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A LARGE CLASS OF DENDRITE MAPS FOR WHICH MÖBIUS DISJOINTNESS PROPERTY OF SARNAK IS FULFILLED.

el HOUCEIN el ABDALAOUI & JOSEPH DEVIANNE

ABSTRACT. We prove that any dendrite map for which the set of endpoints is closed and countable fulfilled Sarnak Möbius disjointness. This extended a result by el Abdalaoui-Askri and Marzougui [1]. We further notice that the Smítal-Ruelle property can be extended to the class of dendrites with closed and countable endpoints.

1. Introduction

Let X be a compact metric space with a metric d and let $f : X \rightarrow X$ be a continuous map. We call for short (X, f) a dynamical system. The Möbius disjointness conjecture of Sarnak [31], [32] assert that for any dynamical system (X, f) with topological entropy zero, for any continuous function $\phi : X \rightarrow \mathbb{C}$, for any point $x \in X$, we have

$$(1.1) \quad S_N(x, \phi) := \frac{1}{N} \sum_{n=1}^N \mu(n) \phi(f^n(x)) = o(1), \text{ as } N \rightarrow +\infty,$$

where μ is the Möbius function defined by

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1 \\ \lambda(n) & \text{if all primes in decomposition of } n \text{ are distinct} \\ 0 & \text{otherwise,} \end{cases}$$

Here, λ stand for the Liouville function λ given by $\lambda(n) = 1$ if the number of prime factors of n is even and -1 otherwise.

We recall that the topological entropy $h(f)$ of a dynamical system (X, f) is defined as

$$h(f) = \lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow +\infty} \frac{1}{n} \log \text{sep}(n, f, \varepsilon),$$

where for n integer and $\varepsilon > 0$, $\text{sep}(n, f, \varepsilon)$ is the maximal possible cardinality of an (n, f, ε) -separated set in X , this later means that for every two points of it, there exists $0 \leq j < n$ with $d(f^j(x), f^j(y)) > \varepsilon$, f^j stand for the j -th iterate of f .

The purpose of this paper is to strengthen the first main result in [1] by proving that the Möbius disjointness of Sarnak is fulfilled for a zero topological map on the dendrites for which the set of endpoints is closed and countable. In our proof, we extend and correct the Askri's proof of the main theorem in [8]. We further fill the gap in the proof of Theorem 4.10 in [1] when the derivative set of the endpoints does not intersect an open connected component of the canonical decomposition of dendrite.

Our result is also related to the conjecture of J. Li, P. Oprocha and G-H. Zhang which assert that the spectrum of such map is discrete [27]. Therefore, under this conjecture, the Möbius disjointness property of Sarnak is fulfilled. Indeed, it is well-known that a dynamical system with discrete spectrum satisfy the Möbius disjointness [19], [3]. Let us further notice that recently, M.

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Nerurkar and the first author proved that the Möbius disjointness of Sarnak holds for a dynamical system with singular spectrum [5]. At this point, let us mention that J. Li, P. Oprocha and G. Zhang proved that any dynamical system with zero topological entropy can be embedded in a Gehman dendrites with zero entropy. It follows that proving the Möbius disjointness of Sarnak will settle the conjecture. We thus restrict ourselves to this setting and for the recent and other results, we refer to [1] and [5].

The principal arithmetical tool of our proof is the Dirichlet prime number theorem (DPNT) [6]. We stress, as pointed out by e.H. el Abdalaoui and M. Nerurkar [5], that the only arithmetical ingredients used until now are the following.

- (1) The prime number theorem (PNT) and the Dirichlet PNT.
- (2) The so-called Daboussi-Katai-Bourgain-Sarnak-Ziegler criterion. This criterion asserts that if a sequence (a_n) satisfies for a large prime p and q , the orthogonality of (a_{np}) and (a_{nq}) , then the orthogonality holds between (a_n) and any bounded multiplicative function. The proof of this ingredient is based on the PNT and it was used first by Bourgain-Sarnak-Ziegler [11] and then by many other authors. Recently, this criterion was generalized by M. Cafferata, A. Perelli and A. Zaccagnini to some class of non bounded multiplicative functions such as the normalized Fourier coefficients λ_f of the normalized Hecke eigenform f and its Dirichlet inverse [14].
- (3) Finally, the result of Matomaki-Radziwiłł-Tao on the validity of average Chowla of order two [28]. This was used to establish the conjecture for systems with discrete spectra. It was used by el Abdalaoui-Lemańczyk-de-la-Rue [2], and Huang-Wang and Zhang [20].

Let us emphasize also that the method of the authors in [5] does not use any of the above techniques. In fact, therein, the authors presented a spectral dissection based on an unpublished result of W. Veech. From the number theory, they used Davenport's estimation combined with Daboussi's characterization of the Besicovitch class of multiplicative functions [16].

We recall that the dynamical system (X, f) is said to have a topological discrete spectrum if the eigenfunctions of its Koopmann operator on the space $C(X)$ span a dense linear subspace of the $C(X)$ ($C(X)$ is the space of continuous functions equipped with the strong topology). It is well-known that this latter property is equivalent to the equicontinuity of the dynamical system [34]. It is also well-known that the system (X, f) has always an invariant probability measure μ , by Krylov-Bogoliubov theorem [34], that is, for any $f \in C(X)$, $\int \phi \circ f d\mu = \int f d\mu$. The measurable dynamical system (X, f, μ) is said to have a discrete spectrum if the eigenfunctions of its Koopmann operator on the space $L^2(X, \mu)$ span a dense linear subspace of the $L^2(X, \mu)$. In [26], Kuznetsov established that the measurable dynamical system has a discrete spectrum if and only if its metric sequence entropy is zero for any sequence. We recall that the system for which the sequence entropy is zero for any sequence are called null systems. It turns out that the notion of null system is related to the so-called tame system. This latter notion was coined by E. Glasner in [17]. The dynamical system (X, T) is tame if the closure of $\{T^n/n \in \mathbb{Z}\}$ in X^X is Rosenthal compact¹. We recall that the set K is Rosenthal compact if and only if there is a Polish space P such that $K \subset \text{Baire-1}(P)$ where $\text{Baire-1}(P)$ is the first class of Baire functions, that is, pointwise limit of continuous functions on P . By Bourgain-Fremelin-Talagrand's theorem [12], K is Rosenthal compact if and only if K is a subset of the Borel functions on P with $K = \overline{\{f_n\}}$, $f_n \in C(P)$.

The precise connection between null systems and tame systems can be stated as follows [18], [23], [21]

$$\mathcal{N} \subset \mathcal{T},$$

where \mathcal{N} is the class of null systems and \mathcal{T} is the class of tame systems.

In [1], using Kuznetsov's criterion, it is observed that the graph map has a discrete spectrum, for any invariant measure. In [27], the authors give an alternative proof. Therein, the authors strengthened the previous result, by establishing that the quasi-graph has a discrete spectrum, for any invariant measure. E. Glasner and M. Megrelishvili proved that every continuous action on

¹ X^X is equipped with the pointwise convergence. This closure is called the enveloping semigroup of Ellis.

dendron is tame [17]. Therein, the authors established also that every action of topological group on metrizable regular continuum is null, hence, tame. Therefore, for all those systems, the Möbius disjointness of Sarnak holds. It is also observed in [1] that the Möbius disjointness property is fulfilled for the monotone maps on a local dendrites. We recall that for every topological space X , a map $f : X \rightarrow X$ is called *monotone* if $f^{-1}(C)$ is connected for any connected subset C of X . In particular, if f is a homeomorphism then it is monotone.

For the connection between the measurable notion of tame and discrete spectrum, we refer to [4].

Recently, G. Askri [9] proved that if the map on a dendrites has no Li-Yorke pairs, then its restriction to $\Lambda(f)$, the union of all ω -limit sets and the set of fixed points, is uniform equicontinuous. It follows that the Möbius disjointness property hold for such maps.

The plan of the paper is as follows. In Section 2, we give some definitions and preliminary properties on dendrites which are useful for the rest of the paper. In section 3, we state our main results and its consequences, we further give the proof of the main topological ingredient need it in its proof. Section 4 is devoted to the proof of our main result. Finally, in section 5, we discuss the reduction of Sarnak Möbius disjointness to the Gehman dendrites with zero topological and its related spectral problems.

2. Preliminaries and some results

Let \mathbb{Z} , \mathbb{Z}_+ and \mathbb{N} be the sets of integers, non-negative integers and positive integers, respectively. For $n \in \mathbb{Z}_+$ denote by f^n the n -th iterate of f ; that is, $f^0 = \text{identity}$ and $f^n = f \circ f^{n-1}$ if $n \in \mathbb{N}$. For any $x \in X$, the subset $\text{Orb}_f(x) = \{f^n(x) : n \in \mathbb{Z}_+\}$ is called the *orbit* of x (under f). A subset $A \subset X$ is called *f -invariant* (resp. strongly *f -invariant*) if $f(A) \subset A$ (resp., $f(A) = A$). It is called a *minimal set of f* if it is non-empty, closed, f -invariant and minimal (in the sense of inclusion) for these properties, this is equivalent to say that it is an orbit closure that contains no smaller one; for example a single finite orbit. When X itself is a minimal set, then we say that f is *minimal*. We define the *ω -limit set* of a point x to be the set $\omega_f(x) = \{y \in X : \exists n_i \in \mathbb{N}, n_i \rightarrow \infty, \lim_{i \rightarrow +\infty} d(f^{n_i}(x), y) = 0\}$. A point $x \in X$ is called

– *periodic* of period $n \in \mathbb{N}$ if $f^n(x) = x$ and $f^i(x) \neq x$ for $1 \leq i \leq n-1$; if $n = 1$, x is called a *fixed point* of f i.e. $f(x) = x$.

– *Almost periodic* if for any neighborhood U of x there is $N \in \mathbb{N}$ such that $\{f^{i+k}(x) : i = 0, 1, \dots, N\} \cap U \neq \emptyset$, for all $k \in \mathbb{N}$. It is well known (see e.g. [10], Chapter V, Proposition 5) that a point x in X is almost periodic if and only if $\overline{\text{Orb}_f(x)}$ is a minimal set of f .

A pair $(x, y) \in X \times X$ is called *proximal* if $\liminf_{n \rightarrow +\infty} d(f^n(x), f^n(y)) = 0$; it is called *asymptotic* if $\lim_{n \rightarrow +\infty} d(f^n(x), f^n(y)) = 0$. A pair $(x, y) \in X \times X$ is said to be a *Li-Yorke pair* of f if it is proximal but not asymptotic.

In this section, we recall some basic properties of graphs, tree and dendrites.

A *continuum* is a compact connected metric space. An *arc* is any space homeomorphic to the compact interval $[0, 1]$. A topological space is *arcwise connected* if any two of its points can be joined by an arc. We use the terminologies from Nadler [29].

By a *graph* X , we mean a continuum which can be written as the union of finitely many arcs such that any two of them are either disjoint or intersect only in one or both of their endpoints. For any point v of X , the *order* of v , denoted by $\text{ord}(v)$, is an integer $r \geq 1$ such that v admits a neighborhood U in X homeomorphic to the set $\{z \in \mathbb{C} : z^r \in [0, 1]\}$ with the natural topology, with the homeomorphism mapping v to 0. If $r \geq 3$ then v is called a *branch point*. If $r = 1$, then we call v an *endpoint* of X . If $r = 2$, v is called a *regular point* of X .

Denote by $B(X)$ and $E(X)$ the sets of branch points and endpoints of X respectively. An edge is the closure of some connected component of $X \setminus B(X)$, it is homeomorphic to $[0, 1]$. A subgraph of X is a subset of X which is a graph itself. Every sub-continuum of a graph is a graph ([29], Corollary 9.10.1). Denote by $S^1 = [0, 1]_{|0 \sim 1}$ the unit circle endowed with the orientation: the counter clockwise sense induced via the natural projection $[0, 1] \rightarrow S^1$. A circle is any space

homeomorphic to S^1 . By a *tree* X , we mean a graph which contains no simple closed curve. A point in X is a non-cut point if and only if it is an endpoint of X ([29], Proposition 9.27), and a continuum X is a tree if and only if X has only finitely many non-cut points ([29], Theorem 9.28). As a consequence, a continuum X is an arc if and only if X has exactly two non cut-points ([29], Corollary 9.29).

By a *dendrite* X , we mean a locally connected continuum containing no homeomorphic copy to a circle. Every sub-continuum of a dendrite is a dendrite ([29], Theorem 10.10) and every connected subset of X is arcwise connected ([29], Proposition 10.9). In addition, any two distinct points x, y of a dendrite X can be joined by a unique arc with endpoints x and y , denote this arc by $[x, y]$ and let us denote by $[x, y) = [x, y] \setminus \{y\}$ (resp. $(x, y] = [x, y] \setminus \{x\}$ and $(x, y) = [x, y] \setminus \{x, y\}$). A point $x \in X$ is called an *endpoint* if $X \setminus \{x\}$ is connected. It is called a *branch point* if $X \setminus \{x\}$ has more than two connected components. The number of connected components of $X \setminus \{x\}$ is called the *order* of x and denoted by $\text{ord}(x)$. The order of x relatively to a subdendrite Y of X is denoted by $\text{ord}_Y(x)$. Denote by $E(X)$ and $B(X)$ the sets of endpoints, and branch points of X , respectively. By ([25], Theorem 6, 304 and Theorem 7, 302), $B(X)$ is at most countable. A point $x \in X \setminus E(X)$ is called a *cut point*. It is known that the set of cut points of X is dense in X ([25], VI, Theorem 8, p. 302). Following ([7], Corollary 3.6), for any dendrite X , we have $B(X)$ is discrete whenever $E(X)$ is closed. An arc I of X is called *free* if $I \cap B(X) = \emptyset$. For a subset A of X , we call *the convex hull* of A , denoted by $[A]$, the intersection of all sub-continua of X containing A , one can write $[A] = \cup_{x,y \in A} [x, y]$.

A continuous map from a dendrite (resp. quasi-graph, graph) into itself is called a *dendrite map* (resp. *quasi-graph map*, resp. *graph map*).

It is well known that every dendrite map has a fixed point (see [29]). If Y is a sub-dendrite of X , define the retraction (or the *first point map*) $r_Y : X \rightarrow Y$ by letting $r_Y(x) = x$, if $x \in Y$, and by letting $r_Y(x)$ to be the unique point $r_Y(x) \in Y$ such that $r_Y(x)$ is a point of any arc in X from x to any point of Y (see [29, Lemma 10.24, p. 176]) Note that the map r_Y is constant on each connected component of $X \setminus Y$.

Furthermore, any dendrite can be approximated by tree, that is, there exist an increasing sequence of tree (Y_i) such that $\lim_i Y_i = X$ and the sequence of first point maps (r_{Y_i}) converges uniformly to the identity map on X (see [29], Theorem 10.27) As customary, for a subset A of

X , denote by \overline{A} the closure of A , A' the derivative set, that is, the set of all accumulation points of A and by $\text{diam}(A)$ the diameter of A . For the proprieties of derivative set, we refer to [24, § 9, pp.75-78].

We need the following results on the structure of dendrites.

Proposition 2.1 ([29]). *Every sub-continuum of a dendrite is a dendrite. Every connected subset of X is arcwise connected.*

Proposition 2.2 ([29], Theorem 10.2). *The set of all branch points is countable.*

Proposition 2.3 ([29], p.187, 10.41). *Let X be a dendrite. The set of points of order 2 is continuumwise dense in X .*

Theorem 2.4 ([29], p. 188, 10.42). *Let X be a dendrite. Then for each $\delta > 0$, there are finitely many open connected open and pairwise disjoint subsets U_1, \dots, U_r such that*

- (1) $X = \bigcup_{i=1}^r \overline{U_i}$
- (2) $\text{diam}(U_i) < \delta$
- (3) $\text{card}(U_i \cap U_j) \leq 1$

For the proof of Theorem 2.4 see also the proof of [25, Theorem 5].

Definition 2.5. *Let X be a dendrite and $F \subset X$. We set $[F]$ the convex hull of F given by the intersection of all subdendrites containing F .*

The following lemma gives a better description of convex hulls in dendrites.

Lemma 2.6 ([8]). *Let X be a dendrite and F a non empty closed subset of X . Let $a \in F$, then*

- $[F] = \bigcup_{z \in F} [a, z]$
- $E([F]) \subset F$ and we have $E([F]) = F$ when $F \subset E(X)$

where $[A]$ stand for the convex hull of a subset A .

We need also the following lemma stated with $E(X)'$ finite in [8]. Its proof can be obtained in the similar manner as in [8].

Lemma 2.7 ([8]). *Let X be a dendrite such that $E(X)$ is closed. Let $Y \subset X$ a subdendrite of X such that $E(Y) \cap E(X)' = \emptyset$. Then Y is a tree and $X \setminus Y$ has finitely many connected components. Furthermore, there are pairwise disjoint subdendrites D_1, \dots, D_n in X such that $X \setminus Y \subset \bigcup_{1 \leq i \leq n} D_i$ and $D_i \cap Y$ is reduced to one point for each i .*

3. MAIN RESULTS

The main result of this paper is the following.

Theorem 3.1. *Let f be a dendrite map on X with zero topological entropy for which $E(X)$ is closed and countable. Then, f satisfy Sarnak Möbius disjointness (1.1).*

As a consequence, we have

Corollary 3.2 ([1], Theorem 4.10.). *Let f be a dendrite map on X with zero topological entropy for which $E(X)$ is closed and $E(X)'$ is finite. Then, f satisfy Sarnak Möbius disjointness (1.1).*

Proof. By the characterization theorem of the dendrites with a countable set of end points [15, Theorem 5], if $E(X)$ is not countable then it is a Cantor set. Therefore $E(X)'$ is not finite. This finished the proof of the corollary. ■

At this point, let us notice that recently, G. Askri [9] proved the following lemma.

Lemma 3.3. *Let X be a dendrite with $End(X)$ countable and closed and let $f : X \rightarrow X$ be a continuous map. Suppose that f has no Li-Yorke pairs. If one of the following assumptions holds*

- (1) *the collection of minimal sets is closed in $(2^X, d_H)$,*
- (2) *$f|_{\Lambda(f)}$ is equicontinuous at every point in $End(X)'$,*
- (3) *$f|_{P(f)}$ is uniform equicontinuous,*

then $f|_{\Lambda(f)}$ is equicontinuous. where $\Lambda(f)$ stand for the union of all ω -limit sets a the set of fixed points, and $P(f)$ is the set of periodic points.

It follows that if the dendrite map has no Li-Yorke pairs and one of the previous assumptions holds the Möbius disjointness property is fulfilled. This gives an alternative proof to the proof of D. Karagulan [22]. We further notice that there exist a dendrite map without Li-York pair such that $\Lambda(f) = P(f)$ and he collection of minimal sets is not closed in $(2^X, d_H)$. Therefore, the restriction of f to $\Lambda(f)$ is not equicontinuous (see [9, Lemma 4]).

For the proof of our main result (Theorem 3.1), we need to extend Smital-Ruette property. This later property is implicitly contained in several papers of Sharkovsky, but its first proof can be found in [33] and [30]. G. Askri extended this property to the class of dendrites for which the set of endpoints is closed and its derivative set is finite. Here, we extend it to the dendrites with closed and countable endpoints. Our extension is largely inspired from that of G. Askri. We state it as follows.

Theorem 3.4. *Let X be a dendrite such that $E(X)$ is closed and countable. Let $f : X \rightarrow X$ be a dendrite map with zero topological entropy. Let $L := \omega_f(x)$ be an uncountable ω -limit set. Then there is a sequence of f -periodic subdendrites $(D_k)_{k \geq 1}$ and a sequence $(n_k)_{k \geq 1}$ of integers with the following properties for any k ,*

- (1) *D_k has period $\alpha_k := n_1 n_2 \dots n_k$.*
- (2) *For $i \neq j \in \{0, \dots, \alpha_k - 1\}$, $f^i(D_k) \cap f^j(D_k) = \emptyset$.*

- (3) $L \subset \bigcup_{i=0}^{\alpha_k-1} f^i(D_k)$.
- (4) $\bigcup_{k=0}^{n_j-1} f^{k\alpha_j-1}(D_j) \subset D_{j-1}$.
- (5) For $i \in \{0, \dots, \alpha_k - 1\}$, $f(L \cap f^i(D_k)) = L \cap f^{i+1}(D_k)$.

For sake of completeness, we present a sketch of the proof of Theorem 3.4. For that, we proceed by induction and as in the proof of Smital-Ruelle property, we are going to prove the result at step $k = 1$ and then extend it by induction.

Proposition 3.5. *With the same assumptions as in Theorem 3.4, there exists a connected subset J of X and an interger $n \geq 2$ such that*

- (i) $J, f(J), \dots, f^{n-1}(J)$ are pairwise disjoint.
- (ii) $f^n(J) = J$.
- (iii) $L \subset \bigcup_{0 \leq i \leq n-1} f^i(J)$.
- (iv) For $i \in \{0, \dots, n - 1\}$, $f(L \cap f^i(J)) = L \cap f^{i+1}(J)$.

The proof of Proposition 3.5 relies on the following Lemma.

Lemma 3.6. *For $x \in X$, if $\omega(f)(x) = L$ is infinite then it contains no periodic point of f .*

As in the proof on the interval, we are going to build horseshoes to get contradictions. the following lemma will be useful.

Lemma 3.7. *Let $f : X \rightarrow X$ a dendrite map such that $E(X)$ is closed and countable. Let a a fixed point for f and $L := \omega_f(x)$ an uncountable ω -limit set such that $L \cap P(f) = \emptyset$ then for any $y \in L$, there is $p, k \geq 0$ such that $[a, f^k(x)] \subset [a, f^p(y)]$.*

Proof. Let $y \in L$. We show first that $\omega_f(y) \subset \omega_f(x)$ is uncountable. Since it is closed and invariant, we can consider a subset $K \subset \omega_f(y)$ which is minimal. Since $L \cap P(f) = \emptyset$, it contains no periodic point, so it is infinite and with no isolated point, hence it is uncountable and so it is for $\omega_f(y)$.

Denote now by $(C_i)_{i \in \mathbb{N}}$ the sequence of connected components of $X \setminus (B(X) \cup E(X))$. By Lemma 2.7, each C_i is an open free arc in X . There is $j \in \mathbb{N}$ such that $\omega_f(y) \cap C_j$ is uncountable. Let u, v in this intersection such that $u \in (a, v)$. Let I_u, I_v two open dsijoint arcs in C_j such that $u \in I_u, v \in I_v$. Then there exist $p, k' \in \mathbb{N}$ such that $f^p(y) \in I_v$ and $f^{k'}(y) \in I_u$. but since $f^{k'}(y) \in L$, there is $k \in \mathbb{N}$ such that $f^k(x) \in I_u$ which leads to the inclusion $[a, f^k(x)] \subset [a, f^p(y)]$. ■

Proof. [Proof of Lemma 3.6.] Let $M = [L]$ the convex hull of L . We only focus on the case $M \cap \text{Fix}(f) \neq \emptyset$. Let $a \in M \cap \text{Fix}(f)$. By the Theorem 3.6, we have that $a \in M \setminus L \subset M \setminus E(M)$ by Lemma 2.7. Hence, $\text{ord}(a, M) \geq 2$. ■

The proof of the first step $k = 1$ for the Smital-Ruelle property is basic since the jumping dynamic from one side of the fixed point to the other can be described. But, here since each branch connected to the fixed point a is also a dendrite complicates the proof. Indeed the natural idea would be to set $\omega_i = \omega_f(x) \cap C_i$ with $\{C_i\}_{i=1, \dots, p}$ the connected components of $X \setminus \{a\}$ and then to apply the same reasoning as in the proof of the first step of the Smital-Ruelle property. It would be then very complicated to prove that $a \notin J$. We need then to take apart the preimages of a , construct sets in the same way we constructed ω_0 and ω_1 , but such that they are disjoint from these preimages and then construct the set J . Let us set $F_a = \bigcup_{n \geq 0} f^{-n}(a)$ and $Y_a = [F_a]$.

We thus have the following lemma.

Lemma 3.8. *$L \subset M \setminus Y_a$ and $M \setminus Y_a$ has finitely many connected components.*

Proof. We start first by proving that $[a, z] \cap L = \emptyset$ for $z \in F_a$. Indeed, if $F_a = \{a\}$, there is nothing to prove. Else, suppose contrary that there is $y \in [a, z] \cap M, f^p(z) = a, p \geq 1$. We set $I = [a, y]$ and $J = [y, z]$ and easily construct an horseshoe with the help of Lemma 3.7. It is actually possible to prove something stronger, that is, $L \cap \overline{F_a} = \emptyset$ (see [8]). This proves

that $L \subset M \setminus \bigcup_{z \in \overline{F_a}} [a, z]$. Hence, by Lemma 2.6, we have $L \subset M \setminus Y_a$. Again by Lemma 2.6, $E(Y_a) \subset \overline{F_a}$ and $E(M)' \subset E(M) \subset L$ then $E(Y_a) \cap E(M)' = \emptyset$. Hence, with Lemma 2.7, Y_a is a tree and $M \setminus Y_a$ has finitely many connected components. ■

We are now able to have a better description of M and understand the action of f on it. In fact we have the following lemma.

Lemma 3.9. *Under above notations, we have*

- (1) *If $F_a \neq \{a\}$, for any $y \in L$, $(a, y] \cap F_a \neq \emptyset$.*
- (2) *$M \setminus F_a$ has finitely many connected components C_1, \dots, C_n intersecting L . We denote by $l_k = L \cap C_k$ each intersection.*
- (3) *For each $k \in \{1, \dots, n\}$, there is a unique $j := \sigma(k) \in \{1, \dots, n\}$ such that $f(C_k) \cap C_j \neq \emptyset$ and we have $f(l_k) = l_{\sigma(k)}$.*
- (4) *Each l_k is clopen relatively to L .*
- (5) *σ is an n -cycle.*

Proof. We give only the proof of (3). Since, the proof of (1), (2), (4) and (5) is the same as the proof of lemma 5.10 in [8]. Let $k \in \{1, \dots, n\}$, then $f(l_k) \subset f(L) = L = \bigcup_{1 \leq i \leq n} l_i$. Therefore, there is j such that $f(l_k) \cap l_j \neq \emptyset$ then $f(C_k) \cap C_j \neq \emptyset$. Suppose now that there is $i \neq j$ such that $f(C_k) \cap C_i \neq \emptyset$. Let $x_j \in f(C_k) \cap C_j$ and $x_i \in f(C_k) \cap C_i$. We have $[x_j, x_i] \subset f(C_k)$ and $f(C_k) \cap F_a = \emptyset$ so $[x_j, x_i] \cap F_a = \emptyset$. Hence $H = C_j \cup [x_j, x_i] \cup C_i$ is a connected subset of M such that $H \cap F_a = \emptyset$. By maximality of C_i and C_j we have $H = C_i = C_j$ which gives a contradiction.

Let $j = \sigma(k)$, then we have

$$f(l_k) \subset l_{\sigma(k)}.$$

Suppose that $f(l_k) \subsetneq l_{\sigma(k)}$. Then there exists $i \neq j$ such that $f(l_i) \cap l_{\sigma(k)}$ since $f(L) = L$. Consequently, $f(l_i) \subset l_{\sigma(k)}$ and $\sigma(i) = \sigma(k)$. Whence, σ is not injective and not surjective and we have $f(L) \subset \bigcup_{r=1}^n l_{\sigma(r)} \subsetneq L$ which gives a contradiction and proves that $f(l_k) = l_{\sigma(k)}$. ■

We are now able to give the proof of Proposition 3.5 and Theorem 3.4.

Proof. [Proof of Proposition 3.5 and Theorem 3.4.] Let $s_1 := [l_1] \subset C_1$ and $J := \bigcup_{k=0}^{\infty} f^{kn}(s_1)$. We proceed to establish that J satisfy the conditions of the Proposition 3.5. We

notice first that J is connected as a limit of an increasing sequence of connected sets. Since s_1 is a subdendrite and $s_1 \subset f^n(s_1)$. For the proof of (i), we use here the construction of F_a and

C_i as maximal connected components to show that $J, f(J), \dots, f^{n-1}(J)$ are pairwise disjoint. Indeed, since $s_1 \cap F_a$ and F_a is backward invariant, $f^i(s_1) \cap F_a = \emptyset$ for $i \geq 0$ and consequently, $f^k(J) \cap F_a = \emptyset$ for $k \geq 0$. Suppose now that $f^i(J) \cap f^j(J) \neq \emptyset$, then $f^i(J) \cap f^j(J)$ is connected.

If we take then $u \in l_{\sigma^i(1)}$ and $v \in l_{\sigma^j(1)}$ then $K = C_{\sigma^i(1)} \cup [u, v] \cup C_{\sigma^j(1)}$ is connected in M and disjoint with F_a by maximality, $K = C_{\sigma^i(1)} = C_{\sigma^j(1)}$ and then, $i = j$. (ii) and (iii) are

immediate. Finally, for (iv), we have $f(L \cap f^i(J)) = f(l_{\sigma^i(1)}) = l_{\sigma^{i+1}(1)} = L \cap f^{i+1}(J)$. We

conclude the proof of the theorem by induction. ■

Let us proceed now to the proof of our first main result.

4. PROOF OF THE MAIN RESULT.

As we mention in introduction, our proof is based essentially on DPNT which we formulated as follows.

Proposition 4.1. *If $(x_n)_{n \in \mathbb{N}}$ is eventually periodic sequence,*

$$\frac{1}{N} \sum_{n=1}^N \mu(n)x_n \xrightarrow{N \rightarrow \infty} 0.$$

As a consequence, we have that if the ω -limit set $\omega_f(x)$ is finite then for any continuous φ , we have that

$$S_N(x, \varphi) \xrightarrow{N \rightarrow +\infty} 0.$$

Indeed, it is easy to see that there exist a periodic point y such that (x, y) is asymptotic pair, that is, $\lim_{n \rightarrow +\infty} d(f^n(x), f^n(y)) = 0$. We thus need to verify (1.1) only when the $\omega_f(x)$ is infinite. For that, We need also the following lemma from [22].

Lemma 4.2. *Let $(a_n) \in [0, 1]^{\mathbb{N}}$ such that there exists $n_0 \in \mathbb{N}$ such that for $n, m \in \mathbb{N}$ if $a_n \neq 0$ and $a_m \neq 0$ then $n = m$ or $|n - m| \geq k$. Then*

$$\limsup_{N \rightarrow \infty} \frac{1}{N} \left| \sum_{n=1}^N a_n \right| \leq \frac{1}{k}$$

At this point, let us assume that $L = \omega_f(x)$ is infinite. We need thus to examine two cases according to L is countable or not. In the first case, it follows that $Y = \overline{\text{Orb}(x)} = \text{Orb}(x) \cup L$ is countable, Hence, $(Y, f|_Y)$ is tame and (1.1) holds by [20, Theorem 1.8]. We can thus suppose L uncountable.

As in the proof given by D. Karagulyan [22] and e. H. el Abdalaoui, G. Askri and H. Marzougui [1], we are going to make a proof by an approximation by step functions. Let $\varepsilon > 0$ and let $\delta > 0$ an uniform continuity modulus adapted to φ . We are going to construct a decomposition of X into sets of diameter $< \delta$ and such that each set shares at most two points with the rest of X .

We use Proposition 2.4 to get a first decomposition. There exists open connected subsets $V_i, i = 1, \dots, r$ such that

- (1) $X = \bigcup_{i=1}^r \overline{V_i}$
- (2) $\text{diam}(V_i) < \delta$
- (3) $\text{card}(V_i \cap V_j) \leq 1$ if $i \neq j$

To perform such decomposition, we apply Proposition 2.3 to see that there is a finite set F of cut points in X such that each component V_i of $X \setminus F$ (2) is satisfied (see [29, p.188]) and (see also the proof of [24, Theorem 5. p 302]).

Now, in order to get the same kind of estimate as in the proof by D. Karagulyan [22] and e. H. el Abdalaoui, G. Askri and H. Marzougui [1], we need to modify the sets V_i such that each V_i share at most two points with the rest of the decomposition.

We set $i \in \{1, \dots, r\}$ and consider the set V_i . We put $\partial V_i = \{f_1, \dots, f_{p_i}\}$ with $f_1, \dots, f_{p_i} \in F$.

We consider now the tree which has endpoints the f_1, \dots, f_{p_i} .

$$T = \bigcup_{1 \leq i, j \leq p} [f_i, f_j] \subset \overline{V_i}.$$

We split T into arcs by writing

$$T = \bigcup_{j=1}^m \overline{a_j},$$

with a_j disjoint open arcs, $a_j \cap F = \emptyset$ and $\text{card}(\overline{a_j} \cap F) = \text{card}(\overline{a_j} \cap \{f_1, \dots, f_p\}) \leq 2$.

We set now $(C_k)_{k \in K}$ the open connected components of $V_i \setminus T$ and let $r : \overline{V_i} \rightarrow T$ the first point map of T in V_i . Since r is locally constant, r is constant on each C_k and we denote by $x_k \in T$

the value on C_k . Since $\{f_1, \dots, f_p\}$ are cutpoints, then $x_k \in T \setminus \{f_1, \dots, f_p\}$. Hence, there exists a unique $j = c(k) \in \{1, \dots, m\}$ such that $x_k \in a_j$. We have then a function $c : K \rightarrow \{1, \dots, m\}$ such that for any $k \in K$, and $y \in C_k$, $r(y) \in a_{c(k)}$.

We finally set for $j \in \{1, \dots, m\}$,

$$b_j = a_j \cup \bigcup_{c(k)=j} C_k.$$

Then every b_j is connected since every point in b_j is connected to $x_{c(j)}$, and they are all pairwise disjoint by construction. We further have that $\overline{b_i}$ and $\overline{b_j}$ are either disjoint or intersect on one of their endpoints. Furthermore, since $\partial C_k = x_{c(k)} \in a_{c(k)}$, we have

$$\partial b_j = \partial a_j$$

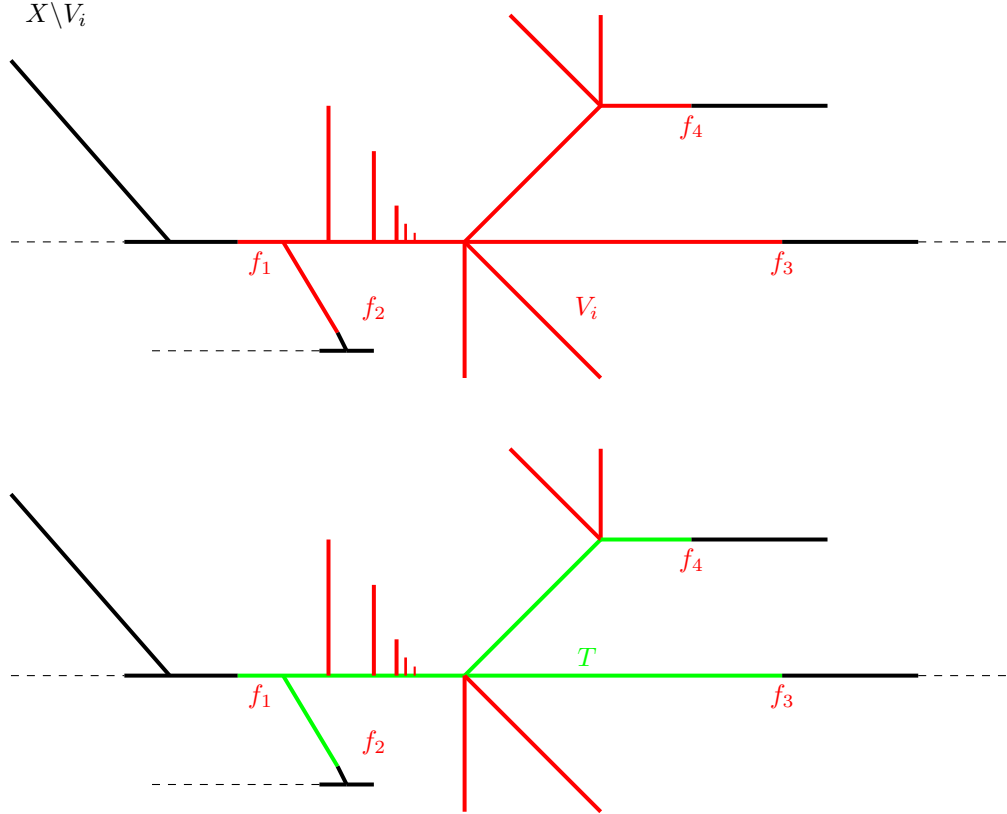
Whence, we have the desired property, that is,

$$\text{card}(\partial b_j) = \text{card}(\partial a_j) \leq 2,$$

and at the same time, we have

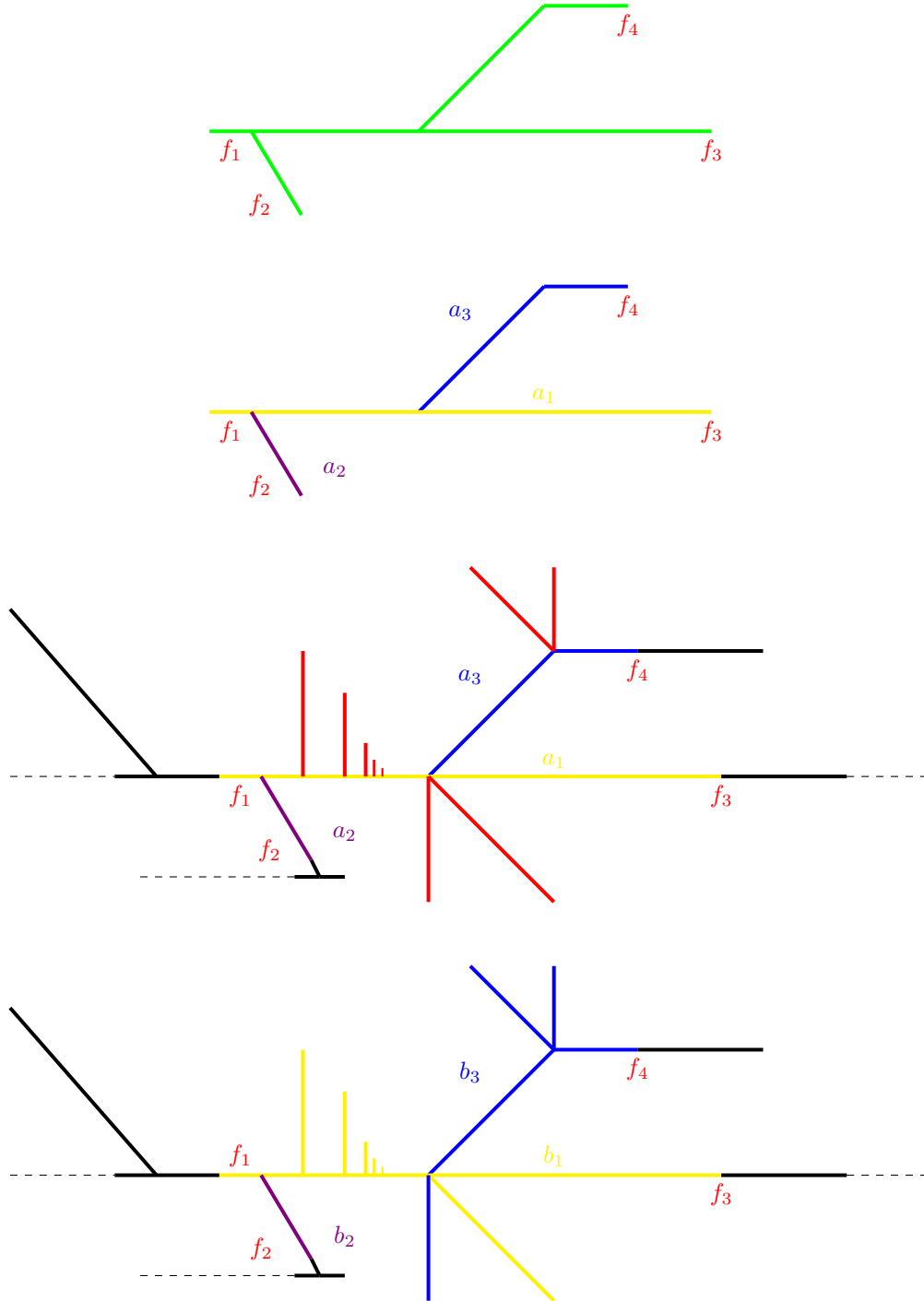
$$\overline{V_i} = \bigcup_{j=1}^m \overline{b_j}$$

We explain the construction with these drawings.



By applying this same decomposition on every V_i , we get a collection $(b_i)_{i=1}^s$ of connected open subsets of \bar{X} such that

$$(1) \quad X = \bigcup_{i=1}^s \overline{b_i},$$



- (2) $\text{diam}(b_i) < \delta$ since each b_i is a subset of one V_j ,
- (3) $\text{card}(\overline{b_j} \cap \overline{b_i}) \leq 1$,
- (4) $\text{card}(b_j \setminus b_j) \leq 2$.

We are now going to make the proof with step functions, and conclude by a density argument. We consider

$$\psi_{(b_i)} : \begin{cases} [0, 1] \rightarrow \mathbb{R} \\ x \mapsto 1 \text{ if } x \in b_i, \\ x \mapsto \frac{1}{\text{ord}(x)} \text{ if } x \in \partial b_i, \\ x \mapsto 0 \text{ otherwise.} \end{cases} .$$

For each b_i , we take $y_i \in b_i$ and set $c_i = \varphi(y_i)$. We set $\varphi_0 = \sum_{i=0}^r c_i \psi_{(b_i)}$ with $c_i, i = 0, \dots, r$. Since $\text{diam}(b_i) < \delta$, we have for $x \in X$

$$|\varphi(x) - \varphi_0(x)| \leq \varepsilon.$$

Since $w_f(x)$ is infinite, for each k , there is one dendrite $f^i(D^k)$ such that

$$w_f(x) \cap \text{int}(f^i(D^k)) \neq \emptyset.$$

Then there exists $n_0 \in \mathbb{N}$ such that

$$f^{n_0}(x) \in \text{int}(f^i(D^k)).$$

But with the properties of $(f^j(D^k))_{j=1, \dots, \alpha_k}$, we have for $m \in \mathbb{N}$

$$f^{n_0+m}(x) \in f^{i+m \bmod \alpha_k}(D^k).$$

We can thus write

$$\begin{aligned} S_N(x, \psi_{(b_i)}) &= \frac{1}{N} \sum_{n < n_0} \mu(n) \psi_{(b_i)}(f^n(x)) + \frac{1}{N} \sum_{n=n_0}^N \mu(n) \psi_{(b_i)}(f^n(x)) \\ &= o(1) + \sum_{j=0}^{\alpha_k-1} \frac{1}{N} \sum_{n_0}^N \mu(n) \psi_{(b_i)}(f^n(x)) \mathbf{1}_{f^j(D^k)}(f^n(x)). \end{aligned}$$

Then we show that for $j \in \{0, \dots, \alpha_k - 1\}$, $x_n^j := \psi_{(b_i)}(f^n(x)) \mathbf{1}_{f^j(D^k)}(f^n(x))$ for $n \geq n_0$ almost behaves like a periodic sequence and control the other terms. Indeed, if $f^j(D^k)$ does not contain any of the points of ∂b_i , then $(x_n^j)_{n \geq n_0}$ is either constantly equal to 0 (if $f^j(D^k) \cap b_i = \emptyset$) or periodic (if $f^j(D^k) \subset b_i$). Then, from Proposition 4.1, in both cases, we have

$$A_N^j := \frac{1}{N} \sum_{n=n_0}^N \mu(n) \psi_{(b_i)}(f^n(x)) = \frac{1}{N} \sum_{n=n_0}^N \mu(n) x_n^j = o(1).$$

If now on the contrary, $f^j(D^k)$ contains one or two points of ∂b_i , we can notice that if $x_n^j \neq 0$ and $x_m^j \neq 0$ then $n = m$ or $|n - m| \geq \alpha_k$. By Lemma 4.2 once again, we have

$$\limsup_{N \rightarrow \infty} A_N^j \leq \frac{1}{\alpha_k}.$$

There at most two intervals $f^j(D^k), f^l(D^k)$ in which this situation can occur since $\text{card}(\partial b_i) \leq 2$. It implies that

$$\limsup_{N \rightarrow \infty} |S_N(x, \psi_{(b_i)})| \leq \limsup_{N \rightarrow \infty} \sum_{j=0}^{\alpha_k-1} |A_N^j| \leq \frac{2}{\alpha_k}.$$

Letting $k \rightarrow \infty$, we obtain the desired result.

5. Remarks on Li-Oprocha-Zhang's reduction theorem.

In this section, we recall the reduction of Sarnak Möbius disjointness to the Gehman dendrites obtained by J. Li, P. Oprocha and G-H. Zhang. Precisely, they established the following

Theorem 5.1 ([27], Corollary 1.7). *The Sarnak's conjecture is true on every dynamical system with zero topological entropy if and only if it is so for all surjective dynamical systems over the Gehman dendrite with zero topological entropy.*

We recall that the Gehman dendrite is the topologically unique dendrite whose set of endpoints is homeomorphic to the Cantor set and whose branch points are all of order 3. The principal ingredient in their proof is the following

Theorem 5.2. *Every one sided-subshift can be embedded to a surjective dynamical system on the Gehman dendrite with the same topological entropy.*

According to the result by M. Boyle, D. Fiebig and U.Fiebig [13], the principal idea is to use the fact that every dynamical system of zero entropy can be seen as a factor of a two sided subshift with zero topological entropy, then get back to a one-side subshift with zero topological entropy. For more details, we refer to [27],

It follows that in the class of Gehman dendrite with zero topological entropy the spectrum can be any spectrum realized by any map with zero topological entropy. Nervelessness, we ask

Question 5.3. *Let (X, f) be a dynamical system with zero topological entropy and endpoints countable, do we have that its spectrum is singular.*

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