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New Conservation Laws Based on Generalised Reynolds Transport Theorems

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Abstract

The Reynolds transport theorem gives a generic conservation law for a conserved quantity carried by fluid flow through a continuous control volume, providing the foundation for all conservation laws of fluid mechanics. We present new spatial and parametric multivariate generalisations of this theorem, each of which provides a continuous mapping of the contents of a domain within a coordinate space. The most general form involves mappings on a manifold, presented in a multivariate extension of exterior calculus. These theorems are used to generate tables of new conservation laws in integral form in different spaces, including Eulerian velocity space and velocity-volume phase space, based on fluid densities defined in these spaces.

Keywords
Reynolds transport theorem, exterior calculus, conservation laws, velocity space, phase space, density

Introduction

The Reynolds transport theorem, given by Reynolds in 1903 [1], provides a generalised conservation equation for a conserved quantity within a body of fluid (the *domain* or *fluid volume*) as it moves through a designated region of space (the *control volume*). Its universal formulation enables the generalised analysis of all conserved quantities, which can be used to extract particular phenomena as needed, e.g., the transport of seven important conserved quantities in fluid flow systems (fluid mass, species mass, linear momentum, angular momentum, energy, charge and thermodynamic entropy) [2, 3]. By the analysis of sources, sinks and fluxes of the conserved quantity in a differential fluid element, the theorem can also be connected to corresponding differential equations for each phenomenon [2, 3, 4, 5, 6, 7]. During the past half-century, the Reynolds transport theorem has been extended to domains with discontinuities [8, 9], moving and smoothly-deforming control volumes [2, 3, 9], irregular and rough domains [10, 11, 12], two-dimensional domains [13, 14, 15, 16, 17], and differentiable manifolds or chains described by patchworks of local coordinate systems [18, 19, 20], the latter expressed in the formalism of exterior calculus. These formulations are restricted to one-parameter (temporal) mappings of the conserved quantity in volumetric space, induced by a velocity vector field. A separate body of research has also been conducted on an analogous *spatial averaging theorem*, based on spatial rather than time derivatives of the fluid volume [21, 22, 23, 24].

Recently, the authors presented a generalised formulation of the Reynolds transport theorem, involving a multivariate mapping of a conserved quantity in a generic coordinate space, induced by a vector or tensor field [25]. Its most general form provides a mapping within a differentiable manifold, using a multivariate extension of exterior calculus. In contrast to the traditional Reynolds transport theorem, which considers the integral curves (pathlines) described in time by a velocity vector field, the new theorems provide parameterised univariate or multivariate integral curves connecting different locations within the coordinate space. They can therefore be referred to as *transformation* theorems, connecting – for example – every spatial position within a velocity gradient field, or every parameterised location within an arbitrary tensor field. The new formulations have been used to define new spatial and parametric variants of the Liouville equation for the conservation of probability, and of the Perron-Frobenius and Koopman operators, providing an assortment of new tools for the analysis of turbulent flow systems [25].

The aim of this work is to present several forms of the new Reynolds transport theorems, both in generic form and applied to several coordinate spaces, including volumetric space, Eulerian velocimetric space and Eulerian phase space. From these formulations, we present tables of integral equations for the seven important conserved quantities identified previously, analogous to those commonly derived from the traditional Reynolds transport theorem [2, 3]. The analyses substantially expand the scope of known conservation laws for the analysis of different types of fluid flow systems.

Generalised Reynolds Transport Theorems

Exterior Calculus Formulation

We first present the general multivariate exterior calculus form of the Reynolds transport theorem. Consider an n-dimensional orientable differentiable manifold M^n represented using a patchwork of local coordinate systems $\boldsymbol{X}$. The manifold contains an r-dimensional oriented compact submanifold Ω^r with boundary $\partial\Omega$, parameterised by an m-dimensional parameter vector $\boldsymbol{C} \in \mathbb{R}^m$. Let $\boldsymbol{V}$ be a smooth vector or tensor field in M^n, a function of local $\boldsymbol{X}$ and also of $\boldsymbol{C}$. Let ω^r represent a field of r-forms in M^n associated with a conserved quantity, which is locally continuous and continuously differentiable with respect to $\boldsymbol{X}$ and $\boldsymbol{C}$ within the manifold. By analysis of the augmented manifold $\Pi^{n+m} = M^n \times \mathbb{R}^m$ defined from M^n and $\boldsymbol{C}$, it can be shown that the integral of ω^r over Ω^r satisfies [25]:

$$
\begin{aligned}
\hat{d} \int\limits_{\Omega(\boldsymbol{C})} \omega^r &= \Bigg[\int\limits_{\Omega(\boldsymbol{C})} \mathcal{L}^{(\boldsymbol{C})}_{\boldsymbol{V} \llcorner \boldsymbol{C}} \omega^r \Bigg] \cdot d\boldsymbol{C} \\
&= \Bigg[\int\limits_{\Omega(\boldsymbol{C})} \nabla_{\boldsymbol{C}} \omega^r + \int\limits_{\Omega(\boldsymbol{C})} i^{(\boldsymbol{C})}_{\boldsymbol{V}} \, d\omega^r + \oint\limits_{\partial\Omega(\boldsymbol{C})} i^{(\boldsymbol{C})}_{\boldsymbol{V}} \, \omega^r \Bigg] \cdot d\boldsymbol{C} \\
&= \Bigg[\int\limits_{\Omega(\boldsymbol{C})} \nabla_{\boldsymbol{C}} \omega^r + i^{(\boldsymbol{C})}_{\boldsymbol{V}} \, d\omega^r + d(i^{(\boldsymbol{C})}_{\boldsymbol{V}} \, \omega^r) \Bigg] \cdot d\boldsymbol{C}
\end{aligned}
\tag{1}
$$

where $\hat{d}$ is the exterior derivative in Π^{n+m}, d is the exterior derivative in M^n, $\mathcal{L}^{(\boldsymbol{C})}_{\boldsymbol{V}\llcorner\boldsymbol{C}}$ is a multivariate Lie derivative with respect to the augmented tensor field $\boldsymbol{V}\llcorner\boldsymbol{C}$ over parameters $\boldsymbol{C}$ [25], $i^{(\boldsymbol{C})}_{\boldsymbol{V}}$ is a multivariate interior product with respect to $\boldsymbol{V}$ over parameters $\boldsymbol{C}$ [25], $\nabla_{\boldsymbol{C}}$ is the gradient with respect to $\boldsymbol{C}$, and "$\cdot$" is the vector scalar product. Eq. (1) provides a quite general theorem applicable to a submanifold of any dimension within a manifold, i.e., to moving surfaces within volumes. For $\boldsymbol{C} = t$, it reduces to the Reynolds transport theorem for differential forms in a time-dependent velocity field [18, 19].

Vector-Tensor Calculus Formulation

To reduce (1), now consider an n-dimensional space M described by global Cartesian coordinates $\boldsymbol{X}$, containing an n-dimensional domain Ω. Let $\boldsymbol{V} = (\nabla_{\boldsymbol{C}}\boldsymbol{X})^\top$ be a smooth vector or tensor field in M (for tensors written in the $\partial(\rightarrow)/\partial(\downarrow)$ convention), a function of $\boldsymbol{X}$ and the m-dimensional parameter vector $\boldsymbol{C}$. Let ψ be the density of a conserved quantity in the coordinate space, which is continuous and continuously differentiable with respect to $\boldsymbol{X}$ and $\boldsymbol{C}$ within M. It can be shown that [25]:

$$\begin{aligned} d \int\limits_{\Omega(\boldsymbol{C})} \psi\, d^n\boldsymbol{X} &= \Bigg[\nabla_{\boldsymbol{C}} \int\limits_{\Omega(\boldsymbol{C})} \psi\, d^n\boldsymbol{X}\Bigg] \cdot d\boldsymbol{C} \\ &= \Bigg[\int\limits_{\Omega(\boldsymbol{C})} \nabla_{\boldsymbol{C}}\psi\, d^n\boldsymbol{X} + \oint\limits_{\partial\Omega(\boldsymbol{C})} \psi\boldsymbol{V}\cdot d^{n-1}\boldsymbol{X}\Bigg] \cdot d\boldsymbol{C} \\ &= \Bigg[\int\limits_{\Omega(\boldsymbol{C})} \big[\nabla_{\boldsymbol{C}}\psi + \nabla_{\boldsymbol{X}}\cdot(\psi\boldsymbol{V})\big] d^n\boldsymbol{X}\Bigg] \cdot d\boldsymbol{C}, \end{aligned} \tag{2}$$

where d is now the differential, $d^n\boldsymbol{X}$ is an n-dimensional element in Ω, $d^{n-1}\boldsymbol{X}$ is a directed $(n-1)$-dimensional element in $\partial\Omega$, and $\nabla_{\boldsymbol{X}}$ is the gradient with respect to $\boldsymbol{X}$.

Eq. (2) provides a general theorem applicable to a domain of identical dimension to its coordinate space (volumes within volumes). Owing to its general theoretical framework, it is applicable to all types of spaces, not just the volumetric space usually considered in theoretical fluid mechanics.

Example Flow Systems

We now apply (2) to three types of flow system, defined respectively in volumetric space, Eulerian velocity space and Eulerian position-velocity (phase) space. In the analyses, we adopt the following specific quantities: specific mass of the cth chemical species χ_c, specific linear momentum $\boldsymbol{u}$ (equivalent to the fluid velocity), specific angular momentum $\boldsymbol{r}\times\boldsymbol{u}$, specific energy e, specific charge z and specific entropy s. To keep track of different types of specific quantities, we denote position dependence by an underbar and velocity dependence by a breve accent, thereby allowing for three possibilities, e.g., $\underline{e}$, $\breve{e}$ and $\underline{\breve{e}}$ (note that for $\boldsymbol{u}$ these labels are implicit).

Volumetric-Temporal Formulation

We first consider a volumetric space with coordinates $\boldsymbol{X} = \boldsymbol{x}$ and time parameter $\boldsymbol{C} = t$, in which the conserved quantity is represented by the generalised density $\alpha(\boldsymbol{x},t) = \rho(\boldsymbol{x},t)\,\underline{\alpha}(\boldsymbol{x},t)$ in volumetric space [units: quantity m^{-3}], where $\rho(\boldsymbol{x},t)$ is the fluid density [kg m^{-3}] and $\underline{\alpha}(\boldsymbol{x},t)$ is a specific quantity [quantity kg^{-1}]. We denote the domain (fluid volume) by Ω and its surface by $\partial\Omega$, both a function of parameter t. The field is given by $\boldsymbol{V} = \partial\boldsymbol{x}/\partial t = \boldsymbol{u}$, the velocity vector field. Eq. (2) gives:

$$\begin{aligned} \frac{DQ}{dt} = \frac{dQ}{dt} = \frac{d}{dt}\iiint\limits_{\Omega(t)} \alpha\, dV &= \iiint\limits_{\Omega(t)} \frac{\partial\alpha}{\partial t} dV + \oiint\limits_{\partial\Omega(t)} \alpha\boldsymbol{u}\cdot\boldsymbol{n}\, dA \\ &= \iiint\limits_{\Omega(t)} \left[\frac{\partial\alpha}{\partial t} + \nabla_{\boldsymbol{x}}\cdot(\alpha\boldsymbol{u})\right] dV \end{aligned} \tag{3}$$

where D/Dt is the substantial derivative (here equivalent to the total derivative d/dt), Q is the total (integrated) conserved quantity, $dV = d^3\boldsymbol{x}$ is a differential three-dimensional volume element in Ω, $dA = |d^2\boldsymbol{x}|$ is a differential area element in $\partial\Omega$, and $\boldsymbol{n}$ is the outwards unit normal. We see that (3) corresponds to the standard Reynolds transport theorem [1, 2, 3, 8, 9].

Using the common scheme, the integral conservation laws obtained from (3) for the seven identified conserved quantities are listed in Table 1 [2, 4, 3, 5, 6, 7]. On the left-hand sides, the source-sink terms DQ/Dt contain the rate of change of mass of the cth chemical species $\dot{m}_c$, the sum of forces $\sum\boldsymbol{F}$, the sum of torques $\sum\boldsymbol{T}$, the total energy E, the inward heat flow rate $\dot{Q}_{in}$, the inward work flow rate $\dot{W}_{in}$, the total charge Z, the inward electrical current I, the charge on the cth species z_c, the thermodynamic entropy S, the entropy production $\dot{\sigma}$, and the non-fluid entropy flux $\dot{S}^{nf}$. In all cases, subscript FV represents calculation over the fluid volume Ω.

Velocimetric-Temporal Formulation

Now consider a velocimetric space with coordinates $\boldsymbol{X} = \boldsymbol{u}$ and time parameter $\boldsymbol{C} = t$, in which the conserved quantity is represented by the generalised density $\beta(\boldsymbol{u},t) = д(\boldsymbol{u},t)\,\breve{\beta}(\boldsymbol{u},t)$ in velocimetric space [quantity (m s^{-1})$^{-3}$], where $д(\boldsymbol{u},t)$ is the fluid velocimetric density [kg (m s^{-1})$^{-3}$] and $\breve{\beta}(\boldsymbol{x},t)$ is a specific quantity [quantity kg^{-1}]. The density $д$ can be interpreted as the mass of fluid carried by fluid elements of velocity $\boldsymbol{u}$, regardless of position, while β gives the corresponding conserved quantity carried by these fluid elements. We denote the velocity domain (the *velocity volume*) by $\mathcal{D}$ and its surface by $\partial\mathcal{D}$, both a function of parameter t. The field is given by $\boldsymbol{V} = \partial\boldsymbol{u}/\partial t = \dot{\boldsymbol{u}}$, the local acceleration vector field. Eq. (2) gives:

$$\begin{aligned} \frac{dQ}{dt} = \frac{d}{dt}\iiint\limits_{\mathcal{D}(t)} \beta\, dU = \iiint\limits_{\mathcal{D}(t)} \frac{\partial\beta}{\partial t} dU + \oiint\limits_{\partial\mathcal{D}(t)} \beta\dot{\boldsymbol{u}}\cdot\boldsymbol{n}_B\, dB \\ = \iiint\limits_{\mathcal{D}(t)} \left[\frac{\partial\beta}{\partial t} + \nabla_{\boldsymbol{u}}\cdot(\beta\dot{\boldsymbol{u}})\right] dU \end{aligned} \tag{4}$$

where $dU = d^3\boldsymbol{u}$ is a differential three-dimensional velocity element in $\mathcal{D}$, $dB = |d^2\boldsymbol{u}|$ is a differential boundary area element in $\partial\mathcal{D}$, $\boldsymbol{n}_B$ is the outwards unit normal, and $\nabla_{\boldsymbol{u}}$ is the gradient with respect to $\boldsymbol{u}$. We see that (4) provides a new velocimetric-temporal formulation of the Reynolds transport theorem [25].

The integral conservation laws obtained from (3) for the seven identified conserved quantities are listed in Table 2. Since these integrals are equal to the total rate of change of the conserved quantity dQ/dt, they equate to the same source-sink terms as for the volumetric-temporal formulation (Table 1). However, due to their different formulation, they are labelled VV to indicate integration over the velocity volume $\mathcal{D}$.

Velocimetric-Spatial (Time-Independent) Formulation

Finally, consider a time-independent volumetric and velocimetric space, which is integrated only with respect to the velocity coordinates $\boldsymbol{X} = \boldsymbol{u}$, and is parameterised by $\boldsymbol{C} = \boldsymbol{x}$. The conserved quantity is represented by the generalised phase space density $\varphi(\boldsymbol{u},\boldsymbol{x}) = \zeta(\boldsymbol{u},\boldsymbol{x})\,\underline{\breve{\varphi}}(\boldsymbol{u},\boldsymbol{x})$ [quantity m^{-3} (m s^{-1})$^{-3}$], where $\zeta(\boldsymbol{u},\boldsymbol{x})$ is the fluid phase space density [kg m^{-3} (m s^{-1})$^{-3}$] and $\underline{\breve{\varphi}}(\boldsymbol{u},\boldsymbol{x})$ is a specific quantity [quantity kg^{-1}]. The density ζ can be interpreted as the mass of fluid carried by a fluid element of velocity $\boldsymbol{u}$ at the location $\boldsymbol{x}$, while φ gives the corresponding conserved quantity carried by this fluid element. We again consider a velocity domain $\mathcal{D}$ with surface $\partial\mathcal{D}$, now functions of the position $\boldsymbol{x}$. The field is given by $\boldsymbol{V} = (\nabla_{\boldsymbol{x}}\boldsymbol{u})^\top = \boldsymbol{G}^\top$, the velocity gradient tensor field. Eq. (2) gives:

$$\begin{aligned} \nabla_{\boldsymbol{x}}\alpha &= \nabla_{\boldsymbol{x}} \iiint\limits_{\mathcal{D}(\boldsymbol{x})} \varphi \, dU \\ &= \iiint\limits_{\mathcal{D}(\boldsymbol{x})} \nabla_{\boldsymbol{x}}\varphi \, dU + \oiint\limits_{\partial\mathcal{D}(\boldsymbol{x})} \varphi \boldsymbol{G}^\top \cdot \boldsymbol{n}_B \, dB \\ &= \iiint\limits_{\mathcal{D}(\boldsymbol{x})} \Big[\nabla_{\boldsymbol{x}}\varphi + \nabla_{\boldsymbol{u}} \cdot (\varphi \boldsymbol{G}^\top) \Big] dU \end{aligned} \tag{5}$$

The integral conservation laws obtained from (5) for the seven identified conserved quantities are listed in Table 3. As evident, the derived relations are of different character to those given previously (Tables 1-2), giving new expressions for the spatial gradient of each generalised volumetric density $\alpha = \rho\underline{\alpha}$, expressed in terms of the corresponding generalised phase space density $\varphi = \zeta\underline{\varphi}$.

Conclusions

Building on previous research [25], we present new generalised forms of the Reynolds transport theorem, involving continuous mapping of the contents of a domain within a coordinate space. The most general form involves mappings on a manifold, employing a multivariate extension of exterior calculus. The new theorems are here used to derive the existing Reynolds transport theorem in volumetric space, as well as new forms in Eulerian velocity space and Eulerian position-velocity phase space. Each formulation is then used to extract the integral conservation laws for the transport of seven conserved quantities (fluid mass, species mass, linear momentum, angular momentum, energy, charge and thermodynamic entropy), based on generalised and fluid densities defined within its formulation. The analyses substantially expand the scope of known conservation laws for the analysis of different types of fluid flow systems.

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Conserved quantity	Density $= \rho\,\underline{\alpha}$	Integral Equation	
Fluid mass	ρ	0	$= \iiint_{\Omega(t)} \left[\frac{\partial \rho}{\partial t} + \nabla_{\boldsymbol{x}} \cdot (\rho\,\boldsymbol{u})\right] dV$
Species mass	$\rho\,\underline{\chi}_c$	$(\dot{m}_c)_{FV}$	$= \iiint_{\Omega(t)} \left[\frac{\partial \rho\,\underline{\chi}_c}{\partial t} + \nabla_{\boldsymbol{x}} \cdot (\rho\,\underline{\chi}_c\,\boldsymbol{u})\right] dV$
Linear momentum	$\rho\,\boldsymbol{u}$	$\sum \boldsymbol{F}_{FV}$	$= \iiint_{\Omega(t)} \left[\frac{\partial \rho\,\boldsymbol{u}}{\partial t} + \nabla_{\boldsymbol{x}} \cdot (\rho\,\boldsymbol{u}\,\boldsymbol{u})\right] dV$
Angular momentum	$\rho\,(\underline{\boldsymbol{r}} \times \boldsymbol{u})$	$\sum \boldsymbol{T}_{FV}$	$= \iiint_{\Omega(t)} \left[\frac{\partial \rho\,(\underline{\boldsymbol{r}} \times \boldsymbol{u})}{\partial t} + \nabla_{\boldsymbol{x}} \cdot (\rho\,(\underline{\boldsymbol{r}} \times \boldsymbol{u})\,\boldsymbol{u})\right] dV$
Energy	$\rho\,\underline{e}$	$\frac{DE_{FV}}{Dt} = (\dot{Q}_{in} + \dot{W}_{in})_{FV}$	$= \iiint_{\Omega(t)} \left[\frac{\partial \rho\,\underline{e}}{\partial t} + \nabla_{\boldsymbol{x}} \cdot (\rho\,\underline{e}\,\boldsymbol{u})\right] dV$
Charge (in solution)	$\rho\,\underline{z}$	$\frac{DZ_{FV}}{Dt} = I_{FV} + \left(\sum_c z_c \dot{m}_c\right)_{FV}$	$= \iiint_{\Omega(t)} \left[\frac{\partial \rho\,\underline{z}}{\partial t} + \nabla_{\boldsymbol{x}} \cdot (\rho\,\underline{z}\,\boldsymbol{u})\right] dV$
Entropy	$\rho\,\underline{s}$	$\frac{DS_{FV}}{Dt} = \dot{\sigma}_{FV} + \dot{S}_{FV}^{nf}$	$= \iiint_{\Omega(t)} \left[\frac{\partial \rho\,\underline{s}}{\partial t} + \nabla_{\boldsymbol{x}} \cdot (\rho\,\underline{s}\,\boldsymbol{u})\right] dV$

Table 1. Conservation Laws for the Volumetric-Temporal Formulation (after [2, 4, 3, 5, 6, 7]).

Conserved quantity	Density $= \text{д}\,\breve{\beta}$	Integral Equation	
Fluid mass	д	0	$= \iiint_{\mathcal{D}(t)} \left[\frac{\partial \text{д}}{\partial t} + \nabla_{\boldsymbol{u}} \cdot (\text{д}\,\dot{\boldsymbol{u}})\right] dU$
Species mass	$\text{д}\,\breve{\chi}_c$	$(\dot{m}_c)_{VV}$	$= \iiint_{\mathcal{D}(t)} \left[\frac{\partial \text{д}\,\breve{\chi}_c}{\partial t} + \nabla_{\boldsymbol{u}} \cdot (\text{д}\,\breve{\chi}_c\,\dot{\boldsymbol{u}})\right] dU$
Linear momentum	$\text{д}\,\boldsymbol{u}$	$\sum \boldsymbol{F}_{VV}$	$= \iiint_{\mathcal{D}(t)} \left[\frac{\partial \text{д}\,\boldsymbol{u}}{\partial t} + \nabla_{\boldsymbol{u}} \cdot (\text{д}\,\boldsymbol{u}\,\dot{\boldsymbol{u}})\right] dU$
Angular momentum	$\text{д}\,(\breve{\boldsymbol{r}} \times \boldsymbol{u})$	$\sum \boldsymbol{T}_{VV}$	$= \iiint_{\mathcal{D}(t)} \left[\frac{\partial \text{д}\,(\breve{\boldsymbol{r}} \times \boldsymbol{u})}{\partial t} + \nabla_{\boldsymbol{u}} \cdot (\text{д}\,(\breve{\boldsymbol{r}} \times \boldsymbol{u})\,\dot{\boldsymbol{u}})\right] dU$
Energy	$\text{д}\,\breve{e}$	$\frac{DE_{VV}}{Dt} = (\dot{Q}_{in} + \dot{W}_{in})_{VV}$	$= \iiint_{\mathcal{D}(t)} \left[\frac{\partial \text{д}\,\breve{e}}{\partial t} + \nabla_{\boldsymbol{u}} \cdot (\text{д}\,\breve{e}\,\dot{\boldsymbol{u}})\right] dU$
Charge (in solution)	$\text{д}\,\breve{z}$	$\frac{DZ_{VV}}{Dt} = I_{VV} + \left(\sum_c z_c \dot{m}_c\right)_{VV}$	$= \iiint_{\mathcal{D}(t)} \left[\frac{\partial \text{д}\,\breve{z}}{\partial t} + \nabla_{\boldsymbol{u}} \cdot (\text{д}\,\breve{z}\,\dot{\boldsymbol{u}})\right] dU$
Entropy	$\text{д}\,\breve{s}$	$\frac{DS_{VV}}{Dt} = \dot{\sigma}_{VV} + \dot{S}_{VV}^{nf}$	$= \iiint_{\mathcal{D}(t)} \left[\frac{\partial \text{д}\,\breve{s}}{\partial t} + \nabla_{\boldsymbol{u}} \cdot (\text{д}\,\breve{s}\,\dot{\boldsymbol{u}})\right] dU$

Table 2. Conservation Laws for the Velocimetric-Temporal Formulation.

Conserved quantity	Density $= \zeta\,\underline{\breve{\varphi}}$	Integral Equation	
Fluid mass	ζ	$\nabla_{\boldsymbol{x}}\rho$	$= \iiint_{\mathcal{D}(\boldsymbol{x})} \left(\nabla_{\boldsymbol{x}}\zeta + \nabla_{\boldsymbol{u}} \cdot (\zeta\,\boldsymbol{G}^{\top})\right) dU$
Species mass	$\zeta\,\underline{\breve{\chi}}_c$	$\nabla_{\boldsymbol{x}}(\rho\,\underline{\chi}_c)$	$= \iiint_{\mathcal{D}(\boldsymbol{x})} \left(\nabla_{\boldsymbol{x}}(\zeta\,\underline{\breve{\chi}}_c) + \nabla_{\boldsymbol{u}} \cdot (\zeta\,\underline{\breve{\chi}}_c\,\boldsymbol{G}^{\top})\right) dU$
Linear momentum	$\zeta\,\boldsymbol{u}$	$\nabla_{\boldsymbol{x}}(\rho\boldsymbol{u})$	$= \iiint_{\mathcal{D}(\boldsymbol{x})} \left(\nabla_{\boldsymbol{x}}(\zeta\,\boldsymbol{u}) + \nabla_{\boldsymbol{u}} \cdot (\zeta\,\boldsymbol{u}\,\boldsymbol{G}^{\top})\right) dU$
Angular momentum	$\zeta\,(\underline{\breve{\boldsymbol{r}}} \times \boldsymbol{u})$	$\nabla_{\boldsymbol{x}}(\rho(\underline{\boldsymbol{r}} \times \boldsymbol{u}))$	$= \iiint_{\mathcal{D}(\boldsymbol{x})} \left(\nabla_{\boldsymbol{x}}(\zeta\,(\underline{\breve{\boldsymbol{r}}} \times \boldsymbol{u})) + \nabla_{\boldsymbol{u}} \cdot (\zeta\,(\underline{\breve{\boldsymbol{r}}} \times \boldsymbol{u})\,\boldsymbol{G}^{\top})\right) dU$
Energy	$\zeta\,\underline{\breve{e}}$	$\nabla_{\boldsymbol{x}}(\rho\underline{e})$	$= \iiint_{\mathcal{D}(\boldsymbol{x})} \left(\nabla_{\boldsymbol{x}}(\zeta\,\underline{\breve{e}}) + \nabla_{\boldsymbol{u}} \cdot (\zeta\,\underline{\breve{e}}\,\boldsymbol{G}^{\top})\right) dU$
Charge (in solution)	$\zeta\,\underline{\breve{z}}$	$\nabla_{\boldsymbol{x}}(\rho\underline{z})$	$= \iiint_{\mathcal{D}(\boldsymbol{x})} \left(\nabla_{\boldsymbol{x}}(\zeta\,\underline{\breve{z}}) + \nabla_{\boldsymbol{u}} \cdot (\zeta\,\underline{\breve{z}}\,\boldsymbol{G}^{\top})\right) dU$
Entropy	$\zeta\,\underline{\breve{s}}$	$\nabla_{\boldsymbol{x}}(\rho\underline{s})$	$= \iiint_{\mathcal{D}(\boldsymbol{x})} \left(\nabla_{\boldsymbol{x}}(\zeta\,\underline{\breve{s}}) + \nabla_{\boldsymbol{u}} \cdot (\zeta\,\underline{\breve{s}}\,\boldsymbol{G}^{\top})\right) dU$

Table 3. Conservation Laws for the Velocimetric-Spatial (Time-Independent) Formulation.