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COMPACT COMPLEX NON-KÄHLER MANIFOLDS ASSOCIATED WITH TOTALLY REAL RECIPROCAL UNITS

CHRISTIAN MIEBACH AND KARL OELJEKLAUS

Abstract. Using the theory of totally real number fields we construct a new class of compact complex non-Kähler manifolds in every even complex dimension and study their analytic and geometric properties.

1. Introduction

In this paper we construct a new class of compact complex non-Kähler manifolds in every complex dimension $n = 2d, d \geqslant 1$, and investigate complex analytic and topological properties.

Using totally real number fields we construct first $4d$-dimensional real solvable Lie groups G admitting irreducible cocompact discrete subgroups Γ. This method works in general to produce *real solv-manifolds*. Next we show that these Lie groups admit left invariant complex structures. The left quotient $X := \Gamma\backslash G$ is then a compact complex manifold. In the case $d = 1$, one recovers the Inoue surfaces noted $S_N^{(+)}$ in the famous paper [4].

In the following sections we prove that the identity component of the holomorphic automorphism group $\mathrm{Aut}^0(X)$ is isomorphic to $(\mathbb{C}^*)^d$ and that the whole group $\mathrm{Aut}(X)$ has infinitely many components if $d \geqslant 2$.

The action of $(\mathbb{C}^*)^d$ is free, induces a holomorphic foliation $\mathcal{F}$ which is transversely hyperbolic, and is preserved by the whole automorphism group. If $d = 2$, the restriction of certain automorphisms of X to the tangent bundle $T\mathcal{F}$ has an Anosov property in the sense that this bundle splits transversely into a stable and an unstable subbundle.

Furthermore we determine some topological invariants, prove that X is non-Kähler and show that the algebraic dimension is zero.

Open questions are whether some of the here constructed manifolds are locally conformally Kähler, whether they admit proper complex subvarieties, as well as whether they admit Anosov diffeomorphisms relative to $\mathcal{F}$ also for $d \geqslant 3$.

2. The construction

In this section we explain in detail the construction of a new class of compact complex manifolds. In the first two subsections we collect for the reader's convenience a number of well-known facts about simply-connected nilpotent Lie groups, their rational structures and cocompact discrete subgroups, and the free 2-step nilpotent Lie algebra. In Section 2.3 we shall see how particular totally real number fields K allow the construction of irreducible rational structures on the d-fold product N of the three-dimensional real Heisenberg group. Then we extend N by an Abelian group in such a way that the corresponding solvable group possesses a cocompact discrete subgroup associated with the group of algebraic units in K, see

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Section 2.4. Finally, we show that the so obtained solv-manifolds carry a complex structure, which completes our construction.

2.1. **Cocompact discrete subgroups of nilpotent Lie groups.** In this section we recall some facts related to cocompact discrete subgroups of simply-connected nilpotent Lie groups. For proofs and more details we refer the reader to [6, Chapter II].

Let N be a simply-connected nilpotent real Lie group with Lie algebra $\mathfrak{n}$. A rational structure on N consists of a rational subalgebra $\mathfrak{n}_{\mathbb{Q}}$ of $\mathfrak{n}$ such that $\mathfrak{n}_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathfrak{n}$. Equivalently, a rational structure on N is given by a basis $\mathcal{B} = (\xi_1, \dots, \xi_n)$ of $\mathfrak{n}$ such that for all $1 \leqslant k < l \leqslant n$ the coordinates of $[\xi_k, \xi_l]$ with respect to $\mathcal{B}$, i.e., the structure constants of $\mathfrak{n}$ with respect to $\mathcal{B}$, are rational.

Two rational structures on N are called isomorphic if the corresponding rational Lie algebras are isomorphic. A rational structure on N is called *irreducible* if $\mathfrak{n}_{\mathbb{Q}}$ is not isomorphic to the direct sum of two non-trivial ideals.

Remark. There are simply-connected nilpotent Lie groups N that do not admit any rational structure. It is also possible that N possesses several non-isomorphic rational structures, see [6, Remarks 2.14 and 2.15].

In order to explain how a rational structure on N yields cocompact discrete subgroups of N, we restate [6, Theorem 2.12] for the reader's convenience. Let $\Lambda \subset \mathfrak{n}$ be any lattice of maximal rank contained in $\mathfrak{n}_{\mathbb{Q}}$. Then the group Γ generated by $\exp(\Lambda)$ in N is a cocompact discrete subgroup of N. Any two discrete subgroups associated with the same rational structure are commensurable. Conversely, if $\Gamma \subset N$ is a cocompact discrete subgroup, then the $\mathbb{Z}$-span of $\exp^{-1}(\Gamma)$ in $\mathfrak{n}$ is a lattice of maximal rank in $\mathfrak{n}$ and any basis of $\mathfrak{n}$ contained in this lattice defines a rational structure on N. If $\widetilde{\Gamma}$ is commensurable with Γ, then the associated rational structures are isomorphic.

Remark. It follows from the preceding considerations that the rational structure on N is irreducible, if and only if the associated cocompact discrete subgroup of N is not commensurable to the direct product of two non-trivial normal subgroups.

2.2. **The free 2-step nilpotent Lie algebra.** Let V be a $2d$-dimensional real vector space and let $\bigwedge^2 V$ denote its exterior algebra. On the vector space $V \oplus \bigwedge^2 V$ we define a Lie bracket by

$$\big[(v, \alpha), (w, \beta)\big] := (0, v \wedge w).$$

The resulting Lie algebra is the free 2-step nilpotent Lie algebra $\mathfrak{f}_{2d}$ of dimension $2d + \binom{2d}{2} = 2d^2 + d$. We have

$$W := \bigwedge^2 V = \mathfrak{f}'_{2d} = \mathcal{Z}(\mathfrak{f}_{2d}),$$

where $\mathcal{Z}(\mathfrak{f}_{2d})$ denotes the center of $\mathfrak{f}_{2d}$.

Let F_{2d} be the simply-connected nilpotent Lie group with Lie algebra $\mathfrak{f}_{2d}$.

Example. The Lie algebra $\mathfrak{f}_2$ is isomorphic to the 3-dimensional Heisenberg algebra $\mathfrak{h}_3$. An explicit isomorphism is given by

$$\mathfrak{h}_3 \to \mathfrak{f}_2, \quad \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} \mapsto (xe_1 + ye_2, ze_1 \wedge e_2) \in \mathbb{R}^2 \oplus \bigwedge^2 \mathbb{R}^2.$$

On the group level, one can realize the 3-dimensional Heisenberg group as

$$H_3 := \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} ;\ x, y, z \in \mathbb{R} \right\}.$$

The map

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mapsto \left(x, y, z - \frac{xy}{2}\right)$$

yields an explicit isomorphism with the realization of the Heisenberg group as the free nilpotent group F_2, with group structure given by

$$(x, y, z)(\tilde{x}, \tilde{y}, \tilde{z}) = \left(x + \tilde{x}, y + \tilde{y}, z + \tilde{z} + \frac{1}{2}(x\tilde{y} - \tilde{x}y)\right).$$

We shall therefore use in the sequel H_3 as a model for F_2.

The choice of any basis of V leads to a rational structure on F_{2d} as follows. Let $(e_1, \dots, e_{2d})$ be a basis of V and put $f_{k,l} := [e_k, e_l]$. Then

$$(e_1, \dots, e_{2d}, f_{1,2}, f_{1,3}, \dots, f_{2d-1,2d})$$

is a basis of $\mathfrak{f}_{2d}$ with respect to which the structure constants of $\mathfrak{f}_{2d}$ are rational. As explained above, this procedure yields cocompact discrete subgroups of F_{2d}.

2.3. **Rational structures associated with totally real number fields.** Let $\mathbb{Q} \subset K$ be a field extension of degree $2d$, $d \geqslant 1$, and let $\mathcal{O}_K \subset K$ be its ring of algebraic integers. Choose elements $\omega_1, \dots, \omega_{2d} \in \mathcal{O}_K$ such that

$$\mathcal{O}_K \cong \mathbb{Z}\omega_1 \oplus \cdots \oplus \mathbb{Z}\omega_{2d}$$

as $\mathbb{Z}$-modules.

We suppose that K is totally real, i.e., that all $2d$ embeddings $\sigma_1, \dots, \sigma_{2d} \colon K \to \mathbb{R} \subset \mathbb{C}$ are real and consider the map $\sigma \colon K \to V := \mathbb{R}^{2d}$ given by $\sigma(x) = \big(\sigma_1(x), \dots, \sigma_{2d}(x)\big)$. It follows that $\Lambda_K := \sigma(\mathcal{O}_K) \subset V$ is a lattice of maximal rank generated by $e_k := \sigma(\omega_k)$ for $1 \leqslant k \leqslant 2d$.

Now, as mentioned above, the basis $\mathcal{B} = \big(e_1, \dots, e_{2d}\big)$ of V yields the basis

$$\widetilde{\mathcal{B}} = (e_1, \dots, e_{2d}, f_{1,2}, f_{1,3}, \dots, f_{2d-1,2d})$$

of $\mathfrak{f}_{2d}$ and induces therefore a rational structure on F_{2d}. In the following, $(\mathfrak{f}_{2d})_{\mathbb{Q}}$ *always* denotes the corresponding rational Lie algebra.

Let $\mathcal{O}_K^*$ be the (multiplicative) group of units in $\mathcal{O}_K$. We say that a unit $u \in \mathcal{O}_K^*$ is totally positive if $\sigma_j(u) > 0$ for all $1 \leqslant j \leqslant 2d$, and we write $\mathcal{O}_K^{*,+}$ for the group of totally positive units. Due to Dirichlet's theorem, the group $\mathcal{O}_K^{*,+}$ is isomorphic to $\mathbb{Z}^{2d-1}$.

The group $\mathcal{O}_K^{*,+}$ acts on Λ_K as a group of $\mathbb{Z}$-module automorphisms via

$$u \cdot \sigma(x) := \sigma(ux).$$

Extending $\sigma(x) \mapsto \sigma(ux)$ to an $\mathbb{R}$-linear map $\rho(u) \colon V \to V$ we obtain a representation

$$\rho \colon \mathcal{O}_K^{*,+} \to \mathrm{SL}(V).$$

Note that we have $\det \rho(u) = 1$ for all $u \in \mathcal{O}_K^{*,+}$ since the eigenvalues of $\rho(u)$ are the conjugates $\lambda_k = \sigma_k(u)$, $k = 1, \dots, 2d$, of u and thus all positive.

Remark. Every element of $\mathrm{GL}(V)$ extends uniquely to an automorphism of the Lie algebra $\mathfrak{f}_{2d}$. If the matrix of $\varphi \in \mathrm{GL}(V)$ with respect to $\mathcal{B}$ has rational coefficients, we obtain an automorphism of $(\mathfrak{f}_{2d})_{\mathbb{Q}}$.

Note that by construction the matrix of $\rho(u)$ with respect to $\mathcal{B}$ lies in $\mathrm{SL}(2d, \mathbb{Z})$ for every $u \in \mathcal{O}_K^{*,+}$. Hence, $\rho(\mathcal{O}_K^{*,+})$ is a discrete subgroup of $\mathrm{SL}(V)$. Furthermore, $\rho(u)$ induces an

automorphism of $\mathfrak{f}_{2d}$ that leaves $(\mathfrak{f}_{2d})_{\mathbb{Q}}$ invariant. We will denote this automorphism of $\mathfrak{f}_{2d}$ as well as its restriction to $(\mathfrak{f}_{2d})_{\mathbb{Q}}$ by $\widehat{\rho}(u)$. Clearly, $\widehat{\rho}(u)$ respects the decomposition

$$(\mathfrak{f}_{2d})_{\mathbb{Q}} = V_{\mathbb{Q}} \oplus W_{\mathbb{Q}}$$

and the eigenvalues of the restriction of $\widehat{\rho}(u)$ to $W_{\mathbb{Q}}$ are the products $\sigma_k(u)\sigma_l(u) = \lambda_k\lambda_l$, $1 \leqslant k < l \leqslant 2d$.

We formulate the following lemma in our setting, but it remains also valid in a more general form. Its proof is a slight adaptation of the proof of [1, Proposition 2.1.4].

Lemma 2.1. *The set of all endomorphisms $\rho(u)$ of $V_{\mathbb{Q}}$, $u \in \mathcal{O}_K^*$ is simultaneously diagonalizable over the Galois closure L of K.*

Proof. Since the number field K is totally real, there is a primitive unit $u_0 \in \mathcal{O}^{*,+}$, see e.g. [8, Theorem 1.4]. Note that $V_{\mathbb{Q}}$ and K are isomorphic as rational vector spaces and that $\rho(u_0)$ corresponds to multiplication by u_0. It follows that the characteristic polynomial of $\rho(u_0)$ coincides with the minimal polynomial of u_0. Hence, $\rho(u_0)$ is diagonalizable over L and each of its eigenvalues has multiplicity 1. Since the group $\mathcal{O}^{*,+}$ is Abelian, it stabilizes every eigenspace, which proves the claim. □

In order to proceed with the construction, we need the following lemma which was communicated to us with proof by Professor A. Dubickas. Recall that a unit $u \in \mathcal{O}_K^{*,+}$ is called *reciprocal* if u and u^{-1} are conjugate, i.e., have the same minimal polynomial, which then is palindromic. Moreover, u is *primitive* if $K = \mathbb{Q}(u)$.

Lemma 2.2. *For every $d \geqslant 1$ there exist totally real number fields of degree $2d$ which admit primitive reciprocal units.*

Proof. Let α be a totally real algebraic integer of degree d and denote its minimal polynomial by $P(X)$. Let L be an integer so large that for each of the d roots α_j, $j = 1, \ldots, d$, of $P(X)$ we have $L + \alpha_j > 2$. Consider then the polynomial $Q(X) = P(X + \frac{1}{X} - L)X^d$. It is clear that Q is a monic palindromic polynomial of degree $2d$ with $2d$ real roots, since $(L + \alpha_j)^2 > 4$, $j = 1, \ldots, d$. Furthermore, for a generic choice of L, the polynomial Q is irreducible and its roots are totally real reciprocal units of degree $2d$. □

From now on we suppose that there exists a primitive reciprocal unit $u_0 \in \mathcal{O}_K^{*,+}$. In other words, we suppose that u_0 and u_0^{-1} are conjugate, as well as $K = \mathbb{Q}(u_0)$.

Proposition 2.3. *Let $u_0 \in \mathcal{O}_K^{*,+}$ be a primitive reciprocal unit. Then there exists a $\widehat{\rho}(u_0)$-invariant rational decomposition*

$$W_{\mathbb{Q}} = (W_1)_{\mathbb{Q}} \oplus (W_2)_{\mathbb{Q}}$$

where W_1 is the d-dimensional subspace of $\widehat{\rho}(u_0)$-fixed points in W.

Proof. Since the characteristic polynomial of $\rho(u_0)$ is palindromic, we can arrange the eigenvalues $\lambda_1, \ldots, \lambda_{2d}$ of $\rho(u_0)$ such that $\lambda_{2k} = \lambda_{2k-1}^{-1}$ for all $1 \leqslant k \leqslant d$. Since the restriction of $\widehat{\rho}(u_0)$ to W is diagonalizable with eigenvalues $\lambda_k\lambda_l$ for $1 \leqslant k < l \leqslant 2d$, we see that the subspace W_1 of $\widehat{\rho}(u_0)$-fixed points is of dimension d. Moreover, it follows that the characteristic polynomial of $\widehat{\rho}(u_0)|_W$ is divisible by $(x-1)^d$ in $\mathbb{Z}[x]$. This gives the desired decomposition of $W_{\mathbb{Q}}$ into two $\widehat{\rho}(u_0)$-stable rational subspaces, see [5, Theorem XI.4.1]. □

Let us consider the rational Lie algebra

$$\mathfrak{n}_{\mathbb{Q}} := (\mathfrak{f}_{2d})_{\mathbb{Q}}/(W_2)_{\mathbb{Q}}.$$

In the following we will view $\mathfrak{n}_{\mathbb{Q}}$ as $V_{\mathbb{Q}} \oplus \big(W_{\mathbb{Q}}/(W_2)_{\mathbb{Q}}\big)$. Note that $W_{\mathbb{Q}}/(W_2)_{\mathbb{Q}}$ coincides with the center of $\mathfrak{n}_{\mathbb{Q}}$.

Proposition 2.4. *We have* $\mathfrak{n} := \mathfrak{n}_{\mathbb{Q}} \otimes \mathbb{R} \cong \mathfrak{h}_3^d$*, i.e., the construction yields a rational structure on* $N := H_3^d$*. Moreover, this rational structure on* N *is irreducible.*

Proof. Let $(v_1, \dots, v_{2d})$ be a basis of V such that $\rho(u_0)v_{2j-1} = \lambda_j v_{2j-1}$ and $\rho(u_0)v_{2j} = \lambda_j^{-1} v_{2j}$ for all $1 \leqslant j \leqslant d$. Set $w_j := [v_{2j-1}, v_{2j}],\ j = 1, \dots, d$. Then

$$\mathbb{R}v_{2j-1} \oplus \mathbb{R}v_{2j} \oplus \mathbb{R}w_j,\ j = 1, \dots, d,$$

is a subalgebra of $\mathfrak{f}_{2d}$ isomorphic to $\mathfrak{h}_3$ which intersects W_2 trivially. Using the decomposition established in Proposition 2.3, one sees that these subalgebras commute pairwise modulo W_2, which proves the first claim.

In order to show that the rational structure on N is irreducible, suppose that we have a decomposition $\mathfrak{n}_{\mathbb{Q}} = \mathfrak{a}_{\mathbb{Q}} \oplus \mathfrak{b}_{\mathbb{Q}}$ where $\mathfrak{a}_{\mathbb{Q}}$ and $\mathfrak{b}_{\mathbb{Q}}$ are non-trivial ideals of $\mathfrak{n}_{\mathbb{Q}}$. Then we have $\mathfrak{n} = \mathfrak{h}_3^d = \mathfrak{a} \oplus \mathfrak{b}$ for $\mathfrak{a} = \mathfrak{a}_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{R}$ and $\mathfrak{b} = \mathfrak{b}_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{R}$.

If an ideal of $\mathfrak{n}$ contains an element of the form

$$\xi = \sum_{j=1}^{d} \big(\kappa_j v_{2j-1} + \mu_j v_{2j} + \nu_j w_j\big)$$

with $(\kappa_{j_0}, \mu_{j_0}) \neq (0,0)$, then this ideal contains also w_{j_0}. Define

$$J_{\mathfrak{a}} := \left\{ 1 \leqslant j_0 \leqslant d;\ \exists\, \xi = \sum_{j=1}^{d} \big(\kappa_j v_{2j-1} + \mu_j v_{2j} + \nu_j w_j\big) \in \mathfrak{a} \text{ with } (\kappa_{j_0}, \mu_{j_0}) \neq (0,0) \right\}$$

and similarly $J_{\mathfrak{b}}$. Since $\mathfrak{n} = \mathfrak{a} \oplus \mathfrak{b}$, the preceding observation implies that $J_{\mathfrak{a}} \cup J_{\mathfrak{b}} = \{1, \dots, d\}$ and that this union is disjoint. Therefore, we can suppose that $J_{\mathfrak{a}} = \{1, \dots, k\}$ and hence get

$$\pi(\mathfrak{a}) = \bigoplus_{j=1}^{k} (\mathbb{R}v_{2j-1} \oplus \mathbb{R}v_{2j}) \quad \text{and} \quad \pi(\mathfrak{b}) = \bigoplus_{j=k+1}^{d} (\mathbb{R}v_{2j-1} \oplus \mathbb{R}v_{2j}),$$

where $\pi\colon \mathfrak{h}_3^d \to V$ is the projection along the center of $\mathfrak{h}_3^d$.

Since $\pi(\mathfrak{a})$ and $\pi(\mathfrak{b})$ are invariant under $\rho(u_0)$ and since π is defined over $\mathbb{Q}$, we obtain the corresponding rational decomposition $V_{\mathbb{Q}} = \pi(\mathfrak{a}_{\mathbb{Q}}) \oplus \pi(\mathfrak{b}_{\mathbb{Q}})$ into two rational $\rho(u_0)$-invariant subspaces. Now the claim follows from the fact that the characteristic polynomial of $\rho(u_0)$ is irreducible over $\mathbb{Q}$. □

As explained in Section 2.1 we can now define a cocompact discrete subgroup of N as follows.

Definition 2.5. Let $\widehat{\Lambda}_K \subset V \oplus W/W_2$ be the full lattice generated by $\sigma(\omega_k) \in V$ for $1 \leqslant k \leqslant 2d$ and the images of $f_{kl} = \big[\sigma(\omega_k), \sigma(\omega_l)\big], k < l$ in $W_{\mathbb{Q}}/(W_2)_{\mathbb{Q}}$ for $1 \leqslant k < l \leqslant 2d$. We define Γ_N to be the discrete cocompact subgroup generated by $\exp(\widehat{\Lambda}_K) \subset N$.

Remark. Due to Proposition 2.4, the group Γ_N is not commensurable to the product of two proper normal subgroups.

2.4. **Solv-manifolds associated with totally real number fields.** In this subsection, we construct an extension of N by an Abelian group which admits a cocompact discrete subgroup containing Γ_N.

Recall that $\mathcal{O}_K^{*,+}$ can be considered as a discrete Abelian subgroup of $\mathrm{SL}(2d, \mathbb{R})$ which leaves the lattice $\Lambda_K \subset V_{\mathbb{Q}}$ invariant. Moreover, its action extends to $V \oplus W$ leaving $W_{\mathbb{Q}}$ invariant.

Now we have

Proposition 2.6. *The group* $\mathcal{O}_K^{*,+}$ *respects the decomposition* $W_{\mathbb{Q}} = (W_1)_{\mathbb{Q}} \oplus (W_2)_{\mathbb{Q}}$*. Consequently, every* $u \in \mathcal{O}_K^{*,+}$ *acts on* $\mathfrak{n}_{\mathbb{Q}}$ *by an automorphism* $\rho_{\mathfrak{n}}(u)$*.*

Proof. Due to Lemma 2.1, the transformations $\rho(u) \in \mathrm{End}(V)$ with $u \in \mathcal{O}_K^{*,+}$ are simultaneously diagonalizable. Thus the same holds for the transformations $\widehat{\rho}(u) \in \mathrm{GL}(V \oplus W)$. The claim follows from the fact that W_1 and W_2 are direct sums of eigenspaces for $\widehat{\rho}(u_0)$. In fact, W_1 is the eigenspace corresponding to the eigenvalue 1 and W_2 is a direct sum of eigenspaces with eigenvalues not equal to 1. □

Since $\mathcal{O}_K^{*,+}$ acts on $\mathfrak{n}_{\mathbb{Q}}$ by automorphisms, it respects the decomposition $\mathfrak{n}_{\mathbb{Q}} = V_{\mathbb{Q}} \oplus \mathcal{Z}(\mathfrak{n}_{\mathbb{Q}})$ where $\mathcal{Z}(\mathfrak{n}_{\mathbb{Q}}) \cong (W_1)_{\mathbb{Q}}$ as rational vector spaces. Consider the homomorphism

$$\varphi\colon \mathcal{O}_K^{*,+} \to \mathrm{SL}\big(\mathcal{Z}(\mathfrak{n}_{\mathbb{Q}})\big), \quad \varphi(u) := \rho_{\mathfrak{n}}(u)|_{\mathcal{Z}(\mathfrak{n}_{\mathbb{Q}})}.$$

We define $\Gamma_A := \ker(\varphi) \subset \mathcal{O}_K^{*,+}$. By construction, we have $u_0 \in \Gamma_A$.

Proposition 2.7. *The subgroup $\Gamma_A \subset \mathcal{O}_K^{*,+}$ has rank d, hence is isomorphic to $\mathbb{Z}^d$, and consists of reciprocal units in $\mathcal{O}_K^{*,+}$.*

Proof. We note A the group of diagonal matrices in $\mathrm{SL}(2d, \mathbb{R})$ and consider $\mathcal{O}_K^{*,+}$ as a discrete subgroup of A, see Lemma 2.1. Let $a \in A$ be the matrix having eigenvalues $t_1, \dots, t_{2d}$. The induced linear transformation on W_1 is again diagonal with eigenvalues $(t_1t_2, t_3t_4, \dots, t_{2d-1}t_{2d})$ and this yields a homomorphism $\widehat{\varphi}\colon A \to \mathrm{SL}(d, \mathbb{R})$ which extends φ.

Due to Dirichlet's theorem, we can view φ as a homomorphism from $\mathbb{Z}^{2d-1}$ to $\mathrm{SL}(d, \mathbb{Z})$ whose image consists of matrices that are simultaneously diagonalizable over $\mathbb{R}$, hence as a homomorphism $\varphi\colon \mathbb{Z}^{2d-1} \to \mathbb{Z}^{d-1}$. This implies that the rank of Γ_A is at least d.

The Lie algebra $\mathfrak{a}$ of A is the set of trace zero diagonal matrices, hence $\mathfrak{a} \cong \mathbb{R}^{2d-1}$. The derivative of $\widehat{\varphi}\colon A \to \mathrm{SL}(d, \mathbb{R})$ can be identified with the map $\mathbb{R}^{2d-1} \to \mathbb{R}^{d-1}$ given by

$$(x_1, \dots, x_{2d}) \mapsto (x_1 + x_2, x_3 + x_4, \dots, x_{2d-1} + x_{2d}),$$

where we suppose that $x_1 + \cdots + x_{2d} = 0$. Since this map is surjective, its kernel is isomorphic to $\mathbb{R}^d$, which implies that the *discrete* subgroup $\Gamma_A \subset A$ is of rank at most d. □

Let us summarize our construction. We have seen that we can view $\mathcal{O}_K^{*,+} \cong \mathbb{Z}^{2d-1}$ as a discrete subgroup of $\mathrm{SL}(2d, \mathbb{R})$ that normalizes Γ_N. The identity component of its real Zariski closure is $\widetilde{A} \cong (\mathbb{R}^{>0})^{2d-1}$ in $\mathrm{SL}(2, \mathbb{R})$, the elements of which are simultaneously diagonalizable. Moreover, there is a subgroup $\Gamma_A \cong \mathbb{Z}^d$ of $\mathcal{O}_K^{*,+}$ and the identity component of its real Zariski closure is

$$A \cong \big\{(a_1, b_1, \dots, a_d, b_d) \in (\mathbb{R}^{>0})^{2d};\ a_1b_1 = \cdots = a_db_d = 1\big\} \cong (\mathbb{R}^{>0})^d.$$

Consequently, Γ_A acts on Γ_N and we obtain the solvable discrete subgroup $\Gamma := \Gamma_A \ltimes \Gamma_N$ which is cocompact in $A \ltimes N \cong (\mathbb{R}^{>0})^d \ltimes N$ and Zariski dense in $(\mathbb{R}^*)^d \ltimes N$.

Since for $a, b \in \mathbb{R}^{>0}$ and $x, y, z \in \mathbb{R}$ we have

$$\begin{pmatrix} ab & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a^{-1}b^{-1} & 0 & 0 \\ 0 & b^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & ax & abz \\ 0 & 1 & by \\ 0 & 0 & 1 \end{pmatrix},$$

one can realise the Lie group $\widetilde{G} := \widetilde{A} \ltimes N \cong (\mathbb{R}^{>0})^{2d-1} \ltimes N$ as a matrix group isomorphic to

$$\left\{ \begin{pmatrix} M_1 & 0 & \cdots & 0 \\ 0 & M_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & M_d \end{pmatrix} \middle| \ M_i = \begin{pmatrix} a_ib_i & x_i & z_i \\ 0 & b_i & y_i \\ 0 & 0 & 1 \end{pmatrix},\ a_i, b_i \in \mathbb{R}^{>0}, x_i, y_i, z_i \in \mathbb{R},\ i = 1, \dots, d \right\}.$$

Under this isomorphism the group $\mathcal{O}_K^{*,+}$ corresponds to

$$\left\{ \begin{pmatrix} D_1 & 0 & \cdots & 0 \\ 0 & D_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & D_d \end{pmatrix} \middle| \; D_i = \begin{pmatrix} \sigma_{2i-1}(u)\sigma_{2i}(u) & 0 & 0 \\ 0 & \sigma_{2i}(u) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; i = 1, \ldots, d, \; u \in \mathcal{O}_K^{*,+} \right\}.$$

Furthermore $G := A \ltimes N \cong (\mathbb{R}^{>0})^d \ltimes N$ is the subgroup of $\widetilde{G} := \widetilde{A} \ltimes N$ given by the equations $a_i b_i = 1, \; i = 1, \ldots, d$ and the subgroup $\Gamma_A \subset \mathcal{O}_K^{*,+}$ corresponds to

$$\left\{ \begin{pmatrix} D_1 & 0 & \cdots & 0 \\ 0 & D_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & D_d \end{pmatrix} \middle| \; D_i = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sigma_{2i}(u) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; i = 1, \ldots, d, \; u \in A \right\}.$$

2.5. **Left-invariant complex structure on G.** The matrix group

$$S := \mathbb{R}^{>0} \ltimes H_3 = \left\{ \begin{pmatrix} 1 & x & v \\ 0 & b & y \\ 0 & 0 & 1 \end{pmatrix} \middle| \; b \in \mathbb{R}^{>0}, \; x, y, v \in \mathbb{R} \right\}$$

acts linearly on $\mathbb{C}^3$. The affine hyperplane $\mathbb{C}^2 \times \{1\}$ of $\mathbb{C}^3$ is invariant under S. A direct calculation shows that the orbit through the point $z = (0, i, 1)$ is open, has trivial isotropy and coincides with $\mathbb{C} \times \mathbb{H}^+ \times \{1\}$ where $\mathbb{H}^+ \subset \mathbb{C}$ is the upper half plane.

This proves the following result.

Proposition 2.8. *The solvable real Lie group S admits a left-invariant complex structure with respect to which it is biholomorphic to $\mathbb{C} \times \mathbb{H}^+$.*

Consider now the natural action of $\widetilde{G}$ on $\mathbb{C}^{3d}$. The orbit of the subgroup G through the point $(0, i, 1, 0, i, 1, \ldots, 0, i, 1) \in \mathbb{C}^{3d}$ has trivial isotropy group, is biholomorphic to $(\mathbb{C} \times \mathbb{H}^+)^d$ and hence gives a left-invariant complex structure on the real Lie group G.

Therefore, the left quotient $X := \Gamma \backslash G$ is a compact complex manifold.

2.6. **A density property of Γ_N.** We continue to consider $N \cong H_3^d$ and $G \cong S^d$ as matrix groups consisting of block diagonal matrices as written down in the closing of Section 2.4.

Let

$$H := \left\{ \begin{pmatrix} M_1 & 0 & \cdots & 0 \\ 0 & M_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & M_d \end{pmatrix} \middle| \; M_i = \begin{pmatrix} 1 & x_i & z_i \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, x_i, v_i \in \mathbb{R}, \; i = 1, \ldots, d \right\}.$$

Proposition 2.9. *The subgroup $\Gamma_N H$ is topologically dense in the Lie group N.*

Proof. Let $K := (\overline{\Gamma_N H}^{\mathrm{top}})^0$ be the identity component of the topological closure of $\Gamma_N H$ in N. Since $\Gamma_N H$ is invariant under conjugation by elements of Γ_A, we see that the same is true for the subgroup K which has also the property that $(K \cap \Gamma_N) \backslash K$ is compact. Therefore the Lie algebra $\mathfrak{k}$ of K is compatible with the rational structure of $\mathfrak{n}$ and the projection V' of $\mathfrak{k}$ along the center $\mathfrak{z}(\mathfrak{n})$ onto the vector space V has the same property. Furthermore, $V' \subset V$ is a u_0-invariant non-trivial subspace compatible with the rational structure and this implies that $V' = V$, see also the proof of Proposition 2.4. The proposition is proven. □

3. Properties of the quotient manifolds

Let $G = A \ltimes N = (\mathbb{R}^{>0} \ltimes H_3)^d = S^d$ be the solvable Lie group equipped with the left-invariant complex structure such that $G \cong (\mathbb{C} \times \mathbb{H})^d$ and let $\Gamma = \Gamma_A \ltimes \Gamma_N$ be the cocompact discrete subgroup constructed above. In this section we establish a number of topological and complex geometric properties of the compact complex manifold $X = \Gamma\backslash G$.

3.1. **The CR-fibration with Levi-flat fibers, the transversally hyperbolic foliation $\mathcal{F}$ and the Kodaira dimension.** Let Z denote the center of G, which is also the center of N. We first remark that X considered as a real solv-manifold admits the following commutative diagram of equivariant fibrations, see [6, Proposition 2.17, Theorem 3.3, and Corollary 3.5]:

$$\begin{tikzcd} & Z \cdot \Gamma\backslash G \arrow[dr, "(S_1)^{2d}"] & \\ p\colon X = \Gamma\backslash G \arrow[ur, "(S_1)^d"] \arrow[rr, "\Gamma_N\backslash N"] & & N \cdot \Gamma\backslash G \cong (S_1)^d \end{tikzcd}$$

The group $\mathbb{C}^d$ acts on $G \cong (\mathbb{C} \times \mathbb{H})^d$ by translation in the $\mathbb{C}$-factors. One shows directly that this action commutes with the left multiplication by G and hence induces a holomorphic action of $\mathbb{C}^d$ on X. The ineffectivity of this action is $\Gamma \cap Z$ and therefore we obtain an inclusion $(\mathbb{C}^*)^d \hookrightarrow \mathrm{Aut}(X)$. The orbits of this $(\mathbb{C}^*)^d$ are exactly the images of the $\mathbb{C}^d$-factors in the universal covering of X. As a consequence, we see that the action of $(\mathbb{C}^*)^d$ on X induces a transversally hyperbolic holomorphic foliation $\mathcal{F}$ of X.

Since the lift of p to the universal covering G of X coincides with the quotient map $G \to G/N \cong \mathbb{R}^d$, we see that p is $(\mathbb{C}^*)^d$-invariant. Moreover, the construction of the left-invariant complex structure on G shows that the N-orbits are generic CR-submanifolds of real dimension $3d$ and CR-dimension d in G. Since the complex tangent space to the N-orbits contains the N'-orbit, they are Levi-flat. It follows that p is a CR-map having Levi-flat fibers.

We determine the topological closure of the orbits of this $(\mathbb{C}^*)^d$-action in the following

Proposition 3.1. *Let $x = \Gamma g \in \Gamma\backslash G = X$. The topological closure of $(\mathbb{C}^*)^d \cdot x$ in X coincides the fiber of the projection p passing through the point x and is therefore isomorphic to the CR-nilmanifold $\Gamma_N\backslash N$. In particular, X does not contain any proper $(\mathbb{C}^*)^d$-invariant analytic subset.*

Proof. For the proof, it suffices to remark that the $(\mathbb{C}^*)^d$-orbits are exactly the right orbits in $\Gamma\backslash G = X$ of the normal subgroup H and to apply Proposition 2.9. □

Corollary 3.2. *Every holomorphic function on $\Gamma_N\backslash G$ is constant.*

Proof. This follows from Proposition 3.1, since $\Gamma_N\backslash N$ is a generic CR-submanifold of $\Gamma_N\backslash G$. □

This corollary implies the following

Corollary 3.3. *The Kodaira dimension of X is $-\infty$.*

Proof. Since Γ acts by affine-linear transformations on $(\mathbb{C} \times \mathbb{H})^d$, the tangent bundle of X and all its induced vector bundles are flat. In particular, the canonical bundle of X and all its powers are flat, i.e., given by representations of Γ in $\mathbb{C}^*$. This implies that the canonical bundle of a finite covering of $\Gamma_N\backslash G$ is holomorphically trivial, since for the commutator group one has $\Gamma' \subset \Gamma_N$. Since every holomorphic function on $\Gamma_N\backslash G$ is constant, we see that $H^0(X, K_X^n) = 0$ for all $n \geqslant 0$. Hence, $\operatorname{kod} X = -\infty$. □

3.2. **The identity component of** $\mathrm{Aut}(X)$ **and the non-Kähler property.** In order to determine explicitely the holomorphic vector fields on X, let us give the action $G = A \ltimes N$ on each factor of $(\mathbb{C} \times \mathbb{H}^+)^d$ explicitely. For $g = \left(\begin{smallmatrix} 1 & a & c \\ 0 & t & b \\ 0 & 0 & 1 \end{smallmatrix}\right) \in \mathbb{R}^{>0} \ltimes H_3$ and $(z, w) \in \mathbb{C} \times \mathbb{H}^+$ we have

$$\begin{pmatrix} 1 & a & c \\ 0 & t & b \\ 0 & 0 & 1 \end{pmatrix} \cdot (z, w) = (z + aw + c, tw + b).$$

Let $\pi\colon G \to \mathrm{Aut}\big((\mathbb{H}^+)^d\big)$ be the natural projection. It follows from Proposition 2.9 that $\pi(\Gamma_N)$ is a countable, topologically dense subgroup of the unipotent radical of the Borel subgroup of affine transformations in $\mathrm{Aut}\big((\mathbb{H}^+)^d\big)$. This observation allows us to carry over the proof of [4, Proposition 3(ii)] in order to obtain the following.

Proposition 3.4. *We have* $H^0(X, \Theta) = \mathbb{C}^d \cong \langle \frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_d} \rangle_{\mathbb{C}}$ *and therefore* $\mathrm{Aut}^0(X) \cong (\mathbb{C}^*)^d$.

Corollary 3.5. *The manifold X is not Kähler.*

Proof. If X was Kähler, then due to [3] the group $\mathrm{Aut}^0(X)$ would act meromorphically on X and consequently its orbits would be locally closed, which is not the case. □

3.3. **Infinitely many connected components for** $d \geqslant 2$**.** In this subsection we show that the whole group $\mathrm{Aut}(X)$ has infinitely many components for $d \geqslant 2$. Note that the automorphism groups of Inoue surfaces $S_N^{(+)}$, (this is the case $d = 1$) have only finitely many components.

First we note that the group $\widetilde{A} \cong (\mathbb{R}^{>0})^{2d-1}$ acts as a group of holomorphic transformations on $(\mathbb{C} \times \mathbb{H}^+)^d$ by

$$(\lambda_1, \mu_1, \dots, \lambda_d, \mu_d) \cdot (z_1, w_1, \dots, z_d, w_d) := (\lambda_1 \mu_1 z_1, \mu_1 w_1, \dots, \lambda_d \mu_d z_d, \mu_d w_d),$$

where we suppose $\lambda_1 \mu_1 \cdots \lambda_d \mu_d = 1$.

This action extends the A-action on $(\mathbb{C} \times \mathbb{H}^+)^d$ where A is embedded in $\widetilde{A}$ by

$$(\mu_1, \dots, \mu_d) \mapsto (\mu_1^{-1}, \mu_1, \dots, \mu_d^{-1}, \mu_d).$$

In the next step we show that the $\widetilde{A}$-action normalizes the simply transitive G-action on $(\mathbb{C} \times \mathbb{H}^+)^d$. Writing $G = A \ltimes N$ as the d-fold product of the matrix group

$$S = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & \alpha & b \\ 0 & 0 & 1 \end{pmatrix};\ \alpha \in \mathbb{R}^{>0}, a, b, c \in \mathbb{R} \right\},$$

and defining $\varphi_{\lambda,\mu}(z, w) := (\lambda \mu z, \mu w)$ for $\lambda, \mu > 0$ and $(z, w) \in \mathbb{C} \times \mathbb{H}^+$, we obtain

$$\varphi_{\lambda,\mu}\big(g \cdot \varphi_{\lambda,\mu}^{-1}(z, w)\big) = (z + \lambda a w + \lambda \mu c, \alpha w + \mu b).$$

Consequently, the induced action of $\widetilde{A}$ on G coincides with the conjugation of $\widetilde{A}$ on the normal subgroup $A \ltimes N$ in $\widetilde{A} \ltimes N$.

It therefore follows that the action of the subgroup $\mathcal{O}_K^{*,+}$ of $\widetilde{A}$ on $(\mathbb{C} \times \mathbb{H}^+)^d$ normalizes $\Gamma = \Gamma_A \ltimes \Gamma_N$. This implies that the action of $\mathcal{O}_K^{*,+}$ descends holomorphically to the compact quotient $X = \Gamma \backslash G$.

3.4. **An Anosov property of the foliation $\mathcal{F}$ in the case $d = 2$.** As we have seen in the previous subsection, the group $\mathcal{O}_K^{*,+}/\Gamma_A$ embeds into $\mathrm{Aut}(X)$. It is clear that this discrete group of automorphisms stabilizes the foliation $\mathcal{F}$ of X. In this subsection we shall see that for $d = 2$ non-trivial elements φ of $\mathcal{O}_K^{*,+}/\Gamma_A$ have an Anosov property relative to $\mathcal{F}$, i.e. that the bundle map φ_* is Anosov when restricted to the involutive subbundle $T\mathcal{F} \subset TX$.

Suppose first that d is arbitrary and consider the bundle map $\varphi_*\colon TX \to TX$ given by the push-forward of tangent vectors. We shall trivialize first TG via left-invariant vector fields which trivialize then TX as well. Concretely, let $g \in G$ and consider $\varphi_*\colon T_gG \to T_{\varphi(g)}G$. Since $T_gG = (\ell_g)_*\mathfrak{g}$, we are led to consider

$$\left(\ell_{\varphi(g)}^{-1} \circ \varphi \circ \ell_g\right)_*\colon \mathfrak{g} \to \mathfrak{g}.$$

The map $G \to \mathrm{GL}(\mathfrak{g})$ given by $g \mapsto \left(\ell_{\varphi(g)}^{-1} \circ \varphi \circ \ell_g\right)_*$ encodes the action of φ_* on TG. Moreover, since φ normalizes the action of Γ by left multiplication on G, it follows that we obtain a well-defined map

$$\rho_\varphi\colon X = \Gamma\backslash G \to \mathrm{GL}(\mathfrak{g})$$

that encodes the action of φ_* on TX. In particular, for $\varphi = \ell_\gamma$ with $\gamma \in \Gamma$ we have $\rho_\varphi(x) = \mathrm{Id}_{\mathfrak{g}}$ for all $x \in X$.

Since the above defined matrix group S is an open subset of an affine subspace of $\mathbb{R}^{3\times 3}$, we have global coordinates on $G = S^d$ with respect to which we can explicitely calculate the map $S^d \to \mathrm{GL}(\mathfrak{g})$. For $\varphi\colon S^d \to S^d$ given by

$$\varphi\begin{pmatrix} 1 & x_i & z_i \\ 0 & a_i & y_i \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \lambda_i\mu_i x_i & \lambda_i\mu_i z_i \\ 0 & \mu_i a_i & \mu y_i \\ 0 & 0 & 1 \end{pmatrix}, \quad i = 1, \dots, d,$$

and $\xi_i = \left(\begin{smallmatrix} 0 & p_i & r_i \\ 0 & t_i & q_i \\ 0 & 0 & 0 \end{smallmatrix}\right) \in \mathfrak{s}$ we obtain

$$\left(\ell_{\varphi(g)}^{-1} \circ \varphi \circ \ell_g\right)_*\xi_i = \begin{pmatrix} 0 & \lambda_i\mu_i p_i & \lambda_i\mu_i r_i \\ 0 & t_i & q_i \\ 0 & 0 & 0 \end{pmatrix}.$$

This shows that for all $\varphi \in \mathcal{O}_K^{*,+}/\Gamma_A$ the action of φ on the bundle $T\mathcal{F}$ is given by multiplication with the $\lambda_i\mu_i$ in the coordiante $\frac{\partial}{\partial z_i}$ for $i = 1, \dots, d$.

If $d = 2$, let $\varphi \in \mathcal{O}_K^{*,+}/\Gamma_A$ be a non-trivial element and let $\lambda_1, \mu_1, \lambda_2, \mu_2 > 0$ be the factors corresponding to φ. We have $\prod_{i=1}^2 \lambda_i\mu_i = 1$. If one of the products $\lambda_i\mu_i$ was equal to 1, the other one would also be equal to 1. Then φ would be an element of Γ_A and, considered as an element of $\mathrm{Aut}(X)$, would be the identity, a contradiction. Therefore the bundle $T\mathcal{F}$ enjoys the mentioned Anosov property with respect to φ.

For $d \geqslant 3$ it seems to be an interesting number theoretic question if there always exists an automorphism having this Anosov property with respect to the foliation $\mathcal{F}$.

3.5. **Topological structure of X.** Since G is simply-connected solvable and since the adjoint operators of A are diagonalizable over $\mathbb{R}$, the real cohomology of X may be computed via the Lie algebra cohomology of $\mathfrak{g}$.

Since $G = A \ltimes N$ is the identity component of the real-algebraic Lie group $(\mathbb{R}^*)^d \ltimes N$ and since Γ is Zariski-dense in $(\mathbb{R}^*)^d \ltimes N$, we may apply [6, Corollary 7.29] in order to determine the deRham cohomology of X.

Proposition 3.6. *We have $H^k(X,\mathbb{R}) \cong H^k(\mathfrak{g})$ for all $k \geqslant 0$. Moreover, we have*

$$H^k(\mathfrak{g}) = \bigoplus_{k_1+\cdots+k_d=k} \big(H^{k_1}(\mathbb{R}\oplus\mathfrak{h}_3)\otimes\cdots\otimes H^{k_d}(\mathbb{R}\otimes\mathfrak{h}_3)\big),$$

where $H^0(\mathbb{R}\otimes\mathfrak{h}_3) \cong H^4(\mathbb{R}\otimes\mathfrak{h}_3) \cong \mathbb{R}$, $H^1(\mathbb{R}\otimes\mathfrak{h}_3) \cong H^3(\mathbb{R}\otimes\mathfrak{h}_3) \cong \mathbb{R}$, and $H^2(\mathbb{R}\otimes\mathfrak{h}_3) = \{0\}$.

Remark. The above result shows that the topological Euler characteristic of X is zero. This can also be directly deduced from the fact that X is diffeomorphic to a tower of torus bundles over $(S^1)^d$. More precisely, the projection $G = A \ltimes N \to A$ induces a real fiber bundle $X \to \Gamma_A\backslash A \cong (S^1)^d$ with typical fiber $\Gamma_N\backslash N$ which in turn has the structure of a smooth fiber bundle over $(S^1)^d$ with fiber $(S^1)^{2d}$, cf. the diagram in Section 3.1.

3.6. **Closed holomorphic 1-forms on X.** In this subsection we give a second proof of the fact that X is not Kähler.

Proposition 3.7. *There is no non-zero closed holomorphic 1-form on X. In particular, X is not Kähler.*

Proof. Let ω be a closed holomorphic 1-form on X and let ξ be a holomorphic vector field on X induced by the $(\mathbb{C}^*)^d$-action. Then we have

$$\mathcal{L}_\xi(\omega) = \iota_\xi d\omega + d\iota_\xi\omega = 0.$$

Hence, every closed holomorphic 1-form on X must be $(\mathbb{C}^*)^d$-invariant. Pulling it back we get a Γ-invariant closed holomorphic 1-form ω on $(\mathbb{C}\times\mathbb{H})^d$ which must be of the form

$$\omega = \sum_{j=1}^d \lambda_j dz_j + \sum_{j=1}^d f_j(w_1,\ldots,w_d)dw_j.$$

Since an element of N acts on dz_j by $dz_j \mapsto dz_j + a dw_j$, we conclude that in fact

$$\omega = \sum_{j=1}^d f_j(w_1,\ldots,w_d)dw_j.$$

Now the claim follows from the fact that $\pi(\Gamma)$ contains a dense subgroup of the unipotent radical of the Borel subgroup of affine transformations in $\mathrm{Aut}\big((\mathbb{H}^+)^d\big)$, see Proposition 2.9. □

3.7. **The algebraic dimension of X.** We conclude by determining the algebraic dimension of X.

Theorem 3.8. *Every meromorphic function on X is constant, i.e., X has algebraic dimension zero.*

Proof. Due to [2, Example 2], there exists a projective complex space Y arednd a holomorphic map $\pi\colon X \to Y$ such that every holomorphic map from X to any projective complex space factorizes through π. Consequently, $(\mathbb{C}^*)^d$ acts holomorphically on Y such that π is equivariant.

We claim that the induced action of $(\mathbb{C}^*)^d$ on $\mathrm{Alb}(Y)$ is trivial. If this was not the case, the composed map $X \to Y \to \mathrm{Alb}(Y)$ would not be constant and hence we would obtain a non-zero closed holomorphic 1-form on X, contradicting Proposition 3.7.

Since $(\mathbb{C}^*)^d$ acts trivially on $\mathrm{Alb}(Y)$, it has a fixed point in Y due to [7]; the π-fiber over this fixed point is a $(\mathbb{C}^*)^d$-invariant analytic subset of X, hence X itself due to Proposition 3.1. It follows that Y is a point, i.e., every holomorphic map from X to a projective complex space is constant.

Now let us consider the algebraic reduction $a\colon X \dashrightarrow Z$ which is a priori only a meromorphic map. Since there are no proper $(\mathbb{C}^*)^d$-invariant analytic subsets of X, we obtain a holomorphic map $X \to \mathbb{P}_N$ by adding sufficiently many meromorphic functions. As we have seen above, this map must be constant, which proves the claim. $\square$

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