On the curl operator and some characterizations of matrix fields in Lipschitz domains

Chérif Amrouche, Bassem Bahouli, El Hacène Ouazar

On the curl operator and some characterizations of matrix fields in Lipschitz domains

Chérif Amrouche [1], Bassem Bahouli [1,2], El Hacène Ouazar [2]

[1]Laboratoire de Mathématiques Appliquées, Université de Pau et des Pays de l'Adour, Avenue de l'Université, 64000 Pau, France

[2]Laboratoire des Équations aux Dérivées Partielles Non Linéaires et Histoire des Mathématiques (EDPNL-HM), ENS Kouba, 16309, Alger, Algérie

January 5, 2021

Abstract: As well-known, De Rham's Theorem is a classical way to characterize vector fields as the gradient of the scalar fields, it is a tool of great importance in the theory of fluids mechanic. The first aim of this paper is to provide a useful rotational version of this theorem to establish several results on boundary value problems in the field of electromagnetism. Gurtin presented the completeness of Beltrami's representation in smooth domains for smooth symmetric matrix. The extension of Beltrami's completeness was obtained by Geymonat and Krasucki when the domain Ω is only Lipschitz and for symmetric matrix in $\mathbb{L}^2_s(\Omega)$. Our second aim is to present new extensions of Beltrami's completeness results on Lipschitz domains, firstly in the case where the data are in $\mathbb{D}_s(\Omega)$ and secondly when the data are in $\mathbb{W}^{m,r}_s(\Omega)$. The third objective is to extend Saint-Venant's Theorem to the case of distributions.

Keywords: De Rham's Theorem, symmetric matrix fields, Beltrami's representation, Beltrami's completeness, Saint Venant's Theorem, Poincaré's Lemma.

1 Introduction

First, all notations and definitions not given in this introduction are mentioned in Section 2. Let Ω be a bounded and connected open set of $\mathbb{R}^3$ with a Lipschitz-continuous boundary. The surjectivity of the operator div : $\boldsymbol{\mathcal{D}}(\Omega) \longrightarrow \mathcal{D}_0(\Omega)$, where $\mathcal{D}_0(\Omega)$ denotes the subspace of functions in $\mathcal{D}(\Omega)$ with zero average, is an important tool in the analysis of Stokes equations. This result has been shown by many authors through different techniques (see [17], [20], [32]) and it provides us with a simple proof for the following usual version of De Rham's Theorem: *let* $\boldsymbol{f} \in \boldsymbol{\mathcal{D}}'(\Omega)$ *satisfying*

$$for\ all\ \boldsymbol{\varphi} \in \boldsymbol{\mathcal{V}}(\Omega), \quad {}_{\boldsymbol{\mathcal{D}}'(\Omega)}\langle \boldsymbol{f}, \boldsymbol{\varphi}\rangle_{\boldsymbol{\mathcal{D}}(\Omega)} = 0,$$

where $\boldsymbol{\mathcal{V}}(\Omega)$ *denotes the subspace of vector fields in* $\boldsymbol{\mathcal{D}}(\Omega)$ *with divergence free, then there exists a scalar field* $p \in \mathcal{D}'(\Omega)$ *such that*

$$\boldsymbol{f} = \boldsymbol{\nabla}\, p \quad \text{in} \quad \Omega.$$

The first aim of this work is to present a new extension of the above theorem that we will call *the rotational version of De Rham's Theorem*. In the case where the open set Ω is star-shaped with respect to an open ball, Costabel et al in [17] and Mitrea in [32] have used the properties of pseudodifferential operators to show that the operator

$$\mathbf{curl} : \boldsymbol{\mathcal{D}}(\Omega) \longrightarrow \boldsymbol{\mathcal{V}}(\Omega), \tag{1}$$

is onto. In Section 3, we will give a new proof of this result by using the theory of singular integrals. Furthermore, we will generalize it in the case where Ω is Lipschitz but not necessarily star-shaped with respect to an open ball. More precisely, we will show that if

$$\boldsymbol{f} \in \boldsymbol{\mathcal{V}}(\Omega) \quad \text{satisfies} \quad \int_\Omega \boldsymbol{f} \cdot \boldsymbol{\varphi}\, dx = 0, \quad \forall \boldsymbol{\varphi} \in \boldsymbol{K}_T(\Omega), \tag{2}$$

where

$$\boldsymbol{K}_T(\Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega),\ \mathbf{curl}\, \boldsymbol{v} = \mathbf{0},\ \text{div}\, \boldsymbol{v} = 0 \text{ in } \Omega,\ \boldsymbol{v} \cdot \boldsymbol{n} = 0 \text{ on } \partial\Omega\} \tag{3}$$

(see Section 2), then there exists $\boldsymbol{\varphi} \in \boldsymbol{\mathcal{D}}(\Omega)$ satisfying $\mathbf{curl}\, \boldsymbol{\varphi} = \boldsymbol{f}$ in Ω. Next, we deduce a rotational version of De Rham's Theorem, and here we state our first main result:

Theorem A. (**Vector Potentials in $\boldsymbol{\mathcal{D}}(\Omega)$**) i) Let r be a real number such that $1 < r < \infty$ and m a nonnegative integer. Then, for any $\boldsymbol{f} \in \boldsymbol{\mathcal{V}}(\Omega)$ satisfying (2), there exists $\boldsymbol{\psi} \in \boldsymbol{\mathcal{D}}(\Omega)$ such that

$$\mathbf{curl}\, \boldsymbol{\psi} = \boldsymbol{f} \quad \text{in} \quad \Omega,$$

and there exists a constant $C(r, m, \Omega)$ such that

$$\|\boldsymbol{\psi}\|_{\boldsymbol{W}^{m+1,r}(\Omega)} \le C(r, m, \Omega) \|\boldsymbol{f}\|_{\boldsymbol{W}^{m,r}(\Omega)}.$$

ii) Let $\boldsymbol{g} \in \boldsymbol{\mathcal{D}}'(\Omega)$ and satisfies

$$\text{for all } \boldsymbol{\varphi} \in \boldsymbol{\mathcal{G}}(\Omega), \quad {}_{\boldsymbol{\mathcal{D}}'(\Omega)}\langle \boldsymbol{g}, \boldsymbol{\varphi} \rangle_{\boldsymbol{\mathcal{D}}(\Omega)} = 0,$$

where $\boldsymbol{\mathcal{G}}(\Omega) = \{\boldsymbol{\varphi} \in \boldsymbol{\mathcal{D}}(\Omega),\ \mathbf{curl}\, \boldsymbol{\varphi} = \mathbf{0} \text{ in } \Omega\}$. Then, there exists $\boldsymbol{\psi} \in \boldsymbol{\mathcal{D}}'(\Omega)$ such that

$$\mathbf{curl}\, \boldsymbol{\psi} = \boldsymbol{g} \quad \text{in} \quad \Omega.$$

As a close result, Borchers and Sohr in [11] have shown that if the open set Ω is of class $\mathcal{C}^{m,1}$, where m is a nonnegative integer, then the operator $\mathbf{curl} : \boldsymbol{W}_0^{m,r}(\Omega) \longrightarrow \boldsymbol{U}^{m,r}(\Omega)$ is onto. In Section 4, we will generalize this result in the case when the open set Ω is only Lipschitz and then we will deduce a weak rotational version of De Rham's Theorem. Our second main result is given in the following theorem:

Theorem B. (**Rotational Version of De Rham's Theorem**) Let r be a real number such that $1 < r < \infty$ and m a nonnegative integer. Then, there exists a constant $C_{r,m}(\Omega)$ such that for any $\boldsymbol{f} \in \boldsymbol{U}^{m,r}(\Omega)$, there exists $\boldsymbol{\psi} \in \boldsymbol{W}_0^{m+1,r}(\Omega)$, that satisfies

$$\mathbf{curl}\, \boldsymbol{\psi} = \boldsymbol{f} \quad \text{in} \quad \Omega,$$

and

$$\|\boldsymbol{\psi}\|_{\boldsymbol{W}^{m+1,r}(\Omega)} \leq C_{r,m}(\Omega) \|\boldsymbol{f}\|_{\boldsymbol{W}^{m,r}(\Omega)}.$$

ii) Let $m \geq 1$ be an integer and $\boldsymbol{g} \in \boldsymbol{W}^{-m,r'}(\Omega)$ satisfying

$$\forall \boldsymbol{\varphi} \in \boldsymbol{\mathcal{G}}^{\boldsymbol{m,r}}(\Omega), \quad {}_{\boldsymbol{W}^{-m,r'}(\Omega)}\langle \boldsymbol{g}, \boldsymbol{\varphi} \rangle_{\boldsymbol{W}_0^{m,r}(\Omega)} = 0.$$

Then there exists $\boldsymbol{\Psi} \in \boldsymbol{W}^{-m+1,r'}(\Omega)$ such that

$$\mathbf{curl}\, \boldsymbol{\Psi} = \boldsymbol{g} \ \text{ in } \ \Omega.$$

The vector potential $\boldsymbol{\psi}$ obtained by Borchers and Sohr satisfies the additional property: $\Delta^{m+1}\text{div}\ \boldsymbol{\psi} = 0$ in Ω, which needs the regularity $\mathcal{C}^{m,1}$ of the domain Ω.

In the absence of body forces the stress equations of equilibrium take the form

$$\mathbf{div}\, \mathbb{S} = \mathbf{0} \quad \text{in} \quad \Omega, \quad \mathbb{S} = \mathbb{S}^T, \tag{4}$$

the second order symmetric tensor field being the stress in the reference configuration Ω of an elastic body. The first stress function solution of the equilibrium equation (4) was presented by Airy in [2] for the two dimensional case. The generalizations for the three dimensional case were obtained by Maxwell in [28], Morera in [31] and Beltrami in [9]. The solutions of Morera and Maxwell are special cases of the Beltrami's solution defined as follows :

$$\mathbb{S} = \mathbf{curl}\,\mathbf{curl}\,\mathbb{A} \quad \text{for all smooth symmetric second order tensor fields } \mathbb{A} \text{ in } \Omega. \tag{5}$$

Gurtin [26] gave an example of a stress field $\mathbb{S}$ satisfying (4) but which is not given by (5). So that this representation is incomplete. However the Beltrami solution is complete in the class of smooth stress fields $\mathbb{S}$ which are *self-equilibrated*, *i.e.* for each closed regular surface $\mathcal{C}$ contained in Ω, the resultant force and the moment vanish. In other words, $\mathbb{S}$ satisfies the following condition:

$$\int_{\mathcal{C}} \mathbb{S}\cdot\boldsymbol{n}\,d\sigma = \int_{\mathcal{C}} \boldsymbol{P}^i \times (\mathbb{S}\cdot\boldsymbol{n})\,d\sigma = \mathbf{0}, \text{ for all } \ 1\leq i\leq 3,$$

such that $\boldsymbol{P}^i = -\varepsilon_{ijk}x_k\boldsymbol{e}^j$. For more details see [19].

An extension of this result can be found in [22] and in [23] as follows: let Ω be a bounded and connected open set of $\mathbb{R}^3$ with Lipschitz-continuous boundary and $\mathbb{S}$ be a symmetric matrix field in $\mathbb{L}^2_s(\Omega)$ and satisfying the following conditions:

$$\mathbf{div}\,\mathbb{S} = \mathbf{0} \text{ in } \Omega \quad \text{and} \quad \int_{\Gamma_k} (\mathbb{S}\cdot\boldsymbol{n})\,\cdot\boldsymbol{e}^i\,d\sigma = \int_{\Gamma_k} (\mathbb{S}\cdot\boldsymbol{n})\,\cdot\boldsymbol{P}^i\,d\sigma = 0, 1\leq i\leq 3,\ 1\leq k\leq K.$$

Then, there exists a symmetric matrix field $\mathbb{A}\in\mathbb{H}^2_s(\Omega)$ such that $\mathbf{curl}\,\mathbf{curl}\,\mathbb{A} = \mathbb{S}$ in Ω. Moreover, P. G Ciarlet et al in [16] observed that if the above symmetric matrix field $\mathbb{S}$ satisfies the following conditions:

$$\mathbb{S}\cdot\boldsymbol{n} = \mathbf{0} \text{ on } \partial\Omega \text{ and } \langle\mathbb{S}\cdot\boldsymbol{n},\boldsymbol{e}^i\rangle_{\Sigma_j} = \langle\mathbb{S}\cdot\boldsymbol{n},\boldsymbol{P}^i\rangle_{\Sigma_j} = 0, \text{ for all } \ 1\leq i\leq 3,\ 1\leq j\leq J,$$

where $\langle\cdot,\cdot\rangle_{\Sigma_j}$ denotes the duality pairing between $\boldsymbol{H}^{-\frac{1}{2}}(\Sigma_j)'$ and $\boldsymbol{H}^{\frac{1}{2}}(\Sigma_j)$, then $\mathbb{A}\in\mathbb{H}^2_{0,s}(\Omega)$.

Our third aim in this paper is to show a new extension of the Beltrami's completeness, in the case where the components of the symmetric matrix $\mathbb{S}$ are in $\mathcal{D}(\Omega)$ and to prove the above observation of P.G. Ciarlet et al in a general case, when the components of $\mathbb{S}$ are in $W^{m,r}_0(\Omega)$, with m a nonnegative integer.

Theorem C. (**Completeness of the Beltrami Solution**) i) Let r be a real number such that $1 < r < \infty$, m a nonnegative integer and $\mathbb{S}$ in $\mathbb{V}_s(\Omega)$ satisfies

$$\int_{\Sigma_j} (\mathbb{S}\cdot\boldsymbol{n})\cdot\boldsymbol{e}^i\,d\sigma = \int_{\Sigma_j} (\mathbb{S}\cdot\boldsymbol{n})\cdot\boldsymbol{P}^i\,d\sigma = 0, \text{ for all } \ 1\leq i\leq 3,\ 1\leq j\leq J.$$

Then, there exists $\mathbb{A}\in\mathbb{D}_s(\Omega)$ such that

$$\mathbf{curl}\,\mathbf{curl}\,\mathbb{A} = \mathbb{S} \quad \text{in} \quad \Omega,$$

and there exists a constant C such that

$$\|\mathbb{A}\|_{\mathbb{W}^{m+2,r}(\Omega)} \leq C\|\mathbb{S}\|_{\mathbb{W}^{m,r}(\Omega)}.$$

ii) Let $\mathbb{S} \in \mathbb{U}^{m,r}(\Omega)$, then there exists $\mathbb{A} \in \mathbb{W}_0^{m+2,r}(\Omega)$ such that $\mathbf{curl}\,\mathbf{curl}\,\mathbb{A} = \mathbb{S}$.
iii) Let $\mathbb{S} \in \mathbb{D}'_s(\Omega)$ and satisfies

$$\text{for all} \quad \mathbb{E} \in \mathbb{G}_s(\Omega), \ {}_{\mathbb{D}'(\Omega)}\langle \mathbb{S}, \mathbb{E}\rangle_{\mathbb{D}(\Omega)} = 0,$$

then, there exists $\mathbb{A} \in \mathbb{D}'_s(\Omega)$ such that

$$\mathbf{curl}\,\mathbf{curl}\,\mathbb{A} = \mathbb{S} \quad \text{in} \quad \Omega.$$

Let $\boldsymbol{v}$ be a smooth vector field defined on Ω and $\mathbb{E} = \boldsymbol{\nabla_s v}$ the corresponding strain field. It satisfies the following compatibility equations:

$$\mathbf{curl}\,\mathbf{curl}\,\mathbb{E} = \mathbf{0} \quad \text{in} \quad \Omega. \tag{6}$$

In 1864, A. J. C. B. de Saint-Venant announced that conversely, if Ω is a simply-connected open set of $\mathbb{R}^3$, then for any symmetric matrix in $\mathbb{E} = (E_{ij})$ with $E_{ij} \in \mathcal{C}^2(\Omega)$ which satisfies the above compatibility equations, there exists $\boldsymbol{v} \in \boldsymbol{\mathcal{C}}^3(\Omega)$ such that

$$\boldsymbol{\nabla}_s \boldsymbol{v} = \mathbb{E} \quad \text{in} \quad \Omega. \tag{7}$$

In fact, the first rigorous proof of sufficiency was given by Beltrami in 1886. More recently, if in addition Ω is Lipschitz, Ciarlet and Ciarlet Jr proved that if $\mathbb{E} \in \mathbb{L}^2(\Omega)$ satisfies the compatibility equations (6), then there exists $\boldsymbol{v} \in \mathbf{H}^1(\Omega)$ such that (7) holds. A similar result, with $\mathbb{E} \in \mathbb{H}^{-1}(\Omega)$ and then $\boldsymbol{v} \in \mathbf{L}^2(\Omega)$, was also obtained by Amrouche et al [5].

In 1890, Donati proved that, if Ω is an open subset of $\mathbb{R}^3$ and $\mathbb{E} \in \boldsymbol{\mathcal{C}}^2(\Omega)$ is such that

$$\int_\Omega \mathbb{E} : \mathbb{M} = 0 \quad \text{forall } \mathbb{M} \in \mathbb{D}_s(\Omega) \quad \text{such that } \mathbf{div}\ \mathbb{M} = \mathbf{0} \text{ in } \Omega, \tag{8}$$

then $\mathbb{E}$ satisfies the compatibility equations (6).

A first extension of Donati's Theorem was given by Ting [38] for symmetric matrix field $\mathbb{E} \in \mathbb{L}^2(\Omega)$ when the domain Ω is bounded and Lipschitz-continuous (not necessarily simply-connected): if $\mathbb{E} \in \mathbb{L}^2(\Omega)$ satisfies (8), then there exists $\boldsymbol{v}$ in $\mathbf{H}^1(\Omega)$ such that $\mathbb{E} = \boldsymbol{\nabla}_s \boldsymbol{v}$ in Ω.

Another extension of Donati's Theorem was given by Moreau [30] in the case of distributions: if $\mathbb{E} \in \boldsymbol{\mathcal{D}}'_s(\Omega)$ satisfies (8), then there exists $\boldsymbol{v}$ in $\boldsymbol{\mathcal{D}}'(\Omega)$ such that $\mathbb{E} = \boldsymbol{\nabla}_s \boldsymbol{v}$ in Ω. Here, Ω is an arbitrary open subset of $\mathbb{R}^3$.

More recently, using different proofs, some variants of Donati's Theorem have been independently obtained by Geymonat and Krasucki [21] for $\mathbb{E} \in \boldsymbol{W}_s^{-1,p}(\Omega)$ and for $\mathbb{E} \in \boldsymbol{L}_s^p(\Omega)$ and by Amrouche et al [5] for $\mathbb{E} \in \boldsymbol{L}_s^2(\Omega)$.

Our fourth main result is to give a general extension of Saint-Venant's Theorem when $\mathbb{E} \in \mathbb{D}'_s(\Omega)$.

Theorem D. (**Saint-Venant's Theorem**) Let Ω be a bounded and simply-connected open set of $\mathbb{R}^3$ with a Lipschitz-continuous boundary and $\mathbb{E} \in \mathbb{D}'_s(\Omega)$ satisfies

$$\mathbf{curl}\,\mathbf{curl}\,\mathbb{E} = \mathbf{0} \quad \text{in} \quad \Omega.$$

Then, there exists $\boldsymbol{v} \in \boldsymbol{\mathcal{D}}'(\Omega)$ such that

$$\boldsymbol{\nabla_s v} = \mathbb{E} \quad \text{in} \quad \Omega.$$

2 Notations and preliminaries

First, we recall some geometry notations. We denote by $|\cdot|$ the euclidean norm in $\mathbb{R}^N$. For $x \in \Omega$ and $d > 0$, we define the ball centred at x with radius d by $B(x, d) = \{y \in \mathbb{R}^N,\, |x - y| < d\}$. The open set Ω is starlike with respect to an open ball $B(x, d)$, if the convex hull of the set $\{y\} \cup B(x, d)$ is contained in Ω for each $y \in \Omega$. This means that it is starlike with respect to each point of this ball: for each $z \in \Omega$ and $y \in B(x, d)$ the segment $[zy]$ is contained in Ω. Now, we can show that a bounded, starlike open set with respect to an open ball is Lipschitz. Conversely, any bounded and connected open set with Lipschitz-continuous boundary is finite union of bounded and connected open sets, each being starlike with respect to an open ball. We refer here, this property is stated in [6], [17] and proved in [29]. Also, let Ω contained in $\mathbb{R}^3$ be a bounded and connected open set, we recall that Ω is pseudo-Lipschitz if for any point x on the boundary $\partial\Omega$ there exist an integer $r(x)$ equal to 1 or 2 and a strictly positive real number ρ_0 such that for all real numbers ρ with $0 < \rho < \rho_0$, the intersection of Ω with the ball with center x and radius ρ, has $r(x)$ connected components, each one being Lipschitz.

Second, we take the following hypothesis. We do not assume that the boundary of Ω is connected. We denote by Γ_k the connected components of the boundary $\partial\Omega$, $1 \leq k \leq K$. There exist J connected open surfaces Σ_j, $1 \leq j \leq J$, called 'cuts', contained in Ω, such that (i) each surface Σ_j is an open part of a smooth manifold $\mathcal{M}$,
(ii) the boundary of Σ is contained in $\partial\Omega$ for $1 \leq j \leq J$,

(iii) the intersection $\overline{\Sigma_i} \cap \overline{\Sigma_j}$ is empty for $i \neq j$,

(iv) the open set

$$\mathring{\Omega} = \Omega \backslash \cup_{j=1}^{J} \Sigma_j$$

is pseudo-Lipschitz and simply-connected.

Finally, we recall some functional spaces and differential operators:
Let $A \in \mathcal{L}(E, F)$, $A' \in \mathcal{L}(F', E')$ the dual operator of A is defined by:

$$_{E'}\langle A'u, v\rangle_E \; = \; _{F'}\langle u, Av\rangle_F, \quad \text{for any } v \in E, u \in F'.$$

Here $\mathcal{D}'(\Omega)$ is the space of distributions on Ω i.e., the topological dual space of $\mathcal{D}(\Omega)$. For any $1 < r < \infty$, r' is its conjugate i.e., $\frac{1}{r} + \frac{1}{r'} = 1$ and for $m \in \mathbb{N}$, $W^{m,r}(\Omega)$ is the usual Sobolev space. The subspace $W_0^{m,r}(\Omega)$ denotes the closure of $\mathcal{D}(\Omega)$ in $W^{m,r}(\Omega)$ and $W^{-m,r'}(\Omega)$ its topological dual.

In the following, the vectors, the vector functions (or distributions) and the spaces of vector-valued functions are represented by bold symbols. For example: $\boldsymbol{\mathcal{D}}(\Omega) := (\mathcal{D}(\Omega))^N$, $\boldsymbol{L}^r(\Omega) := L^r(\Omega)^N$. Moreover, we define the kernel space $\boldsymbol{K}_T(\Omega)$ (or space of harmonic knots) by

$$\boldsymbol{K}_T(\Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega),\, \mathbf{curl}\, \boldsymbol{v} = \mathbf{0},\, \operatorname{div} \boldsymbol{v} = 0 \text{ in } \Omega,\, \boldsymbol{v} \cdot \boldsymbol{n} = 0 \text{ on } \partial\Omega\}, \tag{9}$$

which is of finite dimension and its dimension depends on the geometric of Ω. Cantarella et al in [14] have shown that this space is isomorphic to the homology group $H_1(\Omega, \mathbb{R})$ and also to the relative homology group $H_2(\Omega, \Gamma, \mathbb{R})$. Its dimension is equal to the second Betti number J, which corresponds to the total genus of the boundary Γ (see also Amrouche et al in [3]). We define the space $\boldsymbol{V}^{m,r}(\Omega)$ which represents the closure of $\boldsymbol{\mathcal{V}}(\Omega)$ in $\boldsymbol{W}^{m,r}(\Omega)$, by

$$\boldsymbol{V}^{m,r}(\Omega) = \{\boldsymbol{v} \in \boldsymbol{W}_0^{m,r}(\Omega),\, \operatorname{div} \boldsymbol{v} = 0 \text{ in } \Omega\},$$

for $m \geq 1$ and by

$$\boldsymbol{V}^{0,r}(\Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^r(\Omega),\, \operatorname{div} \boldsymbol{v} = 0 \,\text{in } \Omega,\, \boldsymbol{v} \cdot \boldsymbol{n} = 0 \quad \text{on} \quad \partial\Omega\},$$

for $m = 0$. We set for $m \in \mathbb{N}$

$$\boldsymbol{U}^{m,r}(\Omega) = \{\boldsymbol{v} \in \boldsymbol{V}^{m,r}(\Omega);\; \int_\Omega \boldsymbol{v} \cdot \boldsymbol{\varphi} \, dx = 0 \; for \; all \; \boldsymbol{\varphi} \in \boldsymbol{K}_T(\Omega)\}.$$

Observe the following equivalence, for any function $\boldsymbol{v} \in \boldsymbol{V}^{m,r}(\Omega)$:

$$\forall\, \boldsymbol{\varphi} \in \boldsymbol{K}_T(\Omega),\; \int_\Omega \boldsymbol{v} \cdot \boldsymbol{\varphi} \, dx = 0 \Longleftrightarrow \langle \boldsymbol{v} \cdot \boldsymbol{n}, 1\rangle_{\Sigma_j} = 0 \; for \; any \, 1 \leq j \leq J.$$

Matrix fields and spaces of matrix fields are represented by special Roman capitals. Moreover, spaces of symmetric matrix fields are indexed by the Latin letter s. For example, $\mathbb{D}_s(\Omega) = \mathcal{D}(\Omega;\, \boldsymbol{M}^N_{sym})$. We define $\mathbb{V}_{\boldsymbol{s}}(\Omega)$ by

$$\mathbb{V}_s(\Omega) = \{\mathbb{S} \in \mathbb{D}_s(\Omega),\ \mathbf{div}\,\mathbb{S} = \mathbf{0} \quad \text{in } \Omega\},$$

and the kernel space $\mathbb{K}_T(\Omega)$ by

$$\mathbb{K}_T(\Omega) = \{\mathbb{S} \in \mathbb{L}^2_s(\Omega),\, \mathbf{curl}\,\mathbf{curl}\,\mathbb{S} = \mathbf{0},\, \mathbf{div}\,\mathbb{S} = \mathbf{0} \text{ in } \Omega,\, \mathbb{S}\cdot\boldsymbol{n} = \mathbf{0} \text{ on } \partial\Omega\},$$

which is of finite dimension and its dimension is dependent on the geometrical properties of Ω. Ciarlet et al in [16] and Geymonat et al in [23] have shown that the dimension of $\mathbb{K}_T(\Omega)$ is equal to $6J$. As recalled above, we define the space $\mathbb{U}^{m,r}_s(\Omega)$, for $m \geq 1$ by

$$\mathbb{U}^{m,r}_s(\Omega) = \{\mathbb{S} \in \mathbb{W}^{m,r}_{0,s}(\Omega),\, \mathbf{div}\,\mathbb{S} = \mathbf{0},\ \langle \mathbb{S}\cdot\boldsymbol{n}, \boldsymbol{e}^i\rangle_{\Sigma_j} = \langle \mathbb{S}\cdot\boldsymbol{n}, \boldsymbol{P}^i\rangle_{\Sigma_j} = 0,\ 1 \leq i \leq 3,\ 1 \leq j \leq J\}, \tag{10}$$

and for $m = 0$ by

$$\mathbb{U}^{0,r}_s(\Omega) = \{\mathbb{S} \in \mathbb{L}^r_s(\Omega),\ \mathbf{div}\,\mathbb{S} = \mathbf{0},\ \mathbb{S}\cdot\boldsymbol{n} = \mathbf{0} \text{ on } \partial\Omega,\ \langle\, \mathbb{S}\cdot\boldsymbol{n},\, \boldsymbol{e}^i\rangle_{\Sigma_j} = \langle \mathbb{S}\cdot\boldsymbol{n}, \boldsymbol{P}^i\rangle_{\Sigma_j} = \mathbf{0}\}. \tag{11}$$

We also introduce the following space

$$\mathbb{G}_s(\Omega) = \{\mathbb{A} \in \mathbb{D}_s(\Omega),\, \mathbf{curl}\,\mathbf{curl}\,\mathbb{A} = \mathbf{0} \quad \text{in} \quad \Omega\}.$$

The orientation tensor (ϵ_{ijk}) is defined by

$$\epsilon_{ijk} = \begin{cases} +1 & \text{if } \{i,j,k\} \text{ is an even permutation of } \{1,2,3\}, \\ -1 & \text{if } \{i,j,k\} \text{ is an odd permutation of } \{1,2,3\}, \\ \ \ 0 & \text{if at least two indices are equal.} \end{cases}$$

We use the following differential operators throughout the article: the divergence operator $\operatorname{div} : \boldsymbol{\mathcal{D}}'(\Omega) \longrightarrow \mathcal{D}'(\Omega)$ is defined by

$$\operatorname{div} \boldsymbol{v} = \partial_i v_i \quad \text{for any} \quad \boldsymbol{v} = (v_i) \in \boldsymbol{\mathcal{D}}'(\Omega).$$

The vector rotational operator $\mathbf{curl} : \boldsymbol{\mathcal{D}}'(\Omega) \longrightarrow \boldsymbol{\mathcal{D}}'(\Omega)$ is defined by

$$(\mathbf{curl}\,\boldsymbol{v})_i = \epsilon_{ijk}\,\partial_j v_k \quad \text{for any} \quad \boldsymbol{v} = (v_i) \in \boldsymbol{\mathcal{D}}'(\Omega).$$

The matrix symmetrized gradient operator $\boldsymbol{\nabla_s} : \boldsymbol{\mathcal{D}}'(\Omega) \longrightarrow \mathbb{D}'_s(\Omega)$ is defined by

$$(\boldsymbol{\nabla_s}\,\boldsymbol{v})_{ij} = \frac{1}{2}(\partial_i v_j + \partial_j v_i) \quad \text{for any} \quad \boldsymbol{v} = (v_i) \in \boldsymbol{\mathcal{D}}'(\Omega).$$

The matrix rotational operator $\mathbf{curl} : \mathbb{D}'(\Omega) \longrightarrow \mathbb{D}'(\Omega)$ is defined by

$$(\mathbf{curl}\,\mathbb{S})_{ij} = \epsilon_{i\ell k}\,\partial_\ell S_{jk} \quad \text{for any} \quad \mathbb{S} = (S_{ij}) \in \mathbb{D}'(\Omega).$$

3 The Rotational extension of De Rham's Theorem

Mitrea [32], Costabel and Macintosch [17] have shown that if Ω is bounded and starlike with respect to an open ball, then the operator (1) is onto. In this section, we apply the singular integrals theory to give a detailed proof for this result. Then we generalize it for the case where Ω is a bounded and connected open set of $\mathbb{R}^3$ with a Lipschitz-continuous boundary *i.e.*, we prove that the operator

$$\mathbf{curl} : \boldsymbol{\mathcal{D}}(\Omega) \longrightarrow \boldsymbol{\mathcal{V}}(\Omega) \perp \boldsymbol{K}_T(\Omega) \tag{12}$$

is onto. Here $\boldsymbol{\mathcal{V}}(\Omega) \perp \boldsymbol{K}_T(\Omega)$ denotes the space of functions $\boldsymbol{v} \in \boldsymbol{\mathcal{V}}(\Omega)$ such that $\int_\Omega \boldsymbol{v} \cdot \boldsymbol{\varphi}\, dx = 0$ for all $\boldsymbol{\varphi} \in \boldsymbol{K}_T(\Omega)$. This last result is the main key to prove a rotational extension of De Rham's Theorem.

Lemma 3.1. *Let Ω be a bounded and connected open set of $\mathbb{R}^3$ with a Lipschitz-continuous boundary and θ be a function of $\mathcal{D}(\mathbb{R}^3)$ such that*

$$supp\, \theta \subset \Omega \quad and \quad \int_{\mathbb{R}^3} \theta(y)\, dy = 1.$$

Then, for any $\boldsymbol{f} \in \boldsymbol{\mathcal{V}}(\Omega)$, the vector field $\boldsymbol{T}\boldsymbol{f}$ defined by

$$x \in \Omega,\ \boldsymbol{T}\boldsymbol{f}(x) = \int_\Omega \boldsymbol{f}(y) \times \left((x-y) \int_1^\infty (t-1)t\, \theta(y + t(x-y))\, dt \right) dy, \tag{13}$$

satisfies

$$\mathbf{curl}\, \boldsymbol{T}\boldsymbol{f} = \boldsymbol{f}, \qquad \boldsymbol{T}\boldsymbol{f} \in \boldsymbol{\mathcal{C}}^\infty(\Omega) \tag{14}$$

and there exists a constant $C_r(\Omega)$ depending only on r and Ω, such that

$$\|\boldsymbol{T}\boldsymbol{f}\|_{\boldsymbol{W}^{1,r}(\Omega)} \leq C_r(\Omega) \|\boldsymbol{f}\|_{\boldsymbol{L}^r(\Omega)}. \tag{15}$$

In particular, if Ω is starlike with respect to an open ball B and $supp\, \theta \subset B$, then

$$\boldsymbol{T}\boldsymbol{f} \in \boldsymbol{\mathcal{D}}(\Omega). \tag{16}$$

Proof. Note that $\boldsymbol{T}$ is a Poincaré type operator (see [17]). Let $\boldsymbol{f} \in \boldsymbol{\mathcal{V}}(\Omega)$, we denote by $\widetilde{\boldsymbol{f}}$ its extension by $\mathbf{0}$ outside of Ω.

Step 1. We start by establishing the two properties of (14).

We write $\boldsymbol{T}\boldsymbol{f}$ in the form

$$x \in \Omega,\ \boldsymbol{T}\boldsymbol{f}(x) = \int_\Omega \boldsymbol{f}(y) \times \boldsymbol{K}(x,y)\, dy,$$

where

$$\boldsymbol{K}(x,y) = (x-y)\int_1^\infty (t-1)t\,\theta(y+t(x-y))\,dt.$$

We observe that

$$\boldsymbol{T}\boldsymbol{f}(x) = \lim_{\varepsilon\longrightarrow 0}\int_{|x-y|\geq\varepsilon} \widetilde{\boldsymbol{f}}(y)\times\boldsymbol{K}(x,y)\,dy,$$

then

$$\begin{aligned}
\mathbf{curl}\,(\boldsymbol{T}\boldsymbol{f})(x) = \nabla_x\times\boldsymbol{T}\boldsymbol{f}(x) &= \lim_{\varepsilon\longrightarrow 0}\int_{|x-y|\geq\varepsilon}\nabla_x\times\Big(\widetilde{\boldsymbol{f}}(y)\times\boldsymbol{K}(x,y)\Big)\,dy\\
&+ \lim_{\varepsilon\longrightarrow 0}\int_{|x-y|=\varepsilon}\frac{(x-y)}{|x-y|}\times\Big(\widetilde{\boldsymbol{f}}(y)\times\boldsymbol{K}(x,y)\Big)\,d\sigma_y\\
&:= \lim_{\varepsilon\longrightarrow 0}(\boldsymbol{A}_\varepsilon+\boldsymbol{B}_\varepsilon).
\end{aligned}$$

According to the formula:

$$\mathbf{curl}\,(\boldsymbol{A}\times\boldsymbol{B}) = \nabla\times(\boldsymbol{A}\times\boldsymbol{B}) = (\nabla\cdot\boldsymbol{B})\boldsymbol{A}-(\nabla\cdot\boldsymbol{A})\boldsymbol{B}+(\boldsymbol{B}\cdot\nabla)\boldsymbol{A}-(\boldsymbol{A}\cdot\nabla)\boldsymbol{B},$$

we deduce that

$$\boldsymbol{A}_\varepsilon = \int_{|x-y|\geq\varepsilon}\Big[(\nabla_x\cdot\boldsymbol{K}(x,y))\,\widetilde{\boldsymbol{f}}(y)-(\widetilde{\boldsymbol{f}}(y)\cdot\nabla_x)\boldsymbol{K}(x,y)\Big]\,dy := \boldsymbol{A}_1(\varepsilon)-\boldsymbol{A}_2(\varepsilon).$$

Using now the following formula:

$$\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c}) = \boldsymbol{b}(\boldsymbol{a}\cdot\boldsymbol{c})-\boldsymbol{c}(\boldsymbol{a}\cdot\boldsymbol{b}),$$

we have

$$\boldsymbol{B}_\varepsilon = \int_{|x-y|=\varepsilon}\left[\left(\frac{(x-y)}{|x-y|}\cdot\boldsymbol{K}(x,y)\right)\widetilde{\boldsymbol{f}}(y)\;-\left(\frac{(x-y)}{|x-y|}\cdot\widetilde{\boldsymbol{f}}(y)\right)\boldsymbol{K}(x,y)\right]\,d\sigma_y := \boldsymbol{B}_1(\varepsilon)-\boldsymbol{B}_2(\varepsilon).$$

Thus, we can write

$$\mathbf{curl}\,(\boldsymbol{T}\boldsymbol{f})(x) = \lim_{\varepsilon\longrightarrow 0}\left[\boldsymbol{A}_1(\varepsilon)-\boldsymbol{A}_2(\varepsilon)+\boldsymbol{B}_1(\varepsilon)-\boldsymbol{B}_2(\varepsilon)\right].$$

i) Study of $\boldsymbol{A}_1(\varepsilon)$. We have

$$\boldsymbol{A}_1(\varepsilon) = \int_{|x-y|\geq\varepsilon}\left[(\nabla_x\cdot\boldsymbol{K}_1(x,y))+(\nabla_x\cdot\boldsymbol{K}_2(x,y))\right]\widetilde{\boldsymbol{f}}(y)\,dy, \tag{17}$$

where

$$\boldsymbol{K}_1(x,y) = (x-y)\int_1^\infty t^2\theta(y+t(x-y))dt$$

and

$$\boldsymbol{K}_2(x,y) = -(x-y)\int_1^{\infty} t\,\theta(y+t(x-y))dt. \tag{18}$$

We remark that $\boldsymbol{K}_1(\cdot,\cdot)$ is the kernel of the Bogovskiĭ's operator (see [10]), then

$$\nabla_x \cdot \boldsymbol{K}_1(x,y) = -\theta(x). \tag{19}$$

It is straightforward to see that

$$\begin{aligned}
\nabla_x \cdot \boldsymbol{K}_2(x,y) &= -3\int_1^{\infty} t\,\theta(y+t(x-y))\,dt - \sum_{i=1}^{3}(x_i-y_i)\int_1^{\infty} t^2\partial_i\theta(y+t(x-y))\,dt \\
&= -\int_1^{\infty} t\,\theta(y+t(x-y))\,dt - \int_1^{\infty} \frac{\partial(t^2\theta)}{\partial t}(y+t(x-y))\,dt \\
&= \theta(x) - \int_1^{\infty} t\,\theta(y+t(x-y))\,dt.
\end{aligned} \tag{20}$$

Then, from (17), (19) and (20), we obtain

$$\lim_{\varepsilon\longrightarrow 0}\boldsymbol{A}_1(\varepsilon) = -\int_{\Omega}\boldsymbol{f}(y)\int_1^{\infty} t\,\theta(y+t(x-y))\,dt\,dy. \tag{21}$$

ii) Study of $B_1(\varepsilon)$. It is easy to prove that

$$\lim_{\varepsilon\longrightarrow 0}\int_{|x-y|=\varepsilon}\left(\frac{(x-y)}{|x-y|}\cdot\boldsymbol{K}_1(x,y)\right)\widetilde{\boldsymbol{f}}(y)\,d\sigma_y = \boldsymbol{f}(x).$$

In (18), we use the change of variable $s = t|x-y|$ to get

$$\boldsymbol{K}_2(x,y) = -\frac{(x-y)}{|x-y|}\int_{|x-y|}^{\infty} s\,\theta\left(y + s\frac{(x-y)}{|x-y|}\right)\,ds.$$

Then

$$\begin{aligned}
&\int_{|x-y|=\varepsilon}\frac{(x-y)}{|x-y|}\cdot\boldsymbol{K}_2(x,y)\widetilde{\boldsymbol{f}}(y)\,d\sigma_y \\
&= -\sum_{i=1}^{3}\int_{|x-y|=\varepsilon}^{\infty}\widetilde{\boldsymbol{f}}(y)\left(\frac{x_i-y_i}{|x-y|}\right)^2\int_{|x-y|}^{\infty} s\,\theta(y+s\frac{(x-y)}{|x-y|})\,ds\,d\sigma_y.
\end{aligned}$$

Using now the change of variables $z = \dfrac{x-y}{\varepsilon}$ and $s' = s-\varepsilon$, we obtain

$$\begin{aligned}
&\lim_{\varepsilon\longrightarrow 0}\int_{|x-y|=\varepsilon}\frac{(x-y)}{|x-y|}\cdot\boldsymbol{K}_2(x,y)\widetilde{\boldsymbol{f}}(y)d\sigma_y \\
&= -\lim_{\varepsilon\longrightarrow 0}\sum_{i=1}^{3}\varepsilon\int_{|z|=1}\widetilde{\boldsymbol{f}}(x-\varepsilon z)z_i^2\int_0^{\infty}(s'+\varepsilon)\,\theta(x+s'z)ds'd\sigma_z \\
&= \boldsymbol{0}.
\end{aligned}$$

Consequently,

$$\lim_{\varepsilon \longrightarrow 0} \boldsymbol{B}_1(\varepsilon) = \boldsymbol{f}(x). \tag{22}$$

iii) Study of $\boldsymbol{A}_2(\varepsilon) + \boldsymbol{B}_2(\varepsilon)$. According to the Stokes formula, we obtain

$$\begin{aligned}
\lim_{\varepsilon \longrightarrow 0} (\boldsymbol{A}_2(\varepsilon) + \boldsymbol{B}_2(\varepsilon)) &= \lim_{\varepsilon \longrightarrow 0} \left[\int_{|x-y|\geq \varepsilon} (\widetilde{\boldsymbol{f}}(y) \cdot \nabla_x) \boldsymbol{K}(x,y) + \int_{|x-y|=\varepsilon} \left(\frac{(x-y)}{|x-y|} \cdot \widetilde{\boldsymbol{f}}(y) \right) \boldsymbol{K}(x,y) \right] \\
&= \lim_{\varepsilon \longrightarrow 0} \int_{|x-y|\geq \varepsilon} \Big[(\widetilde{\boldsymbol{f}}(y) \cdot \nabla_x) \boldsymbol{K}(x,y) + \boldsymbol{K}(x,y) \operatorname{div} \widetilde{\boldsymbol{f}}(y) \\
&+ \ (\widetilde{\boldsymbol{f}}(y) \cdot \nabla_y) \, \boldsymbol{K}(x,y) \Big] \, dy \\
&= \lim_{\varepsilon \longrightarrow 0} \int_{|x-y|\geq \varepsilon} \Big[(\widetilde{\boldsymbol{f}}(y) \cdot \nabla_x) \boldsymbol{K}(x,y) \ + (\widetilde{\boldsymbol{f}}(y) \cdot \nabla_y) \, \boldsymbol{K}(x,y) \Big] \, dy.
\end{aligned}$$

For any $1 \leq i, j, \leq 3$, we have

$$\frac{\partial K_j}{\partial x_i}(x,y) = -\frac{\partial K_j}{\partial y_i}(x,y) + (x_j - y_j) \int_1^{\infty} (t-1)t \ \partial_i \theta(y + t(x-y)) dt.$$

Then

$$\lim_{\varepsilon \longrightarrow 0} (A_2(\varepsilon)_j + B_2(\varepsilon)_j) = \lim_{\varepsilon \longrightarrow 0} \int_{|x-y|\geq \varepsilon} \left[\widetilde{\boldsymbol{f}}(y) \cdot \boldsymbol{L}_j(x,y) \right] \, dy, \tag{23}$$

where the i^{th} component of $\boldsymbol{L}_j$ is given by

$$((\boldsymbol{L}_j(x,y))_i = (x_j - y_j) \int_1^{\infty} (t-1)t \ \partial_i \theta(y + t(x-y)) \, dt.$$

iv) Verification of $\operatorname{\mathbf{curl}} \boldsymbol{T}\boldsymbol{f} = \boldsymbol{f}$. From (21)-(23), we conclude that for all $1 \leq j \leq 3$

$$\begin{aligned}
(\operatorname{\mathbf{curl}} \boldsymbol{T}\boldsymbol{f})_j(x) &= f_j(x) - \int_{\Omega} [\boldsymbol{f}(y) \cdot \boldsymbol{L}_j(x,y)] \, dy - \int_{\Omega} f_j(y) \int_1^{\infty} t\theta(y + t(x-y)) \, dt \, dy \\
&= f_j(x) + \int_{\Omega} [\boldsymbol{f}(y) \cdot \boldsymbol{H}_j(x,y)] \, dy,
\end{aligned}$$

with

$$\boldsymbol{H}_j(x,y) = -\boldsymbol{L}_j(x,y) - \boldsymbol{e_j} \int_1^{\infty} t\theta(y + t(x-y)) \, dt$$

and where $\boldsymbol{e_j}$ is the j^{th} vector of the canonical basis of $\mathbb{R}^3$. It is easy to verify that

$$\forall x \in \mathbb{R}^3, \ \boldsymbol{H}_j(x, \cdot) = \operatorname{\mathbf{grad}} \chi_j(x, \cdot),$$

where

$$\chi_j(x,y) = (x_j - y_j) \int_1^{\infty} t\theta(y + t(x-y)) dt.$$

Since $\boldsymbol{f} \in \boldsymbol{\mathcal{V}}(\Omega)$, we deduce

$$\begin{aligned}(\mathbf{curl}\,\boldsymbol{Tf})_j(x) &= f_j(x) - \int_\Omega \operatorname{div} \boldsymbol{f}(y)\, \chi_j(x,y)\, dy \\ &= f_j(x).\end{aligned}$$

v) Verification of $\boldsymbol{Tf} \in \boldsymbol{\mathcal{C}}^\infty(\Omega)$. In (13), we use the changes of variables $z = x - y$ and $s = (t-1)|x-y|$, so we obtain

$$x \in \Omega,\, \boldsymbol{Tf}(x) = \int_{\mathbb{R}^3} \widetilde{\boldsymbol{f}}(x-z) \times z \int_0^\infty \left(\frac{s^2}{|z|^3} + \frac{s}{|z|^2} \right) \theta(x + s\frac{z}{|z|}) ds\, dz. \tag{24}$$

Then, for any $\alpha \in \mathbb{N}^3$, we have

$$\begin{aligned}\partial^\alpha \boldsymbol{Tf}(x) &= \int_{\mathbb{R}^3} \partial^\alpha \widetilde{\boldsymbol{f}}(x-z) \times z \int_0^\infty \left(\frac{s^2}{|z|^3} + \frac{s}{|z|^2} \right) \theta(x + s\frac{z}{|z|}) ds\, dz \\ &+ \int_{\mathbb{R}^3} \widetilde{\boldsymbol{f}}(x-z) \times z \int_0^\infty \left(\frac{s^2}{|z|^3} + \frac{s}{|z|^2} \right) \partial^\alpha \theta(x + s\frac{z}{|z|}) ds\, dz.\end{aligned}$$

Since $\partial^\alpha \boldsymbol{f}$ and $\partial^\alpha \theta$ are continuous in Ω, then $\partial^\alpha \boldsymbol{Tf}$ is continuous and $\boldsymbol{Tf} \in \boldsymbol{C}^\infty(\Omega)$.

Step 2. Now, we establish the estimate (15).

Let $f \in \boldsymbol{\mathcal{V}}(\Omega)$ and $1 \le i, j \le 3$, we have

$$\begin{aligned}\frac{\partial \boldsymbol{Tf}}{\partial x_j}(x) &= \lim_{\epsilon \longrightarrow 0} \left[\int_{|x-y| \ge \epsilon} \frac{\partial}{\partial x_j} (\boldsymbol{f}(y) \times \boldsymbol{K}(x,y))\, dy + \int_{|x-y|=\epsilon} (\boldsymbol{f}(y) \times \boldsymbol{K}(x,y)) \frac{x_j - y_j}{|x-y|} d\sigma \right] \\ &= \lim_{\epsilon \longrightarrow 0} \left[\int_{|x-y| \ge \epsilon} \left(\boldsymbol{f}(y) \times \frac{\partial \boldsymbol{K_1}}{\partial x_j}(x,y) \right) dy + \int_{|x-y|=\epsilon} (\boldsymbol{f}(y) \times \boldsymbol{K_1}(x,y)) \frac{x_j - y_j}{|x-y|} d\sigma \right] \\ &+ \lim_{\epsilon \longrightarrow 0} \left[\int_{|x-y| \ge \epsilon} \left(\boldsymbol{f}(y) \times \frac{\partial \boldsymbol{K_2}}{\partial x_j}(x,y) \right) dy + \int_{|x-y|=\epsilon} (\boldsymbol{f}(y) \times \boldsymbol{K_2}(x,y)) \frac{x_j - y_j}{|x-y|} d\sigma \right].\end{aligned}$$

Since we have shown

$$\lim_{\epsilon \longrightarrow 0} \int_{|x-y|=\epsilon} (\boldsymbol{f}(y) \times \boldsymbol{K_2}(x,y)) \frac{x_j - y_j}{|x-y|} d\sigma = 0,$$

then

$$\begin{aligned}\frac{\partial \boldsymbol{Tf}}{\partial x_j}(x) &= \int_\Omega \left(\boldsymbol{f}(y) \times \frac{\partial \boldsymbol{K_1}}{\partial x_j}(x,y) \right) dy + \int_\Omega \left(\boldsymbol{f}(y) \times \frac{\partial \boldsymbol{K_2}}{\partial x_j}(x,y) \right) dy \\ &+ \left(\boldsymbol{f}(x) \times \int_\Omega (x-y) \left(\theta(y) \frac{x_j - y_j}{|x-y|} \right) dy \right) \\ &:= \boldsymbol{J_1 f}(x) + \boldsymbol{J_2 f}(x) + \boldsymbol{J_3 f}(x).\end{aligned}$$

Also

$$\begin{cases} (\boldsymbol{J_1 f})_{ij}(x) &= \displaystyle\int_\Omega \left(\varepsilon_{imn} f_m(y) \frac{\partial K_{1_n}}{\partial x_j}(x,y)\right) dy \\ (\boldsymbol{J_2 f})_{ij}(x) &= \displaystyle\int_\Omega \left(\varepsilon_{imn} f_m(y) \frac{\partial K_{2_n}}{\partial x_j}(x,y)\right) dy \\ (\boldsymbol{J_3 f})_{ij}(x) &= \varepsilon_{imn} f_m(x) \displaystyle\int_\Omega K_{1_n}(x,y) \frac{x_j - y_j}{|x-y|^2} dy. \end{cases}$$

There exists a constant $C(\Omega)$ (see page 166 of [20]) such that

$$\forall \varphi \in \mathcal{D}(\Omega), \ \| \int_\Omega \varphi(y) \frac{\partial K_{1_n}}{\partial x_j}(\cdot, y)\, dy \|_{L^r(\Omega)} \leq C(\Omega) \|\varphi\|_{L^r(\Omega)}.$$

Then, there exists a constant $C_1(\Omega)$ such that

$$\|\boldsymbol{J_1 f}\|_{L^r(\Omega)} \leq C_1(\Omega) \|\boldsymbol{f}\|_{\boldsymbol{L}^r(\Omega)}. \tag{25}$$

Besides, we have

$$\forall \varphi \in \mathcal{D}(\Omega), \ \int_\Omega \varphi(y) \frac{\partial K_{2_n}}{\partial x_j}(x,y) dy = \frac{1}{2} \int_\Omega \varphi(y) G_{nj}(x,y) dy.$$

Thus, there exists a constant $C_2(\Omega)$ such that (see page 165 of [20])

$$\|\boldsymbol{J_2 f}\|_{L^r(\Omega)} \leq C_2(\Omega) \|\boldsymbol{f}\|_{\boldsymbol{L}^r(\Omega)}. \tag{26}$$

For the last estimate, since for each $x \in \Omega$

$$|(\boldsymbol{J_3 f})_i(x)| \leq \sum_{k=1}^{3} |f_k(x)|,$$

we deduce that

$$\|\boldsymbol{J_3 f}\|_{L^r(\Omega)} \leq C_3(\Omega) \|\boldsymbol{f}\|_{\boldsymbol{L}^r(\Omega)}. \tag{27}$$

Finally, from (25)-(27), there exists a constant $C_r(\Omega)$ such that

$$\|\boldsymbol{T f}\|_{\boldsymbol{W}^{1,r}(\Omega)} \leq C_r(\Omega) \|\boldsymbol{f}\|_{\boldsymbol{L}^r(\Omega)}.$$

Step 3. Now, we suppose that Ω is starlike with respect to an open ball B and that $\operatorname{supp} \theta \subset B$. We will prove the property (16).

Indeed, in what follows we take

$$A = \{z \in \Omega; z = \lambda z_1 + (1-\lambda) z_2, \ z_1 \in \ \operatorname{supp} \boldsymbol{f}, \ z_2 \in \overline{B}, \ \lambda \in [0,1]\}.$$

Since Ω is starlike with respect to an open ball B, the compact set A is included in Ω. Fixing any $x \in \Omega \setminus A$, for any $y \in \operatorname{supp} \boldsymbol{f}$ and $t \geq 1$ we have $y + t(x-y) \notin \overline{B}$. According to (13), we deduce that $\boldsymbol{T f}(x) = \boldsymbol{0}$ and $\operatorname{supp} \boldsymbol{T f} \subset A \subset \Omega$. Consequently $\boldsymbol{T f} \in \boldsymbol{\mathcal{D}}(\Omega)$. $\square$

The following corollary generalizes the estimate (15) in the case where we replace the Lebesgue space $\boldsymbol{L}^r(\Omega)$ by the Sobolev space $\boldsymbol{W}^{m,r}(\Omega)$, for any positive integer m.

Corollary 3.2. *Let Ω be a bounded and connected open set of $\mathbb{R}^3$ with a Lipschitz-continuous boundary, $\boldsymbol{f} \in \boldsymbol{\mathcal{V}}(\Omega)$ and $\boldsymbol{T}$ the operator defined in (13). Then, for any real number $1 < r < \infty$ and for any integer $m \geq 1$, there exists a constant C depending only on r, m and Ω such that*

$$\|\boldsymbol{T}\boldsymbol{f}\|_{\boldsymbol{W}^{m+1,r}(\Omega)} \leq C\|\boldsymbol{f}\|_{\boldsymbol{W}^{m,r}(\Omega)}. \tag{28}$$

Proof. All the constants which appear in the inequalities below are noted by a generic letter C. We consider the case $m = 1$ and we write the operator $\boldsymbol{T}$ as in (24). So for any $j = 1, \cdots, N$, we have

$$\begin{aligned}
\partial_j \boldsymbol{T}\boldsymbol{f}(x) &= \int_{\mathbb{R}^3} \partial_j \widetilde{\boldsymbol{f}}(x-z) \times z \int_0^\infty \left(\frac{s^2}{|z|^3} + \frac{s}{|z|^2}\right) \theta(x + s\frac{z}{|z|}) ds\, dz \\
&+ \int_{\mathbb{R}^3} \widetilde{\boldsymbol{f}}(x-z) \times \int_0^\infty \left(\frac{s^2}{|z|^3} + \frac{s}{|z|^2}\right) \partial_j\theta(x + s\frac{z}{|z|}) ds\, dz \\
&:= \boldsymbol{h}_1(x) + \boldsymbol{h}_2(x).
\end{aligned}$$

Estimate of $\|\partial_k \boldsymbol{h}_1\|_{L^r(\Omega)}$. We observe that $\boldsymbol{h}_1 = \partial_j \boldsymbol{T}\boldsymbol{f}$, then Lemma 3.1 implies

$$\|\partial_k \boldsymbol{h}_1\| \leq C\|\partial_j \boldsymbol{f}\|_{\boldsymbol{L}^r(\Omega)} \leq C\|\boldsymbol{f}\|_{\boldsymbol{W}^{1,r}(\Omega)}. \tag{29}$$

Estimate of $\|\partial_k \boldsymbol{h}_2\|_{L^r(\Omega)}$. We remark that the function $\boldsymbol{h}_2$ has the same form as the function $\boldsymbol{T}\boldsymbol{f}$ with θ replaced by $\partial_j\theta$. Note that, we find the estimate of the point without using the property $\displaystyle\int_\Omega \theta(x)dx = 1$. This means that by the same method, we obtain

$$\|\partial_k \boldsymbol{h}_2\|_{L^r(\Omega)} \leq C\|\boldsymbol{f}\|_{\boldsymbol{L}^r(\Omega)}. \tag{30}$$

From (29) and (30), we deduce the existence of a constant C depending only on r and Ω such that (28) holds. For $m > 1$, we proceed by induction and so we apply the same approach as for the case $m = 1$. □

We have shown that if Ω is a starlike open set with respect to an open ball, then the rotational operator is onto from $\boldsymbol{\mathcal{D}}(\Omega)$ into $\boldsymbol{\mathcal{V}}(\Omega)$. This result can be extended to a bounded and connected open set of $\mathbb{R}^3$ with a Lipschitz-continuous boundary. For that we need the following lemma (see [8]).

Lemma 3.3. *Let Ω be a bounded and connected open set of $\mathbb{R}^N$ with a Lipschitz-continuous boundary. Then, there exist connected open sets Ω_j of $\mathbb{R}^N$, $j \geq 1$, with the following properties:*
i) $\partial\Omega_j$ is of classe $\mathcal{C}^\infty$.
ii) $\overline{\Omega_j} \subset \Omega_{j+1} \subset \Omega$ for each $j \geq 1$, and $\Omega = \cup_{j=1}^\infty \Omega_j$.

Theorem 3.4. *Let Ω be a bounded and connected open set of $\mathbb{R}^3$ with a Lipschitz-continuous boundary. Then, for any $\boldsymbol{f} \in \boldsymbol{\mathcal{V}}(\Omega)$ satisfying the condition (2), there exists $\boldsymbol{\psi} \in \boldsymbol{\mathcal{D}}(\Omega)$ such that*

$$\mathbf{curl}\,\boldsymbol{\psi} = \boldsymbol{f} \quad in \quad \Omega.$$

Moreover, for any $1 < r < \infty$ and for any nonnegative integer m, there exists a constant $C(r, m, \Omega)$ such that

$$\|\boldsymbol{\psi}\|_{\boldsymbol{W}^{m+1,r}(\Omega)} \le C(r, m, \Omega)\|\boldsymbol{f}\|_{\boldsymbol{W}^{m,r}(\Omega)}. \tag{31}$$

Proof. Let $\boldsymbol{f} \in \boldsymbol{\mathcal{V}}(\Omega)$ satisfying the condition (2). Lemma 3.1 and Corollary 3.2 imply that $\boldsymbol{T f} \in \boldsymbol{\mathcal{C}}^\infty(\Omega)$, $\mathbf{curl}\,(\boldsymbol{T f}) = \boldsymbol{f}$ in Ω with the estimate (31). Thanks to Lemma 3.3 there exists an open set Ω_{j_0} which is connected and of class $\mathcal{C}^\infty$, such that $\mathrm{supp}\boldsymbol{f} \subset \overline{\Omega_{j_0}} \subset \Omega$. Define the open set $\Omega' = \Omega \setminus \overline{\Omega_{j_0}}$, which is bounded and connected open set of $\mathbb{R}^3$ with a Lipschitz-continuous boundary. Setting now $\boldsymbol{\psi'} = \boldsymbol{T f}|_{\Omega'}$, it follows from Lemma 3.1 that $\mathbf{curl}\,\boldsymbol{\psi'} = \mathbf{0}$ in Ω' and by Corollary 3.2 that $\boldsymbol{\psi'} \in \bigcap_{\substack{1<r<\infty,\\ m\in\mathbb{N}}} \boldsymbol{W}^{m,r}(\Omega)$. The compatibility condition $\displaystyle\int_\Omega \boldsymbol{f}\cdot\boldsymbol{\varphi}\,dx = 0$ for all $\boldsymbol{\varphi} \in \boldsymbol{K}_T(\Omega)$ implies that for any curves γ_j^* inside Ω' and surrounding Σ_j, we have $\displaystyle\int_{\gamma_j^*} \boldsymbol{\psi'}\cdot\boldsymbol{t} = \int_{\Sigma_j} \boldsymbol{f}\cdot\boldsymbol{n} = 0$. Hence $\boldsymbol{\psi'}$ has no circulations in Ω'. Then, there exists p' satisfying $p' \in \bigcap_{\substack{1<r<\infty,\\ m\in\mathbb{N}}} W^{m,r}(\Omega)$, such that $\mathbf{grad}\,p' = \boldsymbol{\psi'}$ in Ω' (see Corollary 1 page 199 in [36]) and with the estimate

$$\|p'\|_{W^{m+2,r}(\Omega')} \le C\|\boldsymbol{\psi'}\|_{\boldsymbol{W}^{m+1,r}(\Omega')}.$$

Theorem 1.4.3.1 of [25] implies that there exists $\widetilde{p} \in \mathcal{C}^\infty(\mathbb{R}^3)$ such that $\widetilde{p}|_{\Omega'} = p'$ and

$$\|\widetilde{p}\|_{W^{m+2,r}(\mathbb{R}^3)} \le C\|p'\|_{W^{m+2,r}(\Omega')} \le C\|\boldsymbol{\psi'}\|_{\boldsymbol{W}^{m+1,r}(\Omega')}.$$

Setting now $p = \widetilde{p}|_\Omega$ and $\boldsymbol{\psi} = \boldsymbol{T f} - \mathbf{grad}\,p$, we have $\boldsymbol{\psi}|_{\Omega'} = \mathbf{0}$ and then $\boldsymbol{\psi} \in \boldsymbol{\mathcal{D}}(\Omega)$. Furthermore, it is clear that $\mathbf{curl}\,\boldsymbol{\psi} = \boldsymbol{f}$ in Ω and for any $1 < r < \infty$ and $m \in \mathbb{N}$ the estimate (31) holds. $\square$

Now, we use the surjectivity of the rotational operator to show a rotational extension of De Rham's Theorem.

Theorem 3.5. *Let Ω be a bounded and connected open set of $\mathbb{R}^3$ with a Lipschitz-continuous boundary, and $\boldsymbol{f} \in \boldsymbol{\mathcal{D}}'(\Omega)$ satisfying the following condition:*

$$\forall \boldsymbol{\varphi} \in \boldsymbol{\mathcal{G}}(\Omega), \quad {}_{\boldsymbol{\mathcal{D}}'(\Omega)}\langle \boldsymbol{f}, \boldsymbol{\varphi}\rangle_{\boldsymbol{\mathcal{D}}(\Omega)} = 0. \tag{32}$$

Then, there exists $\boldsymbol{\psi} \in \boldsymbol{\mathcal{D}}'(\Omega)$ such that

$$\mathbf{curl}\,\boldsymbol{\psi} = \boldsymbol{f} \quad in \quad \Omega.$$

Remark 3.1. The converse is obvious, because for any $\boldsymbol{\psi} \in \boldsymbol{\mathcal{D}}'(\Omega)$ and any $\boldsymbol{\varphi} \in \boldsymbol{\mathcal{G}}(\Omega)$, we have

$$_{\boldsymbol{\mathcal{D}}'(\Omega)}\langle \mathbf{curl}\,\boldsymbol{\psi}, \boldsymbol{\varphi}\rangle_{\boldsymbol{\mathcal{D}}(\Omega)} = {}_{\boldsymbol{\mathcal{D}}'(\Omega)}\langle \boldsymbol{\psi}, \mathbf{curl}\,\boldsymbol{\varphi}\rangle_{\boldsymbol{\mathcal{D}}(\Omega)} = 0.$$

Proof. Acccording to Theorem 3.4,

$$\mathbf{curl}\ :\ \boldsymbol{\mathcal{D}}(\boldsymbol{\Omega})/\boldsymbol{\mathcal{G}}(\Omega) \longrightarrow \boldsymbol{\mathcal{V}}(\Omega) \perp \boldsymbol{K}_T(\Omega)$$

is one to one and onto. Then, its adjoint

$$\mathbf{curl}\ :\ (\boldsymbol{\mathcal{V}}(\Omega) \perp \boldsymbol{K}_T(\Omega))' \longrightarrow \boldsymbol{\mathcal{D}'}(\boldsymbol{\Omega}) \perp \boldsymbol{\mathcal{G}}(\Omega) \tag{33}$$

is also one to one and onto.

Let $\boldsymbol{L} \in (\boldsymbol{\mathcal{V}}(\Omega) \perp \boldsymbol{K}_T(\Omega))'$. As $\boldsymbol{\mathcal{V}}(\Omega) \perp \boldsymbol{K}_T(\Omega)$ is closed in $\boldsymbol{\mathcal{D}}(\Omega)$, we can extend $\boldsymbol{L}$ by $\widetilde{\boldsymbol{L}} \in \boldsymbol{\mathcal{D}}'(\Omega)$. Two expressions $\boldsymbol{g} \in \boldsymbol{\mathcal{D}}'(\Omega)$ and $\boldsymbol{h} \in \boldsymbol{\mathcal{D}}'(\Omega)$ of $\widetilde{\boldsymbol{L}}$ coincide on $\boldsymbol{\mathcal{V}}(\Omega) \perp \boldsymbol{K}_T(\Omega)$ if and only if

$$\forall\, \boldsymbol{\varphi} \in \boldsymbol{\mathcal{V}}(\Omega) \perp \boldsymbol{K}_T(\Omega),\ {}_{\boldsymbol{\mathcal{D}}'(\Omega)}\langle \boldsymbol{g} - \boldsymbol{h}, \boldsymbol{\varphi}\rangle_{\boldsymbol{\mathcal{D}}(\Omega)} = 0.$$

Using again Theorem 3.4, we have

$$\begin{aligned}
\forall\, \boldsymbol{\varphi} \in \boldsymbol{\mathcal{V}}(\Omega) \perp \boldsymbol{K}_T(\Omega),\ {}_{\boldsymbol{\mathcal{D}}'(\Omega)}\langle \boldsymbol{g} - \boldsymbol{h}, \boldsymbol{\varphi}\rangle_{\boldsymbol{\mathcal{D}}(\Omega)} &= {}_{\boldsymbol{\mathcal{D}}'(\Omega)}\langle \boldsymbol{g} - \boldsymbol{h}, \mathbf{curl}\,\boldsymbol{\psi}\rangle_{\boldsymbol{\mathcal{D}}(\Omega)} \\
&= {}_{\boldsymbol{\mathcal{D}}'(\Omega)}\langle \mathbf{curl}(\boldsymbol{g} - \boldsymbol{h}), \boldsymbol{\psi}\rangle_{\boldsymbol{\mathcal{D}}(\Omega)} \\
&= 0.
\end{aligned}$$

Which means that

$$\boldsymbol{g} - \boldsymbol{h} \in \mathbf{Ker}\,\mathbf{curl}, \quad \text{where} \quad \mathbf{curl}\ :\ \boldsymbol{\mathcal{D}}'(\Omega) \longrightarrow \boldsymbol{\mathcal{D}}'(\Omega),$$

and consequently

$$(\boldsymbol{\mathcal{V}}(\Omega) \perp \boldsymbol{K}_T(\Omega))' = \boldsymbol{\mathcal{D}}'(\Omega)/\mathbf{Ker}\,\mathbf{curl}. \tag{34}$$

Let $\boldsymbol{f} \in \boldsymbol{\mathcal{D}}'(\Omega)$ satisfies (32). In other words, this means that $\boldsymbol{f} \in \boldsymbol{\mathcal{D}}'(\Omega) \perp \boldsymbol{\mathcal{G}}(\Omega)$. From (33) and the characterization (34), there exists $\boldsymbol{\psi} \in \boldsymbol{\mathcal{D}}'(\Omega)$, such that

$$\mathbf{curl}\,\boldsymbol{\psi} = \boldsymbol{f} \quad \text{in} \quad \Omega.$$

$\square$

4 A weak rotational extension of De Rham's Theorem

In this section, we will use Theorem 3.4 to show another surjectivity result of the **curl** operator. Then, we will use this result to prove a weak rotational extension of De Rham's Theorem. First, we need the following lemma:

Lemma 4.1. *Let Ω be a bounded and connected open set of $\mathbb{R}^3$ with a Lipschitz-continuous boundary and m a nonnegative integer. Then, the space $\boldsymbol{\mathcal{V}}(\Omega) \perp \boldsymbol{K}_T(\Omega)$ is dense in $\boldsymbol{U}^{m,r}(\Omega)$.*

Proof. **Step 1**: we show that the linear mapping $\boldsymbol{\mathcal{R}} : \boldsymbol{\mathcal{V}}(\Omega) \longrightarrow \mathbb{R}^J$ defined by

$$(\boldsymbol{\mathcal{R}}(\boldsymbol{v}))_j = \int_{\Sigma_j} \boldsymbol{v} \cdot \boldsymbol{n}\, d\sigma, \quad 1 \leq j \leq J,$$

is onto, where J is the dimension of $\boldsymbol{K}_T(\Omega)$. For that purpose, we proceed by contradiction. We suppose that $\boldsymbol{\mathcal{R}}$ is not onto, which implies that there exists j_0 such that $1 \leq j_0 \leq J$ and a family of numbers $\{\lambda_j\}_{\substack{1\leq j\leq J,\\ j\neq j_0}}$ such that for any $\boldsymbol{v} \in \boldsymbol{\mathcal{V}}(\Omega)$, we have

$$\int_{\Sigma_{j_0}} \boldsymbol{v} \cdot \boldsymbol{n}\, d\sigma = \sum_{\substack{j=1\\ j\neq j_0}}^{J} \lambda_j \int_{\Sigma_j} \boldsymbol{v} \cdot \boldsymbol{n}\, d\sigma.$$

Using the Green's formula of Lemma 3.10 of [3], then

$$\int_{\Sigma_{j_0}} \boldsymbol{v} \cdot \boldsymbol{n}\, d\sigma - \sum_{\substack{j=1\\ j\neq j_0}}^{J} \lambda_j \int_{\Sigma_j} \boldsymbol{v} \cdot \boldsymbol{n}\, d\sigma = \int_{\Omega} \boldsymbol{v} \cdot \left(\widetilde{\mathbf{grad}\, q_{j_0}^T} - \sum_{\substack{j=1\\ j\neq j_0}}^{J} \lambda_j \widetilde{\mathbf{grad}\, q_j^T} \right) dx = 0,$$

where the vector fields $\widetilde{\mathbf{grad}\, q_j^T}$ are the elements of the basis of $\boldsymbol{K}_T(\Omega)$ (see [3]). Then, the usual De Rham's Theorem (see [6]) implies that there exists $p \in H^1(\Omega)$, unique up to an additive constant, such that

$$\widetilde{\mathbf{grad}\, q_{j_0}^T} - \sum_{\substack{j=1\\ j\neq j_0}}^{J} \lambda_j \widetilde{\mathbf{grad}\, q_j^T} = \mathbf{grad}\, p.$$

Consequently, p is harmonic and $\dfrac{\partial p}{\partial \boldsymbol{n}} = 0$ on $\partial\Omega$. So, p is a constant and then the dimension of $\boldsymbol{K}_T(\Omega)$ is less then J, which is a contradiction. We have proved that for any $1 \leq j \leq J$ there exists $\boldsymbol{\varphi}_j \in \boldsymbol{\mathcal{V}}(\Omega)$ such that

$$\text{for all } 1 \leq k \leq J, \int_{\Sigma_k} \boldsymbol{\varphi}_j \cdot \boldsymbol{n}\, d\sigma = \delta_{kj}. \tag{35}$$

Step 2: we show that $\boldsymbol{\mathcal{V}}(\Omega) \perp \boldsymbol{K}_T(\Omega)$ is dense in $\boldsymbol{U}^{m,r}(\Omega)$ (where $\boldsymbol{U}^{m,r}(\Omega)$ is defined in Section 2). Let $\boldsymbol{v} \in \boldsymbol{U}^{m,r}(\Omega)$, then there exists a sequence $(\boldsymbol{v}_k) \in \boldsymbol{\mathcal{V}}(\Omega)$ that converges to $\boldsymbol{v}$ in $\boldsymbol{W}^{m,r}(\Omega)$. For any $1 \leq j \leq J$, let $\boldsymbol{\varphi}_j$ be the function in $\boldsymbol{\mathcal{V}}(\Omega)$ which satisfies (35). Now, setting

$$\boldsymbol{u}_k = \boldsymbol{v}_k - \sum_{j=1}^{J} \Big(\int_{\Sigma_j} \boldsymbol{v}_k \cdot \boldsymbol{n} \, d\sigma \Big) \, \boldsymbol{\varphi}_j,$$

the function $\boldsymbol{u}_k$ belongs to $\boldsymbol{\mathcal{V}}(\Omega) \perp \boldsymbol{K}_T(\Omega)$. Also the sequence $(\boldsymbol{u}_k)$ converges to $\boldsymbol{v}$ in $\boldsymbol{W}^{m,r}(\Omega)$, which is the required result. □

Theorem 4.2. *Let Ω be a bounded and connected open set of $\mathbb{R}^3$ with a Lipschitz-continuous boundary and m a nonnegative integer. For any $\boldsymbol{f} \in \boldsymbol{U}^{m,r}(\Omega)$, there exists $\boldsymbol{\psi} \in \boldsymbol{W}_0^{m+1,r}(\Omega)$ that satisfies*

$$\mathbf{curl}\, \boldsymbol{\psi} = \boldsymbol{f} \quad \textit{in} \quad \Omega,$$

and there exists a constant $C_{r,m}(\Omega)$ such that

$$\|\boldsymbol{\psi}\|_{\boldsymbol{W}^{m+1,r}(\Omega)} \leq C_{r,m}(\Omega) \|\boldsymbol{f}\|_{\boldsymbol{W}^{m,r}(\Omega)}. \tag{36}$$

Proof. Let $\boldsymbol{f} \in \boldsymbol{U}^{m,r}(\Omega)$ and $(\boldsymbol{f}_n)$ a sequence in $\boldsymbol{\mathcal{V}}(\Omega) \perp \boldsymbol{K}_{\boldsymbol{T}}(\Omega)$, such that

$$\boldsymbol{f}_n \longrightarrow \boldsymbol{f} \quad \text{in} \quad \boldsymbol{W}^{m,r}(\Omega).$$

Theorem 3.4 shows that for any $n \in \mathbb{N}$, there exists a vector field $\boldsymbol{\psi}_n \in \boldsymbol{\mathcal{D}}(\Omega)$ such that

$$\boldsymbol{\psi}_n \in \boldsymbol{\mathcal{D}}(\Omega), \quad \mathbf{curl}\, \boldsymbol{\psi}_n = \boldsymbol{f}_n, \ \textit{and} \ \|\boldsymbol{\psi}_n\|_{\boldsymbol{W}^{m+1,r}(\Omega)} \leq C(r,m,\Omega) \|\boldsymbol{f}_n\|_{\boldsymbol{W}^{m,r}(\Omega)}.$$

Clearly $(\boldsymbol{\psi}_n)$ is a Cauchy sequence. Then, there exists an element $\boldsymbol{\psi} \in \boldsymbol{W}_0^{m+1,r}(\Omega)$ such that

$$\boldsymbol{\psi}_n \longrightarrow \boldsymbol{\psi} \quad \text{in} \quad \boldsymbol{W}^{m+1,r}(\Omega),$$

with $\boldsymbol{\psi}$ satisfies (36). □

Remark 4.1.
i) Theorem 4.2 was proved for Ω bounded and simply-connected open set of $\mathbb{R}^3$ with Lipschitz-continuous boundary, $m = 1$ and $r = 2$ by Ciarlet and Ciarlet, Jr (see the proof of Theorem 3.1 in [15]) and for m a nonnegative integer and $r = 2$ by Amrouche, Ciarlet and Ciarlet, Jr (see [4]).
ii) For m nonnegative integer and $r = 2$, as in [4], we can define a vector field $\boldsymbol{\psi}_{\mathbf{0}} \in \boldsymbol{H}_0^{m+1}(\Omega)$ such that $\mathbf{curl}\, \boldsymbol{\psi}_{\mathbf{0}} = \boldsymbol{f}$ in Ω and $\mathrm{div} \boldsymbol{\Delta}^{m+1} \boldsymbol{\psi}_{\mathbf{0}} = 0$ in Ω. For that, it is sufficient to choose $\boldsymbol{\psi}_{\mathbf{0}} = \boldsymbol{\psi} - \mathbf{grad}\, p$, where p is the unique solution in $\boldsymbol{H}_0^{m+2}(\Omega)$ of $\Delta^{m+2} p = \mathrm{div} \boldsymbol{\Delta}^{m+1} \boldsymbol{\psi}$ and $\boldsymbol{\psi}$ is

given by Theorem 4.2.

iii) For Ω bounded and connected open set of $\mathbb{R}^3$ with boundary of class $\mathcal{C}^{m+2}$, Borchers and Sohr in [11] established the same result that Theorem 4.2 with $\operatorname{div} \boldsymbol{\Delta}^{m+1}\boldsymbol{\psi} = 0$. Moreover, for $m = 1$ and Ω of class $\mathcal{C}^{1,1}$, Amrouche, Bernardi, Dauge and Girault in [3] gave another proof of the result established by Borchers and Sohr. Furthermore, they proved that the vector field $\boldsymbol{\psi} \in \boldsymbol{H}_0^1(\Omega)$ is unique, provided that

$$\langle \partial_n(\operatorname{div} \boldsymbol{\psi}), 1\rangle_{\Gamma_i} = 0,\ 1 \leq i \leq I,$$

where Γ_i are the different connected components of $\partial\Omega$.

The following weak rotational extension of De Rham's Theorem is a direct consequence of Theorem 4.2. We define the space $\boldsymbol{\mathcal{G}}^{\boldsymbol{m},\boldsymbol{r}}(\Omega)$ by

$$\boldsymbol{\mathcal{G}}^{\boldsymbol{m},\boldsymbol{r}}(\Omega) = \{\boldsymbol{\varphi} \in \boldsymbol{W}_0^{m,r}(\Omega),\ \mathbf{curl}\,\boldsymbol{\varphi} = \mathbf{0} \quad \text{in} \quad \Omega\}.$$

Theorem 4.3. *Let Ω be a bounded and connected open set of $\mathbb{R}^3$ with a Lipschitz-continuous boundary and m an integer such that $m \geq 1$, $\boldsymbol{f} \in \boldsymbol{W}^{-m,r'}(\Omega)$ and satisfies*

$$\forall \boldsymbol{\varphi} \in \boldsymbol{\mathcal{G}}^{\boldsymbol{m},\boldsymbol{r}}(\Omega),\ \ {}_{\boldsymbol{W}^{-\boldsymbol{m},\boldsymbol{r}'}(\Omega)}\langle \boldsymbol{f}, \boldsymbol{\varphi}\rangle_{\boldsymbol{W}_0^{m,r}(\Omega)} = 0. \tag{37}$$

Then, there exists $\boldsymbol{\Psi} \in \boldsymbol{W}^{-m+1,r'}(\Omega)$, such that

$$\mathbf{curl}\,\boldsymbol{\Psi} = \boldsymbol{f} \ \text{ in } \Omega.$$

Proof. According to Theorem 4.2, the operator

$$\mathbf{curl} : \boldsymbol{W}_0^{m,r}(\Omega)/\boldsymbol{\mathcal{G}}^{\boldsymbol{m},\boldsymbol{r}}(\Omega) \longrightarrow \boldsymbol{U}^{m-1,r}(\Omega),$$

is one to one and onto. Then, its adjoint

$$\mathbf{curl} : (\boldsymbol{U}^{m-1,r}(\Omega))' \longrightarrow \boldsymbol{W}^{-m,r'}(\Omega) \perp \boldsymbol{\mathcal{G}}^{\boldsymbol{m},\boldsymbol{r}}(\Omega), \tag{38}$$

is also one to one and onto. Proceeding as in the proof of Theorem 3.5 and using Theorem 4.2, it is easy to prove that

$$(\boldsymbol{U}^{m-1,r}(\Omega))' = \boldsymbol{W}^{-m+1,r'}(\Omega)/\mathbf{Ker}\,\mathbf{curl}, \tag{39}$$

where

$$\mathbf{curl} :\ \boldsymbol{W}^{-m+1,r'}(\Omega) \longrightarrow \boldsymbol{W}^{-m,r'}(\Omega).$$

Let $\boldsymbol{f} \in \boldsymbol{W}^{-m,r'}(\Omega)$ satisfying (37). In other words, $\boldsymbol{f} \in \boldsymbol{W}^{-m,r'}(\Omega) \perp \boldsymbol{\mathcal{G}}^{\boldsymbol{m},\boldsymbol{r}}(\Omega)$. Since the operator (38) is an isomorphism, the characterization (39) implies that there exists $\boldsymbol{\Psi} \in \boldsymbol{W}^{-m+1,r'}(\Omega)$ such that $\mathbf{curl}\,\boldsymbol{\Psi} = \boldsymbol{f}$ in Ω. $\square$

5 A new proof of the general extension of Poincaré's Lemma

The classical Poincaré's Lemma asserts that if Ω is a simply-connected open set, then for any $\boldsymbol{h} \in \boldsymbol{\mathcal{C}}^{\mathbf{1}}(\Omega)$ which satisfies $\mathbf{curl}\,\boldsymbol{h} = \mathbf{0}$ in Ω, there exists $p \in \mathcal{C}^2(\Omega)$ such that $\boldsymbol{h} = \mathbf{grad}\,p$. This lemma is also true in the general case where $\boldsymbol{h} \in \boldsymbol{L}^2(\Omega)$ and Ω is a bounded and simply-connected open set with a Lipschitz-continuous boundary (see Theorem 2.9 chapter 1 in [24]). A general extension when $\boldsymbol{h} \in \boldsymbol{H}^{-1}(\Omega)$ was proved by Ciarlet and Ciarlet, Jr (see [15]).

In this section, we study the case where $\boldsymbol{h}$ is a distribution. The first proof of this extension, based on differential geometry tools, was given by S. Mardare [27] in 2008 in the case where Ω is a simply-connected open (Schwartz also proved this extension for $\Omega = \mathbb{R}^3$, see Section 3 of [37]). Here, we give a simpler proof, using the characterization of the dual space $\boldsymbol{\mathcal{V}}(\Omega)'$ given in the proof of Theorem 3.5.

Lemma 5.1. *Let Ω be a bounded and simply-connected open set of $\mathbb{R}^3$ with a Lipschitz-continuous boundary and $\boldsymbol{h} \in \boldsymbol{\mathcal{D}}'(\Omega)$. If*

$$\mathbf{curl}\ \boldsymbol{h} = \mathbf{0} \quad in \quad \Omega,$$

then there exists $p \in \mathcal{D}'(\Omega)$ such that

$$\boldsymbol{h} = \mathbf{grad}\,p \quad in \quad \Omega.$$

Proof. Let $\boldsymbol{L} \in \boldsymbol{\mathcal{V}}(\Omega)'$. Since $\boldsymbol{\mathcal{V}}(\Omega)$ is closed in $\boldsymbol{\mathcal{D}}(\Omega)$, we can extend $\boldsymbol{L}$ by $\widetilde{\boldsymbol{L}} \in \boldsymbol{\mathcal{D}}'(\Omega)$. Two expressions $\boldsymbol{g} \in \boldsymbol{\mathcal{D}}'(\Omega)$ and $\boldsymbol{h} \in \boldsymbol{\mathcal{D}}'(\Omega)$ of $\widetilde{\boldsymbol{L}}$ coincide on $\boldsymbol{\mathcal{V}}(\Omega)$ if and only if

$$\forall\, \boldsymbol{\varphi} \in \boldsymbol{\mathcal{V}}(\Omega), \ \ {}_{\boldsymbol{\mathcal{D}}'(\Omega)}\langle \boldsymbol{g} - \boldsymbol{h}, \boldsymbol{\varphi}\rangle_{\boldsymbol{\mathcal{D}}(\Omega)} = 0.$$

According to the usual De Rham's Theorem, there exists $p \in \mathcal{D}'(\Omega)$ such that $\boldsymbol{g} - \boldsymbol{h} = \mathbf{grad}\,p$. This means that we can define $\boldsymbol{\mathcal{V}}(\Omega)'$ as follows:

$$\boldsymbol{\mathcal{V}}(\Omega)' = \boldsymbol{\mathcal{D}}'(\Omega)/\mathbf{Im}\,(\mathbf{grad}) \quad \text{where} \quad \mathbf{grad} : \mathcal{D}'(\Omega) \longrightarrow \boldsymbol{\mathcal{D}}'(\Omega). \tag{40}$$

It has already been shown that

$$\boldsymbol{\mathcal{V}}(\Omega)' = \boldsymbol{\mathcal{D}}'(\Omega)/\mathbf{ker}\,\mathbf{curl} \quad \text{where} \quad \mathbf{curl} : \boldsymbol{\mathcal{D}}'(\Omega) \longrightarrow \boldsymbol{\mathcal{D}}'(\Omega). \tag{41}$$

According to (40) and (41), we conclude that

$$\mathbf{Ker}\,(\mathbf{curl}) = \mathbf{Im}\,(\mathbf{grad}), \tag{42}$$

hence the required result. $\square$

6 Beltrami representation

In Section 3, we have shown that the operator (12) is onto. Then, we have used this surjectivity result to prove a rotational extension of De Rham's Theorem. In this section, we use the same argument to prove an extension of the Beltrami representation. First, we show a completeness of the Beltrami's representation for data in $\mathbb{D}_s(\Omega)$.

Theorem 6.1. *Let Ω be a bounded and connected open set of $\mathbb{R}^3$ with a Lipschitz-continuous boundary, r a real number such that $1 < r < \infty$ and m a nonnegative integer. For any matrix $\mathbb{S} \in \mathbb{V}_s(\Omega)$ satisfies $\int_\Omega \mathbb{S} : \mathbb{M}\, dx = 0$ for all $\mathbb{M} \in \mathbb{K}_T(\Omega)$, there exists $\mathbb{A} \in \mathbb{D}_s(\Omega)$ such that*

$$\mathbf{curl}\,\mathbf{curl}\,\mathbb{A} = \mathbb{S} \quad \textit{in} \quad \Omega.$$

Moreover, there exists a constant C depending only on r, m and Ω such that

$$\|\mathbb{A}\|_{\mathbb{W}^{m+2,r}(\Omega)} \leq C\|\mathbb{S}\|_{\mathbb{W}^{m,r}(\Omega)}. \tag{43}$$

Proof. The proof follows the lines of the proof of Theorem 2.2 in [22]. Let $\mathbb{S} \in \mathbb{V}_s(\Omega) \perp \mathbb{K}_T(\Omega)$ i.e., $\mathbb{S} \in \mathbb{V}_s(\Omega)$ and satisfying $\int_\Omega \mathbb{S} : \mathbb{M}\, dx = 0$ for all $\mathbb{M} \in \mathbb{K}_T(\Omega)$. That means that for any $1 \leq i \leq 3$ and any $1 \leq j \leq J$ (see [16])

$$\mathbf{div}\,\mathbb{S} = \mathbf{0} \ \text{ in } \ \Omega, \tag{44}$$

$$\int_{\Sigma_j} (\mathbb{S} \cdot \boldsymbol{n}) \cdot \boldsymbol{e}^i \, d\sigma = 0, \tag{45}$$

$$\int_{\Sigma_j} (\mathbb{S} \cdot \boldsymbol{n}) \cdot \boldsymbol{P}^i \, d\sigma = 0. \tag{46}$$

Observe that conditions (44), (45) are equivalent to: for each $1 \leq i \leq 3$, $\boldsymbol{S}^i \in \boldsymbol{\mathcal{V}}(\Omega) \perp \boldsymbol{K}_T(\Omega)$ where $\boldsymbol{S}^i$ is the i-th line of matrix $\mathbb{S}$. Then, Theorem 3.4 implies that there exists some vector field $\boldsymbol{W}^i$ in $\boldsymbol{\mathcal{D}}(\Omega)$ such that $\mathbf{curl}\,\boldsymbol{W}^i = \boldsymbol{S}^i$, and satisfying the estimate

$$\|\boldsymbol{W}^i\|_{\boldsymbol{W}^{m+1,r}(\Omega)} \leq C\|\boldsymbol{S}^i\|_{\boldsymbol{W}^{m,r}(\Omega)}.$$

We define $\mathbb{W}$ the matrix field whose lines are the vectors $\boldsymbol{W}^i$. So $\mathbb{W}$ satisfies $\mathbf{curl}\,\mathbb{W} = \mathbb{S}^T$ in Ω and

$$\|\mathbb{W}\|_{\mathbb{W}^{m+1,r}(\Omega)} \leq C\|\mathbb{S}\|_{\mathbb{W}^{m,r}(\Omega)}.$$

Now setting $\mathbb{B} = \mathbb{W}^T - \mathrm{tr}(\mathbb{W})\mathbb{I}$. The symmetry of $\mathbb{S}$ implies that

$$\mathbf{div}\,\mathbb{B} = \mathbf{0} \quad \text{in} \quad \Omega. \tag{47}$$

Indeed, for $i = 1$ for example, we have

$$\begin{aligned}\mathbf{div}\,\boldsymbol{B}^1 &= \partial_2 W_{21} - \partial_1 W_{22} + \partial_3 W_{31} - \partial_1 W_{33}\\ &= 0.\end{aligned}$$

Moreover,

$$\int_{\Sigma_j}((\mathbf{curl}\,\mathbb{W})^T\cdot\boldsymbol{n})\cdot\boldsymbol{P}^i\,d\sigma = \int_{\Sigma_j}((\mathbf{curl}(\mathbb{P}\mathbb{W}))^T\cdot\boldsymbol{n})\cdot\boldsymbol{e}^i\,d\sigma + \int_{\Sigma_j}(\mathbb{W}^T\cdot\boldsymbol{n})\cdot\boldsymbol{e}^i\,d\sigma - \int_{\Sigma_j}(\mathrm{tr}(\mathbb{W})\mathrm{I}\cdot\boldsymbol{n})\cdot\boldsymbol{e}^i\,d\sigma. \tag{48}$$

Because $\mathbb{P}\mathbb{W}\in\mathbb{D}(\Omega)$, we get

$$\int_{\Sigma_j}(\mathbf{curl}(\mathbb{P}\mathbb{W}))^T\cdot\boldsymbol{n})\cdot\boldsymbol{e}^i\,d\sigma = 0. \tag{49}$$

Hence (46), (48) and (49) imply that

$$\int_{\Sigma_j}(\mathbb{B}\cdot\boldsymbol{n})\cdot\boldsymbol{e}^i\,d\sigma = 0. \tag{50}$$

By using (47), (50) and applying again Theorem 3.4, there exists a matrix field $\mathbb{D}$ in $\mathbb{D}(\Omega)$ such that

$$\mathbf{curl}\,\mathbb{D} = \mathbb{B}^T = \mathbb{W} - \mathrm{tr}(\mathbb{W})\mathrm{I}, \tag{51}$$

with

$$\|\mathbb{D}\|_{\mathbb{W}^{m+2,r}(\Omega)} \leq C\|\mathbb{B}\|_{\mathbb{W}^{m+1,r}(\Omega} \leq C\|\mathbb{S}\|_{\mathbb{W}^{m,r}(\Omega)}.$$

Therefore

$$\mathbf{curl}\,\mathbf{curl}\,\mathbb{D} = \mathbb{S}^T - \mathbf{curl}(\mathrm{tr}(\mathbb{W})\boldsymbol{I}). \tag{52}$$

We also have

$$\begin{aligned}\mathbf{curl}\,\mathbf{curl}\,\mathbb{D}^T &= (\mathbf{curl}\,\mathbf{curl}\,\mathbb{D})^T\\ &= \mathbb{S} + \mathbf{curl}(\mathrm{tr}(\mathbb{W})\boldsymbol{I}).\end{aligned} \tag{53}$$

Define $\mathbb{A} = \dfrac{\mathbb{D}+\mathbb{D}^T}{2}$, then (52) and (53) imply

$$\mathbf{curl}\,\mathbf{curl}\,\mathbb{A} = \frac{\mathbb{S}+\mathbb{S}^T}{2} = \mathbb{S},$$

which is the required result. □

P.G. Ciarlet et al in [16] stated the range of the operator $\mathbf{curl}\,\mathbf{curl} : \mathbb{H}^2_{0,s}(\Omega) \longrightarrow \mathbb{L}^2_s(\Omega)$ is the space $\mathbb{U}^{0,2}_s(\Omega)$. In the following, using the Beltrami's completeness given in Theorem 6.1, we generalize this result by showing that the operator $\mathbf{curl}\,\mathbf{curl} : \mathbb{W}^{m+2,p}_{0,s}(\Omega) \longrightarrow \mathbb{U}^{m,p}_s(\Omega)$ is onto, for any $1 < r < \infty$ and any nonnegative integer m.

Using the same argument of the proof of Lemma 4.1, the following result holds:

Lemma 6.2. *Let Ω be a bounded and connected open set of $\mathbb{R}^3$ with a Lipschitz-continuous boundary and m a nonnegative integer. Then the space $\mathbb{V}_s(\Omega) \perp \mathbb{K}_T(\Omega)$ is dense in $\mathbb{U}_s^{m,r}(\Omega)$.*

Theorem 6.3. *Let Ω be a bounded and connected open set of $\mathbb{R}^3$ with a Lipschitz-continuous boundary, r a real number such that $1 < r < \infty$ and m a nonnegative integer. For any matrix $\mathbb{S}$ in $\mathbb{U}_s^{m,r}(\Omega)$, there exists $\mathbb{A} \in \mathbb{W}_{0,s}^{m+2,r}(\Omega)$ such that*

$$\mathbf{curl}\,\mathbf{curl}\,\mathbb{A} = \mathbb{S} \quad \textit{in} \quad \Omega \quad \textit{and} \quad \|\mathbb{A}\|_{\mathbb{W}_s^{m+2,r}(\Omega)} \leq C\|\mathbb{S}\|_{\mathbb{W}_s^{m,r}(\Omega)}.$$

Proof. Let $\mathbb{A} \in \mathbb{U}_s^{m,r}(\Omega)$. Since $\mathbb{V}_s(\Omega) \perp \mathbb{K}_T(\Omega)$ is dense in $\mathbb{U}_s^{m,r}(\Omega)$, there exists a sequence $(\mathbb{S}_k)$ of $\mathbb{V}_s(\Omega) \perp \mathbb{K}_T(\Omega)$ such that

$$\mathbb{S}_k \longrightarrow \mathbb{S} \quad \text{in} \quad \mathbb{W}_s^{m,r}(\Omega) \quad \text{when} \quad k \longrightarrow \infty.$$

From Lemma 6.1, for any $k \in \mathbb{N}$, there exists $\mathbb{A}_k \in \mathbb{D}_s(\Omega)$ such that

$$\mathbf{curl}\,\mathbf{curl}\,\mathbb{A}_k = \mathbb{S}_k \quad \text{with} \quad \|\mathbb{A}_k\|_{\mathbb{W}_s^{m+2,r}(\Omega)} \leq C\|\mathbb{S}_k\|_{\mathbb{W}_s^{m,r}(\Omega)}.$$

Clearly $(\mathbb{A}_k)$ is a Cauchy sequence and there exists $\mathbb{A} \in \mathbb{W}_{0,s}^{m+2,r}(\Omega)$ such that

$$\mathbb{A}_k \longrightarrow \mathbb{A} \quad \text{in} \quad \mathbb{W}_s^{m+2,r}(\Omega).$$

with

$$\mathbf{curl}\,\mathbf{curl}\,\mathbb{A} = \mathbb{S} \quad \text{in} \quad \Omega \quad \text{and} \quad \|\mathbb{A}\|_{\mathbb{W}_s^{m+2,r}(\Omega)} \leq C\|\mathbb{S}\|_{\mathbb{W}_s^{m,r}(\Omega)}.$$

□

Now, we will use Lemma 6.1 to show the following extension of the Beltrami's completeness.

Theorem 6.4. *Let Ω be a bounded and connected open set of $\mathbb{R}^3$ with a Lipschitz-continuous boundary and $\mathbb{E} \in \mathbb{D}_s'(\Omega)$ satisfies*

$$_{\mathbb{D}'(\Omega)}\langle \mathbb{S}, \mathbb{E}\rangle_{\mathbb{D}(\Omega)} = 0 \quad \textit{for all} \quad \mathbb{E} \in \mathbb{G}_s(\Omega). \tag{54}$$

Then there exists $\mathbb{A} \in \mathbb{D}_s'(\Omega)$ such that

$$\mathbf{curl}\,\mathbf{curl}\,\mathbb{A} = \mathbb{S} \quad \textit{in} \quad \Omega.$$

Remark 6.1. The converse is obvious, because for any $\mathbb{S} \in \mathbb{D}_s'(\Omega)$ and any $\mathbb{E} \in \mathbb{G}_s(\Omega)$, we have

$$_{\mathbb{D}'(\Omega)}\langle \mathbf{curl}\,\mathbf{curl}\,\mathbb{S}, \mathbb{E}\rangle_{\mathbb{D}(\Omega)} = {}_{\mathbb{D}'(\Omega)}\langle \mathbb{S}, \mathbf{curl}\,\mathbf{curl}\,\mathbb{E}\rangle_{\mathbb{D}(\Omega)} = 0.$$

Proof. According to Theorem 6.1,

$$\mathbf{curl\,curl}\ :\ \mathbb{D}_s(\Omega)/\mathbb{G}_s(\Omega) \longrightarrow \mathbb{V}_s(\Omega)\perp\mathbb{K}_T(\Omega) \tag{55}$$

is one to one and onto. Then its adjoint

$$\mathbf{curl\,curl}\ :\ (\mathbb{V}_s(\Omega)\perp\mathbb{K}_T(\Omega))' \longrightarrow \mathbb{D}'_s(\Omega)\perp\mathbb{G}_s(\Omega)$$

is one to one and onto.

Let $\boldsymbol{L}\in(\mathbb{V}_s(\Omega)\perp\mathbb{K}_T(\Omega))'$ and $\widetilde{\boldsymbol{L}}$ any extension of $\boldsymbol{L}$ in $\mathbb{D}'_s(\Omega)$. Two expressions $\mathbb{S}$ and $\mathbb{A}$ of $\widetilde{\boldsymbol{L}}$ coincide on $\mathbb{V}_s(\Omega)\perp\mathbb{K}_T(\Omega)$ if and only if,

$$\forall\mathbb{E}\in\mathbb{V}_s(\Omega)\perp\mathbb{K}_T(\Omega),\ {}_{\mathbb{D}'(\Omega)}\langle\mathbb{S}-\mathbb{A},\mathbb{E}\rangle_{\mathbb{D}(\Omega)}=0.$$

Using again Lemma 6.1, we get

$$\forall\mathbb{B}\in\mathbb{D}_s(\Omega),\ {}_{\mathbb{D}'(\Omega)}\langle\mathbb{S}-\mathbb{A},\mathbf{curl\,curl}\,\mathbb{B}\rangle_{\mathbb{D}(\Omega)} \quad=\quad {}_{\mathbb{D}'(\Omega)}\langle\mathbf{curl\,curl}\,(\mathbb{S}-\mathbb{A}),\mathbb{B}\rangle_{\mathbb{D}(\Omega)}=0,$$

which means that $\mathbb{S}-\mathbb{A}\in\mathbf{Ker\,curl\,curl}$, where

$$\mathbf{curl\,curl}\ :\ \mathbb{D}'_s(\Omega)\longrightarrow\mathbb{D}'_s(\Omega).$$

Consequently,

$$(\mathbb{V}_s(\Omega)\perp\mathbb{K}_T(\Omega))'=\mathbb{D}'_s(\Omega)/\mathbf{Ker\,curl\,curl}. \tag{56}$$

Let $\mathbb{S}\in\mathbb{D}'_s(\Omega)$ satisfies (54). In other words, that means that $\mathbb{S}\in\mathbb{D}'_s(\Omega)\perp\mathbb{G}_s(\Omega)$. Since the operator (55) is an isomorphism, the characterization (56) implies that there exists $\mathbb{A}\in\mathbb{D}'_s(\Omega)$ such that $\mathbf{curl\,curl}\,\mathbb{A}=\mathbb{S}\quad\text{in}\quad\Omega.$ □

7 The general extension of Saint-Venant's Theorem

Podio-Guidugli in [35] have used a Beltrami's completeness to show the equivalence between the sufficient conditions of Donati's and Saint-Venant's Theorems: Let Ω be a smooth bounded and simply-connected open set of $\mathbb{R}^3$, then any symmetric matrix field $\mathbb{E}=(E_{ij})$ with $E_{ij}\in\mathcal{C}^N(\Omega)$ $(N\geq 2)$ satisfies

$$\mathbf{curl\,curl}\,\mathbb{E}=\mathbf{0}\quad\text{in }\Omega,$$

if and only if

$$\int_\Omega\mathbb{E}:\mathbb{M}\,dx=0\quad\text{for any}\ \ \mathbb{M}\in\mathbb{V}_s(\Omega).$$

Later, Geymonat and Krasucki in [21] have proved the above equivalence when $\mathbb{E} \in \mathbb{L}^2_s(\Omega)$ and they have used it together with Ting's Theorem to conclude an extension of Saint-Venant's Theorem in $\mathbb{L}^2_s(\Omega)$. In the following, we will use the same idea to present an extension of Saint-Venant's Theorem in $\mathbb{D}'_s(\Omega)$.

Theorem 7.1. *Let Ω be a bounded and simply-connected open set of $\mathbb{R}^3$ with a Lipschitz-continuous boundary and $\mathbb{E} \in \mathbb{D}'_s(\Omega)$ satisfies*

$$\mathbf{curl}\,\mathbf{curl}\,\mathbb{E} = \mathbf{0} \quad in \quad \Omega. \tag{57}$$

Then there exists $\boldsymbol{v} \in \boldsymbol{\mathcal{D}}'(\Omega)$ such that

$$\boldsymbol{\nabla}_{\boldsymbol{s}}\boldsymbol{v} = \mathbb{E} \quad in \quad \Omega.$$

Proof. Let $\mathbb{E}$ be a symmetric matrix field in $\mathbb{D}'_s(\Omega)$ such that $\mathbf{curl}\,\mathbf{curl}\,\mathbb{E} = \mathbf{0}$ in Ω. We have already shown that for any symmetric matrix field $\mathbb{A}$ in $\mathbb{V}_s(\Omega)$, there exists $\mathbb{B} \in \mathbb{D}_s(\Omega)$ such that $\mathbf{curl}\,\mathbf{curl}\,\mathbb{B} = \mathbb{A}$. Then, we have

$$_{\mathbb{D}'(\Omega)}\langle \mathbb{E},\, \mathbb{A}\rangle_{\mathbb{D}(\Omega)} = {}_{\mathbb{D}'(\Omega)}\langle \mathbb{E},\, \mathbf{curl}\,\mathbf{curl}\,\mathbb{B}\rangle_{\mathbb{D}(\Omega)} = {}_{\mathbb{D}'(\Omega)}\langle \mathbf{curl}\,\mathbf{curl}\,\mathbb{E},\, \mathbb{B}\rangle_{\mathbb{D}(\Omega)} = 0.$$

Thus, Moreau's Theorem implies that there exists $\boldsymbol{v} \in \boldsymbol{\mathcal{D}}'(\Omega)$ such that $\boldsymbol{\nabla}_{\boldsymbol{s}}\boldsymbol{v} = \mathbb{E}$ in Ω, which is the required result. □

References

[1] R.A. Adams: Sobolev Spaces, Academic Press, New York, (1975).

[2] B. Airy, IV: On the strains in the interior of beams. *Phil. Trans. Royal. Soc. Lond.* **153**, 49-79, (1863).

[3] C. Amrouche, C. Bernardi, M. Dauge, V. Girault: Vector potentials in three-dimensional non-smooth domains. *Math. Meth. Appl. Sci.* **21**, 823-864, (1998).

[4] C. Amrouche, P.G. Ciarlet, P. Ciarlet, Jr: Weak vector and scalar potentials: applications to Poincaré's theorem and Korn's inequality in Sobolev spaces with negative exponents. *Anal and Appl.* **8**, 1-17, (2010).

[5] C. Amrouche, P.G. Ciarlet, L. Gratie, S. Kesavan: On the characterizations of matrix fields as linearized strain tensor fields. *J. Math. Pures Appl.* **86**, 116-132, (2006).

[6] C. Amrouche, P.G. Ciarlet, C. Mardare: On a lemma of Jacques-Louis Lions and its relations to other fundamental results. *J. Math. Pures Appl.* **104**, 207-226, (2015).

[7] C. Amrouche, V. Girault: Decomposition of vector spaces and application to the Stokes problem in arbitrary dimension. *Czeckoslov. Math. J.* **44**, 109-140, (1994).

[8] J.M. Ball, A. Zarnescu: Partial regularity and smooth topology-preserving approximations of rough domains. Preprint Oxford Center for Nonlinear PDE, Mathematical Institute, University of Oxford. December 16, 2013; also arXiv:1312.5156.

[9] E. Beltrami: Osservazioni sulla nota precedente. *Atti. Accad. Lincei. Rend.* **1**, 141-142, (1892).

[10] M.E. Bogovskiĭ: Solution of the first boundary value problem for the equation of continuity of an incompressible medium. *Sov. Math. Dokl.* **20**, 1094-1098, (1979).

[11] W. Borchers, H. Sohr: On the equation $\mathbf{rot}\, \boldsymbol{v} = \boldsymbol{g}$ and $\operatorname{div} \boldsymbol{u} = f$ with zero boundary conditions. *Hokkaido Math. J.* **19**, 67-87, (1990).

[12] H. Brézis: Functional Analysis, Sobolev Spaces and Partial Differential Equations, Springer, New York, (2011).

[13] A.P. Calderòn, A. Zygmund: On singular integrals. *Amer. J. Math.* **78**, 289-309, (1956).

[14] J. Cantarella, D. De Turck, H. Gluck: Vector calculus and topology of domains in 3-space. *Amer. Math. Monthly.* **109**, 409-442, (2002).

[15] P.G. Ciarlet, P. Ciarlet, Jr: Another approach to linearized elasticity and a new proof of Korn's inequality, *Math. Mod. Meth. Appl. Sci.* $\mathbf{15-2}$, 259-271, (2005).

[16] P.G. Ciarlet, P. Ciarlet Jr, G. Geymonant, F. Krasucki: Characterization of the kernel of the operator CURL CURL, *C. R. Acad. Sci. Paris, Ser.* I **344**, 305-308, (2007).

[17] M. Costabel, A. McIntosh: On Bogovskiĭ and regularized Poincaré integral operators for de Rham complexes on Lipschitz domains. *Math. Z.* **265**, 297-320, (2010).

[18] G. Duvaut, J.L. Lions: Les Inéquations en Mécanique et en Physique, Dunod, Paris, (1972).

[19] R. Fosdick, G. Royer-Carfagni: A Stokes theorem for second-order tensor fields and its implications in continuum mechanics. *Internat. J. Non-Linear Mech,* **40**, 381-386, (2005).

[20] G.P. Galdi: An Introduction to the Mathematical Theory of the Navier-Stokes Equations, Steady-State Problems, Second Edition, Springer, (2011).

[21] G. Geymonat, F. Krasucki: Some remarks on the compatibility conditions in elasticity. *Accad. Naz. Sci. XL, Mem. Mat.Appl,* ***XXIX***, 175–182, (2005).

[22] G. Geymonat, F. Krasucki: Beltrami's solutions of general equilibrium equations in continuum mechanics. *C. R. Acad. Sci. Paris,* **342**, 359-363, (2006).

[23] G. Geymonat, F. Krasucki: Hodge decomposition for symmetric matrix fields and the elasticity complex in Lipschitz domains. *Commun. Pure. Appl. Anal,* **8**, 295-309, (2009).

[24] V. Girault, P.A. Raviart: Finite Element Methods for Navier-Stokes Equations, Springer, Berlin, (1986).

[25] P. Grisvard: Elliptic Boundary Value Problems in Nonsmooth Domains, Pitman, London, (1985).

[26] M. Gurtin: The linear Theory of Elasticity, C. Truesdell (Ed.), Handbuch der Physik, vol. 2, Springer, Berlin, (1972).

[27] S. Mardare: On Poincaré and De Rham's theorem. *Rev. Roum. Math. Pures Appl.* **53**, 523-541, (2008).

[28] J. C. Maxwell: On reciprocal figures, frames, and diagrams of forces. *Earth. Envi. Sci. Trans. Roya. Soci. Edin.* **26**, 1-40, (1870).

[29] V. Maz'ja: Sobolev Spaces, Springer Berlin, Heidelberg, (1985).

[30] J.J. Moreau: Duality characterization of strain tensor distributions in arbitrary open set. *J. Math. Anal. Appl.* **72**, 760-770, (1979).

[31] G. Morera: Soluzione generale delle equazioni indefinite dell'equilibrio di un corpo continuo. *Atti Accad. Naz. Lincei. Rend. Cl. Fis. Mat. Natur.* **1**, 137-141, (1892).

[32] M. Mitrea: Shap Hodge decompositions, Maxwell's equations, and vector Poisson problems on nonsmooth three-dimensional Riemaniann manifolds. *Duke Math. J.* **125 (3)**, 467-547, (2004).

[33] D. Mitrea, M. Mitrea, S. Monniaux: The Poisson problem for the exterior derivative operator with Dirichlet boundary condition on nonsmooth domains. *Comm. Pure Appl. Anal.* **7(6),** 1295-1333, (2008).

[34] J. Nečas: Direct Methods in the Theory of Elliptic Equations, Springer Monographs in Mathematics, Springer, Berlin, (2012). Translated from the 1967 French original by G. Tronel and A. Kufner. Editorial coordination and preface by Š. Nečasová and a contribution by Ch. G. Simader.

[35] P.P. Podio-Guidugli: The compatibility constraint in linear elasticity. *J. Elasticity.* **59**, 393-398, (2000).

[36] L. Schwartz: Cours d'Analyse, Deuxième Partie, Ecole Polytechnique, (1959).

[37] L. Schwartz: Théorie des Distributions, Herman, Paris, (1966).

[38] T.W. Ting: St. Venant's compatibility conditions. *Rock. Moun. Math. Cons.* **28**, 5-12, (1974).