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On automatic bias reduction for extreme expectile estimation

Joint work with
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Introduction

Expectile estimation

Bias reduction

Simulation study and real data example

Expectiles

Quantiles [Koenker and Bassett, 1978] have been recently criticized [Acerbi, 2002], [Artzner et al., 1999] for not being a coherent risk measure.

$$q(\alpha) \in \arg \min_{t \in \mathbb{R}} \mathbb{E} [\rho_\alpha(Y - t) - \rho_\alpha(Y)],$$

where $\rho_\alpha(y) = |\alpha - \mathbb{1}_{\{y \leq 0\}}| |y|$. Some authors thus proposed expectiles [Newey and Powell, 1987] as an alternative :

$$e(\alpha) = \arg \min_{t \in \mathbb{R}} \mathbb{E} [\eta_\alpha(Y - t) - \eta_\alpha(Y)],$$

where $\eta_\alpha(y) = |\alpha - \mathbb{1}_{\{y \leq 0\}}| y^2$.

Expectiles

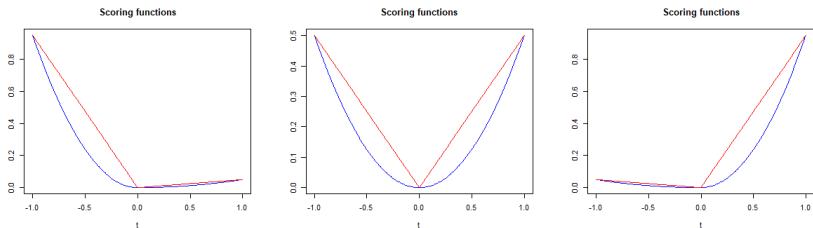


Figure: Quantile (red) and expectile (blue) loss functions for $\alpha = 0.05, 0.5$ and 0.95 .

Expectiles

- According to [Jones, 1994], $e(\alpha)$ is solution of

$$\bar{E}(y) = \frac{\mathbb{E}[(Y - y)\mathbb{1}_{\{Y > y\}}]}{2\mathbb{E}[(Y - y)\mathbb{1}_{\{Y > y\}}] + (y - \mathbb{E}[Y])} = 1 - \alpha.$$

- According to [Bellini et al., 2014], if $\bar{F}(y) = y^{-1/\gamma}\ell(y)$, then

$$\lim_{\alpha \rightarrow 1} \frac{\bar{F}(e(\alpha))}{1 - \alpha} = \gamma^{-1} - 1 ; \quad \lim_{\alpha \rightarrow 1} \frac{e(\alpha)}{q(\alpha)} = (\gamma^{-1} - 1)^{-\gamma},$$

for $\gamma < 1$.

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Intermediate expectiles estimation

Y_1, \dots, Y_n are i.i.d. realizations of Y . If $\alpha_n \ll 1 - 1/n$ (or equivalently $n(1 - \alpha_n) \rightarrow \infty$ as $n \rightarrow \infty$) is an intermediate sequence, two approaches have been considered for expectile estimation:

- The first one, used in [Daouia et al., 2018], directly derives from the definition of expectiles:

$$\hat{e}_n(\alpha_n) = \arg \min_{\theta \in \mathbb{R}} \sum_{i=1}^n \eta_{\alpha_n}(Y_i - \theta).$$

- The second one, introduced in [Girard et al., 2020], uses the property of [Jones, 1994]:

$$\hat{e}_n(\alpha_n) = \inf \left\{ y \in \mathbb{R} \mid \hat{E}_n(y) \leq 1 - \alpha_n \right\},$$

with

$$\hat{E}_n(y) = \frac{\sum_{i=1}^n (Y_i - y) \mathbb{1}_{\{Y_i > y\}}}{\sum_{i=1}^n |Y_i - y|}.$$

Extreme expectiles estimation

Let us assume

$$\lim_{t \rightarrow \infty} \frac{\bar{F}(ty)}{\bar{F}(t)} = y^{-1/\gamma}.$$

In this context, extreme quantiles may be estimated using the Weissman estimator. If $\beta_n \gg 1 - 1/n$ and $\alpha_n \ll 1 - 1/n$,

$$\frac{q(\beta_n)}{q(\alpha_n)} \approx \left(\frac{1 - \beta_n}{1 - \alpha_n} \right)^{-\gamma} \Rightarrow \hat{q}_n^*(\beta_n) = \hat{q}_n(\alpha_n) \left(\frac{1 - \beta_n}{1 - \alpha_n} \right)^{-\hat{\gamma}}.$$

Since quantiles and expectiles are asymptotically proportional, the same approximation holds for extreme expectiles, and

[Daouia et al., 2018] introduced

$$\hat{e}_n^*(\beta_n) = \hat{e}_n(\alpha_n) \left(\frac{1 - \beta_n}{1 - \alpha_n} \right)^{-\hat{\gamma}} \quad \text{or} \quad \tilde{e}_n^*(\beta_n) = \hat{q}_n^*(\beta_n) (\hat{\gamma}^{-1} - 1)^{-\hat{\gamma}}.$$

Tail index estimation

Let us consider the second order assumption (\mathcal{C}_2):

$$\forall y > 0, \lim_{t \rightarrow \infty} \frac{1}{A(1/\bar{F}(t))} \left(\frac{\bar{F}(ty)}{\bar{F}(t)} - y^{-1/\gamma} \right) = y^{-1/\gamma} \frac{y^{\rho/\gamma} - 1}{\gamma\rho}.$$

The most widespread estimator of the tail index γ is the Hill estimator:

$$\hat{\gamma}_{k_n}^H = \frac{1}{k_n} \sum_{i=1}^{k_n} \log \frac{Y_{n-i+1,n}}{Y_{n-k_n,n}},$$

where $k_n \rightarrow \infty$ and $k_n/n \rightarrow 0$ as $n \rightarrow \infty$. Under (\mathcal{C}_2), and if $\sqrt{k_n}A(n/k_n) \rightarrow \lambda \in \mathbb{R}$, then

$$\sqrt{k_n} (\hat{\gamma}_{k_n}^H - \gamma) \rightarrow \mathcal{N} \left(\frac{\lambda}{1-\rho}, \gamma^2 \right).$$

Tail index estimation

Using the asymptotic relationship between quantiles and expectiles, we can introduce the following tail index estimator (see [Girard et al., 2020]):

$$\widehat{\gamma}_{k_n}^E = \left(1 + \frac{n\widehat{F}_n(\widehat{e}_n(1 - k_n/n))}{k_n} \right)^{-1}.$$

Under (\mathcal{C}_2) with $0 < \gamma < 1/2$, and if $\sqrt{k_n}A(n/k_n) \rightarrow \lambda_1 \in \mathbb{R}$ and $\sqrt{k_n}q(1 - k_n/n)^{-1} \rightarrow \lambda_2 \in \mathbb{R}$, then

$$\sqrt{k_n} \left(\widehat{\gamma}_{k_n}^E - \gamma \right) \rightarrow \mathcal{N} \left(\frac{\gamma(\gamma^{-1} - 1)^{1-\rho}}{1 - \rho - \gamma} \lambda_1 + \gamma^2 (\gamma^{-1} - 1)^{\gamma+1} \mathbb{E}[Y] \lambda_2, \frac{\gamma^3(1 - \gamma)}{1 - 2\gamma} \right).$$

Tail index estimation

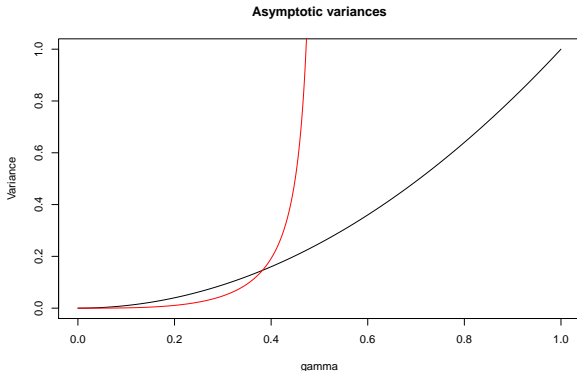


Figure: Asymptotic variances of $\tilde{\gamma}_{k_n}^H$ (black curve) and $\tilde{\gamma}_{k_n}^E$ (red curve) as functions of $\gamma \in (0, 1)$.

Back to extreme expectiles estimation

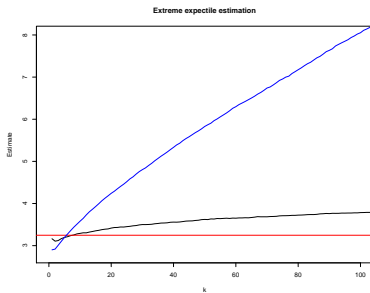


Figure: Mean estimates of 1,000 estimates $\hat{e}_n^*(\beta_n)$ using $\tilde{\gamma}_{k_n}^H$ (black) and $\tilde{\gamma}_{k_n}^E$ (blue) for $k_n = 1, \dots, 100$ in the case of a Burr distribution with $\gamma = 0.25$, $\rho = -5$, $n = 1,000$ and $\beta_n = 1 - 5/n = 0.995$.

Why so much bias ?

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Bias reduction of the tail index estimators

By doing the assumption that $A(t) = b\gamma t^\rho$, the following bias-reduced version of $\hat{\gamma}_{k_n}^H$ is proposed in [Gomes and Martins, 2002]:

$$\tilde{\gamma}_{k_n}^H = \hat{\gamma}_{k_n}^H \left(1 - \frac{\bar{b}}{1 - \bar{\rho}} \left(\frac{n}{k_n} \right)^{\bar{\rho}} \right).$$

We thus propose a similar approach for $\tilde{\gamma}_{k_n}^E$. For that purpose, we notice $\bar{F}(e(\alpha))/(1 - \alpha) = (\gamma^{-1} - 1)(1 + r(\alpha))$, where $1 + r(\alpha) =$

$$\left(1 - \frac{\mathbb{E}[Y]}{e(\alpha)} \right) \frac{1}{2\alpha - 1} \left(1 + A \left(\frac{1}{\bar{F}(e(\alpha))} \right) \frac{1}{\gamma(1 - \gamma - \rho)} (1 + o(1)) \right)^{-1}$$

as $\alpha \uparrow 1$.

Bias reduction of the tail index estimators

We thus introduce the following bias-reduced estimator:

$$\tilde{\gamma}_{k_n}^E = \left(1 + \frac{n\widehat{F}_n(\widehat{e}_n(1 - k_n/n))}{k_n} \frac{1}{1 + \bar{r}(1 - k_n/n)} \right)^{-1},$$

where $1 + \bar{r}(1 - k_n/n) =$

$$\left(1 - \frac{\bar{Y}_n}{\widehat{e}_n(1 - k_n/n)} \right) \frac{1}{1 - 2k_n/n} \left(1 + \frac{\bar{b}[\widehat{F}_n(\widehat{e}_n(1 - k_n/n))]^{-\bar{\rho}}}{1 - \bar{\gamma} - \bar{\rho}} \right)^{-1}.$$

Under some conditions concerning $\bar{\rho}$ and \bar{b} , we can prove

$$\sqrt{k_n} (\tilde{\gamma}_{k_n}^E - \gamma) \rightarrow \mathcal{N} \left(0, \frac{\gamma^3(1 - \gamma)}{1 - 2\gamma} \right).$$

Bias reduction of the extrapolation step

We can find some bias reduction approaches for extreme quantile estimators (see for instance [Gomes and Pestana, 2007]). The second order condition \mathcal{C}_2 giving

$$q(\beta_n) = q \left(1 - \frac{k_n}{n} \right) \left(\frac{n(1-\beta_n)}{k_n} \right)^{-\gamma} \left(1 + \frac{\left(\frac{n(1-\beta_n)}{k_n} \right)^{-\rho} - 1}{\rho} A \left(\frac{n}{k_n} \right) (1 + o(1)) \right),$$

we easily deduce, with $A(t) = b\gamma t^\rho$,

$$\widehat{q}_n^{*,RB}(\beta_n) = \widehat{q}_n^*(\beta_n) \left(1 + \frac{[n(1-\beta_n)/k_n]^{-\bar{\rho}} - 1}{\bar{\rho}} \bar{b}\bar{\gamma}(n/k_n)^{\bar{\rho}} \right).$$

Bias reduction of the extrapolation step

The bias reduction of extreme expectiles is less obvious, and 3 bias terms have to be eliminated, hence

$$\begin{cases} \hat{e}_n^{*,RB}(\beta_n) = \hat{e}_n^*(\beta_n)(1 + \bar{B}_{1,n})(1 + \bar{B}_{2,n})(1 + \bar{B}_{3,n}) \\ \tilde{e}_n^{*,RB}(\beta_n) = \tilde{e}_n^*(\beta_n)(1 + \bar{B}_{1,n})(1 + \bar{B}_{3,n}) \end{cases},$$

where

$$\begin{cases} 1 + \bar{B}_{1,n} = 1 + \frac{[n(1-\beta_n)/k_n]^{-\bar{\rho}} - 1}{\bar{\rho}} \bar{b}\bar{\gamma}(n/k_n)^{\bar{\rho}} \\ 1 + \bar{B}_{2,n} = (1 + \bar{r}(1 - \frac{k_n}{n}))^{\bar{\gamma}} \left(1 + \frac{(\bar{\gamma}^{-1} - 1)^{-\bar{\rho}}(1 + \bar{r}(1 - \frac{k_n}{n}))^{-\bar{\rho}} - 1}{\bar{\rho}} \bar{b}\bar{\gamma}(\frac{n}{k_n})^{\bar{\rho}} \right)^{-1} \\ 1 + \bar{B}_{3,n} = (1 + \bar{r}^*(\beta_n))^{-\bar{\gamma}} \left(1 + \frac{(\bar{\gamma}^{-1} - 1)^{-\bar{\rho}}(1 + \bar{r}^*(\beta_n))^{-\bar{\rho}} - 1}{\bar{\rho}} \bar{b}\bar{\gamma}(1 - \beta_n)^{-\bar{\rho}} \right) \end{cases}.$$

Note that these bias reduced estimators are proposed in the R package `Expectrem`.

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Simulation study

- We simulate $n = 1,000$ independent realizations Y_1, \dots, Y_n of a Burr distribution:

$$\bar{F}(y) = (1 + y^{-\rho/\gamma})^{1/\rho}.$$

- We consider $\rho = -5, -1$ and -0.5 , and $\gamma = 0.1, 0.2, 0.3$ and 0.4 .
- For each case, we estimate the expectile of level $\beta_n = 1 - 5/n = 0.995$.
- How to choose k_n ?

Simulation study

For the Hill based estimators, we choose the k_n which minimizes the following *AMSE* with $A(t) = b\gamma t^\rho$:

$$\frac{A\left(\frac{n}{k_n}\right)^2}{(1-\rho)^2} + \frac{\gamma^2}{k_n} \text{ hence } k_n^H = \left[\left(\frac{(1-\bar{\rho})^2}{-2\bar{\rho}\bar{b}^2} \right)^{1/(1-2\bar{\rho})} n^{-2\bar{\rho}/(1-2\bar{\rho})} \right].$$

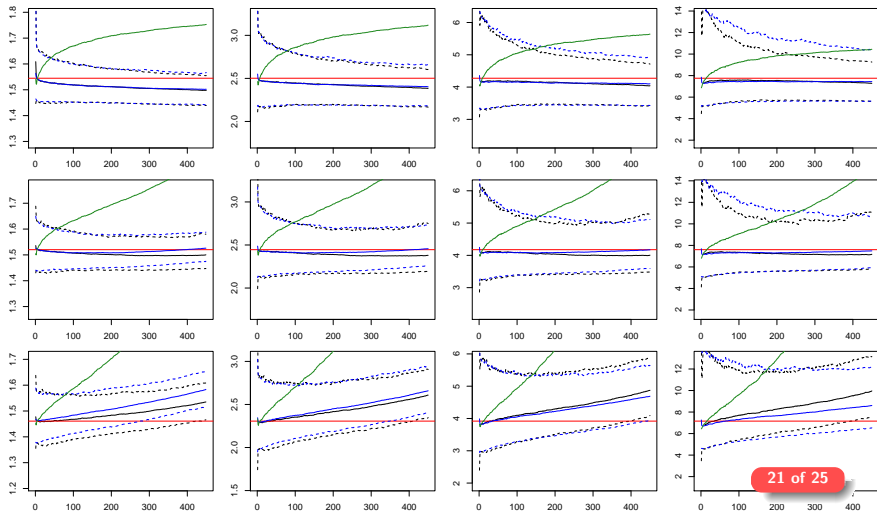
For $\hat{\gamma}_{k_n}^E$, we minimize the following Partial AMSE:

$$\left[\frac{\gamma(\gamma^{-1}-1)^{1-\rho}}{1-\gamma-\rho} A(n/k_n) \right]^2 + \frac{\gamma^3(1-\gamma)}{1-2\gamma} \times \frac{1}{k_n},$$

hence

$$\hat{k}_n^E = \min \left(\left[\left(\frac{(\bar{\gamma}^{-1}-1)^{2\bar{\rho}-1}(1-\bar{\gamma}-\bar{\rho})^2}{-2\bar{\rho}\bar{b}^2(1-2\bar{\gamma})} \right)^{1/(1-2\bar{\rho})} n^{-2\bar{\rho}/(1-2\bar{\rho})} \right], \left\lfloor \frac{n}{2} \right\rfloor - 1 \right).$$

Simulation study



Reinsurance example

Let Y be a claim amount, and R a retention level.

- Insurer pays $\min(Y, R)$.
- Reinsurer pays $\max(Y - R, 0)$.

The reinsurance premium $\Pi(R)$ may be calculated using the distortion principle:

$$\Pi_g(R) = \int_R^\infty g(\bar{F}(x)) dx,$$

where $g : [0, 1] \rightarrow [0, 1]$ is a nondecreasing concave function.

Reinsurance example

If Y is heavy-tailed and $g(1/\cdot)$ regularly varying with index $\delta < -\gamma$, then

$$\lim_{\beta \uparrow 1} \frac{\Pi_g(e(\beta))}{e(\beta) g(1-\beta)} = \frac{(\gamma^{-1} - 1)^{-\delta}}{-\delta/\gamma - 1},$$

by taking an extreme expectile as retention level. We thus use our expectile estimators to approximate the reinsurance premium, and compare our results with those obtained with the data set *secura* in [Vandewalle and Beirlant, 2006], with two distortion functions:

$$\begin{cases} g(x) = x & \text{(Net premium principle)} \\ g(x) = 1 - (1-x)^{\kappa} & \text{(Dual-power principle)} \end{cases}.$$

Reinsurance example

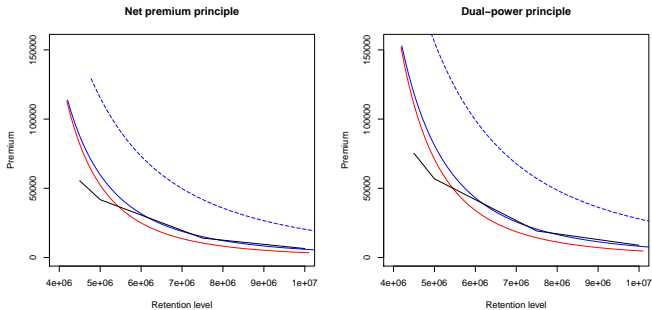


Figure: $\Pi_g(R)$ as function of $R = e(\beta)$ for β ranging from $1 - 10/n \approx 0.973$ to $1 - 1/(8n) \approx 0.9997$ (here $n = 370$). The premiums are estimated using $\tilde{\gamma}_{k_n}^H$ (solid blue curve), $\tilde{\gamma}_{k_n}^E$ (solid red curve) and the naïve estimator (dotted blue curve). The black curve is constructed by linear interpolation using the estimates found in [Vandewalle and Beirlant, 2006]. $\kappa = 1.366$.

Materials

- R package Expectrem, available at <https://github.com/AntoineUC/Expectrem>.
- Girard, S., Stupfler, G. and Usseglio-Carleve, A. (2020) On automatic bias reduction for extreme expectile estimation, preprint.

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