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Translation Surfaces and Moduli Spaces

DIMITRI ZVONKINE

Translation surfaces are surfaces glued from polygons by identifying only parallel sides of equal lengths. Many questions about them can be asked in elementary terms. Sometimes these questions can be answered using elementary methods such as polygonal billiards, monodromies and permutations, cutting and pasting. Other times they require advanced methods of dynamics or intersection theory on moduli spaces.

Translation surfaces

Figure 1 represents two polygons whose sides are divided into pairs of same length and direction. By gluing the polygons along these pairs of sides we obtain a *translation surface*. This is a closed surface with a flat metric (a Riemannian metric with zero curvature, locally isometric to the Euclidean plane) and a finite number of conic singularities whose angles are multiples of 2π . In this example, the surface is of genus 3, and it has 2 conic singularities with angles 4π (white dots) and 8π (black dots).

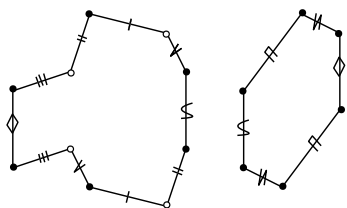


Figure 1. These two polygons can be glued into a translation surface

In general, given a translation surface of genus g with n conic singularities that have angles $2\pi(k_i + 1)$, $1 \leq i \leq n$, the Gauss-Bonnet formula implies $\sum k_i = 2g - 2$.

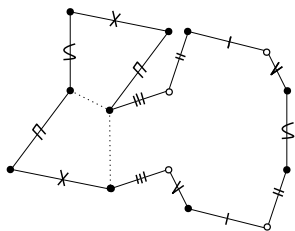


Figure 2. This is the same translation surface as in Figure 1

Any translation surface can be cut into polygons in many ways. We will consider all such cuttings as

equivalent; in other words, we are interested in the surface itself and in its metric, not in a particular cutting. For instance, the polygon in Figure 2 is obtained from those in Figure 1 by cutting the rightmost polygon in two and gluing the pieces along distinguished pairs of sides (the gluings are shown as dotted lines). Both figures represent the same translation surface.

Let C be a translation surface and S^1 the circle of directions on the plane. Then $C \times S^1$ carries a natural time flow: just imagine a ball rolling on the surface C along the geodesic in the given direction. If the ball meets a side of a polygon it reappears from the other side of the distinguished pair. If it hits a vertex, then the flow is not well-defined. Thus to each translation surface corresponds a dynamical system, and we can ask whether the time flow is ergodic, whether there are periodic trajectories, and so on. See, for instance, the study of periodic orbits in the regular pentagonal billiard [2] and the corresponding jewelry at tinyurl.com/yboob8dw. Here, however, we will be more interested in the space $\mathcal{H}_g(k_1, \dots, k_n)$ of all translation surfaces of genus g with conical angles $2\pi(k_i + 1)$, $1 \leq i \leq n$.

The complex numbers represented by the sides of the polygons form local coordinates on this space. Moreover, they endow it with an *integer affine structure*. This means that choosing a different cutting of the surface induces an affine change of local coordinates with integer coefficients. In particular, this allows us to define a canonical volume form on the space $\mathcal{H}_g(k_1, \dots, k_n)$. While the total volume of this space is infinite, the volume of $\mathcal{H}_g^{\leq 1}(k_1, \dots, k_n)$, the space of translation surfaces with area at most 1, was shown to be finite by Masur [7] and Veech [12].

Question 1. What is the volume of $\mathcal{H}_g^{\leq 1}(k_1, \dots, k_n)$?

One of the ways to answer this question is to count the translation surfaces whose coordinates are Gaussian integers $a + bi$, $a, b \in \mathbb{Z}$. Eskin and Okounkov [6] showed that the number of such surfaces with area

up to N grows as $V \cdot N^{2g-1+n}$, where V is the volume of the space $\mathcal{H}_g^{\leq 1}(k_1, \dots, k_n)$ and $2g-1+n$ its complex dimension. Using representation theory and the theory of quasi-modular forms they were able to produce a method to compute the volumes of all spaces $\mathcal{H}_g^{\leq 1}(k_1, \dots, k_n)$. This was later refined to deal with the volumes of connected components of these spaces, classified by Kontsevich and Zorich [9].

Let us illustrate the method with the simplest example of genus 1 translation surfaces. Because such a surface has an infinite automorphism group acting by global translations, it is convenient to mark one point on the surface and consider the space of translation surfaces with one marked point. The marked point could be viewed as a conical point with angle 2π .

Every genus 1 translation surface of area N with one marked point can be cut into a parallelogram, as in Figure 3, in a unique way, d being an integral divisor of N , and $0 \leq a \leq d$ an integer. The marked point is the image of the vertices of the parallelogram under the gluing.

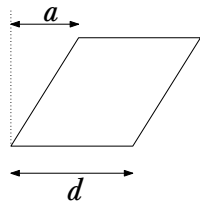


Figure 3. All translation surfaces of $g = 1$ with 1 marked point look like this

Thus the number of such translation surfaces is the sum of divisors of N , and the corresponding generating function is the quasi-modular form

$$G_2(q) = -\frac{1}{24} + \sum_N \sum_{d|N} dq^N.$$

The well-known transformation law for this quasi-modular form reads

$$G_2\left(e^{-2\pi i/\tau}\right) = \tau^2 G_2\left(e^{2\pi i\tau}\right) + \frac{i\pi}{4},$$

where the first term is the standard transformation law for modular forms and the second term is the quasi-modular correction. Plugging $h = 2\pi i/\tau$ and recalling that $G_2(0) = -1/24$, we obtain the asymptotic expansion

$$G_2(e^{-h}) \sim \frac{\pi^2}{6h^2} - \frac{1}{2h}$$

as $h \rightarrow 0^+$. Note that this is the *whole* asymptotic expansion: the difference between the left-hand side and the right-hand side decreases exponentially.

Returning to our original question, we now have

$$G_2(e^{-h}) \sim \frac{\pi^2}{6} \frac{1}{(1-q)^2}, \quad \text{as } h \rightarrow 0^+,$$

so the coefficients of $G_2(q)$ grow as $\pi^2 N/6$, the partial sums up to degree N as $\pi^2 N^2/12$, hence $\pi^2/12$ is the volume of the space $\mathcal{H}_{1,0}^{\leq 1}$.

The group $\text{GL}(2, \mathbb{R})$ acts on plane polygons and thus on the space of translation surfaces in the natural way. This leads to new important questions.

Question 2. Describe the orbit closures of the $\text{GL}(2, \mathbb{R})$ action in the space of translation surfaces.

We will come back to this later.

Question 3. Restrict the action to $\text{SL}(2, \mathbb{R})$. What are the possible stabilizers of translation surfaces? For what surfaces is the stabilizer a lattice in $\text{SL}(2, \mathbb{R})$?

If the stabilizer of a translation surface is a lattice, it is called a *Veech surface*. Veech surfaces have not been classified so far, but many unexpected examples have been constructed.

Example. Consider the translation surface obtained by gluing together the opposite sides of a regular $2n$ -gon. One can show that the matrix

$$\begin{pmatrix} 1 & \cotan \frac{\pi}{2n} \\ 0 & 1 \end{pmatrix}$$

lies in its stabilizer and deduce that the stabilizer is a lattice.

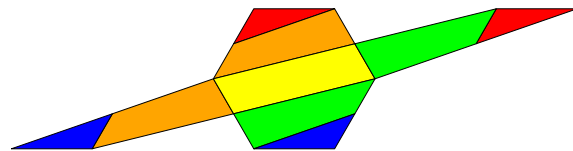


Figure 4. An element of $\text{SL}(2, \mathbb{R})$ that stabilizes the regular $2n$ -gon translation surface

Further restricting the $\text{SL}(2, \mathbb{R})$ action to diagonal matrices

$$\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix},$$

we obtain a flow on the space $\mathcal{H}_g(k_1, \dots, k_n)$, called the *Teichmüller flow*. Masur and Veech [7, 12] showed that this flow is ergodic on every connected component of this space. An important question is: what does this flow do to the homology classes of the

translation surface as $t \rightarrow \infty$? More precisely, if we consider a small open set U in $\mathcal{H}_g(k_1, \dots, k_n)$, we can naturally identify the groups $H_1(C, \mathbb{R})$ for all translation surfaces C parametrized by U , since these surfaces are obtained from each other by small deformations. Now, pick a point in U and suppose that at some moment of time t the flow brings us back to U . The translation surface has undergone some transformation that acts by a linear map $A(t)$ on the homology group $H_1(C, \mathbb{R})$. Denote by

$$0 < \beta_g \leq \dots \leq \beta_1 \leq \alpha_1 \leq \dots \leq \alpha_g \in \mathbb{R}$$

the eigenvalues of $A(t)A^+(t)$. Because $A(t)$ preserves the intersection pairing on $H_1(C, \mathbb{R})$ we actually have $\alpha_i(t)\beta_i(t) = 1$. The *Lyapunov exponents* of the Teichmüller flow are defined as

$$\lambda_i = \lim_{t \rightarrow \infty} \frac{1}{2t} \ln |\alpha_i(t)|.$$

By the Oseledets ergodic theorem, these limits exist and are the same for almost all starting points.

Question 4. What are the Lyapunov exponents $\lambda_1, \dots, \lambda_g$ for a given connected component of $\mathcal{H}_g(k_1, \dots, k_n)$?

Main results on this topic include the proof by Avila and Viana [1] that the Lyapunov spectrum is simple, that is, $0 < \lambda_1 < \dots < \lambda_g$, and the computation of the sum $\lambda_1 + \dots + \lambda_g$ by Eskin, Kontsevich and Zorich [4]. The complete answer, however, is still unknown.

Moduli spaces and Abelian differentials

The space $\mathcal{H}_g(k_1, \dots, k_n)$ can be viewed in a completely different way if we realize that a translation surface is the same thing as a Riemann surface with an Abelian differential (i.e., a holomorphic differential 1-form).

Indeed, since every polygon lies in the complex plane, it inherits the complex coordinate z , that can be taken as a local coordinate on the translation surface. It also induces the holomorphic 1-form dz on the polygon. These local coordinates and holomorphic 1-forms glue nicely along the edges of the polygons: the change of local coordinate from one chart to the other is just a translation. A special treatment is needed at the conical points. If z_i is a conical point with angle $2\pi(k_i + 1)$, we can use $w = (z - z_i)^{1/(k_i+1)}$ as the local coordinate at this point. The Abelian

differential $dz = d(w^{k_i+1})$ has a zero of order k_i at the conical point. Thus we obtain a Riemann surface C with n marked points x_1, \dots, x_n and an Abelian differential α that has zeros of orders k_1, \dots, k_n at x_1, \dots, x_n .

The sides of the polygons are now the *periods* of α , either absolute or relative between two zeros. The complete set of periods on a basis of the group $H_1(C, \{x_1, \dots, x_n\}, \mathbb{Z})$ forms a set of local period coordinates on $\mathcal{H}_g(k_1, \dots, k_n)$ that are called *period coordinates*.

Denote by $\mathcal{M}_{g,n}$ the moduli space of genus g Riemann surfaces with n marked points. This space is well-defined as soon as the Euler characteristic $2 - 2g - n$ of the surface punctured at the marked points is negative. Its points parametrize all possible complex structures on an oriented genus g surface with n distinct marked numbered points, up to a point-preserving isomorphism. For instance, $\mathcal{M}_{0,3}$ is a point, because all genus 0 Riemann surfaces are isomorphic to $\mathbb{C}P^1$ and any three points can be brought to $0, 1, \infty$ in a unique way by an isomorphism $z \mapsto (az + b)/(cz + d)$ of $\mathbb{C}P^1$. Similarly, the space $\mathcal{M}_{0,4}$ is isomorphic to $\mathbb{C}P^1 \setminus \{0, 1, \infty\}$. Indeed, by an isomorphism of $\mathbb{C}P^1$ we can bring four points x_1, x_2, x_3, x_4 to $0, 1, \infty, t$, where

$$t = \frac{x_4 - x_1}{x_2 - x_1} : \frac{x_4 - x_3}{x_2 - x_3} \in \mathbb{C}P^1 \setminus \{0, 1, \infty\} = \mathcal{M}_{0,4}$$

is the cross-ratio of the four points. Further, the space $\mathcal{M}_{1,1}$ is the modular figure $\mathbb{H}/\text{SL}(2, \mathbb{Z})$, where \mathbb{H} is the upper-half plane. Indeed, any elliptic curve can be represented as $\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ for $\tau \in \mathbb{H}$. The $\text{SL}(2, \mathbb{Z})$ action corresponds to replacing the basis $(1, \tau)$ of the lattice $\mathbb{Z} + \tau\mathbb{Z}$ with the basis $(c\tau + d, a\tau + b)$, and then re-scaling the lattice so as to bring the first vector of the basis to 1, the second vector becoming $\tau' = (a\tau + b)/(c\tau + d)$. The three simplest moduli spaces described above are shown in Figure 5.

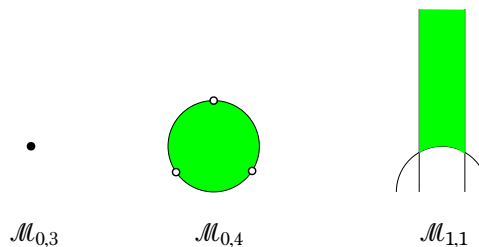


Figure 5. Three simplest moduli spaces

The moduli space $\mathcal{M}_{g,n}$ admits a natural compactification $\overline{\mathcal{M}}_{g,n}$ called the *Deligne-Mumford compactification*.

tion. It is the moduli space of *stable curves*, that is, nodal curves with n distinct smooth marked points that have a finite number of automorphisms. For instance, $\overline{\mathcal{M}}_{0,4}$ is \mathbb{CP}^1 , where the three extra points $0, 1, \infty$ correspond to the three stable curves in the Figure 6.

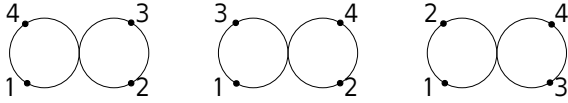


Figure 6. The three nonsmooth stable curves of $g = 0$ with 4 marked points

The moduli space $\overline{\mathcal{M}}_{1,1}$ is obtained from $\mathcal{M}_{1,1}$ by adding one extra point. The corresponding curve is \mathbb{CP}^1 with points $x = 0$ and $x = \infty$ identified into a node and marked point at $x = 1$, see Figure 7.

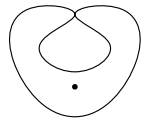


Figure 7. The unique nonsmooth stable curve of $g = 1$ with one marked point

The moduli space $\overline{\mathcal{M}}_{g,n}$ is a smooth Deligne–Mumford stack, or orbifold, of complex dimension $3g - 3 + n$. The points of $\overline{\mathcal{M}}_{g,n} \setminus \mathcal{M}_{g,n}$, that we add to compactify the space, parametrize nonsmooth stable curves. They form a normal crossings divisor in $\overline{\mathcal{M}}_{g,n}$ called the *boundary*.

The *Hodge bundle* $\mathbb{E}_{g,n} \rightarrow \overline{\mathcal{M}}_{g,n}$ is the rank g vector bundle over the moduli space whose fiber over a point $p \in \mathcal{M}_{g,n}$ is the space of Abelian differentials on the corresponding curve C_p . The Hodge bundle extends naturally to $\overline{\mathcal{M}}_{g,n}$. Without going into details of this, let us mention that an Abelian differential on a stable curve is a meromorphic 1-form on each irreducible component of the curve with at most simple poles at the nodes of the curve, the residues at the two branches of the node being opposite to each other. With this definition it is easy to check that the Abelian differentials form a g -dimensional vector space on any stable curve of genus g . For instance, on the genus 1 nodal curve obtained by identifying $x = 0$ and $x = \infty$ on \mathbb{CP}^1 , all Abelian differentials are proportional to dx/x .

Once we have introduced the moduli space and the Hodge bundle, we can remark that every space $\mathcal{H}_g(k_1, \dots, k_n)$ of translation surfaces is naturally embedded into the Hodge bundle $\mathbb{E}_{g,0}$. Specifically,

it is the locus of Abelian differentials with n zeros of orders k_1, \dots, k_n . Actually, these spaces form a stratification of the total space of the Hodge bundle over $\mathcal{M}_{g,n}$. We can also view $\mathcal{H}_g(k_1, \dots, k_n)$ as a subspace of $\mathbb{E}_{g,n}$ if we decide to mark the zeros of the differential. Note that the projectivization $\mathbb{P}\mathbb{E}_{g,n}$ is a compact orbifold, and the projectivizations $\mathbb{P}\overline{\mathcal{H}}_g(k_1, \dots, k_n)$ are its closed sub-orbifolds. This new setting leads to a new series of questions.

Question 5. Do the spaces $\mathcal{H}_g(k_1, \dots, k_n) \subset \mathbb{E}_{g,n}$ have smooth closures?

The answer to this precise question is negative: the closures are not even normal. It is conjectured, however, that their normalizations are smooth. This would provide a natural smooth compactification for every space $\mathcal{H}_g(k_1, \dots, k_n)$.

Question 6. What is the cohomology class Poincaré dual to $\mathbb{P}\overline{\mathcal{H}}_g(k_1, \dots, k_n)$ in $\mathbb{P}\mathbb{E}_{g,n}$? What is the push-forward of this cohomology class to $\overline{\mathcal{M}}_{g,n}$?

As the simplest example, consider the space $\mathcal{H}_2(2)$ of Abelian differentials with a double zero on a genus 2 curve. Every genus 2 curve C carries two linearly independent Abelian differentials α and β , each of which has two simple zeros or a double zero. Their ratio α/β is a degree 2 map from C to \mathbb{CP}^1 or, in more intrinsic terms, to the projectivization of the dual to the space of Abelian differentials on C . This shows that every genus 2 curve is hyperelliptic. The degree 2 map described above has six ramification points, called the *Weierstraß points*. Every Abelian differential on C has either two zeros whose images under the map α/β coincide, or a double zero at a Weierstraß point. Thus the image of $\mathcal{H}_2(2)$ in $\overline{\mathcal{M}}_{2,1}$ is the divisor

$$\{(C, x) \in \overline{\mathcal{M}}_{2,1} \mid x \text{ is a Weierstraß point}\}.$$

The cohomology class Poincaré dual to this divisor was first determined by Eisenbud and Harris [3]. It is equal to

$$3\psi_1 - \frac{6}{5} \left[\text{diagram of two circles with a separating node} \right] - \frac{1}{10} \left[\text{diagram of two circles with a nonseparating node} \right].$$

So what do all these terms mean? The second and the third terms are pictures representing boundary divisors of $\overline{\mathcal{M}}_{2,1}$. The second term is the boundary divisor parametrizing curves with a separating node; the third term is, similarly, the boundary divisor parametrizing curves with a nonseparating node. In both cases we take the Poincaré dual cohomology

classes of these divisors. The class ψ_1 is the first Chern class¹ of a line bundle L_1 over $\overline{\mathcal{M}}_{2,1}$. The fiber of this bundle over a point $(C, x) \in \overline{\mathcal{M}}_{2,1}$ is the cotangent line to C at x . In general, there are n line bundles L_1, \dots, L_n like that over $\overline{\mathcal{M}}_{g,n}$, corresponding to the n marked points.

A general method to compute the cohomology classes Poincaré dual to $\mathcal{P}\overline{\mathcal{H}}_g(k_1, \dots, k_n)$ in $\mathbb{P}\mathbb{E}_{g,n}$ was developed by Sauvaget [10], though it does not lead to a closed formula.

A conjectural closed formula for the cohomology class of the image of $\overline{\mathcal{H}}_g(k_1, \dots, k_n)$ in $\overline{\mathcal{M}}_{g,n}$ was proposed in the appendix of [8]. It involves the ψ -classes introduced above, the classes of boundary strata, while their coefficients are some special values of Bernoulli polynomials.

In conclusion, let us mention two developments that connect the two views on the spaces of translation surfaces.

The first one is a conjecture by Sauvaget relating the cohomology class Poincaré dual to $\mathcal{P}\overline{\mathcal{H}}_g(k_1, \dots, k_n)$ to the volume of $\overline{\mathcal{H}}_g^{\leq 1}(k_1, \dots, k_n)$. For $n = 1$ the conjecture is proved in [11].

The second is a result by Eskin, Mirzakhani and Mohamadi [5] that Anton Zorich, in his review paper [13] called “the magical wand theorem”. It states that any $\mathrm{GL}(2, \mathbb{Z})$ -invariant closed subset of $\overline{\mathcal{H}}_g(k_1, \dots, k_n)$ is an algebraic subvariety of $\mathbb{E}_{g,n}$ defined over $\overline{\mathbb{Q}}$; locally, in period coordinates it is an affine subspace. This is the single most important step towards the classification of all $\mathrm{GL}(2, \mathbb{Z})$ -invariant closed sets.

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¹One of the ways to define the first Chern class of a line bundle L is to construct a meromorphic section of L and take the divisor of its zeros minus the divisor of its poles. The Poincaré dual cohomology class of the difference between these two divisors does not depend on the section and represents the first Chern class of L .