

APODICTIC DISCOURSE IN ANCIENT AND MODERN MATHEMATICS

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ISSUE

- Many important Indian authors, in particular S. Ramanujan, appear not to have justified their results. If it is more than an induction, **there must be an implied derivation.**
- We show that, in some cases, the **derivation was encoded through the discursive structure** itself. This is what we call an **apodictic discourse.**
- Examples: Brahmagupta, Ramanujan, Hardy, Gauss, Tartaglia...

MATHEMATICAL DISCOURSE

...may belong to all categories but

- research mathematicians prefer **apodictic discourse**...
- ...because it enables imparting **motivation**, history and derivation,...
- ...allows reading at **different levels**...
- ...and makes the continuity of knowledge clear.
- For Ramanujan : may be useful to **scrutinize sequences of propositions** even when they are not formally proved.

ARGUMENT

- A. Mathematics made public is a discourse
- B. Case study: Brahmagupta's "triquadrilateral".
[How do you analyze apodictic discourse.]
- C. Ramanujan's discursive strategies
(examples).

For Ramanujan, almost all sources are now available: Berndt (*Notebooks*), Hardy, R.'s Collected papers. But R.'s results have often been **verified**, seldom **derived**.

A. MATHEMATICS AS DISCOURSE

ARGUMENTATION IS DISCURSIVE

- Mathematical activity is in part non-discursive (Hadamard), but precise **communication is a discourse**.
- Non-rigorous discourse (including comments on tables, diagrams etc.) may have **heuristic value** ; can also be misleading.
- **Ex.: a figure** may **suggest** an argument, **it is not an argument**.
- **Ex.: Lists of rules?** Procedures? Tables, memory aids?
None of these is in itself a discourse.

ARGUMENTATION IS DISCURSIVE

- **Mathematical discourse implies an *audience* and a structured *argumentation*:**
 - Is the audience oneself (math'l diaries, *private notes*, etc.)?
 - Is it rigorous [**essential arguments present**] and if so,
 - Is it aimed at imparting some content to a group to which the author does not belong (*dogmatic discourse*)?
 - Or does it indicate motivation and steps of derivations for scholars with about the same level of knowledge (*apodictic discourse*)?

FORMS OF MATHEMATICAL DISCOURSE

1. **Rigorous** : Apodictic or dogmatic discourse
2. **Suggestive**, but non-rigorous (possibly misleading) : **heuristic** d., memory aids etc.
3. **Inconclusive**: arguments based on sense perception (“cas de figure”), analogies,...
4. **Communication** in M. is through discourses.
5. **Analysis of earlier work** is a discourse analysis.

What binds mathematicians is the experience that rigorous discourse is possible.

DOGMATIC AND APODICTIC DISCOURSES ARE COMPLEMENTARY

- Archimedes seems (Wallis, quoted by Heath, HGM II, 20) “as it were of set purpose to have covered up the traces of his investigation, as if he had grudged posterity the secret of his method of inquiry, while he wished to extort from them assent to his results”. “...not only Archimedes but nearly all the ancients so hid from posterity their method of Analysis... that more modern mathematicians found it easier to invent a new Analysis than to seek out the old.” **Dogmatic discourse insufficient.**
- Tartaglia’s discourse in the *Quesiti* not understood by Cardano. **Apodictic discourse requires an audience with the same background as the author.**
- Neither type of discourse is “universal” since either may fail to be understood by some. But **apodictic discourse** may be **explicated** by internal analysis, **dogmatic discourse** may be **verified**. (There are intermediate types.)

EXAMPLES

Refs.: <https://www.normalesup.org/~kichenassamy>

(How **Brahmagupta** obtained his results on the **cyclic quadrilateral**)

- *Historia Mathematica*, 37(1) (2010) 28-61.
- *Historia Mathematica*, 39(4), (2012) 387-404.

(How **Brahmagupta** obtained his results on **congruences**)

- *Gaṇita Bhāratī*, to appear

(How **Tartaglia** obtained his results on **cubic equations**)

- *Historia Mathematica*, 2015, 42 (4), 407-435.

(How **Baudhāyana** obtained his approx. **quadrature of the circle**)

- *Historia Mathematica*, 33, 2006, pp.149-183.

REFERENCES : DISCURSIVITY IN MATHEMATICS AND PHILOSOPHY

- *Comptes-rendus des séances de l'Académie des inscriptions et belles-lettres*, Paris : Durand, 2012-II (4), pp.781-796.
- *Gaṇita Bhāratī*, (2011).
- *Journal Asiatique*, 2018, 306 (1), 85-99.
- « L'emploi métonymique de l'arbre *kallāl* dans la philosophie médiévale en pays tamoul. » P.-S. Filliozat et M. Zink (eds.). *L'Arbre en Asie*, De Boccard, Paris, 2018, pp.279-299.
- « L'irruption de l'infini : la légende de la colonne de lumière » (= *liṅgodbhava*), *Comptes-Rendus des Séances de l'Académie des Inscriptions et Belles-Lettres*, 2018 (4), to appear.

DISCOURSE ANALYSIS

How to analyze an apodictic discourse :

- Need to know it is **not merely heuristic** or borrowed etc.
- Need to have an idea of the **author's** previous **knowledge...**
- ... and **standard practices**.
- If so, perform **close reading** (every element of the text, including its form, must be significant).

Upshot :

- You **solve old problems** in the History of Mathematics.
- You tend to **read** every mathematical work **more closely...**
- ... and may **obtain new results** by following the argument implied by earlier papers.

**B. CASE STUDY:
BRAHMAGUPTA'S TRIQUADRILATERAL**

BRAHMAGUPTA'S DISCOURSE IN B.S.S. XII.21-38

- **Objective** : given a cyclic quadrilateral,
 - **express** in terms of its sides **all the lines** generated by **projection, extension and intersection** of already constructed segments, (XII.21-32) and
 - **parameterize** all possible sets of **sides** (XII.33-38).
- XII.21 states the **area formula**; XII.21-27 give elements of derivation; XII.28 the diagonals, and XII.29-32, all perpendiculars, projections etc.
- **Derivation of area formula** : the triquadrilateral is naturally split into two triangles with the **same circumcenter** and the **same circumradius**.

QUESTIONS

- XII.24 expresses the theorem that the “**trilateral in a semi-circle**” is right.
- XII.25 gives a general result on **similar half-oblongs**.
- XII.30-31 (“**Brahmagupta’s theorem**”) unknown outside India (Chasles, 1837).

This proves that **mathematics is not cumulative**.

Questions: How is the derivation **motivated**? Where did he specify that the quadrilateral was **cyclic** [= inscribed in a circle of unspecified radius] ?

BRAHMAGUPTA'S ARGUMENT

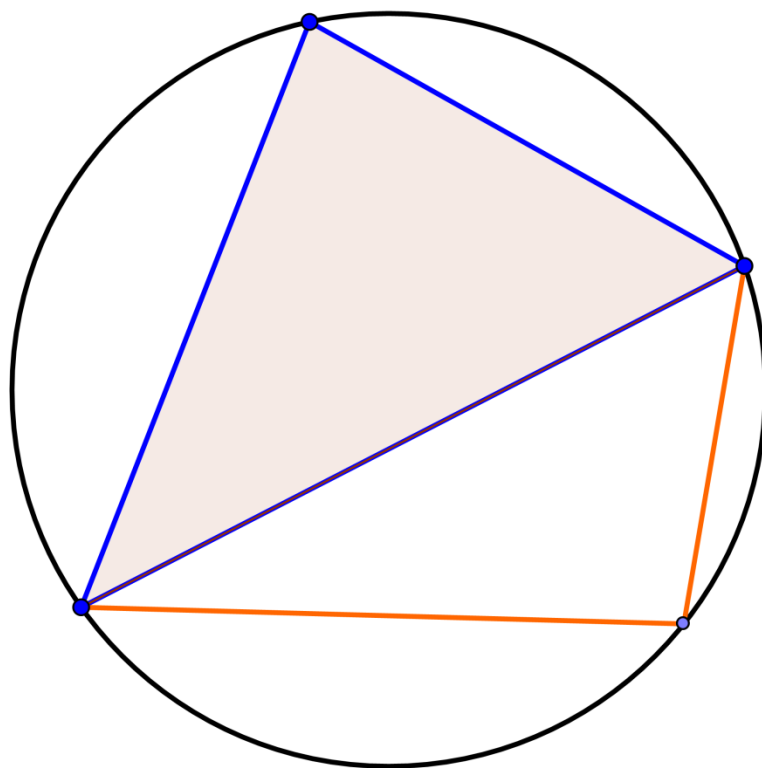
The cyclic character is indicated by a neologism “*tricatorbhuja*”, *lit.* triquadrilateral (XII.21), whose meaning is fixed by XII.27.

Proof:

- Term occurs only twice: XII.21, XII.27 : **specific** to Brahmagupta and to this situation.
- XII.27 tells how to compute the common circumradius of a trilateral and a triquadrilateral: it is **inscribed**, it is not a **trilateral**, and it contains a **distinguished triangle**.

Conclusion : the triquadrilateral is a **quadrilateral** obtained by completing a trilateral by a point arbitrarily taken on its circumcircle.

TRIQUADRILATERAL



12.21

12.21 *sthūlaphalam tricaturbhujabāhupratibāhuyogadalaghātaḥ
bhujayogārdhacatuṣṭayabhujonaghātāt padaṁ sūkṣmam*

*A **crude** value [indeed] of the area of a
triquadrilateral*

*Is the product of the half-sums of **opposite sides** ;*

*Of a group consisting of four half-sums of the
sides, from which*

*The sides have been subtracted [in turn], the root
of the product is the **refined** [value].*

AREA OF A TRIQUADRILATERAL

$$a, b, c, d : \text{sides, } s = (a + b + c + d)/2$$

$$A \cong \frac{a+c}{2} \times \frac{b+d}{2} \quad [(a, c), (b, d) \text{ pairs of } \textit{opposite sides}]$$

$$A = \sqrt{(s-a)(s-b)(s-c)(s-d)}$$

$$s-d = \frac{a+b+c-d}{2} = \frac{a+c}{2} + \frac{b-d}{2}$$

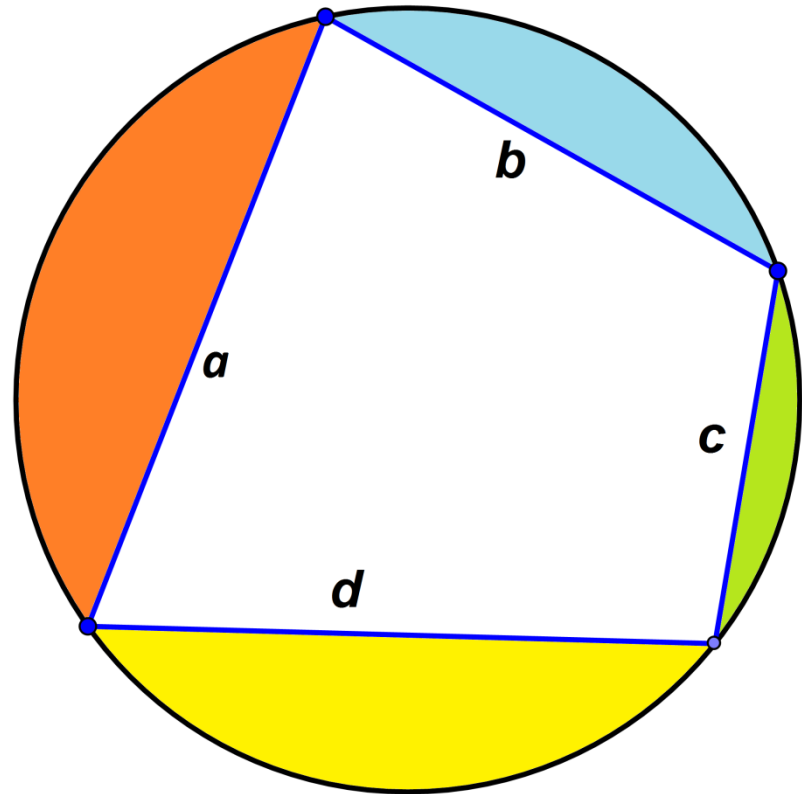
$$s-b = \frac{a-b+c+d}{2} = \frac{a+c}{2} - \frac{b-d}{2}$$

$$(s-b)(s-d) = \left[\left(\frac{a+c}{2} \right)^2 - \left(\frac{b-d}{2} \right)^2 \right], \dots$$

Hence : *gross formula* \geq *exact value*

XII.21 : THE ORDER OF THE SIDES IS IMMATERIAL

Quadrilateral
= circle with
four regions
called
« *pradhi* »s in
the *Śulva-s*



XII.27 : CIRCUMCIRCLE

*tribhujasya vadho bhujayor dviguṇita-lamboddhṛto hṛdayarajjuḥ
sā dviguṇā tricaturbhujakoṇa-spr̥g-vṛtta-viṣkambhaḥ*

*In the **trilateral**, the **product of the sides**
divided by the perpendicular multiplied by two,*

*Is the **cord of the heart**,*

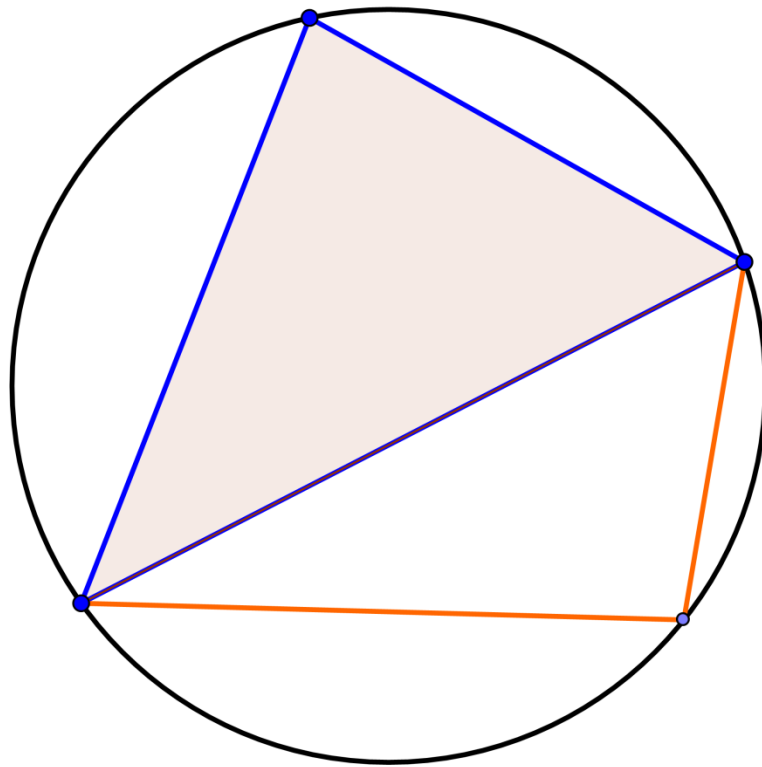
*Which, times two, is, in the **triquadrilateral**, the
diameter of the cercle that touches [its] corners.*

12.21 Triquadrilateral is a **closed figure** (has an area) that has four sides.

12.27 It has a **circumcircle** determined by the height of a **triangle** : the triquadrilateral is obtained by **completing a triangle by adding a fourth point on its circumcircle**.

NB : 12.26 describes two special cases, in which the fourth point is **obtained from one of the vertices by symmetry**.

TRIQUADRILATERAL



To find the perpendicular in XII.27, the triquadrilateral must contain a distinguished triangle.

OTHER REMARKABLE PASSAGES

- XII.24 expresses the theorem that the “**trilateral in a semi-circle**” is right.
- XII.25 gives a general result on **similar half-oblongs**.
- XII.30-31 (“**Brahmagupta’s theorem**”) unknown outside India (Chasles, 1837).
- The last part of XII gives a proof of “**Gauss’ Lemma**” (if $a \mid bc$ and a/b irreducible, then $a \mid c$)
- XVIII **extends operations** from rational numbers to negatives, constructible quadratic irrationals and “non-manifest” quantities (theoretical basis of algebra, developed in his theory of equations).

Mathematics not cumulative. What is forgotten can only be recovered by historical investigation (in the modern sense).

C. RAMANUJAN'S DISCURSIVE STRATEGIES

ORIENTATION

- Compare S. Ramanujan's **discursive strategies** to that of other mathematicians.
- His approach is not unlike those of others, but he was **not part of an extensive community of scholars** [did not undergo regular training], except possibly at the end of his life.
- Upshot: better understanding of **his intended meaning** and possibly carry it **further**.
- **NB : Part of R.'s work is not discursive & may not have been meant for communication: more complicated than the case of Brahmagupta.**

DISCURSIVE STRATEGIES

- Ramanujan: “a rational human being who happened to be a great mathematician” (H. p. 5), “not in the least disposed to make a mystery of his achievements” (p. 11).
- **Defensive: give only results.** « If I had given you my methods of proof I am sure you will follow the London Professor » [who was convinced he did not understand the notion of convergence] (R to H, 27 Feb. 1913),
- **Heuristic** (for himself first and foremost): – **not always definitive results.** L on R, March 1913 : « I **imagine** that he is satisfied if he can convince himself that his results are correct »; H on R (p. 27) “He had a “**proof**”, a definite and very ingenious train of reasoning.” **We want to remove the quotes !**
- Hardy regretted not having asked R. about his reasoning (“I hardly asked him a single question of this kind” p. 11)
- **Carr’s Synopsis** (1880-6) as a model ?

CARR'S SYNOPSIS (PREFACE)

“I have, in many cases, merely indicated the salient points of a demonstration, or merely referred to the theorems by which the proposition is proved. I am convinced that it is more beneficial to the student to recall demonstrations with such aids, than to read and re-read them.”

(i.e.: conducive to active learning)

But: hybrid format (idiosyncratic, preparation for Tripos)

- Structured like an Indian text (!) beginning with units (for physical quantities), putting algebra foremost etc.
- But includes a list of propositions from Books of Euclid for reference.
- Lacks motivation: (a) why is this interesting ? (b) how does one reach this result? (c) why choose this proof ?

« ON HIGHLY COMPOSITE NUMBERS »

- R. adopts the **dogmatic style** (PLMS 2, XIV, 1915, 347-409 = Paper 15, p. 78)
- H. explains the motivation: (ch. III “**Round numbers**”)
[round nb. = “the product of a considerably large number of comparatively small factors”]...
- ... “It is a matter of **common observation** that *round numbers are very rare*; the fact may be **verified** by anyone who will make a habit of factorising numbers, such as numbers of motor cars of railway carriages, which are presented to his attention in a random manner. **Both Ramanujan and I had observed this phenomenon**, which seems at first a little paradoxical.” + footnote: “Half the numbers are divisible by 2, ...one-sixth by both 2 and 3. Surely then we may them expect numbers to have a *large* number of factors? But the facts seem to show the opposite. (H., *ibid*, p. 55)

FROM THE NOTEBOOKS: ENTRIES 20(II-III)

Notebooks, vol. III (Berndt, 1991, p. 199)

Entry 20(i): construction of $\frac{355}{113} \approx \pi$; Cor. (i) [equilateral tr. inscribed in a circle: $\sqrt[3]{31} = 3.14138 \dots \approx \pi$) and (ii)

$$\sqrt[4]{97\frac{1}{2} - \frac{1}{11}} \approx \pi$$

“The appearance below of Entries 20(ii) and 20(iii) is enigmatic indeed; there does not seem to be any connection between these entries and any other result in Chapter 18.”

Entry 20(ii) : “Parametric solutions of the equation $A^3 + B^3 = C^2$ are given by $A = 3n^3 + 6n^2 - n$, $B = -3n^3 + 6n^2 + n$ and $C = 6n^2(3n^2 + 1)$ where n is arbitrary

FROM THE NOTEBOOKS: ENTRY 20(III)

Entry 20(ii) : “Parametric solutions of the equation $A^3 + B^3 = C^2$ are given by $A = 3n^3 + 6n^2 - n$, $B = -3n^3 + 6n^2 + n$ and $C = 6n^2(3n^2 + 1)$ where n is arbitrary

Entry 20(iii) : “Parametric solutions of the equation $A^3 + B^3 + C^3 = D^3$ are given by

$$A = m^7 - 3(p + l)m^4 + (3p^2 + 6p + 2)m,$$

$$B = 2m^6 - 3(2p + 1)m^3 + (3p^2 + 3p + 1),$$

$$C = m^6 - (3p^2 + 3p + 1)$$

$$D = m^7 - 3pm^4 + (3p^2 - 1)m,$$

where m and p denote arbitrary numbers.”

Berndt provides **verification** + many **references**.

FROM THE NOTEBOOKS: ENTRY 20(II-III)

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$$A = m^7 - 3(p + l)m^4 + (3p^2 + 6p + 2)m,$$

$$B = 2m^6 - 3(2p + 1)m^3 + (3p^2 + 3p + 1),$$

$$C = m^6 - (3p^2 + 3p + 1),$$

$$D = m^7 - 3pm^4 + (3p^2 - 1)m,$$

where m and p denote arbitrary numbers.” + List of examples

Parallel formulation : same method ?

FROM THE NOTEBOOKS: A LIST OF SOLUTIONS IN 20(III)

Entry 20(ii) : “Parametric solutions of the equation $A^3 + B^3 = C^2$ are given by $A = 3n^3 + 6n^2 - n$, $B = -3n^3 + 6n^2 + n$ and $C = 6n^2(3n^2 + 1)$ where n is arbitrary

- A and B have the form $an^3 + bn^2 - n$ and $-an^3 + bn^2 + n$
- ...optimized to kill odd powers in sum of cubes., hence
- $C^2 = 6bn^4 \left[1 + \frac{1}{3}(b^2 - 6a)n^2 + (an^2)^2 \right]$
- Need: $6b = q^2$, $b^2 - 6a = 6a$ hence $12a = b^2 = \frac{q^4}{36}$
- Therefore $q = 6r$, $b = 6r^2$...

1. Possibly, similar approach for 20(iii)
2. List of solutions of 20(iii) : last example has different status

CONCLUSION

1. Mathematics, when communicated, is a **discourse**.
2. Rigorous results **need not be made explicit** (when talking to scholars)
3. Important texts (Brahmagupta, Tartaglia, S. Ramanujan,...) **appear non-rigorous** because they are not fully explicit.

Conclusion: **1) There are several types of rigorous discourse.**

2) Textual analysis as practiced by modern historians is an essential part of mathematical activity and leads to new results.

**THANK YOU
FOR
YOUR ATTENTION**