inequalities.** An important tool
 239 to study, in the next section, the existence and the uniqueness of vector potentials, is
 240 the characterization of some kernels of harmonic vector fields. We establish also some
 241 Friedrich's inequalities which are essential to solve some elliptic problems. We give
 242 finally a new Stokes formula in a general pseudo-Lipschitz domain.

243 We assume that for any point x on the boundary $\partial\Omega$ there exists a system of
 244 orthogonal co-ordinates y_j , a hypercube U containing x ($U = \Pi_{i=1}^d] - a_i, a_i[$) and a
 245 function Φ of class $\mathcal{C}^{1,1}$ such that

$$246 \quad \begin{aligned} \Omega \cap U &= \{(y', y_d) \in U \mid y_d < \Phi(y')\}, \\ \partial\Omega \cap U &= \{(y', y_d) \in U \mid y_d = \Phi(y')\}. \end{aligned}$$

248 The next lemma concerns the estimate of vector fields in the Hilbert case when tan-
 249 gential and normal boundary conditions are both applied *ie.* $\mathbf{u} \times \mathbf{n} = \mathbf{0}$ on Γ_D and
 250 $\mathbf{u} \cdot \mathbf{n} = 0$ on Γ_N . In what follows, we assume that Ω is also connected.

251 **LEMMA 3.1.** *Assume that $\mathbf{u} \in \mathbf{H}^1(\Omega)$ with $\mathbf{u} \times \mathbf{n} = \mathbf{0}$ on Γ_D and $\mathbf{u} \cdot \mathbf{n} = 0$ on*
 252 *Γ_N , then the following estimate is satisfied*

$$253 \quad (3.1) \quad \|\nabla \mathbf{u}\|_{\mathbf{L}^2(\Omega)} \leq C \left(\|\operatorname{curl} \mathbf{u}\|_{\mathbf{L}^2(\Omega)} + \|\operatorname{div} \mathbf{u}\|_{L^2(\Omega)} + \|\mathbf{u}\|_{\mathbf{L}^2(\Omega)} \right),$$

254 where C is a constant depending only on Ω .

255 *Proof.* To prove the estimate (3.1), we recall Theorem 3.1.1.2 in [24] which in-
 256 volves the curvature tensor of the boundary denoted by β and defined as

$$257 \quad \beta(\boldsymbol{\zeta}, \boldsymbol{\kappa}) = \sum_{i,j=1}^{d-1} \frac{\partial^2 \Phi}{\partial y_i \partial y_j}(0) \boldsymbol{\zeta}_i \boldsymbol{\kappa}_j,$$

258 and $\text{Tr } \beta$ denotes the trace of this operator. We have the following relation

$$259 \quad \|\nabla \mathbf{u}\|_{\mathbf{L}^2(\Omega)}^2 = \|\mathbf{curl } \mathbf{u}\|_{\mathbf{L}^2(\Omega)}^2 + \|\text{div } \mathbf{u}\|_{L^2(\Omega)}^2 - \int_{\Gamma_D} (\text{Tr } \beta)(\mathbf{u} \cdot \mathbf{n})^2 ds$$

$$260 \quad - \int_{\Gamma_N} \beta(\mathbf{u} \times \mathbf{n}, \mathbf{u} \times \mathbf{n}) ds.$$

261 For the boundary terms, we have

$$262 \quad \left| \int_{\Gamma_N} \beta(\mathbf{u} \times \mathbf{n}, \mathbf{u} \times \mathbf{n}) ds \right| \leq C \int_{\Gamma_N} |\mathbf{u}|^2 ds \leq \frac{1}{4} \|\nabla \mathbf{u}\|_{\mathbf{L}^2(\Omega)}^2 + C' \|\mathbf{u}\|_{\mathbf{L}^2(\Omega)}^2.$$

263 With a similar inequality for the term on Γ_D , we get

$$264 \quad \left| \int_{\Gamma_D} (\text{Tr } \beta)(\mathbf{u} \cdot \mathbf{n})^2 ds \right| \leq \frac{1}{4} \|\nabla \mathbf{u}\|_{\mathbf{L}^2(\Omega)}^2 + C'' \|\mathbf{u}\|_{\mathbf{L}^2(\Omega)}^2.$$

265 We deduce that (3.1) holds. \square

266 **THEOREM 3.2.** *Let $\mathbf{u} \in \mathbf{X}^p(\Omega)$ such that $\mathbf{u} \times \mathbf{n} = \mathbf{0}$ on Γ_D and $\mathbf{u} \cdot \mathbf{n} = 0$ on Γ_N ,
 267 then $\mathbf{u} \in \mathbf{W}^{1,p}(\Omega)$ and satisfies*

$$268 \quad (3.2) \quad \|\mathbf{u}\|_{\mathbf{W}^{1,p}(\Omega)} \leq C(\Omega) \|\mathbf{u}\|_{\mathbf{X}^p(\Omega)},$$

269 where $C(\Omega)$ is a constant depending on p and Ω . The same result holds for the space
 270 $\tilde{\mathbf{X}}^p(\Omega)$.

Proof. Let θ be a function defined in $C_0^\infty(\mathbb{R}^d)$, $0 \leq \theta \leq 1$ and satisfying $\theta = 1$ at the neighborhood of Γ_D and $\theta = 0$ at the neighborhood of Γ_N , we set $\eta = 1 - \theta$. As soon as $\mathbf{u} \times \mathbf{n} = \mathbf{0}$ on Γ_D and $\mathbf{u} \cdot \mathbf{n} = 0$ on Γ_N , we deduce that $\theta \mathbf{u} \times \mathbf{n} = \mathbf{0}$ on Γ and $\eta \mathbf{u} \cdot \mathbf{n} = 0$ on Γ . Then, from Theorem 3.2 of [5] for $\theta \mathbf{u}$ we deduce that

$$\|\theta \mathbf{u}\|_{\mathbf{W}^{1,p}(\Omega)} \leq C_1(\Omega) \|\theta \mathbf{u}\|_{\mathbf{X}^p(\Omega)} \leq C_2(\Omega) \|\mathbf{u}\|_{\mathbf{X}^p(\Omega)},$$

where $C_1(\Omega)$ and $C_2(\Omega)$ depend only on Ω and p . By using Theorem 3.4 of [5] for $\eta \mathbf{u}$, we get

$$\|\eta \mathbf{u}\|_{\mathbf{W}^{1,p}(\Omega)} \leq C_3(\Omega) \|\eta \mathbf{u}\|_{\mathbf{X}^p(\Omega)} \leq C_4(\Omega) \|\mathbf{u}\|_{\mathbf{X}^p(\Omega)}$$

271 where $C_3(\Omega)$ and $C_4(\Omega)$ depend only on Ω and p . Since $\mathbf{u} = \theta \mathbf{u} + \eta \mathbf{u}$, then by
 272 combining the obtained estimates, we obtain

$$273 \quad \|\mathbf{u}\|_{\mathbf{W}^{1,p}(\Omega)} \leq \|\theta \mathbf{u}\|_{\mathbf{W}^{1,p}(\Omega)} + \|\eta \mathbf{u}\|_{\mathbf{W}^{1,p}(\Omega)} \leq C(\Omega) \|\mathbf{u}\|_{\mathbf{X}^p(\Omega)},$$

274 where $C(\Omega) = C_2(\Omega) + C_4(\Omega)$. \square

275 In order to avoid extra difficulties, we start by checking some results for the
 276 Laplace operator

$$277 \quad (3.3) \quad \Delta u = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \Gamma_D \quad \text{and} \quad \frac{\partial u}{\partial \mathbf{n}} = 0 \quad \text{on } \Gamma_N.$$

278 We know that for a given $f \in L^2(\Omega)$, there exists a unique solution $u \in H^1(\Omega)$. It
 279 is clear that the solution u belongs to $H^2(\Omega)$ because of the assumptions on Γ_D and
 280 Γ_N . We will give in the following corollary a brief proof to get this regularity.

281 **COROLLARY 3.3.** *For any $f \in L^2(\Omega)$, the solution $u \in H^1(\Omega)$ of the Problem*
 282 *(3.3) belongs to $H^2(\Omega)$ and satisfies the estimate*

$$283 \quad (3.4) \quad \|u\|_{H^2(\Omega)} \leq C\|f\|_{L^2(\Omega)}.$$

284 *Proof.* We set $\mathbf{z} = \nabla u$, then $\mathbf{z} \in \mathbf{L}^2(\Omega)$, $\operatorname{div} \mathbf{z} \in L^2(\Omega)$, $\operatorname{curl} \mathbf{z} \in \mathbf{L}^2(\Omega)$ with
 285 $\mathbf{z} \times \mathbf{n} = \mathbf{0}$ on Γ_D , and $\mathbf{z} \cdot \mathbf{n} = 0$ on Γ_N . We infer from Theorem 3.2 with $p = 2$ that
 286 $\mathbf{z} \in \mathbf{H}^1(\Omega)$. Since $\mathbf{z} = \nabla u \in \mathbf{H}^1(\Omega)$, therefore $u \in H^2(\Omega)$ and satisfies the estimate
 287 (3.4). \square

288 In the case where the boundary conditions $\mathbf{u} \times \mathbf{n} = \mathbf{0}$ on Γ_D and $\mathbf{u} \cdot \mathbf{n} = 0$ on Γ_N
 289 are replaced by inhomogeneous ones, the estimate (3.2) is generalized in the following
 290 corollary.

291 **COROLLARY 3.4.** *Let $\mathbf{u} \in \mathbf{X}^p(\Omega)$ such that $\mathbf{u} \times \mathbf{n} \in \mathbf{W}^{1-1/p,p}(\Gamma_D)$ and $\mathbf{u} \cdot \mathbf{n} \in$
 292 $W^{1-1/p,p}(\Gamma_N)$. Then $\mathbf{u} \in \mathbf{W}^{1,p}(\Omega)$ and we have the following estimate*
 (3.5)

$$293 \quad \|\mathbf{u}\|_{\mathbf{W}^{1,p}(\Omega)} \leq C \left(\|\mathbf{u}\|_{\mathbf{X}^p(\Omega)} + \|\mathbf{u} \times \mathbf{n}\|_{\mathbf{W}^{1-1/p,p}(\Gamma_D)} + \|\mathbf{u} \cdot \mathbf{n}\|_{W^{1-1/p,p}(\Gamma_N)} \right).$$

294 *Proof.* Arguing similar as in the proof of Theorem 3.2, the first property and the
 295 estimate (3.5) are easily deduced, thanks to Theorem 3.5 and Corollary 5.2 of [5]. \square

296 More generally, we derive the following corollary in the same way.

297 **COROLLARY 3.5.** *Let $m \in \mathbb{N}^*$, Ω of class $\mathcal{C}^{m,1}$ and $\mathbf{u} \in \mathbf{L}^p(\Omega)$ with $\operatorname{div} \mathbf{u} \in$
 298 $W^{m-1,p}(\Omega)$ and $\operatorname{curl} \mathbf{u} \in \mathbf{W}^{m-1,p}(\Omega)$ such that $\mathbf{u} \times \mathbf{n} \in \mathbf{W}^{m-1/p,p}(\Gamma_D)$ and $\mathbf{u} \cdot \mathbf{n} \in$
 299 $W^{m-1/p,p}(\Gamma_N)$. Then $\mathbf{u} \in \mathbf{W}^{m,p}(\Omega)$ and we have the following estimate*

$$300 \quad \|\mathbf{u}\|_{\mathbf{W}^{m,p}(\Omega)} \leq C \left(\|\mathbf{u}\|_{\mathbf{L}^p(\Omega)} + \|\operatorname{div} \mathbf{u}\|_{W^{m-1,p}(\Omega)} + \|\operatorname{curl} \mathbf{u}\|_{\mathbf{W}^{m-1,p}(\Omega)} \right. \\
 301 \quad \left. + \|\mathbf{u} \cdot \mathbf{n}\|_{W^{m-1/p,p}(\Gamma_N)} + \|\mathbf{u} \times \mathbf{n}\|_{\mathbf{W}^{m-1/p,p}(\Gamma_D)} \right).$$

302 The following lemma will serve as an argument in the forthcoming analysis.

303 **LEMMA 3.6.** *Assume that Ω is Lipschitz. Let $\partial\Sigma \subset \Gamma_N$, $\boldsymbol{\psi} \in \mathbf{H}^2(\operatorname{div}, \Omega)$ and
 304 $\boldsymbol{\psi} \cdot \mathbf{n} = 0$ on Γ_N . Then, there exists a sequence $(\boldsymbol{\psi}_k)_k$ of functions in $\mathcal{D}(\tilde{\Omega})$, where
 305 $\tilde{\Omega} = \Omega \cup \left(\bigcup_{\ell=0}^{L_D} \tilde{\Omega}_D^\ell \right)$ and $\tilde{\boldsymbol{\psi}} \in \mathbf{H}^2(\operatorname{div}, \tilde{\Omega})$ satisfying*

$$306 \quad \boldsymbol{\psi}_k \rightarrow \tilde{\boldsymbol{\psi}} \quad \text{in } \mathbf{H}^2(\operatorname{div}, \tilde{\Omega}), \quad \boldsymbol{\psi}_{k|_\Omega} \rightarrow \boldsymbol{\psi} \quad \text{in } \mathbf{H}^2(\operatorname{div}, \Omega).$$

307 *Proof.* For any $0 \leq \ell \leq L_D$, let us consider $\chi_\ell \in H^1(\Omega)$ solution of the problem

$$308 \quad \begin{cases} \Delta \chi_\ell = c_\ell & \text{in } \Omega_D^\ell, \\ \partial_{\mathbf{n}} \chi_\ell = \boldsymbol{\psi} \cdot \mathbf{n} & \text{on } \Gamma_D^\ell. \end{cases}$$

309 where $c_\ell = \frac{1}{|\Omega|} \langle \boldsymbol{\psi} \cdot \mathbf{n}, 1 \rangle_{\Gamma_D^\ell}$. We set $\tilde{\boldsymbol{\psi}} = \boldsymbol{\psi}$ in Ω and $\tilde{\boldsymbol{\psi}} = \nabla \chi_\ell$ in Ω_ℓ . Let $\boldsymbol{\varphi} \in \mathcal{D}(\tilde{\Omega})$,

310 so we have

$$\begin{aligned}
311 \quad \langle \operatorname{div} \tilde{\boldsymbol{\psi}}, \varphi \rangle &= - \int_{\tilde{\Omega}} \tilde{\boldsymbol{\psi}} \cdot \nabla \varphi \, dx = - \int_{\Omega} \boldsymbol{\psi} \cdot \nabla \varphi \, dx - \sum_{\ell=0}^{L_D} \int_{\Omega_D^\ell} \nabla \chi_\ell \cdot \nabla \varphi \, dx \\
312 \quad &= \int_{\Omega} (\operatorname{div} \boldsymbol{\psi}) \varphi \, dx - \sum_{\ell=0}^{L_D} \int_{\Gamma_D^\ell} (\boldsymbol{\psi} \cdot \mathbf{n}) \varphi \, ds + \sum_{\ell=0}^{L_D} \int_{\Omega_\ell} \varphi \Delta \chi_\ell \, dx \\
313 \quad &+ \sum_{\ell=0}^{L_D} \int_{\Gamma_D^\ell} \varphi \partial_{\mathbf{n}} \chi_\ell \, ds = \int_{\Omega} (\operatorname{div} \boldsymbol{\psi}) \varphi \, dx + \sum_{\ell=0}^{L_D} c_\ell \int_{\Omega_D^\ell} \varphi \, dx.
\end{aligned}$$

314 Thus, we obtain

$$315 \quad \left| \langle \operatorname{div} \tilde{\boldsymbol{\psi}}, \varphi \rangle \right| \leq \|\operatorname{div} \boldsymbol{\psi}\|_{L^2(\Omega)} \|\varphi\|_{L^2(\Omega)} + \sum_{\ell=0}^{L_D} |\Omega_D^\ell|^{\frac{1}{2}} |c_\ell| \|\varphi\|_{L^2(\Omega_\ell)}.$$

316 But

$$317 \quad |c_\ell| \leq \frac{C(\Omega)}{|\Omega|} (\|\boldsymbol{\psi}\|_{\mathbf{L}^2(\Omega)} + \|\operatorname{div} \boldsymbol{\psi}\|_{L^2(\Omega)}),$$

318 which implies that $\tilde{\boldsymbol{\psi}} \in \mathbf{H}_0^2(\operatorname{div}, \tilde{\Omega})$. Therefore, there exists $\boldsymbol{\psi}_k \in \mathcal{D}(\tilde{\Omega})$ such that

$$319 \quad \boldsymbol{\psi}_k \rightarrow \tilde{\boldsymbol{\psi}} \text{ in } \mathbf{H}^2(\operatorname{div}, \tilde{\Omega}) \text{ and } \boldsymbol{\psi}_{k|\Omega} \rightarrow \boldsymbol{\psi} \text{ in } \mathbf{H}^2(\operatorname{div}, \Omega),$$

320 which is the required result. \square

321 The next lemma is an extension of the Green's formula (2.4) in the case where
322 $p = 2$ and is the equivalent version of Lemma 3.10 [2] when dealing with mixed
323 boundary conditions. The proof below is more detailed and the dual space $[H^{1/2}(\Sigma_j)]'$
324 in [2] is everywhere replaced by the dual space $[H_{00}^{1/2}(\Sigma_j)]'$ which is more correct.

325 **LEMMA 3.7.** *Assume that Ω is Lipschitz and $\partial\Sigma \subset \Gamma_N$. If $\boldsymbol{\psi} \in \mathbf{H}^2(\operatorname{div}, \Omega)$, then
326 the restriction of $\boldsymbol{\psi} \cdot \mathbf{n}$ to any Σ_j belongs to $[H_{00}^{1/2}(\Sigma_j)]'$ for any $1 \leq j \leq J$ and for
327 any $\chi \in H^1(\Omega^o)$ with $\chi = 0$ on Γ , we have*

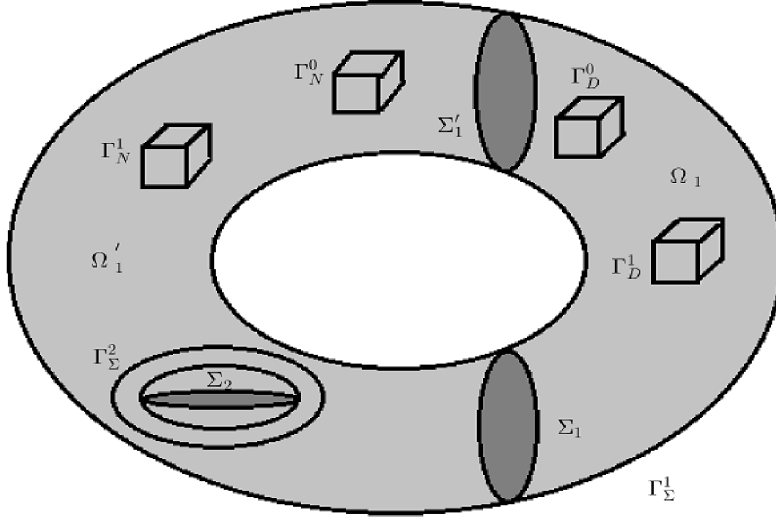
$$328 \quad (3.6) \quad \sum_{j=1}^J \langle \boldsymbol{\psi} \cdot \mathbf{n}, [\chi]_j \rangle_{\Sigma_j} = \int_{\Omega^o} \boldsymbol{\psi} \cdot \nabla \chi \, dx + \int_{\Omega^o} \chi \operatorname{div} \boldsymbol{\psi} \, dx,$$

where

$$H_{00}^{1/2}(\Sigma_j) = \left\{ \mu \in H^{1/2}(\Sigma_j), \tilde{\mu} \in H^{1/2}(\mathcal{M}_j) \right\}.$$

329 Moreover, if $\boldsymbol{\psi} \cdot \mathbf{n} = 0$ on Γ_N then (3.6) holds for any $\chi \in H^1(\Omega^o)$ and $\chi = 0$ on Γ_D .

330 *Proof.* i) Let us consider the case where $\mu \in H_{00}^{1/2}(\Sigma_1)$, we extend the cut Σ_1 by
331 Σ_1' which allows us to divide Ω into two parts Ω_1 and Ω_1' i.e $\Omega = \Omega_1 \cup \Sigma_1 \cup \Omega_1' \cup \Sigma_1'$. We
332 set now $\Omega_1^o = \Omega_1 \cup \Omega_1' \cup (\cup_{j=2}^J \Omega_{\Sigma_j})$. In other words, Ω_1^o is the open set $\Omega \setminus (\Sigma_1 \cup \Sigma_1')$,
333 to which we add the obstacles $\Omega_{\Sigma_2}, \dots, \Omega_{\Sigma_J}$.

FIG. 2. $J = 2$

334 Now, we know that there exists $\varphi_1 \in H^1(\Omega_1)$ satisfying

335
$$\Delta\varphi_1 = 0 \text{ in } \Omega_1, \quad \varphi_1 = 0 \text{ on } \partial\Omega_1 \setminus \Sigma_1, \quad \varphi_1 = \frac{\mu}{2} \text{ on } \Sigma_1,$$

and

$$\|\varphi_1\|_{H^1(\Omega_1)} \leq C\|\mu\|_{H_0^{1/2}(\Sigma_1)}.$$

336 In the same way, there exists $\varphi'_1 \in H^1(\Omega''_1)$ where $\Omega''_1 = \Omega'_1 \cup (\cup_{j=2}^J \Omega_{\Sigma_j})$ satisfying

337
$$\Delta\varphi'_1 = 0 \text{ in } \Omega''_1, \quad \varphi'_1 = 0 \text{ on } \partial\Omega''_1 \setminus \Sigma_1, \quad \varphi'_1 = -\frac{\mu}{2} \text{ on } \Sigma_1,$$

and

$$\|\varphi'_1\|_{H^1(\Omega''_1)} \leq C\|\mu\|_{H_0^{1/2}(\Sigma_1)}.$$

338 Finally, we define the function φ as

339
$$\varphi = \begin{cases} \varphi_1 & \text{in } \Omega_1 \\ \varphi'_1 & \text{in } \Omega''_1 \\ 0 & \text{on } \Sigma'_1. \end{cases}$$

340 Furthermore, it satisfies

341
$$\varphi \in H^1(\Omega_1 \cup \Sigma'_1), \quad [\varphi]_1 = \mu$$

 342
$$\varphi = 0 \quad \text{on } \Gamma_D \cup \Gamma_N, \quad [\varphi]_j = 0 \quad j = 2, \dots, J$$

343 and the estimate

344
$$\|\varphi\|_{H^1(\Omega_1 \cup \Sigma'_1)} \leq C\|\mu\|_{H_0^{1/2}(\Sigma_1)}.$$

345 We take now $\chi = \varphi|_{\Omega^o}$, then

$$346 \quad \chi \in H^1(\Omega^o), \quad [\chi]_1 = \mu, \quad [\chi]_j = 0, \quad j = 2, \dots, J$$

$$347 \quad \chi = 0 \quad \text{on} \quad \partial\Omega$$

$$\|\chi\|_{H^1(\Omega^o)} \leq C\|\mu\|_{H_0^1(\Sigma_1)}.$$

348 We proceed similarly when $\mu \in H_0^1(\Sigma_j)$ with an adapted extension of the cut Σ_j for
349 any $2 \leq j \leq J$.

350 ii) Now, let $\psi \in \mathcal{D}(\bar{\Omega})$, then Green's formula gives for any $1 \leq j \leq J$

$$351 \quad (3.7) \quad \langle \psi \cdot \mathbf{n}, \mu \rangle_{\Sigma_j} = \int_{\Omega^o} \psi \cdot \nabla \chi \, dx + \int_{\Omega^o} \chi \operatorname{div} \psi \, dx.$$

352 Moreover, we have

$$353 \quad |\langle \psi \cdot \mathbf{n}, \mu \rangle_{\Sigma_j}| \leq C\|\psi\|_{\mathbf{H}^2(\operatorname{div}, \Omega)}\|\mu\|_{H_0^1(\Sigma_j)}.$$

As a consequence, $\psi \cdot \mathbf{n} \in [H_0^1(\Sigma_j)]'$ and

$$\|\psi \cdot \mathbf{n}\|_{[H_0^1(\Sigma_j)]'} \leq C\|\psi\|_{\mathbf{H}^2(\operatorname{div}, \Omega)}.$$

354 Because of the density of $\mathcal{D}(\bar{\Omega})$ in $\mathbf{H}^2(\operatorname{div}, \Omega)$, the last inequality holds for any function
355 ψ in $\mathbf{H}^2(\operatorname{div}, \Omega)$. Finally, by using an adapted partition of unity and the Green's
356 formula (3.7), we establish the relation (3.6).

357 Finally, we assume that $\psi \in \mathbf{H}^2(\operatorname{div}, \Omega)$ and $\psi \cdot \mathbf{n} = 0$ on Γ_N , then it is easily
358 checked that the Green's formula (3.6) is valid by means of Lemma 3.6. \square

359 In order to ensure the uniqueness of the first vector potential, we are interested
360 here in the characterization of the kernel $\mathbf{K}_0^2(\Omega)$ in the case where $\partial\Sigma$ is included in
361 Γ_N .

362 **PROPOSITION 3.8.** *Assume that Ω is Lipschitz and $\partial\Sigma \subset \Gamma_N$. Then the dimension*
363 *of the space $\mathbf{K}_0^2(\Omega)$ is equal to $L_D \times J$ and it is spanned by the functions $\widehat{\mathbf{grad}} q_j^\ell$,*
364 *for $1 \leq j \leq J$ and $1 \leq \ell \leq L_D$, where each q_j^ℓ is the unique solution in $\mathbf{H}^1(\Omega^o)$ of the*
365 *problem*

$$366 \quad (3.8) \quad \left\{ \begin{array}{l} -\Delta q_j^\ell = 0 \quad \text{in} \quad \Omega^o, \\ \frac{\partial q_j^\ell}{\partial \mathbf{n}} = 0 \quad \text{on} \quad \Gamma_N, \\ q_j^\ell|_{\Gamma_D^o} = 0, \quad q_j^\ell|_{\Gamma_D^m} = \text{const}, \quad 1 \leq m \leq L_D, \\ [q_j^\ell]_k = \text{const} \quad \text{and} \quad \left[\frac{\partial q_j^\ell}{\partial \mathbf{n}} \right]_k = 0, \quad 1 \leq k \leq J, \\ \left\langle \frac{\partial q_j^\ell}{\partial \mathbf{n}}, 1 \right\rangle_{\Sigma_k} = \delta_{jk}, \quad 1 \leq k \leq J, \\ \left\langle \frac{\partial q_j^\ell}{\partial \mathbf{n}}, 1 \right\rangle_{\Gamma_D^o} = -1 \quad \text{and} \quad \left\langle \frac{\partial q_j^\ell}{\partial \mathbf{n}}, 1 \right\rangle_{\Gamma_D^m} = \delta_{\ell m}, \quad 1 \leq m \leq L_D. \end{array} \right.$$

367 *Proof. Step 1.* We define the space $\Theta^1(\Omega^o)$ as

$$368 \quad \Theta^1(\Omega^o) = \left\{ \begin{array}{l} r \in H^1(\Omega^o); [r]_j = \text{const}, 1 \leq j \leq J, \\ r|_{\Gamma_D^o} = 0 \quad r|_{\Gamma_D^m} = \text{const}, 1 \leq m \leq L_D \end{array} \right\}.$$

369 We look for $q_j^\ell \in \Theta^1(\Omega^o)$ such that

$$370 \quad (3.9) \quad \forall r \in \Theta^1(\Omega^o), \quad \int_{\Omega^o} \nabla q_j^\ell \cdot \nabla r \, dx = [r]_j + r|_{\Gamma_D^\ell}.$$

371 Since $\Theta^1(\Omega^o)$ is a closed subspace of $H^1(\Omega^o)$, using Lax-Milgram lemma, Problem
372 (3.9) has a unique solution.

373 (i) Now let $q_j^\ell \in \Theta^1(\Omega^o)$ be solution of (3.9), by taking $r \in \mathcal{D}(\Omega)$, we get

$$374 \quad \left\langle \operatorname{div}(\widetilde{\mathbf{grad}} q_j^\ell), r \right\rangle = - \int_{\Omega} \widetilde{\mathbf{grad}} q_j^\ell \cdot \nabla r \, dx = - \int_{\Omega^o} \nabla q_j^\ell \cdot \nabla r = 0,$$

375 which implies that $\operatorname{div}(\widetilde{\mathbf{grad}} q_j^\ell) = 0$ in Ω and then $\Delta q_j^\ell = 0$ in Ω^o .

376 (ii) We choose $r \in H_0^1(\Omega)$ and from Green's formula, we obtain

$$377 \quad \int_{\Omega^o} \nabla q_j^\ell \cdot \nabla r \, dx = \sum_{k=1}^J \int_{\Sigma_k} \left[\frac{\partial q_j^\ell}{\partial \mathbf{n}} \right]_k r = [r]_j = 0,$$

378 which means that $\left[\frac{\partial q_j^\ell}{\partial \mathbf{n}} \right]_k = 0$ for any $1 \leq k \leq J$. Furthermore, using (3.9) with
379 $r \in H^1(\Omega)$ such that $r = 0$ on Γ_D , and by applying again Green's formula, we deduce
380 that

$$381 \quad 0 = \int_{\Omega^o} \nabla q_j^\ell \cdot \nabla r = \langle \nabla q_j^\ell \cdot \mathbf{n}, r \rangle_{\Gamma_N}.$$

382 Therefore $\frac{\partial q_j^\ell}{\partial \mathbf{n}} = 0$ on Γ_N .

383 (iii) From Lemma 3.7, we have for any $r \in H^1(\Omega^o)$ such that $r = 0$ on Γ_D

$$384 \quad \sum_{k=1}^J \langle \nabla q_j^\ell \cdot \mathbf{n}, [r]_k \rangle_{\Sigma_k} = \int_{\Omega^o} \nabla q_j^\ell \cdot \nabla r = [r]_j.$$

385 In particular, if we choose $r \in \Theta^1(\Omega^o)$ with $r = 0$ on Γ_D , we get

$$386 \quad \sum_{k=1}^J [r]_k \langle \nabla q_j^\ell \cdot \mathbf{n}, 1 \rangle_{\Sigma_k} = [r]_j,$$

387 from which we easily derive the relations $\langle \nabla q_j^\ell \cdot \mathbf{n}, 1 \rangle_{\Sigma_k} = \delta_{jk}$.

388 (iv) In the same way, if $r \in H^1(\Omega)$ with $r|_{\Gamma_D^m} = \text{const}$, $1 \leq m \leq L_D$ and $r|_{\Gamma_D^0} = 0$,
389 we have

$$390 \quad \sum_{m=1}^{L_D} r|_{\Gamma_D^m} \langle \nabla q_j^\ell \cdot \mathbf{n}, 1 \rangle_{\Gamma_D^m} = r|_{\Gamma_D^\ell},$$

391 from which we deduce the relations $\langle \nabla q_j^\ell \cdot \mathbf{n}, 1 \rangle_{\Gamma_D^m} = \delta_{\ell m}$ for any $1 \leq \ell \leq L_D$ and
392 then $\langle \nabla q_j^\ell \cdot \mathbf{n}, 1 \rangle_{\Gamma_D^0} = -1$.

393 **Step 2.** Conversely, it is easy to check that every solution of Problem (3.8) also solves
 394 (3.9).

395 **Step 3.** Since $q_j^\ell \in H^1(\Omega^o)$ and $[q_j^\ell]_k = \text{const}$, for any $1 \leq k \leq J$, we deduce from
 396 Lemma 3.11 of [2] that $\widetilde{\text{curl grad}} q_j^\ell = \mathbf{0}$ in Ω and then $\widetilde{\text{grad}} q_j^\ell \in \mathbf{K}_0^2(\Omega)$. From the
 397 last properties in (3.8), it is readily checked that the functions $\widetilde{\text{grad}} q_j^\ell$ are linearly
 398 independent for $1 \leq j \leq J$ and $1 \leq \ell \leq L_D$.

399 It remains to show that they span $\mathbf{K}_0^2(\Omega)$. Let $\mathbf{w} \in \mathbf{K}_0^2(\Omega)$ and consider the
 400 function

$$401 \quad (3.10) \quad \mathbf{u} = \mathbf{w} - \sum_{\ell=1}^{L_D} \sum_{j=1}^J \left(\frac{1}{L_D} \langle \mathbf{w} \cdot \mathbf{n}, 1 \rangle_{\Sigma_j} + \frac{1}{J} \langle \mathbf{w} \cdot \mathbf{n}, 1 \rangle_{\Gamma_D^\ell} \right) \widetilde{\text{grad}} q_j^\ell.$$

402 Since $\mathbf{w} \in \mathbf{K}_0^2(\Omega)$, then

$$403 \quad (3.11) \quad \langle \mathbf{w} \cdot \mathbf{n}, 1 \rangle_{\Gamma_D} = \langle \mathbf{w} \cdot \mathbf{n}, 1 \rangle_{\Gamma} = \int_{\Omega} \text{div } \mathbf{w} \, dx = 0.$$

404 Therefore using (3.10), we infer that for any $1 \leq m \leq L_D$

$$405 \quad \langle \mathbf{u} \cdot \mathbf{n}, 1 \rangle_{\Gamma_D^m} = \langle \mathbf{w} \cdot \mathbf{n}, 1 \rangle_{\Gamma_D^m} - \sum_{j=1}^J \left(\frac{1}{L_D} \langle \mathbf{w} \cdot \mathbf{n}, 1 \rangle_{\Sigma_j} + \frac{1}{J} \langle \mathbf{w} \cdot \mathbf{n}, 1 \rangle_{\Gamma_D^m} \right)$$

$$406 \quad = -\frac{1}{L_D} \sum_{j=1}^J \langle \mathbf{w} \cdot \mathbf{n}, 1 \rangle_{\Sigma_j}$$

Clearly from this relation, we get after summing

$$0 = \sum_{m=1}^{L_D} \langle \mathbf{u} \cdot \mathbf{n}, 1 \rangle_{\Gamma_D^m} = -\sum_{j=1}^J \langle \mathbf{w} \cdot \mathbf{n}, 1 \rangle_{\Sigma_j},$$

407 which implies that $\langle \mathbf{u} \cdot \mathbf{n}, 1 \rangle_{\Gamma_D^m} = 0$ for any $1 \leq m \leq L_D$. In the same way for any
 408 $1 \leq j \leq J$, we deduce from (3.11) and (3.10) that

$$409 \quad \langle \mathbf{u} \cdot \mathbf{n}, 1 \rangle_{\Sigma_k} = \langle \mathbf{w} \cdot \mathbf{n}, 1 \rangle_{\Sigma_k} - \sum_{\ell=1}^{L_D} \left(\frac{1}{L_D} \langle \mathbf{w} \cdot \mathbf{n}, 1 \rangle_{\Sigma_k} + \frac{1}{J} \langle \mathbf{w} \cdot \mathbf{n}, 1 \rangle_{\Gamma_D^\ell} \right)$$

$$410 \quad = \langle \mathbf{w} \cdot \mathbf{n}, 1 \rangle_{\Sigma_k} - \langle \mathbf{w} \cdot \mathbf{n}, 1 \rangle_{\Sigma_k} = 0.$$

411 From the above properties, it is obvious that \mathbf{u} belongs to $\mathbf{K}_0^2(\Omega)$. Furthermore, it
 412 satisfies

$$413 \quad (3.12) \quad \langle \mathbf{u} \cdot \mathbf{n}, 1 \rangle_{\Gamma_D^m} = 0, \quad \forall 0 \leq m \leq L_D \quad \text{and} \quad \langle \mathbf{u} \cdot \mathbf{n}, 1 \rangle_{\Sigma_k} = 0, \quad \forall 1 \leq k \leq J.$$

414 Since Ω^o is simply connected and $\text{curl } \mathbf{u} = \mathbf{0}$ in Ω^o then $\mathbf{u} = \nabla q$, where $q \in H^1(\Omega^o)$.
 415 Furthermore, $\text{div } \mathbf{u} = 0$ in Ω then $\Delta q = 0$ in Ω^o . Because $\mathbf{u} \cdot \mathbf{n} = 0$ on Γ_N , we get
 416 $\frac{\partial q}{\partial \mathbf{n}} = 0$ on Γ_N . As $\mathbf{u} \in \mathbf{L}^2(\Omega)$ and $\text{div } \mathbf{u} = 0$ in Ω then $\left[\frac{\partial q}{\partial \mathbf{n}} \right]_j = 0$ for any $1 \leq j \leq J$.
 417 As $\text{curl } \mathbf{u} = \mathbf{0}$ also in Ω , Lemma 3.11 of [2] implies that $[q]_j = \text{const}$ for any $1 \leq j \leq J$.
 418 Therefore for any $r \in \Theta^1(\Omega^o)$, we have by (3.6) and (3.12) that

$$419 \quad \int_{\Omega^o} \nabla q \cdot \nabla r \, dx = \sum_{j=1}^J [r]_j \left\langle \frac{\partial q}{\partial \mathbf{n}}, 1 \right\rangle_{\Sigma_j} - \sum_{m=1}^{L_D} r|_{\Gamma_D^m} \left\langle \frac{\partial q}{\partial \mathbf{n}}, 1 \right\rangle_{\Gamma_D^m} = 0.$$

420 This implies that q is solution of (3.9) with a second hand side equal to zero, which
 421 means that $q = 0$ and then \mathbf{u} is zero and this ends the proof. \square

422 Let us state an immediate consequence of Proposition 3.8.

423 COROLLARY 3.9. Assume that Ω is Lipschitz (resp. $\mathcal{C}^{1,1}$) and $\partial\Sigma \subset \Gamma_N$. On the
 424 space $\mathbf{X}_0^p(\Omega)$, the semi-norm

$$425 \quad (3.13) \quad \mathbf{u} \mapsto \|\operatorname{div} \mathbf{u}\|_{L^p(\Omega)} + \|\operatorname{curl} \mathbf{u}\|_{L^p(\Omega)} + \sum_{\ell=1}^{L_D} |\langle \mathbf{u} \cdot \mathbf{n}, 1 \rangle_{\Gamma_D^\ell}| + \sum_{j=1}^J |\langle \mathbf{u} \cdot \mathbf{n}, 1 \rangle_{\Sigma_j}|$$

426 is a norm equivalent to the norm $\|\cdot\|_{\mathbf{X}^p(\Omega)}$ (resp. $\|\cdot\|_{\mathbf{W}^{1,p}(\Omega)}$).

Proof. The proof consists in applying Peetre-Tartar theorem (cf. Ref. [23]), with the following correspondence: $\mathbf{E}_1 = \mathbf{X}_0^p(\Omega)$ equipped with the graph norm, $\mathbf{E}_2 = L^p(\Omega) \times L^p(\Omega)$, $\mathbf{E}_3 = L^p(\Omega)$, $A\mathbf{u} = (\operatorname{div} \mathbf{u}, \operatorname{curl} \mathbf{u})$ and $B = Id$, the identity operator of \mathbf{E}_1 into \mathbf{E}_3 . Then $\|\mathbf{u}\|_{\mathbf{E}_1} \simeq \|A\mathbf{u}\|_{\mathbf{E}_2} + \|\mathbf{u}\|_{\mathbf{E}_3}$ since $\mathbf{X}_0^p(\Omega) \hookrightarrow \mathbf{W}^{1,p}(\Omega)$. Note that the imbedding of $\mathbf{X}_0^p(\Omega)$ into $L^p(\Omega)$ is compact and the canonical imbedding Id of \mathbf{E}_1 into \mathbf{E}_3 is also compact. Let $M : \mathbf{X}_0^p(\Omega) \mapsto \mathbf{K}_0^p(\Omega)$ be the following continuous linear mapping

$$M\mathbf{u} = \sum_{\ell=1}^{L_D} \sum_{j=1}^J \left(\frac{1}{L_D} \langle \mathbf{u} \cdot \mathbf{n}, 1 \rangle_{\Sigma_j} + \frac{1}{J} \langle \mathbf{u} \cdot \mathbf{n}, 1 \rangle_{\Gamma_D^\ell} \right) \widetilde{\operatorname{grad}} q_j^\ell.$$

We set

$$\|M\mathbf{u}\|_{\mathbf{K}_0^p(\Omega)} = \sum_{\ell=1}^{L_D} \left| \langle \mathbf{u} \cdot \mathbf{n}, 1 \rangle_{\Gamma_D^\ell} \right| + \sum_{j=1}^J \left| \langle \mathbf{u} \cdot \mathbf{n}, 1 \rangle_{\Sigma_j} \right|.$$

Let us check that if $\mathbf{u} \in \operatorname{Ker} A = \mathbf{K}_0^p(\Omega)$, then $M\mathbf{u} = \mathbf{0}$ if and only if $\langle \mathbf{u} \cdot \mathbf{n}, 1 \rangle_{\Gamma_D^\ell} = 0$ for any $1 \leq \ell \leq L_D$ and $\langle \mathbf{u} \cdot \mathbf{n}, 1 \rangle_{\Sigma_j} = 0$, for any $1 \leq j \leq J$, which means that $\mathbf{u} = \mathbf{0}$. So by Peetre-Tartar theorem we deduce that

$$\|\mathbf{u}\|_{\mathbf{X}^p(\Omega)} \leq C \left(\|A\mathbf{u}\|_{\mathbf{E}_2} + \|M\mathbf{u}\|_{\mathbf{K}_0^p(\Omega)} \right)$$

427 and then estimate (3.13). \square

428 We introduce the following space

$$429 \quad \Theta^1(\Omega) = \left\{ r \in H^1(\Omega), \quad r|_{\Gamma_D^0} = 0 \quad \text{and} \quad r|_{\Gamma_D^m} = \operatorname{const}, \quad 1 \leq m \leq L_D \right\}.$$

430 In the case where $\partial\Sigma$ is included in Γ_D , the characterization of the kernel $\mathbf{K}_0^2(\Omega)$
 431 is considered in the following proposition.

432 PROPOSITION 3.10. Assume that Ω is Lipschitz and $\partial\Sigma \subset \Gamma_D$. Then the di-
 433 mension of the space $\mathbf{K}_0^2(\Omega)$ is equal to L_D and it is spanned by the functions ∇q_ℓ ,
 434 $1 \leq \ell \leq L_D$ where each q_ℓ is the unique solution in $H^1(\Omega)$, of the problem

$$435 \quad (3.14) \quad \begin{cases} -\Delta q_\ell = 0 & \text{in } \Omega, \\ \frac{\partial q_\ell}{\partial \mathbf{n}} = 0 & \text{on } \Gamma_N, \\ q_\ell|_{\Gamma_D^0} = 0 \quad \text{and} \quad q_\ell|_{\Gamma_D^m} = \operatorname{const}, & 1 \leq m \leq L_D, \\ \left\langle \frac{\partial q_\ell}{\partial \mathbf{n}}, 1 \right\rangle_{\Gamma_D^0} = -1 \quad \text{and} \quad \left\langle \frac{\partial q_\ell}{\partial \mathbf{n}}, 1 \right\rangle_{\Gamma_D^m} = \delta_{\ell m}, & 1 \leq m \leq L_D. \end{cases}$$

436 *Proof.* It is obvious that the problem: find $q_\ell \in \Theta^1(\Omega)$ such that

$$437 \quad (3.15) \quad \forall r \in \Theta^1(\Omega), \quad \int_{\Omega} \nabla q_\ell \cdot \nabla r \, dx = r|_{\Gamma_D^\ell}$$

438 has a unique solution and each solution q_ℓ of (3.14) also solves (3.15). Conversely,
439 using (3.15) with $r \in \mathcal{D}(\Omega)$, we obtain $\Delta q_\ell = 0$ in Ω . By using Green's formula in
440 (3.15) with $r \in H^1(\Omega)$ and $r = 0$ on Γ_D , thus $\frac{\partial q_\ell}{\partial \mathbf{n}} = 0$ on Γ_N . By taking $r \in \Theta^1(\Omega)$,
441 we have

$$442 \quad \sum_{m=1}^{L_D} r|_{\Gamma_D^m} \langle \nabla q_\ell \cdot \mathbf{n}, 1 \rangle_{\Gamma_D^m} = r|_{\Gamma_D^\ell}$$

443 and then we derive the last equalities in (3.14). The functions ∇q_ℓ are linearly inde-
444 pendent and belong to $\mathbf{K}_0^2(\Omega)$. To prove that they span $\mathbf{K}_0^2(\Omega)$, we take a function
445 $\mathbf{w} \in \mathbf{K}_0^2(\Omega)$ and we consider the function

$$446 \quad \mathbf{u} = \mathbf{w} - \sum_{\ell=1}^{L_D} \langle \mathbf{w} \cdot \mathbf{n}, 1 \rangle_{\Gamma_D^\ell} \nabla q_\ell$$

447 which remains in $\mathbf{K}_0^2(\Omega)$ and satisfies $\langle \mathbf{u} \cdot \mathbf{n}, 1 \rangle_{\Gamma_D^m} = 0$ for any $1 \leq m \leq L_D$ and also
448 for $m = 0$ since $\operatorname{div} \mathbf{u} = 0$ in Ω . Note that $\mathbf{w} = \nabla q$ with $q \in H^1(\Omega^o)$. But if we take
449 another admissible set of cuts denoted by Σ'_j , $1 \leq j \leq J$, we will obtain that $\mathbf{w} = \nabla q'$
450 with $q' \in H^1(\Omega^o)$. But, for any fixed $1 \leq j \leq J$, the function $q' \in H^1(W_j)$ where W_j
451 is a neighborhood of Σ_j . Since $\nabla q = \nabla q'$ in $W_j \setminus \Sigma_j$, we deduce that there exist two
452 constants c_j^+ and c_j^- such that $q' = q + c_j^+$ in $W_j^+ \setminus \Sigma_j$ and $q' = q + c_j^-$ in $W_j^- \setminus \Sigma_j$,
453 where W_j^+ (resp W_j^-) is a part of W_j located on one side of Σ_j (resp on the other
454 side). This means that $[q]_j = \text{const}$. Since $\Delta \mathbf{w} = 0$ in Ω , we have that $\mathbf{w} \in \mathcal{C}^\infty(\Omega)$.
455 Furthermore, q is constant on any connected component Γ_D^ℓ and $\partial \Sigma \subset \Gamma_D$, we infer
456 that $c_j^+ = c_j^-$ i.e $[q]_j = 0$ and then $q \in H^1(\Omega)$. Since $\operatorname{div} \mathbf{u} = 0$ in Ω and $\mathbf{u} \cdot \mathbf{n} = 0$ on
457 Γ_N we have

$$458 \quad \int_{\Omega} \mathbf{u} \cdot \mathbf{u} \, dx = \int_{\Omega} \mathbf{u} \cdot \nabla q \, dx = \sum_{m=1}^{L_D} q|_{\Gamma_D^m} \langle \mathbf{u} \cdot \mathbf{n}, 1 \rangle_{\Gamma_D^m} = 0.$$

459 thus \mathbf{u} is zero and this ends the proof. \square

460 As previously, Proposition 3.10 has a corollary about equivalent norms.

461 **COROLLARY 3.11.** *Assume that Ω is Lipschitz (resp. $\mathcal{C}^{1,1}$) and $\partial \Sigma \subset \Gamma_D$. On the*
462 *space $\mathbf{X}_0^p(\Omega)$, the semi-norm*

$$463 \quad (3.16) \quad \mathbf{u} \mapsto \|\operatorname{div} \mathbf{u}\|_{L^p(\Omega)} + \|\operatorname{curl} \mathbf{u}\|_{L^p(\Omega)} + \sum_{\ell=1}^{L_D} |\langle \mathbf{u} \cdot \mathbf{n}, 1 \rangle_{\Gamma_D^\ell}|$$

464 *is a norm equivalent to the norm $\|\cdot\|_{\mathbf{X}^p(\Omega)}$ (resp. $\|\cdot\|_{\mathbf{W}^{1,p}(\Omega)}$).*

Proof. By applying again the Peetre-Tartar theorem with the same correspondences of \mathbf{E}_1 , \mathbf{E}_2 and \mathbf{E}_3 as in Corollary 3.9. Let $M : \mathbf{X}_0^p(\Omega) \mapsto \mathbf{K}_0^p(\Omega)$ be the following mapping

$$M\mathbf{u} = \sum_{\ell=1}^{L_D} \langle \mathbf{u} \cdot \mathbf{n}, 1 \rangle_{\Gamma_D^\ell} \nabla q_\ell.$$

We set

$$\|M\mathbf{u}\|_{\mathbf{K}_0^p(\Omega)} = \sum_{\ell=1}^{L_D} | \langle \mathbf{u} \cdot \mathbf{n}, 1 \rangle_{\Gamma_D^\ell} |.$$

It is clear that if $\mathbf{u} \in \text{Ker}A = \mathbf{K}_0^p(\Omega)$, then $M\mathbf{u} = \mathbf{0}$ if and only if $\langle \mathbf{u} \cdot \mathbf{n}, 1 \rangle_{\Gamma_D^\ell} = 0$ for any $1 \leq \ell \leq L_D$ which means that $\mathbf{u} = \mathbf{0}$. So by Peetre-Tartar theorem, we deduce that

$$\|\mathbf{u}\|_{\mathbf{X}^p(\Omega)} \leq C \left(\|A\mathbf{u}\|_{\mathbf{E}_2} + \|M\mathbf{u}\|_{\mathbf{K}_0^p(\Omega)} \right)$$

465 and this finishes the proof. \square

466 *Remark 3.12.* Assume that Ω is Lipschitz (resp. $\mathcal{C}^{1,1}$) and $\partial\Sigma \subset \Gamma_N$, then on the
467 space $\tilde{\mathbf{X}}_0^p(\Omega)$, the following semi-norm

$$468 \quad (3.17) \quad \mathbf{u} \mapsto \|\text{div } \mathbf{u}\|_{L^p(\Omega)} + \|\text{curl } \mathbf{u}\|_{L^p(\Omega)} + \sum_{\ell=1}^{L_N} | \langle \mathbf{u} \cdot \mathbf{n}, 1 \rangle_{\Gamma_N^\ell} |$$

469 is a norm equivalent to the norm $\|\cdot\|_{\mathbf{X}^p(\Omega)}$ (resp. $\|\cdot\|_{\mathbf{W}^{1,p}(\Omega)}$). Similarly when
470 $\partial\Sigma \subset \Gamma_D$, the following semi-norm

$$471 \quad (3.18) \quad u \mapsto \|\text{div } \mathbf{u}\|_{L^p(\Omega)} + \|\text{curl } \mathbf{u}\|_{L^p(\Omega)} + \sum_{\ell=1}^{L_N} | \langle \mathbf{u} \cdot \mathbf{n}, 1 \rangle_{\Gamma_N^\ell} | + \sum_{j=1}^J | \langle \mathbf{u} \cdot \mathbf{n}, 1 \rangle_{\Sigma_j},$$

472 is a norm equivalent to the norm $\|\cdot\|_{\mathbf{X}^p(\Omega)}$ (resp. $\|\cdot\|_{\mathbf{W}^{1,p}(\Omega)}$) on $\tilde{\mathbf{X}}_0^p(\Omega)$.

473 The following propositions concern the characterization of the kernel $\tilde{\mathbf{K}}_0^2(\Omega)$ where
474 Γ_N and Γ_D are swapped. The proofs are exactly the same as in Proposition 3.8 and
475 Proposition 3.10 respectively.

476 **PROPOSITION 3.13.** *Assume that Ω is Lipschitz and $\partial\Sigma \subset \Gamma_D$. Then the dimension*
477 *of the space $\tilde{\mathbf{K}}_0^2(\Omega)$ is equal to $L_N \times J$ and it is spanned by the functions $\widetilde{\text{grad}} s_j^\ell$,*
478 *$1 \leq j \leq J$ and $1 \leq \ell \leq L_N$ where each s_j^ℓ is the unique solution in $\mathbf{H}^1(\Omega^o)$ of the*
479 *problem*

$$480 \quad (3.19) \quad \begin{cases} -\Delta s_j^\ell = 0 & \text{in } \Omega^o, \\ \frac{\partial s_j^\ell}{\partial \mathbf{n}} = 0 & \text{on } \Gamma_D, \\ s_j^\ell|_{\Gamma_N^o} = 0 & \text{and } s_j^\ell|_{\Gamma_N^m} = \text{const}, \quad 1 \leq m \leq L_N, \\ [s_j^\ell]_k = \text{const} & \text{and } \left[\frac{\partial s_j^\ell}{\partial \mathbf{n}} \right]_k = 0, \quad 1 \leq k \leq J, \\ \left\langle \frac{\partial s_j^\ell}{\partial \mathbf{n}}, 1 \right\rangle_{\Sigma_k} = \delta_{jk}, & 1 \leq k \leq J, \\ \left\langle \frac{\partial s_j^\ell}{\partial \mathbf{n}}, 1 \right\rangle_{\Gamma_N^o} = -1 & \text{and } \left\langle \frac{\partial s_j^\ell}{\partial \mathbf{n}}, 1 \right\rangle_{\Gamma_N^m} = \delta_{\ell m}, \quad 1 \leq m \leq L_N. \end{cases}$$

481 **PROPOSITION 3.14.** *Assume that Ω is Lipschitz and $\partial\Sigma \subset \Gamma_N$. Then the dimension*
482 *of the space $\tilde{\mathbf{K}}_0^2(\Omega)$ is equal to L_N and it is spanned by the functions ∇s_ℓ ,*
483 *$1 \leq \ell \leq L_N$ where each s_ℓ is the unique solution in $H^1(\Omega)$, of the problem*

$$484 \quad (3.20) \quad \begin{cases} -\Delta s_\ell = 0 & \text{in } \Omega, \\ \frac{\partial s_\ell}{\partial \mathbf{n}} = 0 & \text{on } \Gamma_D, \\ s_\ell|_{\Gamma_N^o} = 0 & \text{and } s_\ell|_{\Gamma_N^m} = \text{const}, \quad 1 \leq m \leq L_N, \\ \left\langle \frac{\partial s_\ell}{\partial \mathbf{n}}, 1 \right\rangle_{\Gamma_N^o} = -1 & \text{and } \left\langle \frac{\partial s_\ell}{\partial \mathbf{n}}, 1 \right\rangle_{\Gamma_N^m} = \delta_{\ell m}, \quad 1 \leq m \leq L_N. \end{cases}$$

Remark 3.15. Observe that if Ω is of class $\mathcal{C}^{1,1}$, then for any $1 < p < \infty$, we have

$$\mathbf{K}_0^p(\Omega) \hookrightarrow \bigcap_{q \geq 1} \mathbf{W}^{1,q}(\Omega).$$

485 We prove this result for any $1 < p < 3$. Let $\mathbf{u} \in \mathbf{K}_0^p(\Omega)$, we know that $\mathbf{u} \in \mathbf{W}^{1,1}(\Omega) \hookrightarrow$
 486 $\mathbf{L}^{3/2}(\Omega)$. Then, $\mathbf{u} \in \mathbf{K}_0^{3/2}(\Omega)$. By using Theorem 3.2, we infer that $\mathbf{u} \in \mathbf{K}_0^{3/2}(\Omega) \hookrightarrow$
 487 $\mathbf{L}^3(\Omega)$. Now, we assume that $p \geq 3$ and due to Theorem 3.2 again, we have $\mathbf{u} \in$
 488 $\mathbf{W}^{1,3}(\Omega) \hookrightarrow \mathbf{L}^q(\Omega)$ for any $q < \infty$. Thanks to Theorem 3.2, $\mathbf{u} \in \mathbf{W}^{1,q}(\Omega)$ and then
 489 the kernel $\mathbf{K}_0^p(\Omega)$ does not depend on p .

490 Now, we state in the following lemma another preliminary result (which was
 491 proven in a different form by Mitrea, Lemma 4.1 p.144 [29]), that is necessary in the
 492 next section.

493 **LEMMA 3.16.** *Let $\boldsymbol{\varphi} \in \mathbf{H}^2(\mathbf{curl}, \Omega)$ with $\boldsymbol{\varphi} \times \mathbf{n} \in \mathbf{L}^2(\Gamma)$. Then*

$$494 \quad (3.21) \quad \operatorname{div}_\Gamma(\boldsymbol{\varphi} \times \mathbf{n}) = \mathbf{curl} \boldsymbol{\varphi} \cdot \mathbf{n} \quad \text{in } H^{-1/2}(\Gamma).$$

495 *In particular if $\boldsymbol{\varphi} \times \mathbf{n} = \mathbf{0}$ on Γ_D , we have $\mathbf{curl} \boldsymbol{\varphi} \cdot \mathbf{n} = 0$ on Γ_D .*

496 *Proof.* For any $\chi \in H^1(\Omega)$ and $\boldsymbol{\varphi} \in \mathbf{H}^2(\mathbf{curl}, \Omega)$, we have from Green's formula

$$497 \quad \int_\Omega \mathbf{curl} \boldsymbol{\varphi} \cdot \nabla \chi \, dx = \langle \mathbf{curl} \boldsymbol{\varphi} \cdot \mathbf{n}, \chi \rangle_{H^{-\frac{1}{2}}(\Gamma) \times H^{\frac{1}{2}}(\Gamma)}.$$

Let us introduce the following Hilbert space:

$$E(\Omega) = \{ \chi \in H^1(\Omega); \chi|_\Gamma \in H^1(\Gamma) \}.$$

498 For any $\chi \in E(\Omega)$ and $\boldsymbol{\varphi} \in \mathbf{H}^2(\mathbf{curl}, \Omega)$, we have the following relation

$$499 \quad (3.22) \quad \int_\Omega \mathbf{curl} \boldsymbol{\varphi} \cdot \nabla \chi \, dx = - \int_\Gamma (\boldsymbol{\varphi} \times \mathbf{n}) \cdot \nabla_\tau \chi,$$

that we prove by using the fact that (see [8])

$$\mathcal{D}(\overline{\Omega}) \quad \text{is dense in } E(\Omega).$$

500 That implies that

$$501 \quad \langle \operatorname{div}_\Gamma(\boldsymbol{\varphi} \times \mathbf{n}), \chi \rangle_{H^{-1}(\Gamma) \times H^1(\Gamma)} = \langle \mathbf{curl} \boldsymbol{\varphi} \cdot \mathbf{n}, \chi \rangle_{H^{-\frac{1}{2}}(\Gamma) \times H^{\frac{1}{2}}(\Gamma)}$$

502 and

$$503 \quad | \langle \operatorname{div}_\Gamma(\boldsymbol{\varphi} \times \mathbf{n}), \chi \rangle_{H^{-1}(\Gamma) \times H^1(\Gamma)} | \leq C(\Omega) \| \mathbf{curl} \boldsymbol{\varphi} \|_{\mathbf{L}^2(\Omega)} \| \chi \|_{H^1(\Omega)}.$$

504 Now, let $\mu \in H^1(\Gamma)$. We know that there exists $\chi \in H^1(\Omega)$ (in fact $\chi \in H^{3/2}(\Omega)$)
 505 such that $\chi = \mu$ on Γ with the estimate $\| \chi \|_{H^1(\Omega)} \leq C(\Omega) \| \mu \|_{H^{1/2}(\Gamma)}$. As $H^1(\Gamma)$ is
 506 dense in $H^{1/2}(\Gamma)$, we deduce that $\operatorname{div}_\Gamma(\boldsymbol{\varphi} \times \mathbf{n}) \in H^{-1/2}(\Gamma)$ and

$$507 \quad \operatorname{div}_\Gamma(\boldsymbol{\varphi} \times \mathbf{n}) = \mathbf{curl} \boldsymbol{\varphi} \cdot \mathbf{n} \quad \text{with} \quad \| \operatorname{div}_\Gamma(\boldsymbol{\varphi} \times \mathbf{n}) \|_{H^{-1/2}(\Gamma)} \leq C(\Omega) \| \mathbf{curl} \boldsymbol{\varphi} \|_{\mathbf{L}^2(\Omega)}. \quad \square$$

508 **4. Vector potentials.** This section presents the first main results of this paper
 509 related to the existence and uniqueness of vector potentials satisfying mixed boundary
 510 conditions in the Hilbert case and then in the L^p -theory, when $\partial\Sigma$ is included in Γ_N
 511 or in Γ_D .

512 We define the following Banach space:

$$513 \quad \tilde{\mathbf{V}}_0^p(\Omega) = \left\{ \mathbf{v} \in \tilde{\mathbf{X}}_0^p(\Omega), \operatorname{div} \mathbf{v} = 0 \text{ in } \Omega, \langle \mathbf{v} \cdot \mathbf{n}, 1 \rangle_{\Gamma_N^\ell} = 0, 1 \leq \ell \leq L_N \right\}.$$

514 **4.1. The Hilbert case** ($p = 2$). The following theorem is an extension of
 515 Theorem 3.12 of [2] when $\Gamma_D \neq \emptyset$.

516 **THEOREM 4.1.** *Assume that Ω is Lipschitz and $\partial\Sigma \subset \Gamma_N$. A function $\mathbf{u} \in \mathbf{L}^2(\Omega)$
 517 satisfies*

$$518 \quad (4.1) \quad \begin{aligned} \operatorname{div} \mathbf{u} &= 0 \quad \text{in } \Omega, & \mathbf{u} \cdot \mathbf{n} &= 0 \quad \text{on } \Gamma_D, \\ \langle \mathbf{u} \cdot \mathbf{n}, 1 \rangle_{\Gamma_N^\ell} &= 0, & 0 \leq \ell &\leq L_N, \end{aligned}$$

519 *if and only if there exists a vector potential $\boldsymbol{\psi} \in \mathbf{X}^2(\Omega)$ such that*

$$520 \quad (4.2) \quad \begin{aligned} \mathbf{u} &= \operatorname{curl} \boldsymbol{\psi} \quad \text{and} \quad \operatorname{div} \boldsymbol{\psi} = 0 \quad \text{in } \Omega, \\ \boldsymbol{\psi} \times \mathbf{n} &= \mathbf{0} \quad \text{on } \Gamma_D \quad \text{and} \quad \boldsymbol{\psi} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_N, \\ \langle \boldsymbol{\psi} \cdot \mathbf{n}, 1 \rangle_{\Gamma_D^\ell} &= 0, \quad 0 \leq \ell \leq L_D, \\ \langle \boldsymbol{\psi} \cdot \mathbf{n}, 1 \rangle_{\Sigma_j} &= 0, \quad 1 \leq j \leq J. \end{aligned}$$

521 *The function $\boldsymbol{\psi}$ is unique and satisfies the estimate*

$$522 \quad (4.3) \quad \|\boldsymbol{\psi}\|_{\mathbf{X}^2(\Omega)} \leq C \|\mathbf{u}\|_{\mathbf{L}^2(\Omega)}.$$

523 *Proof. Step 1. Uniqueness.* Clearly, the uniqueness of the function $\boldsymbol{\psi}$ will
 524 follow from the characterization of the kernel $\mathbf{K}_0^2(\Omega)$ given in Proposition 3.8. Suppose
 525 that $\boldsymbol{\psi} = \boldsymbol{\psi}_1 - \boldsymbol{\psi}_2$ where $\boldsymbol{\psi}_1$ and $\boldsymbol{\psi}_2$ satisfy (4.2), thus $\boldsymbol{\psi}$ belongs to $\mathbf{K}_0^2(\Omega)$ and from
 526 the last properties in (4.2), we deduce that $\boldsymbol{\psi} = \mathbf{0}$.

527 **Step 2. Necessary conditions.** Let us prove that (4.2) implies (4.1). It is obvious
 528 that if $\mathbf{u} = \operatorname{curl} \boldsymbol{\psi}$ then $\operatorname{div} \mathbf{u} = 0$ in Ω . Since $\boldsymbol{\psi} \times \mathbf{n} = \mathbf{0}$ on Γ_D then due to Lemma
 529 3.16, we have $\mathbf{u} \cdot \mathbf{n} = 0$ on Γ_D . For $0 \leq \ell \leq L_D$, let μ_ℓ be a function of $\mathcal{C}^\infty(\bar{\Omega})$ which is
 530 equal to 1 in the neighborhood of Γ_D^ℓ and vanishes in the neighborhood of Γ_N^m where
 531 $0 \leq m \leq L_N$ and $\ell \neq m$ and in the neighborhood of Γ_D . Proceeding as in the proof
 532 of Lemma 3.5 [2], we have

$$533 \quad \langle \mathbf{u} \cdot \mathbf{n}, 1 \rangle_{\Gamma_N^\ell} = \langle \operatorname{curl}(\mu_\ell \boldsymbol{\psi}) \cdot \mathbf{n}, 1 \rangle_{H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)} = \int_{\Omega} \operatorname{div} \operatorname{curl}(\mu_\ell \boldsymbol{\psi}) \, dx = 0.$$

Step 3. Existence. We know that there exists (see Lemma 3.5 in [2]) $\boldsymbol{\psi}_0 \in \mathbf{H}^1(\Omega)$
 such that $\mathbf{u} = \operatorname{curl} \boldsymbol{\psi}_0$ and $\operatorname{div} \boldsymbol{\psi}_0 = 0$ in Ω . Let $\chi \in H^1(\Omega)$ such that

$$\Delta \chi = 0 \quad \text{in } \Omega, \quad \chi = 0 \quad \text{on } \Gamma_D \quad \text{and} \quad \frac{\partial \chi}{\partial \mathbf{n}} = \boldsymbol{\psi}_0 \cdot \mathbf{n} \quad \text{on } \Gamma_N.$$

Setting now $\boldsymbol{\psi}_1 = \boldsymbol{\psi}_0 - \nabla \chi$, then $\operatorname{curl} \boldsymbol{\psi}_1 = \mathbf{u}$ and $\operatorname{div} \boldsymbol{\psi}_1 = 0$ in Ω with $\boldsymbol{\psi}_1 \cdot \mathbf{n} = 0$
 on Γ_N . We define the bilinear form $a(\cdot, \cdot)$ as

$$a(\boldsymbol{\xi}, \boldsymbol{\varphi}) = \int_{\Omega} \operatorname{curl} \boldsymbol{\xi} \cdot \operatorname{curl} \boldsymbol{\varphi} \, dx.$$

534 From (3.17), the bilinear form a is coercive on $\tilde{\mathbf{V}}_0^2(\Omega)$ and the following problem:

$$535 \quad (4.4) \quad \text{Find } \boldsymbol{\xi} \in \tilde{\mathbf{V}}_0^2(\Omega) \text{ such that for any } \boldsymbol{\varphi} \in \tilde{\mathbf{V}}_0^2(\Omega), \\ \int_{\Omega} \mathbf{curl} \boldsymbol{\xi} \cdot \mathbf{curl} \boldsymbol{\varphi} \, dx = \int_{\Omega} \boldsymbol{\psi}_0 \cdot \mathbf{curl} \boldsymbol{\varphi} \, dx - \int_{\Omega} \mathbf{curl} \boldsymbol{\psi}_0 \cdot \boldsymbol{\varphi} \, dx$$

536 admits a unique solution. Next, we want to extend (4.4) to any test function in $\tilde{\mathbf{X}}_0^2(\Omega)$:

$$537 \quad (4.5) \quad \text{Find } \boldsymbol{\xi} \in \tilde{\mathbf{V}}_0^2(\Omega) \text{ such that for any } \boldsymbol{\varphi} \in \tilde{\mathbf{X}}_0^2(\Omega), \\ \int_{\Omega} \mathbf{curl} \boldsymbol{\xi} \cdot \mathbf{curl} \boldsymbol{\varphi} \, dx = \int_{\Omega} \boldsymbol{\psi}_0 \cdot \mathbf{curl} \boldsymbol{\varphi} \, dx - \int_{\Omega} \mathbf{curl} \boldsymbol{\psi}_0 \cdot \boldsymbol{\varphi} \, dx.$$

538 Indeed, it is easy to check that any solution of (4.5) also solves (4.4). On the other
539 side, let $\boldsymbol{\xi} \in \tilde{\mathbf{V}}_0^2(\Omega)$ solution of (4.4) and $\boldsymbol{\varphi} \in \tilde{\mathbf{X}}_0^2(\Omega)$. Then, there exists a unique
540 $\theta \in H^1(\Omega)$ satisfying

$$541 \quad (4.6) \quad \Delta \theta = \operatorname{div} \boldsymbol{\varphi} \quad \text{in } \Omega, \quad \theta = 0 \quad \text{on } \Gamma_N \quad \text{and} \quad \frac{\partial \theta}{\partial \mathbf{n}} = 0 \quad \text{on } \Gamma_D.$$

542 We set

$$543 \quad (4.7) \quad \tilde{\boldsymbol{\varphi}} = \boldsymbol{\varphi} - \nabla \theta - \sum_{\ell=1}^{L_N} \langle (\boldsymbol{\varphi} - \nabla \theta) \cdot \mathbf{n}, 1 \rangle_{\Gamma_N^\ell} \nabla s_\ell.$$

544 Therefore $\tilde{\boldsymbol{\varphi}} \in \tilde{\mathbf{V}}_0^2(\Omega)$, and we observe then that

$$545 \quad \int_{\Omega} \mathbf{curl} \boldsymbol{\xi} \cdot \mathbf{curl} \boldsymbol{\varphi} \, dx = \int_{\Omega} \mathbf{curl} \boldsymbol{\xi} \cdot \mathbf{curl} \tilde{\boldsymbol{\varphi}} \, dx = \int_{\Omega} \boldsymbol{\psi}_0 \cdot \mathbf{curl} \tilde{\boldsymbol{\varphi}} \, dx - \int_{\Omega} \mathbf{curl} \boldsymbol{\psi}_0 \cdot \tilde{\boldsymbol{\varphi}} \, dx \\ 546 \quad \quad \quad = \int_{\Omega} \boldsymbol{\psi}_0 \cdot \mathbf{curl} \boldsymbol{\varphi} \, dx - \int_{\Omega} \mathbf{curl} \boldsymbol{\psi}_0 \cdot \boldsymbol{\varphi} \, dx,$$

547 where we observe that

$$548 \quad (4.8) \quad \int_{\Omega} \mathbf{curl} \boldsymbol{\psi}_0 \cdot \nabla \theta \, dx = \langle \mathbf{u} \cdot \mathbf{n}, \theta \rangle_{\Gamma} = 0$$

549 since $\mathbf{u} \cdot \mathbf{n} = 0$ on Γ_D and $\theta = 0$ on Γ_N and thanks to (4.1)

$$550 \quad \int_{\Omega} \mathbf{curl} \boldsymbol{\psi}_0 \cdot \nabla s_\ell \, dx = \langle \mathbf{u} \cdot \mathbf{n}, s_\ell \rangle_{\Gamma_N} = \sum_{\ell=1}^{L_N} s_\ell \langle \mathbf{u} \cdot \mathbf{n}, 1 \rangle_{\Gamma_N^\ell} = 0.$$

551 From (4.5), we deduce that $\mathbf{curl} \mathbf{curl} \boldsymbol{\xi} = \mathbf{0}$ in Ω and $(\mathbf{curl} \boldsymbol{\xi} - \boldsymbol{\psi}_0) \times \mathbf{n} = \mathbf{0}$ on Γ_D .
552 It follows that the function

$$553 \quad (4.9) \quad \boldsymbol{\psi} = \tilde{\boldsymbol{\psi}} - \sum_{\ell=1}^{L_D} \sum_{j=1}^J \left(\frac{1}{L_D} \langle \tilde{\boldsymbol{\psi}} \cdot \mathbf{n}, 1 \rangle_{\Sigma_j} + \frac{1}{J} \langle \tilde{\boldsymbol{\psi}} \cdot \mathbf{n}, 1 \rangle_{\Gamma_D^\ell} \right) \widetilde{\mathbf{grad}} q_j^\ell,$$

554 with $\tilde{\boldsymbol{\psi}} = \boldsymbol{\psi}_1 - \mathbf{curl} \boldsymbol{\xi}$ satisfies the properties (4.2) of Theorem 4.1. Finally, it is easy
555 to get the estimate (4.3). \square

556 *Remark 4.2.* If Ω is of class $\mathcal{C}^{1,1}$, the vector potential $\boldsymbol{\psi}$ belongs to $\mathbf{H}^1(\Omega)$. Indeed,
557 $\mathbf{z} = \mathbf{curl} \boldsymbol{\xi} \in \mathbf{L}^2(\Omega)$, $\operatorname{div} \mathbf{z} = 0$, $\mathbf{curl} \mathbf{z} = \mathbf{0}$ in Ω and $\mathbf{z} \times \mathbf{n} = \boldsymbol{\psi}_0 \times \mathbf{n}$ on Γ_D . Since
558 $\boldsymbol{\xi} \times \mathbf{n} = \mathbf{0}$ on Γ_N we have $\mathbf{z} \cdot \mathbf{n} = 0$ on Γ_N which implies that $\mathbf{curl} \boldsymbol{\xi}$ belongs to $\mathbf{H}^1(\Omega)$.

559 We consider the following space

$$560 \quad \widetilde{\mathbf{W}}_{\Sigma}^p(\Omega) = \left\{ \mathbf{v} \in \widetilde{\mathbf{X}}_0^p(\Omega), \operatorname{div} \mathbf{v} = 0 \text{ in } \Omega, \langle \mathbf{v} \cdot \mathbf{n}, 1 \rangle_{\Sigma_j} = 0, 1 \leq j \leq J \right. \\ 561 \quad \left. \text{and } \langle \mathbf{v} \cdot \mathbf{n}, 1 \rangle_{\Gamma_N^\ell} = 0, \quad 0 \leq \ell \leq L_N \right\}.$$

562 The following theorem, which is an extension of Theorem 3.17 of [2] when $\Gamma_N \neq \emptyset$,
563 consists on the existence and the uniqueness of a vector potential when $\partial\Sigma$ is included
564 in Γ_D .

565 **THEOREM 4.3.** *Assume that Ω is Lipschitz and $\partial\Sigma \subset \Gamma_D$. A function $\mathbf{u} \in \mathbf{L}^2(\Omega)$*
566 *satisfies*

$$567 \quad (4.10) \quad \begin{aligned} & \operatorname{div} \mathbf{u} = 0 \text{ in } \Omega, \quad \mathbf{u} \cdot \mathbf{n} = 0 \text{ on } \Gamma_D, \\ & \langle \mathbf{u} \cdot \mathbf{n}, 1 \rangle_{\Gamma_N^\ell} = 0, \quad 0 \leq \ell \leq L_N, \\ & \langle \mathbf{u} \cdot \mathbf{n}, 1 \rangle_{\Sigma_j} = 0, \quad 1 \leq j \leq J, \end{aligned}$$

568 *if and only if there exists a vector potential $\boldsymbol{\psi} \in \mathbf{X}^2(\Omega)$ such that*

$$569 \quad (4.11) \quad \begin{aligned} & \mathbf{u} = \operatorname{curl} \boldsymbol{\psi}, \quad \operatorname{div} \boldsymbol{\psi} = 0 \text{ in } \Omega, \\ & \boldsymbol{\psi} \times \mathbf{n} = \mathbf{0} \text{ on } \Gamma_D \text{ and } \boldsymbol{\psi} \cdot \mathbf{n} = 0 \text{ on } \Gamma_N, \\ & \langle \boldsymbol{\psi} \cdot \mathbf{n}, 1 \rangle_{\Gamma_D^\ell} = 0, \quad 0 \leq \ell \leq L_D. \end{aligned}$$

This function $\boldsymbol{\psi}$ is unique and it satisfies

$$\|\boldsymbol{\psi}\|_{\mathbf{X}^2(\Omega)} \leq C \|\mathbf{u}\|_{\mathbf{L}^2(\Omega)}.$$

570 *Proof. Step 1. Uniqueness.* The uniqueness of the vector potential $\boldsymbol{\psi}$ is a
571 consequence of the characterization of the kernel $\mathbf{K}_0^2(\Omega)$ given in Proposition 3.10.

572 **Step 2. Necessary conditions.** As in Step 2 of the proof of Theorem 4.1, if $\boldsymbol{\psi}$
573 satisfies (4.11), we check that $\mathbf{u} = \operatorname{curl} \boldsymbol{\psi}$ satisfies (4.10). Clearly, the fluxes over Γ_N^ℓ
574 are equal to zero and by Lemma 3.16, $\operatorname{curl} \boldsymbol{\psi} \cdot \mathbf{n} = 0$ on Γ_D . Hence $\operatorname{curl} \boldsymbol{\psi}$ satisfies the
575 assumptions of Lemma 3.7 where Γ_N is replaced by Γ_D and then $\operatorname{curl} \boldsymbol{\psi} \cdot \mathbf{n} \in [H^{\frac{1}{2}}(\Sigma_j)]'$
576 for any $1 \leq j \leq J$. Moreover, we have

$$577 \quad \forall \boldsymbol{\varphi} \in \mathcal{D}(\Omega), \quad \forall \mu \in L^2(\Sigma_j), \quad \langle \operatorname{curl} \boldsymbol{\varphi} \cdot \mathbf{n}, \mu \rangle_{\Sigma_j} = \langle \operatorname{grad} \mu \times \mathbf{n}, \boldsymbol{\varphi} \rangle_{\Sigma_j}.$$

578 By choosing $\mu = 1$, we get

$$579 \quad \langle \operatorname{curl} \boldsymbol{\varphi} \cdot \mathbf{n}, 1 \rangle_{\Sigma_j} = 0, \quad 1 \leq j \leq J,$$

580 and by the density of $\mathcal{D}(\Sigma_j)$ in $[H^{\frac{1}{2}}(\Sigma_j)]'$, this last relation holds for $\boldsymbol{\varphi} = \boldsymbol{\psi}$, which
581 proves the last equality of (4.10).

582 **Step 3. Existence.** As in Step 3 of the proof of Theorem 4.1, we set $\boldsymbol{\psi}_1 = \boldsymbol{\psi}_0 - \nabla \chi$
583 and we consider the same bilinear form a which is coercive on $\widetilde{\mathbf{W}}_{\Sigma}^2(\Omega)$ thanks to (3.18).
584 Consequently, the following problem

$$585 \quad (4.12) \quad \begin{aligned} & \text{Find } \boldsymbol{\xi} \in \widetilde{\mathbf{W}}_{\Sigma}^2(\Omega) \text{ such that for any } \boldsymbol{\varphi} \in \widetilde{\mathbf{W}}_{\Sigma}^2(\Omega), \\ & \int_{\Omega} \operatorname{curl} \boldsymbol{\xi} \cdot \operatorname{curl} \boldsymbol{\varphi} \, dx = \int_{\Omega} \boldsymbol{\psi}_0 \cdot \operatorname{curl} \boldsymbol{\varphi} \, dx - \int_{\Omega} \operatorname{curl} \boldsymbol{\psi}_0 \cdot \boldsymbol{\varphi} \, dx, \end{aligned}$$

586 admits a unique solution. We will now extend (4.12) to any test function in $\widetilde{\mathbf{X}}_0^2(\Omega)$:

$$587 \quad (4.13) \quad \text{Find } \boldsymbol{\xi} \in \widetilde{\mathbf{W}}_\Sigma^2(\Omega) \text{ such that for any } \boldsymbol{\varphi} \in \widetilde{\mathbf{X}}_0^2(\Omega), \\ \int_{\Omega} \mathbf{curl} \boldsymbol{\xi} \cdot \mathbf{curl} \boldsymbol{\varphi} \, dx = \int_{\Omega} \boldsymbol{\psi}_0 \cdot \mathbf{curl} \boldsymbol{\varphi} \, dx - \int_{\Omega} \mathbf{curl} \boldsymbol{\psi}_0 \cdot \boldsymbol{\varphi} \, dx.$$

588 Indeed, it is easy to check that any solution of (4.13) also solves (4.12). On the other
589 side, let $\boldsymbol{\xi} \in \widetilde{\mathbf{W}}_\Sigma^2(\Omega)$ solution of (4.12) and $\boldsymbol{\varphi} \in \widetilde{\mathbf{X}}_0^2(\Omega)$. Setting $\widetilde{\boldsymbol{\varphi}} = \boldsymbol{\varphi} - \nabla \theta$ with θ
590 defined in (4.6), we verify easily that the following function

$$591 \quad (4.14) \quad \widetilde{\boldsymbol{\varphi}} = \boldsymbol{\varphi} - \sum_{\ell=1}^{L_N} \sum_{j=1}^J \left(\frac{1}{L_N} \langle \widetilde{\boldsymbol{\varphi}} \cdot \mathbf{n}, 1 \rangle_{\Sigma_j} + \frac{1}{J} \langle \widetilde{\boldsymbol{\varphi}} \cdot \mathbf{n}, 1 \rangle_{\Gamma_N^\ell} \right) \widetilde{\mathbf{grad}} s_j^\ell$$

592 belongs to $\widetilde{\mathbf{W}}_\Sigma^2(\Omega)$ and as in the proof of Theorem 4.1 we have

$$593 \quad \int_{\Omega} \mathbf{curl} \boldsymbol{\xi} \cdot \mathbf{curl} \boldsymbol{\varphi} \, dx = \int_{\Omega} \boldsymbol{\psi}_0 \cdot \mathbf{curl} \boldsymbol{\varphi} \, dx - \int_{\Omega} \mathbf{curl} \boldsymbol{\psi}_0 \cdot \boldsymbol{\varphi} \, dx.$$

594 It follows from this equality that the function

$$595 \quad \boldsymbol{\psi} = \boldsymbol{\psi}_1 - \mathbf{curl} \boldsymbol{\xi} - \sum_{\ell=1}^{L_D} \langle (\boldsymbol{\psi}_1 - \mathbf{curl} \boldsymbol{\xi}) \cdot \mathbf{n}, 1 \rangle_{\Gamma_D^\ell} \nabla q_\ell$$

596 belongs to $\mathbf{X}^2(\Omega)$ and we can verify that $\boldsymbol{\psi}$ satisfies the properties (4.11) of Theorem
597 4.3. \square

598 *Remark 4.4.* As previously, if Ω is of class $\mathcal{C}^{1,1}$ then the obtained vector potential
599 belongs to $\mathbf{H}^1(\Omega)$.

600 **4.2. Other potentials.** In this subsection, we turn our attention to another
601 kind of vector potentials. Indeed, we assume that $\operatorname{div} \mathbf{u} = 0$ in Ω and we look for the
602 conditions to impose on \mathbf{u} such that $\mathbf{u} = \mathbf{curl} \boldsymbol{\psi}$ in Ω and $\boldsymbol{\psi} = \mathbf{0}$ on a part of the
603 boundary. As previously, we consider the case where $\partial\Sigma$ is included in Γ_N or in Γ_D .
604 In the next, we require the following preliminaries.

605 We define the space

$$606 \quad \mathbf{H}^2(\operatorname{div}, \Delta; \Omega) = \{ \mathbf{v} \in \mathbf{H}^2(\operatorname{div}, \Omega); \Delta(\operatorname{div} \mathbf{v}) \in L^2(\Omega) \},$$

607 endowed with the scalar product

$$608 \quad ((\mathbf{u}, \mathbf{v}))_{\mathbf{H}^2(\operatorname{div}, \Delta; \Omega)} = \int_{\Omega} \mathbf{u} \cdot \mathbf{v} \, dx + \int_{\Omega} (\operatorname{div} \mathbf{u})(\operatorname{div} \mathbf{v}) \, dx + \int_{\Omega} \Delta(\operatorname{div} \mathbf{u}) \Delta(\operatorname{div} \mathbf{v}) \, dx,$$

609 which is a Hilbert space.

LEMMA 4.5. *Assume that Ω is Lipschitz. Then*

$$\mathcal{D}(\overline{\Omega}) \text{ is dense in the space } \mathbf{H}^2(\operatorname{div}, \Delta; \Omega).$$

610 *Proof.* Let $\ell \in [\mathbf{H}^2(\operatorname{div}, \Delta; \Omega)]'$ such that for any $\mathbf{v} \in \mathcal{D}(\overline{\Omega})$, $\langle \ell, \mathbf{v} \rangle = 0$. Since
611 $\mathbf{H}^2(\operatorname{div}, \Delta; \Omega)$ is a Hilbert space, we can associate to ℓ a function \mathbf{f} in $\mathbf{H}^2(\operatorname{div}, \Delta; \Omega)$
612 such that for any $\mathbf{v} \in \mathbf{H}^2(\operatorname{div}, \Delta; \Omega)$, we have

$$613 \quad \langle \ell, \mathbf{v} \rangle = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \, dx + \int_{\Omega} (\operatorname{div} \mathbf{f})(\operatorname{div} \mathbf{v}) \, dx + \int_{\Omega} \Delta(\operatorname{div} \mathbf{f}) \Delta(\operatorname{div} \mathbf{v}) \, dx.$$

614 We set now $F = \operatorname{div} \mathbf{f}$, and $G = \Delta F$ and we denote by $\tilde{\mathbf{f}}$, \tilde{F} and \tilde{G} the extensions
 615 of \mathbf{f} , F and G respectively to \mathbb{R}^3 . Assume now that $\ell = \mathbf{0}$ in $\mathcal{D}(\overline{\Omega})$, then for any
 616 $\varphi \in \mathcal{D}(\overline{\Omega})$, we have

$$617 \quad \int_{\mathbb{R}^3} \tilde{\mathbf{f}} \cdot \varphi + \int_{\mathbb{R}^3} \tilde{F} \operatorname{div} \varphi + \int_{\mathbb{R}^3} \tilde{G} \Delta \operatorname{div} \varphi = 0,$$

618 which means that

$$619 \quad \tilde{\mathbf{f}} = \nabla(\tilde{F} + \Delta \tilde{G}) \quad \text{in } \mathbb{R}^3.$$

620 Since $\tilde{\mathbf{f}} \in \mathbf{L}^2(\mathbb{R}^3)$ and $\nabla \tilde{F} \in \mathbf{H}^{-1}(\mathbb{R}^3)$ then $\nabla(\Delta \tilde{G}) \in \mathbf{H}^{-1}(\mathbb{R}^3)$ and $\Delta \tilde{G} \in L^2(\mathbb{R}^3)$.
 621 As $\tilde{G} \in L^2(\mathbb{R}^3)$, we deduce that $\tilde{G} \in H^2(\mathbb{R}^3)$ and thus $G \in H_0^2(\Omega)$. So there exists
 622 ψ_k in $\mathcal{D}(\Omega)$ such that $\psi_k \rightarrow G$ in $H^2(\Omega)$. Furthermore, since $\Delta \tilde{G} = \widetilde{\Delta G}$, we have
 623 $F + \Delta G \in H^1(\mathbb{R}^3)$. In other words,

$$624 \quad F + \Delta G \in H_0^1(\Omega).$$

625 Then, for any \mathbf{v} in $\mathbf{H}^2(\operatorname{div}, \Delta; \Omega)$ we have

$$\begin{aligned} 626 \quad \langle \ell, \mathbf{v} \rangle &= \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \, dx + \int_{\Omega} F \operatorname{div} \mathbf{v} \, dx + \lim_{k \rightarrow \infty} \int_{\Omega} \psi_k \Delta \operatorname{div} \mathbf{v} \, dx \\ 627 \quad &= \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \, dx + \int_{\Omega} F \operatorname{div} \mathbf{v} \, dx + \lim_{k \rightarrow \infty} \int_{\Omega} (\Delta \psi_k) \operatorname{div} \mathbf{v} \, dx \\ 628 \quad &= \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \, dx + \int_{\Omega} (F + \Delta G) \operatorname{div} \mathbf{v} \, dx \\ 629 \quad &= \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \, dx - \int_{\Omega} \nabla(F + \Delta G) \cdot \mathbf{v} \, dx = 0. \end{aligned}$$

630 This ends the proof. □

631 **LEMMA 4.6.** *Let $\psi \in \mathbf{H}^2(\operatorname{div}, \Delta, \Omega)$.*

632 *i) Then, $\partial_n(\operatorname{div} \psi) \in [H_{00}^{3/2}(\Sigma_j)]'$ for any $1 \leq j \leq J$ and we have the following*
 633 *Green's formula for any $r \in H^2(\Omega^o)$ such that $r = \partial_n r = 0$ on Γ and $[\partial_n r]_k = 0$ for*
 634 *any $1 \leq k \leq J$:*

$$635 \quad (4.15) \quad \int_{\Omega^o} r \Delta(\operatorname{div} \psi) \, dx - \int_{\Omega^o} (\operatorname{div} \psi) \Delta r \, dx = \sum_{k=1}^J \langle \partial_n(\operatorname{div} \psi), [r]_k \rangle_{\Sigma_k}.$$

636 *ii) Moreover the following Green's formula holds for any $r \in H^2(\Omega^o)$ such that $\partial_n r = 0$*
 637 *on Γ and $[\partial_n r]_k = 0$ for any $1 \leq k \leq J$:*

$$\begin{aligned} 638 \quad \int_{\Omega^o} r \Delta(\operatorname{div} \psi) \, dx - \int_{\Omega^o} (\operatorname{div} \psi) \Delta r \, dx &= \langle \partial_n(\operatorname{div} \psi), r \rangle_{\Gamma} + \\ 639 \quad (4.16) & \\ 640 &+ \sum_{k=1}^J \langle \partial_n(\operatorname{div} \psi), [r]_k \rangle_{\Sigma_k}. \end{aligned}$$

641 *Proof. i)* Let $\mu \in H_{00}^{3/2}(\Sigma_j)$, then there exists $\varphi \in H^2(\Omega^o)$ such that

$$642 \quad [\varphi]_k = \mu \delta_{jk}, \quad [\partial_n \varphi]_k = 0 \quad \text{for all } k = 1, \dots, J \quad \text{and} \quad \varphi = \partial_n \varphi = 0 \quad \text{on } \Gamma.$$

643 Furthermore, it satisfies

$$644 \quad \|\varphi\|_{H^2(\Omega^o)} \leq C\|\mu\|_{H_{00}^{3/2}(\Sigma_j)}.$$

645 Let $\boldsymbol{\psi} \in \mathcal{D}(\overline{\Omega})$. Then, the Green's formula gives

$$646 \quad (4.17) \quad \int_{\Omega^o} \varphi \Delta(\operatorname{div} \boldsymbol{\psi}) \, dx - \int_{\Omega^o} (\operatorname{div} \boldsymbol{\psi}) \Delta \varphi \, dx = \langle \partial_n(\operatorname{div} \boldsymbol{\psi}), \mu \rangle_{\Sigma_j}.$$

647 Therefore

$$648 \quad |\langle \partial_n(\operatorname{div} \boldsymbol{\psi}), \mu \rangle_{\Sigma_j}| \leq C\|\mu\|_{H_{00}^{3/2}(\Sigma_j)}\|\boldsymbol{\psi}\|_{\mathbf{H}^2(\operatorname{div}, \Delta; \Omega)},$$

649 which proves that $\partial_n(\operatorname{div} \boldsymbol{\psi})|_{\Sigma_j} \in [H_{00}^{3/2}(\Sigma_j)]'$ and

$$650 \quad \|\partial_n(\operatorname{div} \boldsymbol{\psi})\|_{[H_{00}^{3/2}(\Sigma_j)]'} \leq C\|\boldsymbol{\psi}\|_{\mathbf{H}^2(\operatorname{div}, \Delta; \Omega)}.$$

651 We deduce from the density of $\mathcal{D}(\overline{\Omega})$ in $\mathbf{H}^2(\operatorname{div}, \Delta; \Omega)$, that the last inequality holds
652 for any $\boldsymbol{\psi}$ in $\mathbf{H}^2(\operatorname{div}, \Delta; \Omega)$ and we get the formula (4.17). Finally, by an adequate
653 partition of unity, we obtain the required formula (4.15).

654 **ii)** As a consequence, using the density of $\mathcal{D}(\overline{\Omega})$ in $\mathbf{H}^2(\operatorname{div}, \Delta; \Omega)$, we deduce now the
655 following Green's formula: for any $r \in H^2(\Omega^o)$ such that $\partial_n r = 0$ on Γ and $[\partial_n r]_k = 0$
656 for any $1 \leq k \leq J$ and for any $\boldsymbol{\psi} \in \mathbf{H}^2(\operatorname{div}, \Delta; \Omega)$:

$$657 \quad \int_{\Omega^o} r \Delta(\operatorname{div} \boldsymbol{\psi}) \, dx - \int_{\Omega^o} (\operatorname{div} \boldsymbol{\psi}) \Delta r \, dx = \langle \partial_n(\operatorname{div} \boldsymbol{\psi}), r \rangle_{\Gamma} +$$

$$658 \quad (4.18) \quad + \sum_{k=1}^J \langle \partial_n(\operatorname{div} \boldsymbol{\psi}), [r]_k \rangle_{\Sigma_k}.$$

660 Observe that the regularity $\mathcal{C}^{1,1}$ of the domain Ω implies that

$$661 \quad \partial_n(\operatorname{div} \boldsymbol{\psi}) \in H^{-3/2}(\Gamma).$$

662 This finishes the proof. □

663 Let us define the kernel

$$664 \quad \mathbf{B}_0^p(\Omega) = \left\{ \begin{array}{l} \mathbf{w} \in \mathbf{W}^{1,p}(\Omega); \operatorname{div}(\Delta \mathbf{w}) = 0, \operatorname{curl} \mathbf{w} = \mathbf{0} \text{ in } \Omega, \\ \mathbf{w} = \mathbf{0} \text{ on } \Gamma_D, \mathbf{w} \cdot \mathbf{n} = 0 \text{ and } \partial_n(\operatorname{div} \mathbf{w}) = 0 \text{ on } \Gamma_N \end{array} \right\},$$

665 and the space $\Theta^2(\Omega^o)$ by

$$666 \quad \Theta^2(\Omega^o) = \left\{ \begin{array}{l} r \in H^2(\Omega^o); \quad r|_{\Gamma_D^o} = 0, \quad r|_{\Gamma_D^m} = \operatorname{const}, \quad 1 \leq m \leq L_D \\ [r]_j = \operatorname{const}, \quad [\partial_n r]_j = 0, \quad 1 \leq j \leq J, \quad \frac{\partial r}{\partial \mathbf{n}} = 0 \quad \text{on } \Gamma \end{array} \right\}.$$

Remark 4.7. Suppose that

$$r \in H^2(\Omega^o); \quad [r]_j = \operatorname{const} \quad \text{and} \quad [\partial_n r]_j = 0 \quad \text{for any } 1 \leq j \leq J.$$

667 Since for any $1 \leq j \leq J$, we have $[\nabla r \times \mathbf{n}]_j = 0$ then $[\nabla r]_j = 0$, which means that
668 $\widetilde{\operatorname{grad}} r \in \mathbf{H}^1(\Omega)$.

669 The next proposition states the characterization of the kernel $\mathbf{B}_0^2(\Omega)$ when $\partial\Sigma$ is
670 included in Γ_N .

671 PROPOSITION 4.8. *If $\partial\Sigma \subset \Gamma_N$, the dimension of the space $\mathbf{B}_0^2(\Omega)$ is equal to*
672 *$L_D \times J$ and it is spanned by the functions $\widetilde{\mathbf{grad}} \chi_j^\ell$, $1 \leq j \leq J$ and $1 \leq \ell \leq L_D$, where*
673 *each χ_j^ℓ is the unique solution in $H^2(\Omega^\circ)$ of the problem*

$$674 \quad (4.19) \quad \begin{cases} \Delta^2 \chi_j^\ell = 0 & \text{in } \Omega^\circ, \\ \frac{\partial \chi_j^\ell}{\partial \mathbf{n}} = 0 & \text{on } \Gamma, \quad \partial_{\mathbf{n}}(\Delta \chi_j^\ell) = 0 & \text{on } \Gamma_N \\ \chi_j^\ell|_{\Gamma_D^0} = 0, \quad \chi_j^\ell|_{\Gamma_D^m} = \text{const}, & 1 \leq m \leq L_D, \\ [\chi_j^\ell]_k = \text{const}, \quad [\partial_{\mathbf{n}} \chi_j^\ell]_k = [\Delta \chi_j^\ell]_k = [\partial_{\mathbf{n}}(\Delta \chi_j^\ell)]_k = 0, & 1 \leq k \leq J, \\ \langle \partial_{\mathbf{n}}(\Delta \chi_j^\ell), 1 \rangle_{\Sigma_k} = \delta_{jk}, & 1 \leq k \leq J, \\ \langle \partial_{\mathbf{n}}(\Delta \chi_j^\ell), 1 \rangle_{\Gamma_D^0} = -1 & \text{and } \langle \partial_{\mathbf{n}}(\Delta \chi_j^\ell), 1 \rangle_{\Gamma_D^m} = \delta_{\ell m}, & 1 \leq m \leq L_D. \end{cases}$$

675 ii) *Moreover if Ω is of class $\mathcal{C}^{2,1}$, then $\widetilde{\mathbf{grad}} \chi_j^\ell \in \mathbf{H}^2(\Omega)$.*

676 *Proof. Step 1.* Note that $\Theta^2(\Omega^\circ)$ is a closed subspace of $H^2(\Omega^\circ)$. Then from
677 Lax Milgram theorem the problem

$$678 \quad (4.20) \quad \begin{aligned} & \text{Find } \chi_j^\ell \in \Theta^2(\Omega^\circ) \text{ such that} \\ & \forall r \in \Theta^2(\Omega^\circ), \quad \int_{\Omega^\circ} \Delta \chi_j^\ell \Delta r \, dx = -[r]_j - r|_{\Gamma_D^0} \end{aligned}$$

679 admits a unique solution. Moreover, for any $r \in \mathcal{D}(\Omega)$, we have

$$680 \quad \left\langle \text{div} \Delta(\widetilde{\mathbf{grad}} \chi_j^\ell), r \right\rangle = - \int_{\Omega} \text{div}(\widetilde{\mathbf{grad}} \chi_j^\ell) \Delta r \, dx = - \int_{\Omega^\circ} \Delta \chi_j^\ell \Delta r \, dx = 0,$$

681 in other words $\text{div} \Delta(\widetilde{\mathbf{grad}} \chi_j^\ell) = 0$ in Ω and thus $\Delta^2 \chi_j^\ell = 0$ in Ω° .

682 **Step 2.** It remains to show the properties concerning the jumps of $\Delta \chi_j^\ell$ and $\partial_{\mathbf{n}}(\Delta \chi_j^\ell)$
683 over Σ_j and those concerning the fluxes. Taking $r \in H_0^2(\Omega)$, then

$$684 \quad 0 = \int_{\Omega^\circ} \Delta \chi_j^\ell \Delta r \, dx = - \sum_{k=1}^J \left\langle [\partial_{\mathbf{n}}(\Delta \chi_j^\ell)]_{\Sigma_k}, r \right\rangle_k + \sum_{k=1}^J \left\langle [\Delta \chi_j^\ell]_k, \partial_{\mathbf{n}} r \right\rangle_{\Sigma_k}.$$

685 Consequently

$$686 \quad [\partial_{\mathbf{n}}(\Delta \chi_j^\ell)]_k = [\Delta \chi_j^\ell]_k = 0, \quad 1 \leq k \leq J.$$

687 Taking now $r \in H^2(\Omega)$ with $r = 0$ on Γ_D and $\partial_{\mathbf{n}} r = 0$ on Γ and Green's formula
688 leads to

$$689 \quad 0 = \int_{\Omega^\circ} \Delta \chi_j^\ell \Delta r \, dx = - \langle \partial_{\mathbf{n}} \Delta \chi_j^\ell, r \rangle_{\Gamma_N},$$

690 i.e. $\partial_{\mathbf{n}} \Delta \chi_j^\ell = 0$ on Γ_N .

691 Choosing now $r \in H^2(\Omega) \cap \Theta^2(\Omega^\circ)$, we deduce that

$$692 \quad (4.21) \quad \sum_{m=1}^{L_D} r|_{\Gamma_D^m} \langle \partial_{\mathbf{n}} \Delta \chi_j^\ell, 1 \rangle_{\Gamma_D^m} = r|_{\Gamma_D^0}$$

693 and then

$$694 \quad (4.22) \quad \langle \partial_{\mathbf{n}} \Delta \chi_j^\ell, 1 \rangle_{\Gamma_D^m} = \delta_{m\ell}, \quad 1 \leq m \leq L_D.$$

695 Since $\operatorname{div} \Delta(\widetilde{\mathbf{grad}} \chi_j^\ell) = 0$ in Ω then $\langle \partial_{\mathbf{n}} \Delta \chi_j^\ell, 1 \rangle_{\Gamma_D^0} = -1$. Observe now that for any
696 $r \in \Theta^2(\Omega^\circ)$, we have from Lemma 4.6

$$697 \quad \sum_{m=1}^{L_D} r|_{\Gamma_D^m} \langle \partial_{\mathbf{n}} \Delta \chi_j^\ell, 1 \rangle_{\Gamma_D^m} + \sum_{k=1}^J [r]_k \langle \partial_{\mathbf{n}} \Delta \chi_j^\ell, 1 \rangle_{\Sigma_k} = r|_{\Gamma_D^0} + [r]_j.$$

698 Then due to (4.22), we have

$$699 \quad \sum_{k=1}^J [r]_k \langle \partial_{\mathbf{n}} \Delta \chi_j^\ell, 1 \rangle_{\Sigma_k} = [r]_j.$$

700 We finally infer that $\langle \partial_{\mathbf{n}} \Delta \chi_j^\ell, 1 \rangle_{\Sigma_k} = \delta_{jk}$.

701 **Step 3.** It is obvious that any solution of (4.19) also solves (4.20).

702

703 **Step 4.** It is readily checked that the functions $\widetilde{\mathbf{grad}} \chi_j^\ell$ are linearly independent
704 for any $1 \leq j \leq J$ and $1 \leq \ell \leq L_D$. To prove that they span $\mathbf{B}_0^2(\Omega)$, we consider
705 $\mathbf{w} \in \mathbf{B}_0^2(\Omega)$ and the function

$$706 \quad \mathbf{u} = \mathbf{w} - \sum_{\ell=1}^{L_D} \sum_{j=1}^J \left(\frac{1}{L_D} \langle \partial_{\mathbf{n}}(\operatorname{div} \mathbf{w}), 1 \rangle_{\Sigma_j} + \frac{1}{J} \langle \partial_{\mathbf{n}}(\operatorname{div} \mathbf{w}), 1 \rangle_{\Gamma_D^\ell} \right) \widetilde{\mathbf{grad}} \chi_j^\ell$$

707 remains in $\mathbf{B}_0^2(\Omega)$ and satisfies $\langle \partial_{\mathbf{n}}(\operatorname{div} \mathbf{u}), 1 \rangle_{\Gamma_D^m} = 0$ for any $1 \leq m \leq L_D$ and
708 $\langle \partial_{\mathbf{n}}(\operatorname{div} \mathbf{u}), 1 \rangle_{\Sigma_j} = 0$ for any $1 \leq j \leq J$.

709 As $\operatorname{curl} \mathbf{u} = \mathbf{0}$ in Ω° , there exists a function $q \in H^2(\Omega^\circ)$ such that $\mathbf{u} = \nabla q$ in
710 Ω° , with $\Delta^2 q = 0$ in Ω° since $\Delta(\operatorname{div} \mathbf{u}) = 0$ in Ω . Since $\mathbf{u} \in \mathbf{H}^1(\Omega)$ with $\mathbf{u} = \mathbf{0}$ on
711 Γ_D and $\mathbf{u} \cdot \mathbf{n} = 0$ on Γ_N , we deduce that $\partial_{\mathbf{n}} q = 0$ on Γ and $q = \operatorname{const}$ on Γ_D^ℓ for
712 any $0 \leq \ell \leq L_D$. Moreover, we can take the constant equal to zero on Γ_D^0 . Now, we
713 choose the extension of ∇q denoted $\widetilde{\mathbf{grad}} q$ such that $\widetilde{\mathbf{grad}} q = \mathbf{u}$ in Ω . As $\operatorname{curl} \mathbf{u} = \mathbf{0}$
714 in Ω then $\operatorname{curl} \widetilde{\mathbf{grad}} q = 0$ and thus the jump of q is zero almost everywhere across
715 each cut Σ_j (see Lemma 3.11 [2]), which means that $q \in H^1(\Omega)$ and $\mathbf{u} = \widetilde{\mathbf{grad}} q = \nabla q$
716 in Ω . As \mathbf{u} belongs to $\mathbf{H}^1(\Omega)$, we infer that $q \in H^2(\Omega)$ and that $\Delta^2 q = 0$ in Ω due
717 to the fact that $\Delta(\operatorname{div} \mathbf{u}) = 0$ in Ω . Since $\partial_{\mathbf{n}} q = 0$ on Γ , $\partial_{\mathbf{n}}(\Delta q) = 0$ on Γ_N and
718 $q = \operatorname{const}$ on Γ_D^ℓ for any $0 \leq \ell \leq L_D$, we have by using the Green formula

$$719 \quad 0 = \int_{\Omega} |\Delta q|^2 dx - \langle \Delta q, \partial_{\mathbf{n}} q \rangle_{H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)} + \langle \partial_{\mathbf{n}} \Delta q, q \rangle_{H^{-3/2}(\Gamma) \times H^{3/2}(\Gamma)}$$

$$720 \quad = \int_{\Omega} |\Delta q|^2 dx + \sum_{m=1}^{L_D} q|_{\Gamma_D^m} \langle \partial_{\mathbf{n}} \Delta q, 1 \rangle_{\Gamma_D^m}.$$

721 As $\langle \partial_{\mathbf{n}}(\operatorname{div} \mathbf{u}), 1 \rangle_{\Gamma_D^m} = \langle \partial_{\mathbf{n}} \Delta q, 1 \rangle_{\Gamma_D^m} = 0$ for any $1 \leq m \leq L_D$, we deduce that $\Delta q = 0$
722 in Ω which means that q is constant because $\partial_{\mathbf{n}} q = 0$ on Γ and consequently \mathbf{u} is
723 equal to zero.

724 To finish the proof, the point ii) is an immediate consequence of Corollary 3.5. \square

725 THEOREM 4.9. Assume that $\partial\Sigma \subset \Gamma_N$, a function $\mathbf{u} \in \mathbf{L}^2(\Omega)$ satisfies

$$726 \quad (4.23) \quad \begin{aligned} \operatorname{div} \mathbf{u} &= 0 \quad \text{in } \Omega, \quad \mathbf{u} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_D, \\ \langle \mathbf{u} \cdot \mathbf{n}, 1 \rangle_{\Gamma_N^\ell} &= 0, \quad 0 \leq \ell \leq L_N, \end{aligned}$$

727 if and only if there exists a vector potential $\boldsymbol{\psi} \in \mathbf{H}^1(\Omega)$ such that

$$728 \quad (4.24) \quad \begin{aligned} \mathbf{u} &= \operatorname{curl} \boldsymbol{\psi} \quad \text{and} \quad \operatorname{div}(\Delta \boldsymbol{\psi}) = 0 \quad \text{in } \Omega, \\ \boldsymbol{\psi} &= \mathbf{0} \quad \text{on } \Gamma_D \quad \text{and} \quad \boldsymbol{\psi} \cdot \mathbf{n} = \partial_{\mathbf{n}}(\operatorname{div} \boldsymbol{\psi}) = 0 \quad \text{on } \Gamma_N, \\ \langle \partial_{\mathbf{n}}(\operatorname{div} \boldsymbol{\psi}), 1 \rangle_{\Gamma_D^\ell} &= 0, \quad 0 \leq \ell \leq L_D, \\ \langle \partial_{\mathbf{n}}(\operatorname{div} \boldsymbol{\psi}), 1 \rangle_{\Sigma_j} &= 0, \quad 1 \leq j \leq J. \end{aligned}$$

729 This function $\boldsymbol{\psi}$ is unique.

730 *Remark 4.10.* Since $\boldsymbol{\psi} \in \mathbf{H}^1(\Omega)$ and $\operatorname{div}(\Delta \boldsymbol{\psi}) = 0$ in Ω , then by using Lemma
731 4.6, the quantities $\langle \partial_{\mathbf{n}}(\operatorname{div} \boldsymbol{\psi}), 1 \rangle_{\Gamma_D^\ell}$ and $\langle \partial_{\mathbf{n}}(\operatorname{div} \boldsymbol{\psi}), 1 \rangle_{\Sigma_j}$ make sense.

732 *Proof.* The uniqueness is deduced from the characterization of the kernel $\mathbf{B}_0^2(\Omega)$
733 and the necessary conditions are proved in the same way as in the proof of Theorem
734 4.1.

735 Let us consider a function $\mathbf{u} \in \mathbf{L}^2(\Omega)$ satisfying (4.23) to which we associate the vector
736 potential $\boldsymbol{\psi}$ defined in Theorem 4.1 that we will denote hereinafter by $\tilde{\boldsymbol{\psi}}$. We consider
737 now the following problem

$$738 \quad \begin{cases} \Delta^2 \lambda = 0 \quad \text{in } \Omega, \\ \lambda = 0 \quad \text{on } \Gamma_D \quad \text{and} \quad \partial_{\mathbf{n}}(\Delta \lambda) = 0 \quad \text{on } \Gamma_N, \\ \frac{\partial \lambda}{\partial \mathbf{n}} = \tilde{\boldsymbol{\psi}} \cdot \mathbf{n} \quad \text{on } \Gamma. \end{cases}$$

739 This problem admits a solution in $H^2(\Omega)$ since $\tilde{\boldsymbol{\psi}} \cdot \mathbf{n} \in H^{\frac{1}{2}}(\Gamma)$ and the following
740 function

$$741 \quad \boldsymbol{\psi} = \tilde{\boldsymbol{\psi}} - \nabla \lambda + \sum_{\ell=1}^{L_D} \sum_{j=1}^J \left(\frac{1}{L_D} \langle \partial_{\mathbf{n}}(\Delta \chi_\ell), 1 \rangle_{\Sigma_j} + \frac{1}{J} \langle \partial_{\mathbf{n}}(\Delta \chi_\ell), 1 \rangle_{\Gamma_D^\ell} \right) \widetilde{\operatorname{grad}} \chi_j^\ell$$

742 satisfies the properties (4.24) of Theorem 4.9. \square

743 We define the space $\Theta^2(\Omega)$ by

$$744 \quad \Theta^2(\Omega) = \left\{ r \in H^2(\Omega); r|_{\Gamma_D^0} = 0, r|_{\Gamma_D^m} = \operatorname{const}, 1 \leq m \leq L_D, \frac{\partial r}{\partial \mathbf{n}} = 0 \quad \text{on } \Gamma \right\}.$$

745 Let us consider in the next proposition the dimension of the kernel $\mathbf{B}_0^2(\Omega)$ in the case
746 where $\partial\Sigma$ is included in Γ_D .

747 PROPOSITION 4.11. If $\partial\Sigma \subset \Gamma_D$, the dimension of the space $\mathbf{B}_0^2(\Omega)$ is equal to
748 L_D and it is spanned by the function $\nabla \chi_\ell$, $1 \leq \ell \leq L_D$ where each χ_ℓ is the unique
749 solution in $H^2(\Omega)$, of the problem

$$750 \quad (4.25) \quad \begin{cases} \Delta^2 \chi_\ell = 0 \quad \text{in } \Omega, \\ \frac{\partial \chi_\ell}{\partial \mathbf{n}} = 0 \quad \text{on } \Gamma, \quad \partial_{\mathbf{n}}(\Delta \chi_\ell) = 0 \quad \text{on } \Gamma_N, \\ \chi_\ell|_{\Gamma_D^0} = 0 \quad \text{and} \quad \chi_\ell|_{\Gamma_D^m} = \operatorname{const}, \quad 1 \leq m \leq L_D, \\ \langle \partial_{\mathbf{n}}(\Delta \chi_\ell), 1 \rangle_{\Gamma_D^0} = -1 \quad \text{and} \quad \langle \partial_{\mathbf{n}}(\Delta \chi_\ell), 1 \rangle_{\Gamma_D^m} = \delta_{\ell m}, \quad 1 \leq m \leq L_D. \end{cases}$$

751 *Proof.* We look for $\chi_\ell \in \Theta^2(\Omega)$ such that

$$752 \quad (4.26) \quad \forall r \in \Theta^2(\Omega), \quad \int_{\Omega} \Delta \chi_\ell \Delta r \, dx = -r|_{\Gamma_D^\ell}.$$

753 This problem admits a unique solution because the form

$$754 \quad a(\chi_\ell, r) = \int_{\Omega} \Delta \chi_\ell \Delta r \, dx$$

755 is coercive on $\Theta^2(\Omega)$ according to the fact that $\|r\|_{H^2(\Omega)} \leq C\|\Delta r\|_{L^2(\Omega)}$ when $\partial_n r = 0$
756 on Γ . Moreover, due to the density of $\mathcal{D}(\bar{\Omega})$ in the space of functions which belong to
757 $H^2(\Omega)$ and their bi-laplacian operator belongs to $L^2(\Omega)$, we can prove the following
758 Green's formula, for any χ_ℓ and r in $\Theta^2(\Omega)$ such that $\Delta^2 \chi_\ell \in L^2(\Omega)$

$$759 \quad \int_{\Omega} (\Delta^2 \chi_\ell) r \, dx = \int_{\Omega} \Delta \chi_\ell \Delta r \, dx + \sum_{\ell=1}^{L_D} r|_{\Gamma_D^\ell} \langle \partial_n(\Delta \chi_\ell), 1 \rangle_{\Gamma_D^\ell}.$$

760 It is readily checked that if $\chi_\ell \in \Theta^2(\Omega)$ satisfies (4.26) then χ_ℓ is solution of (4.25).
761 By taking $r \in \Theta^2(\Omega)$ and by using Green's formula and the fact that $\Delta^2 \chi_\ell = 0$ in Ω ,
762 we deduce that

$$763 \quad \langle \partial_n \Delta \chi_\ell, r \rangle_{\Gamma} = -r|_{\Gamma_D^\ell},$$

764 Hence, $\partial_n \Delta \chi_\ell = 0$ on Γ_N .

765 Furthermore, the functions $\nabla \chi_\ell$ are linearly independent for any $1 \leq \ell \leq L_D$.
766 One has to prove that they span $\mathbf{B}_0^2(\Omega)$. Let $\mathbf{w} \in \mathbf{B}_0^2(\Omega)$ and consider the function

$$767 \quad \mathbf{u} = \mathbf{w} - \sum_{\ell=1}^{L_D} \langle \partial_n(\operatorname{div} \mathbf{w}), 1 \rangle_{\Gamma_D^\ell} \nabla \chi_\ell.$$

768 which remains in $\mathbf{B}_0^2(\Omega)$ and satisfies $\langle \partial_n(\operatorname{div} \mathbf{u}), 1 \rangle_{\Gamma_D^m} = 0$ for any $1 \leq m \leq L_D$.
769 We follow the same approach as in the fourth step of the proof of Proposition 4.8 to
770 show that $\mathbf{u} = \mathbf{0}$ in Ω . Indeed, there exists a function $q \in H^2(\Omega^\circ)$ such that $\mathbf{u} = \nabla q$
771 in Ω° due to the fact that $\operatorname{curl} \mathbf{u} = \mathbf{0}$ in Ω and thus in Ω° . The remainder of the
772 proof is exactly the same because $\Delta^2 q = 0$ in Ω . \square

773 The following theorem is an extension of Theorem 3.20 of [2] when $\Gamma_N \neq \emptyset$.

774 **THEOREM 4.12.** *If $\partial\Sigma \subset \Gamma_D$, a function $\mathbf{u} \in \mathbf{L}^2(\Omega)$ satisfies*

$$775 \quad (4.27) \quad \begin{aligned} & \operatorname{div} \mathbf{u} = 0 \quad \text{in } \Omega, \quad \mathbf{u} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_D, \\ & \langle \mathbf{u} \cdot \mathbf{n}, 1 \rangle_{\Gamma_N^\ell} = 0, \quad 0 \leq \ell \leq L_N, \\ & \langle \mathbf{u} \cdot \mathbf{n}, 1 \rangle_{\Sigma_j} = 0, \quad 1 \leq j \leq J, \end{aligned}$$

776 *if and only if there exists a vector potential $\boldsymbol{\psi} \in \mathbf{H}^1(\Omega)$ such that*

$$777 \quad (4.28) \quad \begin{aligned} & \mathbf{u} = \operatorname{curl} \boldsymbol{\psi}, \quad \operatorname{div}(\Delta \boldsymbol{\psi}) = 0 \quad \text{in } \Omega, \\ & \boldsymbol{\psi} = \mathbf{0} \quad \text{on } \Gamma_D \quad \text{and} \quad \boldsymbol{\psi} \cdot \mathbf{n} = \partial_n(\operatorname{div} \boldsymbol{\psi}) = 0 \quad \text{on } \Gamma_N, \\ & \langle \partial_n(\operatorname{div} \boldsymbol{\psi}), 1 \rangle_{\Gamma_D^\ell} = 0, \quad 0 \leq \ell \leq L_D. \end{aligned}$$

778 *This function $\boldsymbol{\psi}$ is unique.*

779 *Proof.* The uniqueness of the vector potential is deduced from the characterization
 780 of the kernel $\mathbf{B}_0^2(\Omega)$ and the necessary conditions are proved in the same way as in the
 781 proof of Theorem 4.3. Note that a function \mathbf{u} satisfies (4.27) if and only if there exists
 782 a unique vector potential $\boldsymbol{\psi}$ defined in Theorem 4.3 that we will denote hereinafter
 783 by $\overline{\boldsymbol{\psi}}$. We consider now the following problem

$$784 \quad \begin{cases} \Delta^2 \lambda = 0 & \text{in } \Omega, \\ \lambda = 0 & \text{on } \Gamma_D \text{ and } \partial_n(\Delta \lambda) = 0 & \text{on } \Gamma_N, \\ \frac{\partial \lambda}{\partial \mathbf{n}} = \overline{\boldsymbol{\psi}} \cdot \mathbf{n} & \text{on } \Gamma. \end{cases}$$

785 This problem admits a solution in $H^2(\Omega)$ and the following function

$$786 \quad \boldsymbol{\psi} = \overline{\boldsymbol{\psi}} - \nabla \lambda + \sum_{\ell=1}^{L_D} \langle \partial_n(\Delta \chi), 1 \rangle_{\Gamma_D^\ell} \nabla \chi_\ell$$

787 satisfies the properties (4.28) of Theorem 4.12. \square

788 The next result is an extension of Theorem 3.20 in [2] when $\Gamma_D \neq \emptyset$. We skip the
 789 proof in this paper.

790 **THEOREM 4.13.** *If $\partial\Sigma \subset \Gamma_N$, a function $\mathbf{u} \in \mathbf{L}^2(\Omega)$ satisfies*

$$791 \quad (4.29) \quad \begin{aligned} \operatorname{div} \mathbf{u} &= 0 & \text{in } \Omega, & \quad \mathbf{u} \cdot \mathbf{n} = 0 & \text{on } \Gamma_D \cup \Gamma_N, \\ \langle \mathbf{u} \cdot \mathbf{n}, 1 \rangle_{\Sigma_j} &= 0, & 0 \leq j \leq J, \end{aligned}$$

792 *if and only if there exists a vector potential $\boldsymbol{\psi} \in \mathbf{H}^1(\Omega)$ such that*

$$793 \quad (4.30) \quad \begin{aligned} \mathbf{u} &= \operatorname{curl} \boldsymbol{\psi} & \text{and } \operatorname{div}(\Delta \boldsymbol{\psi}) &= 0 & \text{in } \Omega, \\ \boldsymbol{\psi} \times \mathbf{n} &= \mathbf{0} & \text{on } \Gamma_D & \text{and } \boldsymbol{\psi} = 0 & \text{on } \Gamma_N, \\ \langle \partial_n(\operatorname{div} \boldsymbol{\psi}), 1 \rangle_{\Gamma_D^\ell} &= 0, & 0 \leq \ell \leq L_D, \\ \langle \partial_n(\operatorname{div} \boldsymbol{\psi}), 1 \rangle_{\Gamma_N^\ell} &= 0, & 0 \leq \ell \leq L_N. \end{aligned}$$

794 **4.3. L^p -theory.** In this subsection, we investigate the L^p -theory of the vector
 795 potentials obtained in Theorems 4.1 and 4.3 for any $1 < p < \infty$. The general case
 796 $p \neq 2$ is not as easy as the case $p = 2$ and requires extra work. The following theorems
 797 are about the case where $p > 2$ which is a straightforward consequence of Theorems
 798 4.1 and 4.3.

THEOREM 4.14. *If $\partial\Sigma$ is included in Γ_N and $\mathbf{u} \in \mathbf{L}^p(\Omega)$ with $p > 2$ satisfies (4.1),
 then the vector potential $\boldsymbol{\psi}$ given in Theorem 4.1 belongs to $\mathbf{W}^{1,p}(\Omega)$ and satisfies the
 following estimate*

$$\|\boldsymbol{\psi}\|_{\mathbf{W}^{1,p}(\Omega)} \leq C \|\mathbf{u}\|_{\mathbf{L}^p(\Omega)}.$$

799 *Proof.* The proof of this theorem is immediately deduced from Theorem 4.1 and
 800 Theorem 3.2. \square

801 In the same way, we generalize the results of Theorem 4.3 for any $p > 2$

THEOREM 4.15. *If $\partial\Sigma$ is included in Γ_D and $\mathbf{u} \in \mathbf{L}^p(\Omega)$ with $p > 2$ satisfies
 (4.10), then the vector potential $\boldsymbol{\psi}$ given in Theorem 4.3 belongs to $\mathbf{W}^{1,p}(\Omega)$ and
 satisfies the following estimate*

$$\|\boldsymbol{\psi}\|_{\mathbf{W}^{1,p}(\Omega)} \leq C \|\mathbf{u}\|_{\mathbf{L}^p(\Omega)}.$$

802 We will later on see how to extend the previous results to the case $p < 2$ in
 803 Theorems 4.18 and 4.21. The major task consists on proving two Inf-Sup conditions.

804 LEMMA 4.16. *If $\partial\Sigma \subset \Gamma_N$, there exists a constant $\beta > 0$ depending only on Ω*
 805 *and p , such that the following Inf-Sup condition holds*

$$806 \quad (4.31) \quad \inf_{\substack{\varphi \in \tilde{\mathbf{V}}_0^{p'}(\Omega) \\ \varphi \neq 0}} \sup_{\substack{\xi \in \tilde{\mathbf{V}}_0^p(\Omega) \\ \xi \neq 0}} \frac{|\int_{\Omega} \mathbf{curl} \xi \cdot \mathbf{curl} \varphi|}{\|\xi\|_{\mathbf{W}^{1,p}(\Omega)} \|\varphi\|_{\mathbf{W}^{1,p'}(\Omega)}} \geq \beta.$$

Proof. We use here the following Helmholtz decomposition: every $\mathbf{g} \in \mathbf{L}^p(\Omega)$ can be decomposed as $\mathbf{g} = \nabla\chi + \mathbf{z}$ where $\mathbf{z} \in \mathbf{L}^p(\Omega)$ with $\operatorname{div} \mathbf{z} = 0$ and χ belongs to $W^{1,p}(\Omega)$ with $\chi = 0$ on Γ_D and $(\nabla\chi - \mathbf{g}) \cdot \mathbf{n}$ on Γ_N . Furthermore, it satisfies the estimate

$$\|\nabla\chi\|_{\mathbf{L}^p(\Omega)} + \|\mathbf{z}\|_{\mathbf{L}^p(\Omega)} \leq C\|\mathbf{g}\|_{\mathbf{L}^p(\Omega)}.$$

807 Let φ be a function of $\tilde{\mathbf{V}}_0^{p'}(\Omega)$. From (3.17) of Remark 3.12, we deduce that

$$808 \quad \|\varphi\|_{\mathbf{W}^{1,p'}(\Omega)} \leq C\|\mathbf{curl} \varphi\|_{\mathbf{L}^{p'}(\Omega)} = C \sup_{\substack{\mathbf{g} \in \mathbf{L}^p(\Omega) \\ \mathbf{g} \neq 0}} \frac{|\int_{\Omega} \mathbf{curl} \varphi \cdot \mathbf{g}|}{\|\mathbf{g}\|_{\mathbf{L}^p(\Omega)}}.$$

809 We set

$$810 \quad \tilde{\mathbf{z}} = \mathbf{z} - \sum_{\ell=1}^{L_D} \sum_{j=1}^J \left(\frac{1}{L_D} \langle \mathbf{z} \cdot \mathbf{n}, 1 \rangle_{\Sigma_j} + \frac{1}{J} \langle \mathbf{z} \cdot \mathbf{n}, 1 \rangle_{\Gamma_D^\ell} \right) \widetilde{\mathbf{grad}} q_j^\ell.$$

811 Thus $\tilde{\mathbf{z}} \in \mathbf{L}^p(\Omega)$, $\operatorname{div} \tilde{\mathbf{z}} = 0$ in Ω , and satisfies $\tilde{\mathbf{z}} \cdot \mathbf{n} = 0$ on Γ_N , $\langle \tilde{\mathbf{z}} \cdot \mathbf{n}, 1 \rangle_{\Gamma_D^m} = 0$, for
 812 any $1 \leq m \leq L_D$ and $\langle \tilde{\mathbf{z}} \cdot \mathbf{n}, 1 \rangle_{\Sigma_j} = 0$, for any $1 \leq j \leq J$. Due to Theorem 4.15 where
 813 Γ_D and Γ_N are switched (see Theorem 4.3), there exists a vector potential $\boldsymbol{\psi} \in \tilde{\mathbf{V}}_0^p(\Omega)$
 814 with $p \geq 2$ such that $\tilde{\mathbf{z}} = \mathbf{curl} \boldsymbol{\psi}$ and satisfying (4.11) where Γ_D and Γ_N are switched.
 815 This implies that

$$816 \quad \int_{\Omega} \mathbf{curl} \varphi \cdot \mathbf{g} \, dx = \int_{\Omega} \mathbf{curl} \varphi \cdot \mathbf{z} \, dx = \int_{\Omega} \mathbf{curl} \varphi \cdot \tilde{\mathbf{z}} \, dx,$$

817 because $\int_{\Omega} \mathbf{curl} \varphi \cdot \nabla\chi \, dx = \int_{\Omega} \mathbf{curl} \varphi \cdot \widetilde{\mathbf{grad}} q_j^\ell \, dx = 0$. Furthermore, we have

$$818 \quad \|\tilde{\mathbf{z}}\|_{\mathbf{L}^p(\Omega)} \leq \|\mathbf{z}\|_{\mathbf{L}^p(\Omega)} + \sum_{\ell=1}^{L_D} \sum_{j=1}^J \left| \frac{1}{L_D} \langle \mathbf{z} \cdot \mathbf{n}, 1 \rangle_{\Gamma_D^\ell} + \frac{1}{J} \langle \mathbf{z} \cdot \mathbf{n}, 1 \rangle_{\Sigma_j} \right| \|\widetilde{\mathbf{grad}} q_j^\ell\|_{\mathbf{L}^p(\Omega)}$$

$$819 \quad \leq \|\mathbf{z}\|_{\mathbf{L}^p(\Omega)} + C \left(\sum_{\ell=1}^{L_D} |\langle \mathbf{z} \cdot \mathbf{n}, 1 \rangle_{\Gamma_D^\ell}| + \sum_{j=1}^J |\langle \mathbf{z} \cdot \mathbf{n}, 1 \rangle_{\Sigma_j}| \right)$$

$$820 \quad \leq C\|\mathbf{z}\|_{\mathbf{L}^p(\Omega)} \leq C\|\mathbf{g}\|_{\mathbf{L}^p(\Omega)}.$$

821 We can write now

$$822 \quad \frac{|\int_{\Omega} \mathbf{curl} \varphi \cdot \mathbf{g}|}{\|\mathbf{g}\|_{\mathbf{L}^p(\Omega)}} \leq C \frac{|\int_{\Omega} \mathbf{curl} \varphi \cdot \tilde{\mathbf{z}}|}{\|\tilde{\mathbf{z}}\|_{\mathbf{L}^p(\Omega)}} = C \frac{|\int_{\Omega} \mathbf{curl} \varphi \cdot \mathbf{curl} \boldsymbol{\psi}|}{\|\mathbf{curl} \boldsymbol{\psi}\|_{\mathbf{L}^p(\Omega)}}.$$

823 But from (3.17) of Remark 3.12, we have that $\|\boldsymbol{\psi}\|_{\mathbf{W}^{1,p}(\Omega)} \simeq \|\mathbf{curl} \boldsymbol{\psi}\|_{\mathbf{L}^p(\Omega)}$. Finally

$$824 \quad \frac{|\int_{\Omega} \mathbf{curl} \varphi \cdot \mathbf{g}|}{\|\mathbf{g}\|_{\mathbf{L}^p(\Omega)}} \leq C \frac{|\int_{\Omega} \mathbf{curl} \varphi \cdot \mathbf{curl} \boldsymbol{\psi}|}{\|\boldsymbol{\psi}\|_{\mathbf{W}^{1,p}(\Omega)}}.$$

825 Therefore, we obtain the required Inf-Sup condition for $p \geq 2$. By a symmetry
826 argument, it holds also for $p < 2$. \square

827 **4.4. First elliptic problem with mixed boundary conditions.** The role of
828 the first Inf-Sup condition (4.31) is illustrated in the next proposition as it is used to
829 solve the first elliptic problem.

830 PROPOSITION 4.17. *Assume that $\partial\Sigma \subset \Gamma_N$ and \mathbf{v} belongs to $\mathbf{L}^p(\Omega)$. Then the*
831 *following problem*

$$832 \quad (4.32) \quad \begin{cases} -\Delta \boldsymbol{\xi} = \mathbf{curl} \mathbf{v} & \text{and } \operatorname{div} \boldsymbol{\xi} = 0 & \text{in } \Omega, \\ \boldsymbol{\xi} \cdot \mathbf{n} = 0, & (\mathbf{curl} \boldsymbol{\xi} - \mathbf{v}) \times \mathbf{n} = \mathbf{0} & \text{on } \Gamma_D & \text{and } \boldsymbol{\xi} \times \mathbf{n} = \mathbf{0} & \text{on } \Gamma_N, \\ \langle \boldsymbol{\xi} \cdot \mathbf{n}, 1 \rangle_{\Gamma_N^\ell} = 0, & 1 \leq \ell \leq L_N, \end{cases}$$

833 *has a unique solution in $\mathbf{W}^{1,p}(\Omega)$ and satisfies*

$$834 \quad (4.33) \quad \|\boldsymbol{\xi}\|_{\mathbf{W}^{1,p}(\Omega)} \leq C \|\mathbf{v}\|_{\mathbf{L}^p(\Omega)}.$$

835 *Proof. i)* We consider the following problem:

$$836 \quad (4.34) \quad \begin{aligned} & \text{Find } \boldsymbol{\xi} \in \tilde{\mathbf{V}}_0^p(\Omega) \quad \text{such that for any } \boldsymbol{\varphi} \in \tilde{\mathbf{V}}_0^{p'}(\Omega), \\ & \int_{\Omega} \mathbf{curl} \boldsymbol{\xi} \cdot \mathbf{curl} \boldsymbol{\varphi} \, dx = \int_{\Omega} \mathbf{v} \cdot \mathbf{curl} \boldsymbol{\varphi} \, dx. \end{aligned}$$

Using the Inf-Sup condition (4.31), Problem (4.34) admits a unique solution $\boldsymbol{\xi} \in \tilde{\mathbf{V}}_0^p(\Omega) \hookrightarrow \mathbf{W}^{1,p}(\Omega)$. Next, we want to extend (4.34) to any test function in $\tilde{\mathbf{X}}_0^{p'}(\Omega)$. Let $\boldsymbol{\varphi} \in \tilde{\mathbf{X}}_0^{p'}(\Omega)$ and $\chi \in W^{1,p}(\Omega)$ be the unique solution of the following mixed problem

$$\Delta \chi = \operatorname{div} \boldsymbol{\varphi} \quad \text{in } \Omega, \quad \chi = 0 \quad \text{on } \Gamma_N \quad \text{and} \quad \frac{\partial \chi}{\partial \mathbf{n}} = 0 \quad \text{on } \Gamma_D.$$

837 We set

$$838 \quad (4.35) \quad \tilde{\boldsymbol{\varphi}} = \boldsymbol{\varphi} - \nabla \chi - \sum_{\ell=1}^{L_N} \langle (\boldsymbol{\varphi} - \nabla \chi) \cdot \mathbf{n}, 1 \rangle_{\Gamma_N^\ell} \nabla s_\ell.$$

839 Note that $\tilde{\boldsymbol{\varphi}}$ belongs to $\tilde{\mathbf{V}}_0^{p'}(\Omega)$ and $\mathbf{curl} \tilde{\boldsymbol{\varphi}} = \mathbf{curl} \boldsymbol{\varphi}$, so Problem (4.34) is equivalent
840 to

$$841 \quad (4.36) \quad \begin{aligned} & \text{Find } \boldsymbol{\xi} \in \tilde{\mathbf{V}}_0^p(\Omega) \quad \text{such that for any } \boldsymbol{\varphi} \in \tilde{\mathbf{X}}_0^{p'}(\Omega), \\ & \int_{\Omega} \mathbf{curl} \boldsymbol{\xi} \cdot \mathbf{curl} \boldsymbol{\varphi} \, dx = \int_{\Omega} \mathbf{v} \cdot \mathbf{curl} \boldsymbol{\varphi} \, dx. \end{aligned}$$

842 **ii)** Now, we will give the interpretation of Problem (4.36). More precisely, we will
843 prove that Problem (4.36) is equivalent to find $\boldsymbol{\xi} \in \mathbf{W}^{1,p}(\Omega)$ solution of (4.32). By
844 choosing $\boldsymbol{\varphi} \in \mathcal{D}(\Omega)$, we deduce that $-\Delta \boldsymbol{\xi} = \mathbf{curl} \mathbf{v}$ in Ω . Moreover, because $\boldsymbol{\xi} \in$
845 $\tilde{\mathbf{V}}_0^p(\Omega)$ then $\operatorname{div} \boldsymbol{\xi} = 0$ in Ω and it satisfies $\boldsymbol{\xi} \times \mathbf{n} = \mathbf{0}$ on Γ_N , $\boldsymbol{\xi} \cdot \mathbf{n} = 0$ on Γ_D ,
846 $\langle \boldsymbol{\xi} \cdot \mathbf{n}, 1 \rangle_{\Gamma_N^\ell} = 0$ for any $1 \leq \ell \leq L_N$. The last point to prove is that $(\mathbf{curl} \boldsymbol{\xi} - \mathbf{v}) \times \mathbf{n} = \mathbf{0}$
847 on Γ_D . The function $\mathbf{z} = \mathbf{curl} \boldsymbol{\xi} - \mathbf{v}$ belongs to $\tilde{\mathbf{X}}^p(\Omega)$ and $\mathbf{curl} \mathbf{z} = \mathbf{0}$ in Ω . Therefore,
848 for any $\boldsymbol{\varphi} \in \tilde{\mathbf{X}}_0^{p'}(\Omega)$ we have

$$849 \quad \int_{\Omega} \mathbf{z} \cdot \mathbf{curl} \boldsymbol{\varphi} \, dx - \langle \mathbf{z} \times \mathbf{n}, \boldsymbol{\varphi} \rangle_{\mathbf{W}^{-1/p,p}(\Gamma) \times \mathbf{W}^{1/p,p'}(\Gamma)} = \int_{\Omega} \mathbf{curl} \mathbf{z} \cdot \boldsymbol{\varphi} \, dx = 0.$$

850 Using (4.36), we deduce that

$$851 \quad \forall \varphi \in \tilde{\mathbf{X}}_0^{p'}(\Omega), \quad \langle \mathbf{z} \times \mathbf{n}, \varphi \rangle_{\mathbf{W}^{-1/p,p}(\Gamma_D) \times \mathbf{W}^{1/p,p'}(\Gamma_D)} = 0$$

852 Let $\boldsymbol{\mu}$ any element of $\mathbf{W}^{1-1/p',p'}(\Gamma_D)$. So, there exists φ of $\mathbf{W}^{1,p'}(\Omega)$ such that
 853 $\varphi = \boldsymbol{\mu}_\tau$ on Γ_D and $\varphi = \mathbf{0}$ on Γ_N . It is obvious that φ belongs to $\tilde{\mathbf{X}}_0^p(\Omega)$ and it
 854 satisfies

$$855 \quad \langle \mathbf{z} \times \mathbf{n}, \boldsymbol{\mu} \rangle_{\Gamma_D} = \langle \mathbf{z} \times \mathbf{n}, \boldsymbol{\mu}_\tau \rangle_{\Gamma_D} = \langle \mathbf{z} \times \mathbf{n}, \varphi \rangle_{\Gamma_D} = 0.$$

856 This implies that $\mathbf{z} \times \mathbf{n} = \mathbf{0}$ on Γ_D , which is the required property.

857 **iii)** Let $\mathbf{B} \in \mathcal{L}(\tilde{\mathbf{V}}_0^p(\Omega), (\tilde{\mathbf{V}}_0^{p'}(\Omega))')$ be the following operator:

$$858 \quad \forall \boldsymbol{\psi} \in \tilde{\mathbf{V}}_0^p(\Omega), \quad \forall \varphi \in \tilde{\mathbf{V}}_0^{p'}(\Omega), \quad \langle \mathbf{B}\boldsymbol{\psi}, \varphi \rangle = \int_{\Omega} \mathbf{curl} \boldsymbol{\psi} \cdot \mathbf{curl} \varphi \, dx.$$

859 Thanks to (4.31), the operator \mathbf{B} is an isomorphism from $\tilde{\mathbf{V}}_0^p(\Omega)$ into $(\tilde{\mathbf{V}}_0^{p'}(\Omega))'$ and

$$860 \quad \|\boldsymbol{\psi}\|_{\tilde{\mathbf{X}}_0^p(\Omega)} \simeq \|\mathbf{B}\boldsymbol{\psi}\|_{(\tilde{\mathbf{V}}_0^{p'}(\Omega))'}.$$

861 Hence, since $\boldsymbol{\xi}$ is solution of Problem (4.32), we have

$$862 \quad \|\mathbf{B}\boldsymbol{\xi}\|_{(\tilde{\mathbf{V}}_0^{p'}(\Omega))'} = \sup_{\substack{\varphi \in \tilde{\mathbf{V}}_0^{p'}(\Omega) \\ \varphi \neq 0}} \frac{|\langle \mathbf{B}\boldsymbol{\xi}, \varphi \rangle|}{\|\varphi\|_{\tilde{\mathbf{X}}_0^p(\Omega)}} = \sup_{\substack{\varphi \in \tilde{\mathbf{V}}_0^{p'}(\Omega) \\ \varphi \neq 0}} \frac{\left| \int_{\Omega} \mathbf{v} \cdot \mathbf{curl} \varphi \, dx \right|}{\|\varphi\|_{\tilde{\mathbf{X}}_0^p(\Omega)}}$$

863 Therefore

$$864 \quad \|\mathbf{B}\boldsymbol{\xi}\|_{(\tilde{\mathbf{V}}_0^{p'}(\Omega))'} \leq \|\mathbf{v}\|_{\mathbf{L}^p(\Omega)}$$

865 Thus the estimate (4.33) holds. \square

866 We are now in position to extend Theorem 4.14 to the case $1 < p < 2$. In fact,
 867 the proof of the following theorem is given for any $1 < p < \infty$.

868 **THEOREM 4.18.** *Suppose that $\partial\Sigma$ is included in Γ_N and $\mathbf{u} \in \mathbf{L}^p(\Omega)$ satisfies (4.1)
 869 with $1 < p < \infty$. Then there exists a unique vector potential $\boldsymbol{\psi} \in \mathbf{W}^{1,p}(\Omega)$ satisfying
 870 (4.2) with the estimate*

$$871 \quad (4.37) \quad \|\boldsymbol{\psi}\|_{\mathbf{W}^{1,p}(\Omega)} \leq C \|\mathbf{u}\|_{\mathbf{L}^p(\Omega)}.$$

872 *Proof. Step 1. Uniqueness.* Let $\boldsymbol{\psi}_1$ and $\boldsymbol{\psi}_2$ be two vector potentials and
 873 $\boldsymbol{\psi} = \boldsymbol{\psi}_1 - \boldsymbol{\psi}_2$. Then $\boldsymbol{\psi}$ belongs to $\mathbf{K}_0^p(\Omega)$ and $\langle \boldsymbol{\psi} \cdot \mathbf{n}, 1 \rangle_{\Gamma_D^m} = 0$ for any $1 \leq m \leq L_D$
 874 and $\langle \boldsymbol{\psi} \cdot \mathbf{n}, 1 \rangle_{\Sigma_j} = 0$ for any $1 \leq j \leq J$. Hence, from the characterization of the kernel
 875 $\mathbf{K}_0^p(\Omega)$ we deduce that $\boldsymbol{\psi} = \mathbf{0}$.

Step 2. Existence. Let $\boldsymbol{\psi}_0 \in \mathbf{W}^{1,p}(\Omega)$ such that $\mathbf{u} = \mathbf{curl} \boldsymbol{\psi}_0$ and $\operatorname{div} \boldsymbol{\psi}_0 = 0$ in Ω
 (see Lemma 4.1 of [5]). Let $\chi \in W^{1,p}(\Omega)$ such that

$$\Delta \chi = 0 \quad \text{in } \Omega, \quad \chi = 0 \quad \text{on } \Gamma_D \quad \text{and} \quad \frac{\partial \chi}{\partial \mathbf{n}} = \boldsymbol{\psi}_0 \cdot \mathbf{n} \quad \text{on } \Gamma_N,$$

with

$$\|\chi\|_{W^{1,p}(\Omega)} \leq C \|\psi_0 \cdot \mathbf{n}\|_{W^{-1/p,p}(\Gamma_N)} \leq C \|\psi_0\|_{\mathbf{L}^p(\Omega)}.$$

876 Setting now $\psi_1 = \psi_0 - \nabla\chi$, then $\mathbf{curl} \psi_1 = \mathbf{u}$ and $\operatorname{div} \psi_1 = 0$ in Ω with $\psi_1 \cdot \mathbf{n} = 0$
877 on Γ_N . Due to the Inf-Sup condition (4.31), the following problem

$$878 \quad (4.38) \quad \text{Find } \xi \in \widetilde{\mathbf{V}}_0^p(\Omega) \text{ such that for any } \varphi \in \widetilde{\mathbf{V}}_0^{p'}(\Omega), \\ \int_{\Omega} \mathbf{curl} \xi \cdot \mathbf{curl} \varphi \, dx = \int_{\Omega} \psi_0 \cdot \mathbf{curl} \varphi \, dx - \int_{\Omega} \mathbf{curl} \psi_0 \cdot \varphi \, dx,$$

879 admits a unique solution in $\widetilde{\mathbf{V}}_0^p(\Omega)$ and this solution belongs to $\mathbf{W}^{1,p}(\Omega)$. As previ-
880 ously in the proof of Theorem 4.1, Problem (4.38) is equivalent to

$$881 \quad (4.39) \quad \text{Find } \xi \in \widetilde{\mathbf{V}}_0^p(\Omega) \text{ such that for any } \varphi \in \widetilde{\mathbf{X}}_0^{p'}(\Omega), \\ \int_{\Omega} \mathbf{curl} \xi \cdot \mathbf{curl} \varphi \, dx = \int_{\Omega} \psi_0 \cdot \mathbf{curl} \varphi \, dx - \int_{\Omega} \mathbf{curl} \psi_0 \cdot \varphi \, dx.$$

882 The rest of the proof is similar to that of Theorem 4.1. The required vector poten-
883 tial ψ given by (4.9) belongs to $\mathbf{W}^{1,p}(\Omega)$ since $\mathbf{curl} \xi$, ψ_1 and $\mathbf{grad} q_j^{\ell} \in \mathbf{W}^{1,p}(\Omega)$.
884 Furthermore, it satisfies the estimate (4.37). \square

885 In the case where $\partial\Sigma \subset \Gamma_D$, we also need to establish an Inf-Sup condition in
886 order to solve the second elliptic problem.

887 LEMMA 4.19. *If $\partial\Sigma \subset \Gamma_D$, there exists a constant $\beta > 0$ depending only on Ω*
888 *and p , such that the following Inf-Sup condition holds*

$$889 \quad (4.40) \quad \inf_{\substack{\varphi \in \widetilde{\mathbf{W}}_{\Sigma}^{p'}(\Omega) \\ \varphi \neq 0}} \sup_{\substack{\xi \in \widetilde{\mathbf{W}}_{\Sigma}^p(\Omega) \\ \xi \neq 0}} \frac{\left| \int_{\Omega} \mathbf{curl} \xi \cdot \mathbf{curl} \varphi \right|}{\|\xi\|_{\mathbf{W}^{1,p}(\Omega)} \|\varphi\|_{\mathbf{W}^{1,p'}(\Omega)}} \geq \beta.$$

890 *Proof.* We use here the same Helmholtz decomposition as in the proof of Lemma
891 4.16. Let φ be a function of $\widetilde{\mathbf{W}}_{\Sigma}^{p'}(\Omega)$. From (3.18) of Remark 3.12, we deduce that

$$892 \quad \|\varphi\|_{\mathbf{W}^{1,p'}(\Omega)} \leq C \|\mathbf{curl} \varphi\|_{\mathbf{L}^{p'}(\Omega)} = C \sup_{\substack{\mathbf{g} \in \mathbf{L}^p(\Omega) \\ \mathbf{g} \neq 0}} \frac{\left| \int_{\Omega} \mathbf{curl} \varphi \cdot \mathbf{g} \right|}{\|\mathbf{g}\|_{\mathbf{L}^p(\Omega)}}.$$

893 We set

$$894 \quad \tilde{z} = z - \sum_{\ell=1}^{L_D} \langle z \cdot \mathbf{n}, 1 \rangle_{\Gamma_D^{\ell}} \nabla q_{\ell}.$$

895 Thus $\tilde{z} \in \mathbf{L}^p(\Omega)$, $\operatorname{div} \tilde{z} = 0$ in Ω , and satisfies $\tilde{z} \cdot \mathbf{n} = 0$ on Γ_N and $\langle \tilde{z} \cdot \mathbf{n}, 1 \rangle_{\Gamma_D^m} = 0$,
896 for any $1 \leq m \leq L_D$. Due to Theorem 4.18 when Γ_D and Γ_N are switched, there
897 exists a vector potential $\psi \in \widetilde{\mathbf{W}}_{\Sigma}^p(\Omega)$ such that $\tilde{z} = \mathbf{curl} \psi$. This implies that

$$898 \quad \int_{\Omega} \mathbf{curl} \varphi \cdot \mathbf{g} \, dx = \int_{\Omega} \mathbf{curl} \varphi \cdot z \, dx = \int_{\Omega} \mathbf{curl} \varphi \cdot \tilde{z} \, dx,$$

899 because $\int_{\Omega} \mathbf{curl} \varphi \cdot \nabla\chi \, dx = \int_{\Omega} \mathbf{curl} \varphi \cdot \nabla q_{\ell} \, dx = 0$. The rest of the proof is similar
900 to that of Lemma 4.16. \square

901 **4.5. Second elliptic problem with mixed boundary conditions.**

902 PROPOSITION 4.20. *Assume that $\partial\Sigma \subset \Gamma_D$ and \mathbf{v} belongs to $\mathbf{L}^p(\Omega)$. Then the*
 903 *following problem*

$$904 \quad (4.41) \quad \begin{cases} -\Delta \boldsymbol{\xi} = \mathbf{curl} \mathbf{v} & \text{and } \operatorname{div} \boldsymbol{\xi} = 0 & \text{in } \Omega, \\ \boldsymbol{\xi} \cdot \mathbf{n} = 0, \quad (\mathbf{curl} \boldsymbol{\xi} - \mathbf{v}) \times \mathbf{n} = \mathbf{0} & \text{on } \Gamma_D & \text{and } \boldsymbol{\xi} \times \mathbf{n} = \mathbf{0} & \text{on } \Gamma_N, \\ \langle \boldsymbol{\xi} \cdot \mathbf{n}, 1 \rangle_{\Gamma_N^\ell} = 0, \quad 1 \leq \ell \leq L_N & \text{and } \langle \boldsymbol{\xi} \cdot \mathbf{n}, 1 \rangle_{\Sigma_j} = 0, \quad 1 \leq j \leq J, \end{cases}$$

905 *has a unique solution in $\mathbf{W}^{1,p}(\Omega)$ and satisfies*

$$906 \quad (4.42) \quad \|\boldsymbol{\xi}\|_{\mathbf{W}^{1,p}(\Omega)} \leq C \|\mathbf{v}\|_{\mathbf{L}^p(\Omega)}.$$

907 *Proof. i)* We consider the following problem:

$$908 \quad (4.43) \quad \begin{aligned} &\text{Find } \boldsymbol{\xi} \in \widetilde{\mathbf{W}}_\Sigma^p(\Omega) \text{ such that for any } \varphi \in \widetilde{\mathbf{W}}_\Sigma^{p'}(\Omega), \\ &\int_\Omega \mathbf{curl} \boldsymbol{\xi} \cdot \mathbf{curl} \varphi \, dx = \int_\Omega \mathbf{v} \cdot \mathbf{curl} \varphi \, dx. \end{aligned}$$

909 Using the Inf-Sup condition (4.40), Problem (4.43) admits a unique solution $\boldsymbol{\xi} \in$
 910 $\widetilde{\mathbf{W}}_\Sigma^p(\Omega) \hookrightarrow \mathbf{W}^{1,p}(\Omega)$. As in Theorem 4.3, we show that Problem (4.43) is equivalent
 911 to the following one

$$912 \quad (4.44) \quad \begin{aligned} &\text{Find } \boldsymbol{\xi} \in \widetilde{\mathbf{W}}_\Sigma^p(\Omega) \text{ such that for any } \varphi \in \widetilde{\mathbf{X}}_0^{p'}(\Omega), \\ &\int_\Omega \mathbf{curl} \boldsymbol{\xi} \cdot \mathbf{curl} \varphi \, dx = \int_\Omega \mathbf{v} \cdot \mathbf{curl} \varphi \, dx. \end{aligned}$$

913 **ii)** By taking $\varphi \in \mathbf{D}(\Omega)$, we deduce that $-\Delta \boldsymbol{\xi} = \mathbf{curl} \mathbf{v}$. It is clear that since
 914 $\boldsymbol{\xi} \in \widetilde{\mathbf{W}}_\Sigma^p(\Omega)$ then $\operatorname{div} \boldsymbol{\xi} = 0$ in Ω and it satisfies $\boldsymbol{\xi} \times \mathbf{n} = \mathbf{0}$ on Γ_N , $\boldsymbol{\xi} \cdot \mathbf{n} = 0$ on Γ_D ,
 915 $\langle \boldsymbol{\xi} \cdot \mathbf{n}, 1 \rangle_{\Gamma_N^\ell} = 0$ for any $1 \leq \ell \leq L_N$ and $\langle \boldsymbol{\xi} \cdot \mathbf{n}, 1 \rangle_{\Sigma_j} = 0$ for any $1 \leq j \leq J$. To prove
 916 that $(\mathbf{curl} \boldsymbol{\xi} - \mathbf{v}) \times \mathbf{n} = \mathbf{0}$ on Γ_D and that the estimate (4.42) holds, we use the same
 917 argument as in Proposition 4.17. \square

918 By using the existence and uniqueness result of the second elliptic problem with
 919 mixed boundary conditions, we prove the existence and uniqueness of the following
 920 vector potential for any $1 < p < \infty$

921 THEOREM 4.21. *Suppose that $\partial\Sigma$ is included in Γ_D and $\mathbf{u} \in \mathbf{L}^p(\Omega)$ satisfies*
 922 *(4.10). Then there exists a unique vector potential $\boldsymbol{\psi} \in \mathbf{W}^{1,p}(\Omega)$ satisfying (4.11)*
 923 *and the estimates*

$$924 \quad (4.45) \quad \|\boldsymbol{\psi}\|_{\mathbf{W}^{1,p}(\Omega)} \leq C \|\mathbf{u}\|_{\mathbf{L}^p(\Omega)}.$$

925 *Proof. Step 1. Uniqueness.* It is based on the characterization of the kernel
 926 $\mathbf{K}_0^p(\Omega)$ when $\partial\Sigma \subset \Gamma_D$.

927 **Step 2. Existence.** Setting again $\boldsymbol{\psi}_1 = \boldsymbol{\psi}_0 - \nabla \chi$ with the same $\boldsymbol{\psi}_0$ and χ as in the
 928 proof of Theorem 4.1. Due to the Inf-Sup condition (4.40), the following problem

$$929 \quad (4.46) \quad \begin{aligned} &\text{Find } \boldsymbol{\xi} \in \widetilde{\mathbf{W}}_\Sigma^p(\Omega) \text{ such that for any } \varphi \in \widetilde{\mathbf{W}}_\Sigma^{p'}(\Omega), \\ &\int_\Omega \mathbf{curl} \boldsymbol{\xi} \cdot \mathbf{curl} \varphi \, dx = \int_\Omega \boldsymbol{\psi}_0 \cdot \mathbf{curl} \varphi \, dx - \int_\Omega \mathbf{curl} \boldsymbol{\psi}_0 \cdot \varphi \, dx, \end{aligned}$$

930 admits a unique solution in $\widetilde{\mathbf{W}}_{\Sigma}^p(\Omega)$ and this solution belongs to $\mathbf{W}^{1,p}(\Omega)$. Next, as
 931 previously Problem (4.46) is equivalent to the following one

$$932 \quad (4.47) \quad \text{Find } \boldsymbol{\xi} \in \widetilde{\mathbf{W}}_{\Sigma}^p(\Omega) \text{ such that for any } \boldsymbol{\varphi} \in \widetilde{\mathbf{X}}_0^{p'}(\Omega), \\ \int_{\Omega} \mathbf{curl} \boldsymbol{\xi} \cdot \mathbf{curl} \boldsymbol{\varphi} \, dx = \int_{\Omega} \boldsymbol{\psi}_0 \cdot \mathbf{curl} \boldsymbol{\varphi} \, dx - \int_{\Omega} \mathbf{curl} \boldsymbol{\psi}_0 \cdot \boldsymbol{\varphi} \, dx.$$

933 Finally, the potential we take is given by

$$934 \quad \boldsymbol{\psi} = \boldsymbol{\psi}_1 - \mathbf{curl} \boldsymbol{\xi} - \sum_{\ell=1}^{L_D} \langle (\boldsymbol{\psi}_1 - \mathbf{curl} \boldsymbol{\xi}) \cdot \mathbf{n}, 1 \rangle_{\Gamma_D^{\ell}} \nabla q_{\ell},$$

935 and it satisfies the properties (4.11) together with the estimate (4.45). \square

936 *Remark 4.22.* As we managed to generalize the first vector potentials for any
 937 $1 < p < \infty$, we can handle the L^p theory of the less standard ones mentioned in
 938 Theorems 4.9 and 4.12. We omit the proofs in this paper.

939 *Remark 4.23.* In some particular geometries, one part of $\partial\Sigma$ may be included in
 940 Γ_D and the other part in Γ_N , the existence and uniqueness of vector potentials is still
 941 an open question in this case.

942 **5. Stokes problem.** We consider the Stokes problem subjected to Navier-type
 943 boundary condition on some part of the boundary and a pressure boundary condition
 944 on the other part. Assume that $\partial\Sigma \subset \Gamma_D$

$$945 \quad (S) \quad \begin{cases} -\Delta \mathbf{u} + \nabla \pi = \mathbf{f}, & \text{div } \mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} \times \mathbf{n} = \mathbf{0}, & \pi = \pi_0 & \text{on } \Gamma_N, \\ \mathbf{u} \cdot \mathbf{n} = 0, & \mathbf{curl} \mathbf{u} \times \mathbf{n} = \mathbf{h} \times \mathbf{n} & \text{on } \Gamma_D, \\ \langle \mathbf{u} \cdot \mathbf{n}, 1 \rangle_{\Gamma_N^{\ell}} = 0, & 1 \leq \ell \leq L_N, & \langle \mathbf{u} \cdot \mathbf{n}, 1 \rangle_{\Sigma_j} = 0, \quad 1 \leq j \leq J, \end{cases}$$

946 where \mathbf{f} , \mathbf{h} , π_0 are given functions or distributions. Our aim is to prove the existence
 947 and uniqueness of weak solutions of the system (S). To achieve this result, we solve
 948 the following auxiliary problem where $\partial\Sigma \subset \Gamma_D$:

$$949 \quad (S_1) \quad \begin{cases} -\Delta \boldsymbol{\xi} = \mathbf{f}, & \text{div } \boldsymbol{\xi} = 0 & \text{in } \Omega, \\ \boldsymbol{\xi} \times \mathbf{n} = \mathbf{0} & \text{on } \Gamma_N, \\ \boldsymbol{\xi} \cdot \mathbf{n} = 0, & \mathbf{curl} \boldsymbol{\xi} \times \mathbf{n} = \mathbf{h} \times \mathbf{n} & \text{on } \Gamma_D, \\ \langle \boldsymbol{\xi} \cdot \mathbf{n}, 1 \rangle_{\Gamma_N^{\ell}} = 0, & 1 \leq \ell \leq L_N, & \langle \boldsymbol{\xi} \cdot \mathbf{n}, 1 \rangle_{\Sigma_j} = 0, \quad 1 \leq j \leq J. \end{cases}$$

950 We define $r(p)$ by

$$951 \quad r(p) = \begin{cases} 1/p + \frac{1}{3} & \text{if } p > \frac{3}{2} \\ 1 - \varepsilon & \text{if } p = \frac{3}{2} \\ 1 & \text{if } 1 \leq p < \frac{3}{2}. \end{cases}$$

952 **PROPOSITION 5.1.** *Assume that $\partial\Sigma \subset \Gamma_D$. Let $\mathbf{f} \in \mathbf{L}^{r(p)}(\Omega)$, $\text{div } \mathbf{f} = 0$, $\mathbf{h} \times \mathbf{n} \in$
 953 $\mathbf{W}^{-1/p,p}(\Gamma_D)$ satisfying the following compatibility conditions for any $\boldsymbol{\varphi} \in \widetilde{\mathbf{K}}_0^{p'}(\Omega)$:*

$$954 \quad (5.1) \quad \int_{\Omega} \mathbf{f} \cdot \boldsymbol{\varphi} \, dx + \langle \mathbf{h} \times \mathbf{n}, \boldsymbol{\varphi} \rangle_{\mathbf{W}^{-1/p,p}(\Gamma_D) \times \mathbf{W}^{1/p,p'}(\Gamma_D)} = 0,$$

955

$$956 \quad (5.2) \quad \mathbf{f} \cdot \mathbf{n} = \text{div}_{\Gamma_D}(\mathbf{h} \times \mathbf{n}) \quad \text{on } \Gamma_D,$$

957 where $\operatorname{div}_{\Gamma_D}$ is the surface divergence on Γ_D . Then Problem (S_1) has a unique solution
 958 $\boldsymbol{\xi} \in \mathbf{W}^{1,p}(\Omega)$ satisfying the estimate

$$959 \quad (5.3) \quad \|\boldsymbol{\xi}\|_{\mathbf{W}^{1,p}(\Omega)} \leq C \left(\|\mathbf{f}\|_{\mathbf{L}^{r(p)}(\Omega)} + \|\mathbf{h} \times \mathbf{n}\|_{\mathbf{W}^{-1/p,p}(\Gamma_D)} \right).$$

960 Furthermore, if Ω is of class $\mathcal{C}^{2,1}$, $\mathbf{f} \in \mathbf{L}^p(\Omega)$ and $\mathbf{h} \times \mathbf{n} \in \mathbf{W}^{-1/p,p}(\Gamma_D)$, then $\boldsymbol{\xi}$
 961 belongs to $\mathbf{W}^{2,p}(\Omega)$.

962 *Proof. i) Uniqueness.* To prove the uniqueness of $\boldsymbol{\xi}$, we take $\mathbf{f} = \mathbf{0}$ and $\mathbf{h} = \mathbf{0}$
 963 in (S_1) . Then the function $\mathbf{z} = \operatorname{curl} \boldsymbol{\xi}$ belongs to $\mathbf{L}^p(\Omega)$ and

$$964 \quad \operatorname{div} \mathbf{z} = 0, \quad \operatorname{curl} \mathbf{z} = \mathbf{0} \quad \text{in } \Omega \quad \text{and} \quad \mathbf{z} \times \mathbf{n} = \mathbf{0} \quad \text{on } \Gamma_D, \quad \mathbf{z} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_N.$$

965 This implies that $\mathbf{z} \in \mathbf{K}_0^p(\Omega)$. Thus, we can write \mathbf{z} as

$$966 \quad \mathbf{z} = \sum_{\ell=1}^{L_D} \langle \mathbf{z} \cdot \mathbf{n}, 1 \rangle_{\Gamma_D^\ell} \nabla q_\ell.$$

967 Since $\mathbf{K}_0^p(\Omega) \subset \mathbf{W}^{1,q}(\Omega)$ for any $q \geq 1$, in particular \mathbf{z} belongs to $\mathbf{L}^2(\Omega)$ and we have

$$968 \quad \int_{\Omega} |\mathbf{z}|^2 dx = \int_{\Omega} \mathbf{z} \cdot \operatorname{curl} \boldsymbol{\xi} dx = \sum_{\ell=1}^{L_D} \langle \mathbf{z} \cdot \mathbf{n}, 1 \rangle_{\Gamma_D^\ell} \int_{\Omega} \operatorname{curl} \boldsymbol{\xi} \cdot \nabla q_\ell dx = 0,$$

969 which means that $\operatorname{curl} \boldsymbol{\xi} = \mathbf{0}$. Then $\boldsymbol{\xi}$ belongs to $\widetilde{\mathbf{K}}_0^p(\Omega)$. As the fluxes of $\boldsymbol{\xi}$ on the
 970 connected components of Γ_N and on the cuts Σ_j , with $1 \leq j \leq J$, are equal to zero,
 971 we conclude that $\boldsymbol{\xi} = \mathbf{0}$ and this completes the uniqueness proof.

972 **ii) Compatibility conditions.** The weak formulation of (S_1) is given as follow:

$$973 \quad \text{Find } \boldsymbol{\xi} \in \widetilde{\mathbf{W}}_{\Sigma}^p(\Omega) \quad \text{such that for any } \boldsymbol{\varphi} \in \widetilde{\mathbf{W}}_{\Sigma}^{p'}(\Omega),$$

$$974 \quad (5.4) \quad \int_{\Omega} \operatorname{curl} \boldsymbol{\xi} \cdot \operatorname{curl} \boldsymbol{\varphi} dx = \int_{\Omega} \mathbf{f} \cdot \boldsymbol{\varphi} dx + \langle \mathbf{h} \times \mathbf{n}, \boldsymbol{\varphi} \rangle_{\Gamma_D}.$$

975 So the first compatibility condition (5.1) appears directly by taking $\boldsymbol{\varphi} \in \widetilde{\mathbf{K}}_0^{p'}(\Omega)$.
 976 Setting $\mathbf{z} = \operatorname{curl} \boldsymbol{\xi}$, it is clear that

$$977 \quad \forall \boldsymbol{\varphi} \in W^{2,p'}(\Omega); \quad \langle \operatorname{curl} \mathbf{z} \cdot \mathbf{n}, \boldsymbol{\varphi} \rangle_{\Gamma_D} = - \langle \mathbf{z} \times \mathbf{n}, \nabla \boldsymbol{\varphi} \rangle_{\Gamma_D},$$

978 where $\langle \cdot, \cdot \rangle_{\Gamma_D}$ denotes the duality product between $W^{1/p,p}(\Gamma_D)$ and $W^{-1/p,p'}(\Gamma_D)$.
 979 So since $\mathbf{z} = \operatorname{curl} \boldsymbol{\xi}$, we have

$$980 \quad \langle \mathbf{f} \cdot \mathbf{n}, \boldsymbol{\varphi} \rangle_{\Gamma_D} = - \langle \mathbf{h} \times \mathbf{n}, \nabla \boldsymbol{\varphi} \rangle_{\Gamma_D} = \langle \operatorname{div}_{\Gamma_D}(\mathbf{h} \times \mathbf{n}), \boldsymbol{\varphi} \rangle_{\Gamma_D}.$$

981 Hence $\mathbf{f} \cdot \mathbf{n} = \operatorname{div}_{\Gamma_D}(\mathbf{h} \times \mathbf{n})$ in the sense of $W^{-1-1/p,p}(\Gamma_D)$ (and also in the sense of
 982 $W^{-\frac{1}{r(p)},r(p)}(\Gamma_D)$).

983 **iii) Existence.** Using the Inf-Sup condition (4.40), we know that Problem (5.4)
 984 admits a unique solution $\mathbf{u} \in \widetilde{\mathbf{W}}_{\Sigma}^p(\Omega) \hookrightarrow \mathbf{W}^{1,p}(\Omega)$. In order to extend (5.4) to any
 985 test function in $\widetilde{\mathbf{X}}_0^{p'}(\Omega)$, we use the same argument as in Proposition 4.20 which enable
 986 us to prove that every solution of (5.4) also solves (S_1) .

987 **iv) Estimate.** The estimate (5.3) is obtained by using the same tools as in Proposi-
988 tion 4.17.

989 **v) Regularity.** We set $\mathbf{z} = \mathbf{curl} \boldsymbol{\xi}$. Hence $\mathbf{z} \in \mathbf{L}^p(\Omega)$, $\mathbf{curl} \mathbf{z} \in \mathbf{L}^p(\Omega)$, $\operatorname{div} \mathbf{z} = 0$
990 in Ω , $\mathbf{z} \times \mathbf{n} = \mathbf{h} \times \mathbf{n}$ on Γ_D and $\mathbf{z} \cdot \mathbf{n} = 0$ on Γ_N . Due to Corollary 3.4, \mathbf{z} belongs
991 to $\mathbf{W}^{1,p}(\Omega)$. Since $\boldsymbol{\xi} \in \mathbf{L}^p(\Omega)$, $\operatorname{div} \boldsymbol{\xi} = 0$ in Ω , $\mathbf{curl} \boldsymbol{\xi} \in \mathbf{W}^{1,p}(\Omega)$, $\boldsymbol{\xi} \times \mathbf{n} = \mathbf{0}$ on Γ_N
992 and $\boldsymbol{\xi} \cdot \mathbf{n} = 0$ on Γ_D , then according to Corollary 3.5, we deduce that $\boldsymbol{\xi}$ belongs to
993 $\mathbf{W}^{2,p}(\Omega)$. \square

994 *Remark 5.2.* Assume that $\mathbf{h} \times \mathbf{n} = \mathbf{0}$ on Γ_D and suppose that (5.1)-(5.2) hold.
995 Then we have $\mathbf{f} \cdot \mathbf{n} = 0$ on Γ_D with $\langle \mathbf{f} \cdot \mathbf{n}, 1 \rangle_{\Gamma_N^\ell} = 0$ for any $0 \leq \ell \leq L_N$ and
996 $\langle \mathbf{f} \cdot \mathbf{n}, 1 \rangle_{\Sigma_j} = 0$ for any $1 \leq j \leq J$ (see the proof of Proposition 3.8). Then due to
997 Theorem 4.21, there exists a unique $\mathbf{z} \in \mathbf{W}^{1,r(p)}(\Omega) \hookrightarrow \mathbf{L}^p(\Omega)$ such that $\mathbf{f} = \mathbf{curl} \mathbf{z}$,
998 $\operatorname{div} \mathbf{z} = 0$ in Ω satisfying $\mathbf{z} \times \mathbf{n} = \mathbf{0}$ on Γ_D and $\mathbf{z} \cdot \mathbf{n} = 0$ on Γ_N . Moreover,
999 $\langle \mathbf{z} \cdot \mathbf{n}, 1 \rangle_{\Gamma_D^\ell} = 0$ for any $0 \leq \ell \leq L_D$. Now, according to Theorem 4.18 where we
1000 interchange Γ_D and Γ_N , there exists a unique $\boldsymbol{\xi} \in \mathbf{W}^{1,p}(\Omega)$ such that $\mathbf{z} = \mathbf{curl} \boldsymbol{\xi}$
1001 and $\operatorname{div} \boldsymbol{\xi} = 0$ in Ω satisfying $\boldsymbol{\xi} \times \mathbf{n} = \mathbf{0}$ on Γ_N and $\boldsymbol{\xi} \cdot \mathbf{n} = 0$ on Γ_D . Moreover,
1002 $\langle \boldsymbol{\xi} \cdot \mathbf{n}, 1 \rangle_{\Gamma_N^\ell} = 0$ for any $0 \leq \ell \leq L_N$ and $\langle \boldsymbol{\xi} \cdot \mathbf{n}, 1 \rangle_{\Sigma_j} = 0$ for any $1 \leq j \leq J$, thus $\boldsymbol{\xi}$ is
1003 the unique solution of Problem (S₁).

1004 We state in the following theorem the existence and uniqueness of weak solutions
1005 to Problem (S). Furthermore, we give more regularity properties to that solution,
1006 which is the last main result of this work.

1007 **THEOREM 5.3.** Assume that $\mathbf{f} \in \mathbf{L}^{r(p)}(\Omega)$, $\mathbf{h} \times \mathbf{n} \in \mathbf{W}^{-1/p,p}(\Gamma_D)$, $\operatorname{div}_{\Gamma_D}(\mathbf{h} \times \mathbf{n}) \in$
1008 $W^{-\frac{1}{r(p)},r(p)}(\Gamma_D)$ and $\pi_0 \in W^{1-\frac{1}{r(p)},r(p)}(\Gamma_N)$ satisfying the compatibility condition for
1009 any $\boldsymbol{\varphi} \in \widetilde{\mathbf{K}}_0^{p'}(\Omega)$

$$1010 \quad (5.5) \quad \int_{\Omega} \mathbf{f} \cdot \boldsymbol{\varphi} \, dx - \int_{\Gamma_N} \pi_0 \boldsymbol{\varphi} \cdot \mathbf{n} \, ds + \langle \mathbf{h} \times \mathbf{n}, \boldsymbol{\varphi} \rangle_{\mathbf{W}^{-1/p,p}(\Gamma_D) \times \mathbf{W}^{1/p,p'}(\Gamma_D)} = 0.$$

1011 Then Problem (S) has a unique solution $(\mathbf{u}, \pi) \in \mathbf{W}^{1,p}(\Omega) \times W^{1,r(p)}(\Omega)$ satisfying the
1012 estimate

$$1013 \quad \|\mathbf{u}\|_{\mathbf{W}^{1,p}(\Omega)} + \|\pi\|_{W^{1,r(p)}(\Omega)} \leq C \left(\|\mathbf{f}\|_{\mathbf{L}^{r(p)}(\Omega)} + \|\mathbf{h} \times \mathbf{n}\|_{\mathbf{W}^{-1/p,p}(\Gamma_D)} \right. \\ 1014 \quad \left. + \|\operatorname{div}_{\Gamma_D}(\mathbf{h} \times \mathbf{n})\|_{W^{-\frac{1}{r(p)},r(p)}(\Gamma_D)} + \|\pi_0\|_{W^{1-\frac{1}{r(p)},r(p)}(\Gamma_N)} \right).$$

1015 Furthermore, if Ω is of class $\mathcal{C}^{2,1}$, $\mathbf{f} \in \mathbf{L}^p(\Omega)$, $\mathbf{h} \times \mathbf{n} \in \mathbf{W}^{-1/p,p}(\Gamma_D)$ and $\pi_0 \in$
1016 $W^{1-1/p,p}(\Gamma_N)$ then the solution (\mathbf{u}, π) belongs to $\mathbf{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$ and the following
1017 estimate holds

$$1018 \quad \|\mathbf{u}\|_{\mathbf{W}^{2,p}(\Omega)} + \|\pi\|_{W^{1,p}(\Omega)} \leq C \left(\|\mathbf{f}\|_{\mathbf{L}^p(\Omega)} + \|\mathbf{h} \times \mathbf{n}\|_{\mathbf{W}^{-1/p,p}(\Gamma_D)} \right. \\ 1019 \quad \left. + \|\pi_0\|_{W^{1-1/p,p}(\Gamma_N)} \right).$$

1020 *Proof.* **i)** To get the compatibility condition, we give the weak formulation of (S)

$$1021 \quad \text{Find } \mathbf{u} \in \widetilde{\mathbf{W}}_{\Sigma}^p(\Omega) \text{ such that for any } \boldsymbol{\varphi} \in \widetilde{\mathbf{W}}_{\Sigma}^{p'}(\Omega), \\ 1022 \quad \int_{\Omega} \mathbf{curl} \mathbf{u} \cdot \mathbf{curl} \boldsymbol{\varphi} \, dx - \int_{\Omega} \pi \operatorname{div} \boldsymbol{\varphi} \, dx = \int_{\Omega} \mathbf{f} \cdot \boldsymbol{\varphi} \, dx + \\ 1023 \quad (5.8) \quad + \langle \mathbf{h} \times \mathbf{n}, \boldsymbol{\varphi} \rangle_{\mathbf{W}^{-1/p,p}(\Gamma_D) \times \mathbf{W}^{1/p,p'}(\Gamma_D)} - \int_{\Gamma_N} \pi_0 \boldsymbol{\varphi} \cdot \mathbf{n} \, ds.$$

1024 By taking $\varphi \in \widetilde{\mathbf{K}}_0^{p'}(\Omega)$, we deduce that (5.5) holds.

1025 **ii)** Note that applying the divergence operator to the Stokes equation leads to

$$1026 \quad \Delta\pi = \operatorname{div} \mathbf{f} \quad \text{in } \Omega.$$

1027 Setting then $\boldsymbol{\psi} = \operatorname{curl} \mathbf{u}$, we have

$$1028 \quad -\Delta\mathbf{u} = \operatorname{curl} \boldsymbol{\psi} \quad \text{in } \Omega,$$

1029 and

$$1030 \quad -\Delta\mathbf{u} \cdot \mathbf{n} = \operatorname{curl} \boldsymbol{\psi} \cdot \mathbf{n} = (\mathbf{f} - \nabla\pi) \cdot \mathbf{n}.$$

1031 So the pressure satisfies the following boundary conditions

$$1032 \quad (\nabla\pi - \mathbf{f}) \cdot \mathbf{n} = \operatorname{div}_{\Gamma_D}(\mathbf{h} \times \mathbf{n}) \quad \text{on } \Gamma_D, \quad \pi = \pi_0 \quad \text{on } \Gamma_N.$$

1033 We infer that the pressure can be found independently of the velocity field. We solve
1034 now the following elliptic problem subjected to Dirichlet and Neumann boundary
1035 conditions

$$1036 \quad (5.9) \quad \begin{cases} \Delta\pi = \operatorname{div} \mathbf{f} & \text{in } \Omega \\ (\nabla\pi - \mathbf{f}) \cdot \mathbf{n} = \operatorname{div}_{\Gamma_D}(\mathbf{h} \times \mathbf{n}) & \text{on } \Gamma_D, \quad \pi = \pi_0 \quad \text{on } \Gamma_N. \end{cases}$$

1037 Let $\theta \in W^{1,r(p)}(\Omega)$ be the unique solution of

$$1038 \quad \begin{cases} \Delta\theta = 0 & \text{in } \Omega, \\ \theta = \pi_0 & \text{on } \Gamma_N, \quad \theta = 0 \quad \text{on } \Gamma_D \end{cases}$$

1039 and $\chi \in W^{1,r(p)}(\Omega)$ be the unique solution of

$$1040 \quad \begin{cases} \Delta\chi = \operatorname{div} \mathbf{f} & \text{in } \Omega, \\ (\nabla\chi - \mathbf{f}) \cdot \mathbf{n} = \operatorname{div}_{\Gamma_D}(\mathbf{h} \times \mathbf{n}) - \frac{\partial\theta}{\partial\mathbf{n}} & \text{on } \Gamma_D, \quad \chi = 0 \quad \text{on } \Gamma_N. \end{cases}$$

1041 Moreover θ and χ satisfy respectively the following estimates

$$1042 \quad \|\theta\|_{W^{1,r(p)}(\Omega)} \leq C \|\pi_0\|_{W^{1-\frac{1}{r(p)},r(p)}(\Gamma_N)},$$

1043 and

$$1044 \quad \|\chi\|_{W^{1,r(p)}(\Omega)} \leq C \left(\|f\|_{\mathbf{L}^{r(p)}(\Omega)} + \|\pi_0\|_{W^{1-\frac{1}{r(p)},r(p)}(\Gamma_N)} \right. \\ 1045 \quad \left. + \|\operatorname{div}_{\Gamma_D}(\mathbf{h} \times \mathbf{n})\|_{W^{-\frac{1}{r(p)},r(p)}(\Gamma_D)} \right).$$

1046 Setting $\pi = \chi + \theta$, we have

$$1047 \quad \begin{cases} \Delta\pi = \operatorname{div} \mathbf{f} & \text{in } \Omega, \\ (\nabla\pi - \mathbf{f}) \cdot \mathbf{n} = \operatorname{div}_{\Gamma_D}(\mathbf{h} \times \mathbf{n}) & \text{on } \Gamma_D, \quad \pi = \pi_0 \quad \text{on } \Gamma_N. \end{cases}$$

1048 This implies the existence and uniqueness of $\pi \in W^{1,r(p)}(\Omega)$ solution of (5.9) satisfying
1049 the estimate

$$1050 \quad \|\pi\|_{W^{1,r(p)}(\Omega)} \leq C \left(\|f\|_{\mathbf{L}^{r(p)}(\Omega)} + \|\pi_0\|_{W^{1-\frac{1}{r(p)},r(p)}(\Gamma_N)} + \|\operatorname{div}_{\Gamma_D}(\mathbf{h} \times \mathbf{n})\|_{W^{-\frac{1}{r(p)},r(p)}(\Gamma_D)} \right).$$

1051 **iii)** Setting $\mathbf{F} = \mathbf{f} - \nabla\pi$, since the conditions (5.1)-(5.2) hold, we know from Proposi-
 1052 tion 5.1, that there exists a unique $\mathbf{u} \in \mathbf{W}^{1,p}(\Omega)$ satisfying $-\Delta\mathbf{u} = \mathbf{F}$ and $\operatorname{div} \mathbf{u} = 0$
 1053 in Ω , $\mathbf{u} \times \mathbf{n} = \mathbf{0}$ in Γ_N , $\mathbf{u} \cdot \mathbf{n} = 0$, $\operatorname{curl} \mathbf{u} \times \mathbf{n} = \mathbf{h} \times \mathbf{n}$ on Γ_D , $\langle \mathbf{u} \cdot \mathbf{n}, 1 \rangle_{\Gamma_N^\ell} = 0$, for
 1054 any $0 \leq \ell \leq L_N$, $\langle \mathbf{u} \cdot \mathbf{n}, 1 \rangle_{\Sigma_j} = 0$, for any $1 \leq j \leq J$. Moreover, we have the following
 1055 estimate

$$1056 \quad \|\mathbf{u}\|_{\mathbf{W}^{1,p}(\Omega)} \leq C(\|\mathbf{F}\|_{\mathbf{L}^{r(p)}(\Omega)} + \|\mathbf{h} \times \mathbf{n}\|_{\mathbf{W}^{-1/p,p}(\Gamma_D)}).$$

1057 Hence the problem (S) admits a unique solution $(\mathbf{u}, \pi) \in \mathbf{W}^{1,p}(\Omega) \times W^{1,r(p)}(\Omega)$ sat-
 1058 isfying the required estimate (5.6).

1059 According to Proposition 5.1, we know that if Ω is of class $\mathcal{C}^{2,1}$, $\mathbf{f} \in \mathbf{L}^p(\Omega)$, $\mathbf{h} \times \mathbf{n} \in$
 1060 $\mathbf{W}^{1-1/p,p}(\Gamma_D)$ and $\pi_0 \in W^{1-1/p,p}(\Gamma_N)$ then \mathbf{u} belongs to $\mathbf{W}^{2,p}(\Omega)$ and $\pi \in W^{1,p}(\Omega)$.
 1061 The estimate (5.7) is readily deduced. \square

1062 *Remark 5.4.* We also can consider the case where $\partial\Sigma \subset \Gamma_N$ in the Stokes problem
 1063 (S) which can be solved by using the first Inf-Sup condition and the first elliptic
 1064 problem.

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REFERENCES

- 1066 [1] P. ACEVEDO, C. AMROUCHE, C. CONCA, AND A. GHOSH, *Stokes and Navier–Stokes equations*
 1067 *with Navier boundary condition*, C. R. Math, 357 (2019), pp. 115–119.
 1068 [2] C. AMROUCHE, C. BERNARDI, M. DAUGE, AND V. GIRAULT, *Vector potentials in three-*
 1069 *dimensional non-smooth domains*, Math. Meth. Appl. Sci, 21 (1998), pp. 823–864.
 1070 [3] C. AMROUCHE AND A. REJAIBIA, *L^p -theory for Stokes and Navier–Stokes equations with Navier*
 1071 *boundary condition*, J. Differential Equations, 256 (2014), pp. 1551–1547.
 1072 [4] C. AMROUCHE AND N. SELOULA, *On the Stokes equations with the Navier-type boundary con-*
 1073 *ditions*, Differ. Equ. Appl, 3 (2011), pp. 581–607.
 1074 [5] C. AMROUCHE AND N. SELOULA, *L^p -theory for vector potentials and Sobolev’s inequalities for*
 1075 *vector fields. Applications to the Stokes equations with pressure boundary conditions*, Math.
 1076 *Mod. and Meth. in App*, 23 (2013), pp. 37–92.
 1077 [6] G. BEAVERS AND D. JOSEPH, *Boundary conditions at a naturally permeable wall*, J. Fluid Mech,
 1078 30 (1967), pp. 197–207.
 1079 [7] H. BEIRÃO DA VEIGA, *Regularity for Stokes and generalized Stokes system under non-*
 1080 *homogeneous slip-type boundary conditions*, Adv Differential Equ, 9 (2004), pp. 1079–1114.
 1081 [8] F. BEN BELGASSEM, C. BERNARDI, M. COSTABEL, AND M. DAUGE, *Un résultat de densité pour*
 1082 *les équations de maxwell*, C. R. Acad. Sci. Paris, t. 324, Serie I, (1997), pp. 731–736.
 1083 [9] A. BENDALI, J. DOMINGUEZ, AND S. GALLIC, *A variational approach for the vector potential*
 1084 *formulation of the Stokes and Navier–Stokes problems in three dimensional domains*, J.
 1085 *Math. Anal. App*, 107 (1985), pp. 537–560.
 1086 [10] M. BENEŠ AND P. KUČERA, *Solutions to the Navier–Stokes equations with mixed boundary*
 1087 *conditions in two-dimensional bounded domains*, Math. Nachr, 289 (2016), pp. 194–212.
 1088 [11] J. BERNARD, *Non-standard Stokes and Navier–Stokes problem: existence and regularity in*
 1089 *stationary case*, Math. Meth. Appl. Sci, 25 (2002), pp. 627–661.
 1090 [12] J. BERNARD, *Time-dependent Stokes and Navier–Stokes problems with boundary conditions*
 1091 *involving the pressure, existence and regularity*, Nonlinear Anal. Real World Appl, 4 (2003),
 1092 pp. 805–839.
 1093 [13] S. BERNARDI, T. CHACON REBELLO, AND D. YAKOUBI, *Finite element discretization of the*
 1094 *Stokes and Navier–Stokes equations with boundary condition on pressure*, SIAM J. Numer.
 1095 *Anal*, 53 (2015), pp. 826–850.
 1096 [14] S. BERTOLUZZA, V. CHABANNES, C. PRUD’HOMME, AND M. SZOPOS, *Boundary conditions in-*
 1097 *volving pressure for the Stokes problem and applications in computational hemodynamics*,
 1098 *Comp. Meth. Appl. Mech. Engin*, 322 (2017), pp. 58–80.
 1099 [15] M. BOUKROUCHE, I. BOUSSETOUAN, AND L. PAOLI, *Non-isothermal Navier–Stokes system with*
 1100 *mixed boundary conditions and friction law: Uniqueness and regularity properties*, Non-
 1101 *linear Anal Theory Methods Appl*, 102 (2014), pp. 168–185.
 1102 [16] M. BOUKROUCHE, I. BOUSSETOUAN, AND L. PAOLI, *Existence for non-isothermal fluid flows*
 1103 *with Tresca’s friction and Cattaneo’s heat law*, J. Math. Anal. Appl, 427 (2015), pp. 499–
 1104 514.

- 1105 [17] M. BOUKROUCHE, I. BOUSSETOUAN, AND L. PAOLI, *Existence and approximation for Navier-*
 1106 *Stokes system with Tresca's friction at the boundary and heat transfer governed by Cat-*
 1107 *taneo's law*, Math. Mech. solids, 23 (2018), pp. 519–540.
- 1108 [18] C. CONCA, F. MURAT, AND O. PIRONNEAU, *The Stokes and Navier-Stokes equations with*
 1109 *boundary conditions involving the pressure*, Japan. J. Math. New series, 20 (1994), pp. 279–
 1110 318.
- 1111 [19] P. A. DURBIN, *Considerations on the moving contact-line singularity, with application to fric-*
 1112 *tional drag on a slender drop*, J. Fluid Mech., 197 (1988), pp. 157–169.
- 1113 [20] J. FOUCHER-INCAUX, *Artificial boundaries and formulations for the incompressible Navier-*
 1114 *Stokes equations : applications to air and blood flows*, SeMA Journal, 64 (2014), pp. 1–40.
- 1115 [21] H. FUJITA, *A mathematical analysis of motions of viscous incompressible fluid under leak*
 1116 *or slip boundary conditions*, Mathematical Fluid Mechanics and Modeling, 888 (1994),
 1117 pp. 199–216.
- 1118 [22] V. GIRAULT, *Incompressible finite element methods for Navier-Stokes equations with non-*
 1119 *standard boundary conditions in \mathbb{R}^3* , Math. Comput, 51 (1988), pp. 55–74.
- 1120 [23] V. GIRAULT AND P.-A. RAVIART, *Finite element methods for Navier-Stokes equations, theory*
 1121 *and algorithms*, Springer-Verlag, 1986.
- 1122 [24] P. GRISVARD, *Elliptic problems in nonsmooth domains*, Pitman Publishing Inc, 2nd ed., 1985.
- 1123 [25] D. IFTIMIE, E. RAUGEL, AND G. R. SELL, *Navier-Stokes equations in thin 3D domains with*
 1124 *Navier boundary conditions*, Indiana Univ. Math. J, 56 (2007), pp. 1083–1156.
- 1125 [26] J. K. KELLIHER, *Navier-Stokes equations with Navier boundary conditions for a bounded do-*
 1126 *main in the plane*, SIAM J. Math. Anal, 38 (2004), pp. 210–232.
- 1127 [27] J. L. LIONS, R. TEMAM, AND S. WANG, *Mathematical theory for the coupled atmosphere-ocean*
 1128 *models (CAO III)*, J. Math. Pures Appl., 74 (1995), pp. 105–163.
- 1129 [28] R. MAZYA AND J. ROSSMAN, *Elliptic equations in polyhedral domains (mathematical surveys*
 1130 *and monographs)*, 162 (2010).
- 1131 [29] D. MITREA, M. MITREA, AND J. PIPHER, *Vector potential theory on nonsmooth domains in \mathbb{R}^3*
 1132 *and applications to electromagnetic scattering*, Fourier Anal. Appl., 3 (1997), pp. 131–192.
- 1133 [30] C. NAVIER, *Sur les lois d'équilibre et du mouvement des corps élastiques*, Mém. Acad. Sci., 7
 1134 (1827), pp. 375–394.
- 1135 [31] K. R. RAJAGOPAL AND P. N. KALONI, *Some remarks on boundary conditions for flows of fluids*
 1136 *of the differential type*, Cont. Mech. and its Applications, (Hemisphere Press, New York),
 1137 (1989), pp. 935–942.
- 1138 [32] V. A. SOLONNIKOV AND V. E. SCADLOV, *A certain boundary value problem for the stationary*
 1139 *system of Navier-Stokes equations, (Russian)*, Trudy Mat. Inst. Steklov, (1973), pp. 1996–
 1140 2100.
- 1141 [33] W. VON WAHL, *Estimating ∇u by $\operatorname{div} u$, $\operatorname{curl} u$* , Math. Methods Appl. Sci, 15 (1992), pp. 123–
 1142 143.