



HAL
open science

UNCONDITIONAL CHEBYSHEV BIASES IN NUMBER FIELDS

Daniel Fiorilli, Florent Jouve

► **To cite this version:**

Daniel Fiorilli, Florent Jouve. UNCONDITIONAL CHEBYSHEV BIASES IN NUMBER FIELDS. 2022. hal-03088071v2

HAL Id: hal-03088071

<https://hal.science/hal-03088071v2>

Preprint submitted on 4 Apr 2022

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

UNCONDITIONAL CHEBYSHEV BIASES IN NUMBER FIELDS

DANIEL FIORILLI AND FLORENT JOUVE

ABSTRACT. Prime counting functions are believed to exhibit, in various contexts, discrepancies beyond what famous equidistribution results predict; this phenomenon is known as Chebyshev's bias. Rubinstein and Sarnak have developed a framework which allows to conditionally quantify biases in the distribution of primes in general arithmetic progressions. Their analysis has been generalized by Ng to the context of the Chebotarev density theorem, under the assumption of the Artin holomorphy conjecture, the Generalized Riemann Hypothesis, as well as a linear independence hypothesis on the zeros of Artin L -functions. In this paper we show unconditionally the occurrence of extreme biases in this context. These biases lie far beyond what the strongest effective forms of the Chebotarev density theorem can predict. More precisely, we prove the existence of an infinite family of Galois extensions and associated conjugacy classes $C_1, C_2 \subset \text{Gal}(L/K)$ of same size such that the number of prime ideals of norm up to x with Frobenius conjugacy class C_1 always exceeds that of Frobenius conjugacy class C_2 , for every large enough x . A key argument in our proof relies on features of certain subgroups of symmetric groups which enable us to circumvent the need for unproven properties of zeros of Artin L -functions.

1. INTRODUCTION AND STATEMENT OF RESULTS

In 1853, Chebyshev noticed in a letter to Fuss [C] that there seems to exist a bias in the distribution of primes modulo 4, that is in most intervals of the form $[2, x]$, there appears to be more primes of the form $4n + 3$ than of the form $4n + 1$. It turns out that the specific statements made in Chebyshev's letter are quite deep: the second is equivalent to the Riemann hypothesis for $L(s, \chi_{-4})$, and the first can be made explicit under an additional linear independence hypothesis on the zeros of $L(s, \chi_{-4})$. Chebyshev's observation has been widely generalized over the years; notably, Rubinstein and Sarnak [RS] have shown that for two invertible residue classes a and b modulo q , there exists a bias towards a (that is $\pi(x; q, a) > \pi(x; q, b)$ is true more often than $\pi(x; q, a) < \pi(x; q, b)$) if and only if b is a quadratic residue and a is a non-quadratic residue. These theoretical results, as well as the numerical determinations of the bias in the paper, are conditional on the generalized Riemann hypothesis and a linear independence hypothesis on the non-trivial zeros of Dirichlet L -functions. In the same paper [RS, §5], the authors mention several possible generalizations including biases in the distribution of prime ideals in Galois extensions of number fields. This context was explored by Ng in his Ph.D. thesis [N]. Consider a Galois extension L/K of number fields, a conjugacy class $C \subset G = \text{Gal}(L/K)$, and define the Frobenius counting function

$$\pi(x; L/K, C) := \sum_{\substack{\mathfrak{p} \in \mathcal{O}_K \text{ unram.} \\ \mathcal{N}\mathfrak{p} \leq x \\ \text{Frob}_{\mathfrak{p}} = C}} 1,$$

Date: April 1, 2022.

where $\text{Frob}_{\mathfrak{p}}$ denotes the Frobenius conjugacy class associated to the unramified prime ideal \mathfrak{p} , and $\mathcal{N}\mathfrak{p} = |\mathcal{O}_K/\mathfrak{p}|$ denotes its norm. The Chebotarev density theorem asserts that $\pi(x; L/K, C) \sim \frac{|C|}{|G|} \int_2^x \frac{dt}{\log t}$. More precisely, one is interested in understanding the size of the sets

$$P_{L/K; C_1, C_2} := \{x \in \mathbb{R}_{\geq 1} : |C_2|\pi(x; L/K, C_1) > |C_1|\pi(x; L/K, C_2)\}.$$

Ng [N] has shown under Artin's holomorphy conjecture, GRH, as well as a linear independence hypothesis on the set of zeros of Artin L -functions, that the set $P_{L/K; C_1, C_2}$ admits a logarithmic density, that is the limit

$$\delta(P_{L/K; C_1, C_2}) := \lim_{X \rightarrow \infty} \frac{1}{\log X} \int_{\substack{1 \leq x \leq X \\ x \in P_{L/K; C_1, C_2}}} \frac{dx}{x}$$

exists. Moreover, he computed this density in several explicit extensions, under the same hypotheses.

The goal of this paper is to show *unconditionally* the existence of the density $\delta(P_{L/K; C_1, C_2})$ in some families of extensions and for specific conjugacy classes. More precisely, we will exhibit a sufficient group-theoretic criterion on $G = \text{Gal}(L/K)$ which implies in particular that $\delta(P_{L/K; C_1, C_2}) = 1$. This will involve the class function $r_G : G \rightarrow \mathbb{C}$ defined by

$$r_G(g) := \#\{h \in G : h^2 = g\}.$$

We will require L/\mathbb{Q} to be Galois, and for a conjugacy class $C \subset G$ we will denote by C^+ the unique conjugacy class of $G^+ := \text{Gal}(L/\mathbb{Q})$ which contains C . Explicitly,

$$C^+ := \bigcup_{a \in G^+} aCa^{-1}. \tag{1}$$

Theorem 1.1. *Let L/K be an extension of number fields for which L/\mathbb{Q} is Galois. Assume that the conjugacy classes $C_1, C_2 \subset G = \text{Gal}(L/K)$ are such that $C_1^+ = C_2^+$, but $r_G(gc_1) < r_G(gc_2)$, where gc_i is a representative of C_i . Then, for all large enough x we have the inequality $|C_2|\pi(x, L/K, C_1) > |C_1|\pi(x, L/K, C_2)$. In particular, the set $P_{L/K; C_1, C_2}$ has natural (and logarithmic) density equal to 1.*

Remark. The fact that the natural density of $P_{L/K; C_1, C_2}$ exists in Theorem 1.1 is remarkable since it is widely believed that in the classical case of primes in arithmetic progressions as well as in the more general case of Galois extensions of number fields, the logarithmic density is the appropriate notion to work with. In general one cannot expect natural densities to exist (see [K], as well as [RS, p. 174] and the references therein).

Note also that in Theorem 1.1, one can further impose C_1 and C_2 to have the same size. Indeed, we will see in the proof of Proposition 1.2 (see §2) that there exists families of examples in which the group G is abelian.

Next we state a group theoretic result showing that the hypotheses of Theorem 1.1 are satisfied by infinitely many couples (G, G^+) and associated conjugacy classes $C_1, C_2 \subset G$.

Proposition 1.2. *For $n \geq 8$ the symmetric group $G^+ = S_n$ admits a subgroup G which contains conjugacy classes C_1, C_2 satisfying $C_1^+ = C_2^+$, but $r_G(gc_1) < r_G(gc_2)$, where $gc_i \in C_i$ ($i = 1, 2$).*

The combination of Theorem 1.1, Proposition 1.2 and the fact going back to Hilbert that the inverse Galois problem over \mathbb{Q} is solved for the symmetric group S_n immediately yields the following consequence.

Corollary 1.3. *There exists infinitely many Galois extensions L/K and conjugacy classes $C_1, C_2 \subset \text{Gal}(L/K)$ for which $\delta(P_{L/K;C_1,C_2}) = 1$.*

The paper is organized as follows. Section 2 is devoted to the group theoretic aspects of our main result. In particular we prove Proposition 1.2 and discuss generalizations and related questions. In Section 3, we prove Theorem 1.1. We conclude the paper with Section 4 which is devoted to numerical computations and illustrations of Theorem 1.1.

ACKNOWLEDGMENTS

Experiments presented in this paper were carried out using the PlaFRIM experimental testbed, supported by Inria, CNRS (LABRI and IMB), Université de Bordeaux, Bordeaux INP and Conseil Régional d'Aquitaine (see <https://www.plafrim.fr/>). We thank Bill Allombert for his insights and for providing us with the `pari/gp` code and the data needed for this project. We also thank Mounir Hayani for very inspiring remarks. Finally we thank the referee and editors for a thorough reading and for suggestions which led to significant improvements in the presentation of the paper. The work of both authors was partly funded by the ANR through project FLAIR (ANR-17-CE40-0012).

2. GROUP THEORETICAL RESULTS

The goal of this section is to construct families of abelian extensions L/K satisfying the hypotheses of Theorem 1.1.

Proof of Proposition 1.2. For $n \geq 8$, consider the permutations $g_1 := (12)(34)$ and $g_2 := (57)(68)$ as elements of S_n . Let $G := \langle (12)(34), (5678) \rangle < S_n$. We claim that the choices $C_1 = g_1$ and $C_2 = g_2$ satisfy the required properties. Indeed, $C_1^+ = C_2^+ = C_{(2,2)}$, where $C_{(2,2)}$ is the set of elements of S_n of cycle type $(2, 2)$. Moreover, an enumeration of the elements of G shows that $r_G(g_1) = 0$ and $r_G(g_2) = 4$. \square

The next lemma gives a group theoretical criterion which generalizes the construction in the proof of Proposition 1.2 and which implies the conditions of Theorem 1.1. (Here and later in the paper we make a slight abuse of notation by denoting $r_G(C)$ the common value $r_G(g)$ as g runs over the G -conjugacy class C .)

Lemma 2.1. *Let G^+ be a group and let H and K be subgroups having trivial intersection and such that H centralizes K . Let $h \in H$ be a non-square (in H), and let $k \in K$ be a square (in K) which is a conjugate of h in G^+ . Then, the conjugacy classes $C_1 = C_h$ and $C_2 = C_k$ in the group $G = HK$ are such that $r_G(C_2) > r_G(C_1)$; in other words, the conditions of Theorem 1.1 hold.*

Proof. The fact that H centralizes K guarantees that $G = HK = KH$ is a subgroup of G^+ . Moreover, any $x \in G$ such that $x^2 = k$ can be written $x = st$ with $s \in H$ and $t \in K$ (and in this decomposition there is a unique (s, t) corresponding to each x since $H \cap K = \{1\}$). Thus $k = s^2 t^2$, which implies that $s^2 \in H \cap K$. Therefore $s^2 = 1$, and as a result

$$\#\{x \in G : x^2 = k\} = \#\{x \in K : x^2 = k\} \cdot \#\{x \in H : x^2 = 1\} > 0.$$

By symmetry, we also have that

$$\#\{x \in G : x^2 = h\} = \#\{x \in H : x^2 = h\} \cdot \#\{x \in K : x^2 = 1\} = 0.$$

□

In order to apply Lemma 2.1, take for instance $G^+ = S_n$, and let $\sigma, \tau \in S_n$ be permutations of order divisible by 4 which have the same cycle type, but have disjoint supports. Consider the subgroups $H = \langle \sigma^2 \rangle$ and $K = \langle \tau \rangle$, and the elements $h = \sigma^2$ and $k = \tau^2$. We clearly have that $r_K(k) \geq 1$ and $r_H(h) = 0$, and Lemma 2.1 applies.

Remark. From a group theoretical point of view, it would be interesting to classify the tuples (G, G^+, C_1, C_2) such that G^+ is a finite group, $G < G^+$ and C_1, C_2 are conjugacy classes of G such that $r_G(C_1) \neq r_G(C_2)$ and $C_1^+ = C_2^+$ in G^+ (recall (1)). For example, one notices that no such tuple exists where G is a normal subgroup of G^+ (see [FJ, Proof of Lemma 3.13]). Beyond this case, one may ask the following questions: how rare is the property enjoyed by these tuples? What are the “minimal” examples? Such questions are the subject of Mounir Hayani’s forthcoming Ph.D. thesis.

3. PROOF OF THEOREM 1.1

To introduce the natural framework of Theorem 1.1, we will work in the setting of [B], that is we will consider general class functions $t : \text{Gal}(L/K) \rightarrow \mathbb{C}$, and define¹

$$\psi(x; L/K, t) := \sum_{\substack{\mathfrak{p} \triangleleft \mathcal{O}_K \\ \mathcal{N}\mathfrak{p} \leq x \\ k \geq 1}} t(\text{Frob}_{\mathfrak{p}}^k) \log(\mathcal{N}\mathfrak{p}); \quad \theta(x; L/K, t) := \sum_{\substack{\mathfrak{p} \triangleleft \mathcal{O}_K \\ \mathcal{N}\mathfrak{p} \leq x}} t(\text{Frob}_{\mathfrak{p}}) \log(\mathcal{N}\mathfrak{p});$$

$$\pi(x; L/K, t) := \sum_{\substack{\mathfrak{p} \triangleleft \mathcal{O}_K \\ \mathcal{N}\mathfrak{p} \leq x \\ \mathfrak{p} \text{ unram.}}} t(\text{Frob}_{\mathfrak{p}}).$$

When L/\mathbb{Q} is Galois, we will use the shorthands $G := \text{Gal}(L/K)$, $G^+ := \text{Gal}(L/\mathbb{Q})$, as well as

$$t^+ = \text{Ind}_G^{G^+} t : G^+ \rightarrow \mathbb{C}, \quad g \mapsto \sum_{\substack{aG \in G^+/G : \\ a^{-1}ga \in G}} t(a^{-1}ga).$$

Finally, we recall that the inner product of class functions $t_1, t_2 : G \rightarrow \mathbb{C}$ is defined by

$$\langle t_1, t_2 \rangle_G := \frac{1}{|G|} \sum_{g \in G} t_1(g) \overline{t_2(g)}.$$

(We will simply write $\langle t_1, t_2 \rangle$, dropping the subscript G , where the underlying group is clear from context.)

Lemma 3.1. *Let L/K be an extension of number fields for which L/\mathbb{Q} is Galois, and let $t : \text{Gal}(L/K) \rightarrow \mathbb{C}$ be a class function. We have the estimate*

$$\pi(x; L/K, t) = \int_{2^-}^x \frac{d\psi(u; L/\mathbb{Q}, t^+)}{\log u} - \langle t, r_G \rangle \frac{x^{\frac{1}{2}}}{\log x} + o\left(\frac{x^{\frac{1}{2}}}{\log x}\right).$$

¹See for instance [M, Chap. 1 §4] for a definition of $\text{Frob}_{\mathfrak{p}}$ in the case where \mathfrak{p} is ramified.

Proof. For any integer $\ell \geq 2$, denote by $f_\ell: G \rightarrow G$ the class function defined by $f_\ell(g) = g^\ell$. Let μ denote the Möbius function; inclusion-exclusion implies that

$$\begin{aligned}\theta(x; L/K, t) &= \psi(x; L/K, t) + \sum_{\ell \geq 2} \mu(\ell) \psi(x^{\frac{1}{\ell}}; L/K, t \circ f_\ell) \\ &= \psi(x; L/K, t) - \langle t, r_G \rangle x^{\frac{1}{2}} (1 + o(1)) + O(x^{\frac{1}{3}}),\end{aligned}$$

by the Chebotarev density theorem and the identity $\frac{1}{|G|} \sum_{g \in G} t(g^2) = \langle t, r_G \rangle$. The claimed estimate follows from a summation by parts and an application of the identity

$$\psi(u; L/K, t) = \psi(u; L/\mathbb{Q}, t^+),$$

which is a consequence of the invariance of Artin L -functions under induction ([A, §2], under the form used in [FJ, Proposition 3.11]). \square

Proof of Theorem 1.1. We first compute, for any conjugacy class C of G , any fixed $g_C \in C$ and any irreducible character χ of G^+ ,

$$\langle 1_C^+, \chi \rangle_{G^+} = \langle 1_C, \chi|_G \rangle_G = \frac{|C|}{|G|} \overline{\chi(g_C)} = \frac{|C||G^+|}{|G||C^+|} \langle 1_{C^+}, \chi \rangle_{G^+},$$

where the first step uses Frobenius reciprocity. Therefore, denoting $t_{C_1, C_2}: \text{Gal}(L/K) \rightarrow \mathbb{C}$ the class function $t_{C_1, C_2} = \frac{|G|}{|C_1|} 1_{C_1} - \frac{|G|}{|C_2|} 1_{C_2}$, one has $t_{C_1, C_2}^+ = \frac{|G^+|}{|C_1^+|} 1_{C_1^+} - \frac{|G^+|}{|C_2^+|} 1_{C_2^+} \equiv 0$. Hence, Lemma 3.1 implies that

$$\pi(x; L/K, t_{C_1, C_2}) = -\langle t_{C_1, C_2}, r_G \rangle \frac{x^{\frac{1}{2}}}{\log x} + o\left(\frac{x^{\frac{1}{2}}}{\log x}\right).$$

However, $-\langle t_{C_1, C_2}, r_G \rangle = r_G(g_{C_2}) - r_G(g_{C_1}) > 0$, and thus $\pi(x; L/K, t_{C_1, C_2}) > 0$ for all large enough values of x . \square

We now discuss more precisely the oscillations of $\pi(x; L/K, C_1) - \pi(x; L/K, C_2)$ for triples $(L/K, C_1, C_2)$ chosen as in the proof of Proposition 1.2 and Corollary 1.3 (an explicit example of such a Galois extension produces Figure 1, and the purpose here is to discuss the rate of convergence of the function plotted to its asymptotic value). We recall that in the proof of Proposition 1.2, we have chosen $G = \langle (12)(34), (5678) \rangle$ and $t = 1_{C_1} - 1_{C_2}$, where $C_1 = \{(12)(34)\}$ and $C_2 = \{(57)(68)\}$. Since G is abelian of order 8, the class function $r_m(g) := \#\{h \in G: h^m = g\}$ is identically equal to 1 for all odd $m \geq 1$, and in particular, $\langle t \circ f_3, 1 \rangle = 0$ (where we recall that f_ℓ is the function on G raising elements to their ℓ -th power). The identity $\psi(x; L/K, t) = \psi(x; L/\mathbb{Q}, t^+) \equiv 0$ and the Riemann Hypothesis for Artin L -functions then imply that

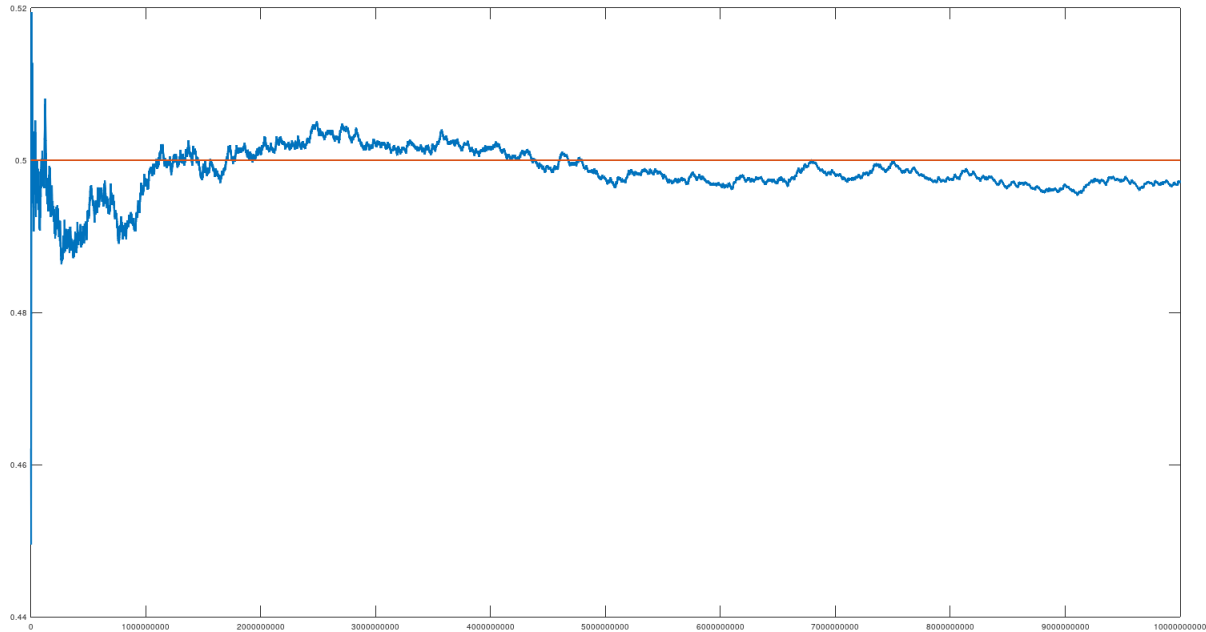
$$\begin{aligned}\theta(x; L/K, t) &= \psi(x; L/K, t) - \psi(x^{\frac{1}{2}}; L/K, t \circ f_2) - \psi(x^{\frac{1}{3}}; L/K, t \circ f_3) + O(x^{\frac{1}{5}}) \\ &= -\langle t, r_G \rangle x^{\frac{1}{2}} + \sum_{\chi \in \text{Irr}(G)} \overline{\langle \chi, t \circ f_2 \rangle} \sum_{\rho_\chi} \frac{x^{\frac{1}{4} + \frac{1}{2} \Im(\rho_\chi) i}}{\rho_\chi} + O(x^{\frac{1}{5}}),\end{aligned}$$

by the explicit formula (see for instance [N, Theorem 3.4.9]). Here, $\text{Irr}(G)$ denotes the set of irreducible characters of G , and ρ_χ runs through the non-trivial zeros of the Artin L -function $L(s, L/K, \chi)$. Now, in this particular example $8t \circ f_2 = 1_{\{(5678)\}} + 1_{\{(5678)(12)(34)\}} + 1_{\{(5876)\}} + 1_{\{(5876)(12)(34)\}}$, and thus $\langle \chi, t \circ f_2 \rangle = \chi((5678)) + \chi((5678)(12)(34)) + \chi((5876)) + \chi((5876)(12)(34))$ (which is not identically zero). This explains why we expect the difference

between the solid line and the data in Figure 1 to be roughly of order $x^{-\frac{1}{4}}$. (More precisely, we expect order $x^{-\frac{1}{4}}$ almost everywhere, and maximal order $x^{-\frac{1}{4}}(\log \log \log x)^2$.)

4. NUMERICAL EXAMPLES

FIGURE 1. The normalized difference $(\pi(x; L/K, C_1) - \pi(x; L/K, C_2))/R(x)$ with $1 \leq x \leq 10^{10}$ (data due to B. Allombert)



In this section we discuss our numerical verification of Theorem 1.1 and Proposition 1.2. It would be computationally very expensive to work with the full group S_8 . However, it turns out that one can replace S_8 with a relatively small subgroup which has the required properties. Consider $G^+ := \langle (12)(34), (5678), (15)(27)(36)(48) \rangle$; let us show that G^+ is isomorphic to the wreath product of $\mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$, which is of order 32. Denote the permutations appearing in the generating set of G^+ by τ , σ , and γ , respectively, and note that $G^+ = \langle \sigma, \gamma\sigma\gamma, \gamma \rangle$ (since $\gamma\sigma\gamma = (1324)$, and thus $(\gamma\sigma\gamma)^2 = \tau$). The subgroup $\langle \sigma, \gamma\sigma\gamma \rangle$ is clearly isomorphic to $(\mathbb{Z}/4\mathbb{Z}) \times (\mathbb{Z}/4\mathbb{Z})$. Moreover, conjugating by γ on $\langle \sigma, \gamma\sigma\gamma \rangle$ amounts to exchanging the two factors $\mathbb{Z}/4\mathbb{Z}$, which is the definition of the wreath product.

Consider also the abelian subgroup $G := \langle (12)(34), (5678) \rangle < G^+$ as well as the conjugacy classes $C_1 := \{(12)(34)\}$ and $C_2 := \{(57)(68)\}$. In the group G^+ , one has that $\gamma^{-1}C_1\gamma = C_2$, that is $C_1^+ = C_2^+$. It follows from Theorem 1.1 that for any Galois number field L/\mathbb{Q} such that $\text{Gal}(L/\mathbb{Q}) \simeq G^+$, the sub-extension $K = L^G$ has the property that for all large enough x ,

$$\pi(x; L/K, C_1) > \pi(x; L/K, C_2)$$

(recall that $|C_1| = |C_2| = 1$). Bill Allombert has kindly provided us with the `pari/gp` code allowing for a numerical check of this inequality up to $x = 10^{10}$, for a particular number field

L/\mathbb{Q} of Galois group G^+ . Explicitly, $L = \mathbb{Q}[x]/(f(x))$, where²

$$\begin{aligned} f(x) = & x^{32} - 128x^{30} + 5680x^{28} - 120576x^{26} + 1386352x^{24} - 9267712x^{22} + 38233408x^{20} \\ & - 101305344x^{18} + 176213088x^{16} - 202610688x^{14} + 152933632x^{12} - 74141696x^{10} \\ & + 22181632x^8 - 3858432x^6 + 363520x^4 - 16384x^2 + 256. \end{aligned}$$

In Figure 1 we have plotted the difference $\pi(x; L/K, C_1) - \pi(x; L/K, C_2)$, normalized by the function

$$R(x) := \frac{x^{\frac{1}{2}}}{\log x} + \int_2^x \frac{du}{u^{\frac{1}{2}}(\log u)^2} \sim \frac{x^{\frac{1}{2}}}{\log x},$$

which can be shown following the proof of Lemma 3.1 to be the “natural approximation” for the order of magnitude of this difference. As expected, we see that the plotted function converges to $\frac{1}{2}$, and to illustrate this we have added the solid line $y = \frac{1}{2}$ on the plot. Finally, we see that as predicted in Section 3, the difference between the graph and the solid line is of order $x^{\frac{1}{4}}$.

REFERENCES

- [A] E. Artin, *Zur theorie der L-reihen mit allgemeinen gruppencharakteren*, Abh. Math. Sem. Univ. Hamburg, 8, 1931, 1, 292–306,
- [B] J. Bellaïche, *Théorème de Chebotarev et complexité de Littlewood*. Ann. Sci. Éc. Norm. Supér. (4) **49** (2016), no. 3, 579–632.
- [C] P. L. Chebyshev, *Lettre de M. le Professeur Tchébychev à M. Fuss sur un nouveau théorème relatif aux nombres premiers contenus dans les formes $4n + 1$ et $4n + 3$* , Bull. Classe Phys. Acad. Imp. Sci. St. Petersburg, 11 (1853), 208.
- [FJ] F. Jouve and D. Fiorilli, *Distribution of Frobenius elements in Galois extensions*. arXiv:2001.05428
- [K] J. Kaczorowski, *On the distribution of primes (mod 4)*. Analysis **15** (1995), no. 2, 159–171.
- [M] J. Martinet, *Character theory and Artin L-functions*. Algebraic number fields: L-functions and Galois properties (Proc. Sympos., Univ. Durham, Durham, 1975), pp. 1–87. Academic Press, London, 1977.
- [N] N. Ng, *Limiting distributions and zeros of Artin L-functions*. Ph.D. thesis, University of British Columbia, 2000.
- [RS] M. Rubinstein and P. Sarnak, *Chebyshev’s bias*. Experiment. Math. **3** (1994), no. 3, 173–197.

UNIV. PARIS-SACLAY, CNRS, LABORATOIRE DE MATHÉMATIQUES D’ORSAY, 91405, ORSAY, FRANCE.
E-mail address: daniel.fiorilli@universite-paris-saclay.fr

UNIV. BORDEAUX, CNRS, BORDEAUX INP, IMB, UMR 5251, F-33400, TALENCE, FRANCE.
E-mail address: florent.jouve@math.u-bordeaux.fr

² For the full code, click here or visit <https://www.math.u-bordeaux.fr/~fjouve001/UnconditionalBiasCode.gp>.