



HAL
open science

Unimodular gauge and ADM gravity path integral

Laurent Baulieu

► **To cite this version:**

Laurent Baulieu. Unimodular gauge and ADM gravity path integral. Phys.Lett.B, 2022, 831, pp.137179. 10.1016/j.physletb.2022.137179 . hal-03086783

HAL Id: hal-03086783

<https://hal.science/hal-03086783>

Submitted on 22 Jul 2024

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons Attribution - NonCommercial 4.0 International License

Unimodular Gauge and ADM Gravity Path Integral

Laurent Baulieu *

*LPTHE, Sorbonne Université, CNRS
4 Place Jussieu, 75005 Paris, France*

Abstract

This paper proposes a definition of gravitational observables and of their path integral formula within the framework of ADM foliation and the choice of unimodular gauge classes. The method enforces a BRST invariant gauge fixing of the lapse and shift fields. It yields the quantum level extension of the known classical property that the conformal classes of internal metrics of constant Lorentz time leafs define the gravitational physical degrees of freedom.

*baulieu@lpthe.jussieu.fr

1 Introduction

The purpose of this note is to enlighten the definition of gravitational observables and of the quantum gravity path integral. To do so, one uses the choice of unimodular gauge classes within the framework of the ADM foliation [1].

Giving a central role to the unimodular gauge is inspired by the observation that what really matters in classical gravity is the propagation of conformal classes of spatial metrics. To the best of our knowledge, this classical and precise property was firstly advocated in the physical literature in [2] where part of Einstein equations of motion is cornered out as describing only physically irrelevant propagation of constraints. [2] shows that the truly relevant initial physical data for solving Einstein equations of motion are the conformal classes of spatial metrics. In fact, mathematicians found already in 1925 the relevance of Weyl symmetry for solving Einstein equations [3, 4, 5], a property that Einstein has foreseen even earlier [6]. Using the Weyl invariance in the quantization of gravity turns out to be very natural in the stochastic quantisation framework [7] and a BRST invariant unimodular gauge fixing has been recently defined such that the conformal factor can be gauge fixed for having no physical propagation, while the theory maintains a BRST invariance [8]. However [8] is not using the ADM paradigm. This is perfectly fine and of interest for some specific questions (such as the perturbative propagation of gravitational quanta). However, the ADM parametrization is most relevant in some cosmological problems due to its visualisation of the space time as a space being foliated by the Lorentz time rather than as a more disorganized set of points. Therefore it makes sense to address the question of how to formulate the unimodular gauge in the ADM foliation framework.

Both terms “unimodular gauge” and “unimodular gravity” should not be confused. The point is that Einstein recommended as early as in 1916 that one uses systems of coordinates to solve his gravity equation such that the determinant of the space-time metric $g_{\mu\nu}$ is locally unimodular [6]. This has been the source of many subsequent works and the word “unimodular gravity” has become quite common. But the so-called “unimodular gravity” has a different physical content than the Einstein theory gauge fixed in a standard BRST invariant way (in particular in the case of the unimodular gauge). The non-exhaustive series of papers [9][10][11][12][13][14][15][16][17][18][19] and references therein address questions related to this domain. When looking at the literature, there are basically two formulations: one that imposes $g = -\det(g_{\mu\nu}) = 1$ as a gauge choice and another one that imposes the constraint $g = 1$ from the beginning. Some confusion is spread around these formulations, although their difference is quite clear.

Working out the Einstein theory in the class of the “unimodular gauges” where the metric determinant g is fixed in a BRST invariant way is nothing but a possible gauge choice, perhaps unfamiliar and difficult to enforce, but formally equivalent to any other (consistent) gauge choice, as can be proven when one uses consistently the BRST methodology to enforce both gauge choices. In fact, the present paper pushes further the BRST framework to explain how the unimodular gauge choice can be done within the ADM metric parametrization. One motivation is that the combination of using unimodular gauge and the ADM paradigm makes more transparent the extension at a path integral level of the classical property the gravity observables can be represented as functionals of the unimodular part of the metric. This latter point relies on the physical degeneracy between the ADM space like leafs related by a Weyl transformation when they are used as an initial condition for solving the Cauchy problem expressed by the Einstein equations, as explained long time ago by York [2]. Getting a proper definition of the observables is an important component of the BRST paradigm. It is a quite sophisticated concept in gravity since its local reparametrization symmetry associated to the propagation of the metric cannot be treated *stricto sensu* as a gauge symmetry and certainly not on the same footing as its local Lorentz symmetry*. But a manifestly BRST invariant method ensures that observables, once they are well defined at the classical level, are independent of gauge artefacts at the quantum level..

In contrast, “unimodular gravity” means changing the theory by varying classically the Einstein–Hilbert action with nothing but variation of metrics with $\sqrt{g} = 1$. A given motivation of “unimodular gravity” has been that it introduces the cosmological constant as a constant of integration to be chosen at will while the “unimodular gauge” fixes the cosmological constant as a parameter of the Lagrangian from the beginning. In fact, the “unimodular gravity” path integral is supposed to be a genuine summation over unimodular metric. It postulates that the symmetry group is reduced from $\text{Diff}(M_d)$ to “transverse” $\text{Diff}(M_d)$. This restriction makes sense at the classical level, but its brute force enforcement at the quantum level doesn’t come with a corresponding Faddeev-Popov ghosts and ghost of ghosts compensating ghost action.

*The involved subtleties share analogies with those occurring in the definition of observables in topological quantum field theories.

It is yet quite unclear if the “unimodular gravity” approach is a consistent one. Indeed, it comes in contradiction with the by now rather well accepted principle that: (i) the local quantum field theory one may associate to given a classical theory with a local symmetry must be built while taking into account the BRST symmetry associated to the whole classical local symmetry; (ii) its gauge-fixing must be done by building a ghost number zero BRST invariant action that is a local functional of all interacting geometrical ghosts (and possibly ghosts of ghosts) with no ghost zero modes as well as of all the trivial BRST doublets of antighosts and Lagrange multipliers that are adapted to the desired gauge choice - in the case of gravity the ghosts and antighosts must be introduced to consistently define the BRST symmetry based on the complete $\text{Diff}(M_d)$ and in the case of the unimodular gauge the ghost and antighost structure is more elaborated than for instance in the deDonder gauge -; (iii) the definition of the physical observables must be done in a way that is compatible with both requirement of BRST symmetry and of unitarity[†].

In fact, the ways one formulates both “unimodular gauge gravity” and “unimodular gravity” are so different that they have a very good chance to provide non equivalent quantum versions. Higher order perturbative computations of quantum gravity might be the clue to reveal their differences, but, in our knowledge, no such computations have yet appeared in the literature.

The logics of “unimodular gauge gravity” was explained in [8] in the context of the BRST quantization method when the fundamental fields that appear in the gravitational path integral measure are the genuine d metric components $g_{\mu\nu}$. [8] relies on the correct property that the condition $\sqrt{g} = 1$ is a well-defined classical local gauge condition for gravity that is allowed by the reparametrization invariance of the classical theory. As a matter of principle, no ambiguity exists in this gauge to define a semi classical quantum field theory of gravity, provided that it is consistently enforced in a BRST invariant way in the path integral, as well as the remaining of the reparametrization is also consistently gauge fixed for the unimodular part $\hat{g}_{\mu\nu}$ of the d metric. [8] solved the question of building a ghost antighost system such that no zero modes is created in the path integral.

The present work extends the proof of [8] within the ADM paradigm. It builds a class of BRST exact actions that enforces consistently the unimodular gauge condition. The latter, which are local functions of the ADM fields and their ghosts, must be added to the classical ADM Einstein action. This provides a definition of the gravity path integral within the ADM formalism in the “unimodular gauge gravity” that is a consistent one (at least semi classically). No possible anomaly is expected in this process but the standard gravitational anomalies that can possibly exist from a purely geometrical point view, due to the structure of the $SO(d-1, 1)$ Lie algebra.

The Lagrangian foliation framework separates the handling of the gravitational constraints from the quantum gauge-fixing of the d dimensional reparametrization symmetry. At the classical level, the ADM lapse and shift have indeed no dynamics. They are basically spectators. At the quantum level, the BRST invariant gauge fixing for the unimodular gauge in the ADM framework is quite relevant to clarify how this property extends at the quantum level. It justifies quite transparently the definition of the quantum gravity observables as functionals of the unimodular parts of the Euclidean $d-1$ dimensional leaf internal metric, as fundamentally inspired by York classical gravity results [2].

Thus, calling \bar{g}_{ij} the unimodular part of the spacelike $d-1$ dimensional internal metrics g_{ij} of the ADM leaves that foliate with respect to the Lorentz time the d dimensional space with pseudo Riemannian d metric $g_{\mu\nu}$, this paper suggests that the physical content of quantum gravity is described by the expectation values

$$\langle \mathcal{O}[\bar{g}_{ij}] \rangle = \text{as it will be defined in this work,} \quad (1)$$

where $\mathcal{O}[\bar{g}_{ij}]$ stands for all possible functionals of \bar{g}_{ij} , expressed in any given set of coordinates. The number of degrees of freedom of \bar{g}_{ij} modulo the internal reparametrization symmetry is $\frac{d(d-3)}{2}$. As it should, this number corresponds to that of a physical graviton in dimension d . The right hand side of (1) will be expressed by an appropriate use of the BRST symmetry for the unimodular gauge choice within the ADM foliation framework (see the precise expression (18)). Therefore, the use of the ADM decomposition of the pseudo Riemannian d metric field for $d > 2$ clarifies enormously the task of representing the gravitational observables,

[†]More must be said about the stability of a given choice of a classical gauge function. In Yang–Mills, the currently used linear gauges are stable because one can complete the gauge symmetry BRST Ward identities by additional ones that are consequences of the antighost equations of motion. The latter ensure the stability of linear gauge fixing. But the non linearity of gravity makes it more difficult to prove that the unimodular gauge fixing is stable under radiative corrections when one introduced order by order in perturbation theory the needed counterterms that are compatible with the Ward identities in this gauge, although its physicality is a good argument for this stability to be true. This question deserves further investigations.

a bit analogously as the use of the Beltrami parametrization of Euclidean $2d$ -metrics clarifies the handling of 2-dimensional quantum gravity and of its conformal properties [21].

Eventually, the cohomology selecting the observables (1) is made of the classes of equivalences of space time metrics $g_{\mu\nu}$ represented by the leaf metrics g_{ij} defined modulo Weyl and leaf reparametrization symmetry. Making explicit its representants is made possible by the separations of the lapse and shift functions introduced in [1] and the use of the unimodular gauge. This definition of observables is compatible with the definition of the classical degrees of freedom in [2]. So, the gauge fixing exhibited in this work conveniently selects the appropriate quantum field theory field variables including the relevant ghosts. Moreover, the paper shows that the chosen gauges for the ADM shift fields share analogies with the regularized Coulomb gauge of the Yang–Mills theory [20].

2 Preliminary remarks

The Einstein action including a cosmological term and matter interactions is $S = \int d^d x \sqrt{g} [R^d(g_{\mu\nu}) + \Lambda + \text{matter field interactions}]$ in a d -dimensional Lorentzian curved space-time \mathcal{M}_d with metric $g_{\mu\nu}$ and scalar curvature $R^d(g_{\mu\nu})$. The proposition of York [2] is that the physics of classical gravity is carried by the conformal classes of the metrics of spacelike $d - 1$ dimensional leafs Σ_{d-1} in of \mathcal{M}_d , a property that is quite transparent by defining à la ADM the possible foliations of \mathcal{M}_d in a coordinate reparametrization invariant way.

The geometry of the pseudo Riemannian manifold \mathcal{M}_d is encoded in the brute force knowledge of the $\frac{d(d+1)}{2}$ component of its metric tensor $g_{\mu\nu}(x^i, t)$ (minuscule latin indices such as i denote spatial coordinates and greek ones world coordinates, with $x^0 \equiv t$). The reparametrization invariance allows one to use any given consistent set of coordinates $\{x^\mu\}$. However it also implies a certain analytical fuzziness that is often harmful to define physics. It makes it not obvious to formulate the proper definition of the physical observables at the quantum level (even assuming that the quantisation scheme itself were well defined).

The $\frac{d(d+1)}{2}$ field equations of $g_{\mu\nu}$ build a non linear second order in time t Cauchy problem. Given suitable initial conditions, only $\frac{d(d-1)}{2}$ independent combinations of the $\frac{d(d+1)}{2}$ solutions truly propagate a physical dynamics. The remaining independent d combinations are constraining identities with no dynamics. This property can be explained in the Hamiltonian framework: only the $\frac{d(d-1)}{2}$ components g_{ij} of each foliated leaf metric have classically propagating Hamiltonian momenta π_{ij} ; the d other components $g_{\mu 0}$ have no Hamiltonian momentum. This coincides with the fact that the $\frac{d(d+1)}{2}$ solutions for $g_{\mu\nu}$ are not independent. The difficulty of dealing with the constraints in the gravity theory surpasses the analogous one in the Yang–Mills theory. The later is also a constrained system. The temporal component A_0 (the formal analog of $g_{\mu 0}$) of the Yang–Mills field has no momentum, but the problems that it causes can be solved by a rather elementary gauge fixing (including a simple recourse to the notion of BRST symmetry). The Yang–Mills observables can be then clearly defined as gauge covariant functionals of the gauge fields. In the gravity case, the constraints mix with the dynamics, leading to the time problem widely discussed in the literature. In fact, with no preferred time, it is not an obvious task to define the notion of a time ordered path integral independent of the choice of the system of coordinates, contrary to what happens in the Yang–Mills situation. Many of the difficulties that physicists encounter when facing the quantum gravity theory are partly due to the fact that they push often too far the comparison between the notions of the reparametrization symmetry and that of a gauge symmetry, that is, beyond the level of local Lorentz symmetry.

Solving the complexity of the constraint problem in theories with a field reparametrization invariance often requires a more sophisticated determination of the BRST invariance of the quantized theory than in theories with a standard gauge invariance. Topological quantum field theories, where one starts from a classical theory defined by a purely boundary term, and thus with a vanishing Hamiltonian, are in fact a good example of this sophistication. New fields belonging to unphysical trivial quartets of the BRST symmetry must be often introduced to solve the quantization of systems with imbricated local symmetries. This causes a non trivial departure from the purely classical concepts. These unphysical fields undergo themselves a non trivial propagation to ensure a unitary behaviour of the physical sector in the context of a local quantum field theory.

To summarize these remarks, the d gravitational constraints stemming from the gravity action are reparametrization covariant. When they are expressed in terms of the independent $\frac{d(d+1)}{2}$ field components $g_{\mu\nu}$, their expression is rather complicated, so they mix non trivially with the equations for the genuine propagation of physical

degrees of freedom. The way it occurs strongly depends on the chosen gauge fixing. This is basically what makes the identification of the physical degrees of gravity quite subtle. In the Yang–Mills case, the constraints amount to the quite simple Gauss law and the definition of the observables shows up immediately as all possible correlators of the electric and magnetic fields (more generally as all gauge covariant functionals of the gauge field). The complexity of the methods required in order to define the observables in the Einstein theory contrasts with the relative simplicity of those that one can use successfully in the Yang–Mills theory.

3 ADM parametrization and some notations

The ADM foliation representation [1] of the $\frac{d(d+1)}{2}$ components of the d dimensional metric $g_{\mu\nu}$ of the Lorentzian space \mathcal{M}_d is most appropriate to compute its foliation into internal spacelike $d-1$ dimensional leaves. Classically, the Lorentz time is the foliation parameter for describing the dynamics led by the d -dimensional Einstein equations of motions. This is at the basis of the interesting notion of “geometrodynamics” of the early 60’s. [2] gave a proof that the classical observables of gravity are in fact the conformal classes of $d-1$ dimensional metrics of constant t leaves in \mathcal{M}_d . This is a remarkable result since the theory is based on the non Weyl invariant Einstein equations of motion. To unravel the possible Weyl covariance properties of the theory, and implement unimodular decompositions, the recent work of [22] is extremely useful in the spirit of [3, 4, 5]. In [7] the unimodular decompositions have been shown in the different context of stochastic quantization to be implemented by a “golden rule” that makes easier the decomposition of all tensors quantities as function of Weyl invariant Σ -tensors plus terms depending on the conformal factor.

The ADM parametrization [1] consists to express as follows the squared invariant Lorentz length

$$ds^2 = -N^2 dt^2 + (dx^i + N^i dt)g_{ij}(dx^j + N^j dt) \quad (2)$$

of any given infinitesimal line element in the space \mathcal{M}_d with coordinates $\{x^\mu\}$. The d -dimensional metric ${}^d g_{\mu\nu}$ with $ds^2 = {}^d g_{\mu\nu} dx^\mu dx^\nu$ has Lorentz signature $(-, +, \dots, +)$. The lower and upper indices components ${}^d g_{\mu\nu}$ and ${}^d g^{\mu\nu}$ of the metric are related to the foliation framework parametrization as follows

$${}^d g_{\mu\nu} = \begin{pmatrix} g_{ij} & N_i \\ N_j & -N^2 + N_i N^j \end{pmatrix} \quad {}^d g^{\mu\nu} = \begin{pmatrix} g^{ij} - \frac{N^i N^j}{N^2} & \frac{N^i}{N^2} \\ \frac{N^j}{N^2} & -\frac{1}{N^2} \end{pmatrix}. \quad (3)$$

$N^i(x^j, t)$ is the shift vector, $N_i \equiv g_{ij}N^j$ and $N(x^j, t)$ is the lapse function. N^i is Weyl invariant because $dx^i + N^i dt$ has Weyl weight zero. The d -dimensional metric determinant is $g \equiv \det({}^d g_{\mu\nu}) = -N\sqrt{\det g_{ij}}$, $\det(g_{ij}) > 0$.

In the ADM framework, one considers (g_{ij}, N, N^i) as fundamental fields. Each $d-1$ dimensional leaf Σ_t of the foliation is defined for a fixed given value of $t = x^0$. Σ_t has internal metric $g_{ij}(x, t)$ with Euclidean signature. One has a normal vector $\mathbf{N} \equiv N^\mu \partial_\mu = \frac{1}{N} \partial_0 - \frac{N^i}{N} \partial_i$ at each point x^i of the leaf, pointing out in the extrinsic space of Σ_t in \mathcal{M}_d , and a one-form $\mathbf{n} \equiv N_\mu dx^\mu = -N dt$. The normalisation is $N_\mu N^\mu = -1$. [23, 24] define a useful projector $P^\mu{}_\nu$ on Σ that extends covariantly to the space \mathcal{M}_d all objects defined on Σ_t . It is defined as

$$P^\mu{}_\nu \equiv \delta^\mu{}_\nu + N^\mu N_\nu, \quad P^\mu{}_i g_{\mu j} = g_{ij}, \quad P^\mu{}_\nu N_\mu = 0. \quad (4)$$

3.1 Leaf covariant tensors

The extrinsic curvature $K_{ij} \equiv \frac{1}{2} \text{Lie}_{\mathbf{N}} g_{ij}$ at each point of a leaf Σ_t is

$$K_{ij} = \frac{1}{2} (N^\rho \partial_\rho g_{ij} + g_{\rho i} \partial_j N^\rho + g_{\rho j} \partial_i N^\rho) = \frac{1}{2N} (\partial_0 g_{ij} - N^k \partial_k g_{ij} - g_{kj} \partial_i N^k - g_{ki} \partial_j N^k) = \frac{1}{2N} (\partial_0 g_{ij} - \nabla_i g_{jt} - \nabla_j g_{it}). \quad (5)$$

The dependance of the extrinsic curvature K_{ij} on $\nabla_{N^k} g_{ij} \equiv \text{Lie}_{N^k} g_{ij} = N^k \partial_k g_{ij} + g_{kj} \partial_i N^k + g_{ki} \partial_j N^k$ makes explicit that it is a covariant entity in the leaf orthogonal to \mathbf{N} . Using $P^\mu{}_\nu$, K_{ij} extends in \mathcal{M}_d as

$$K_{\mu\nu} \equiv P^\mu{}_\mu P^\nu{}_\nu K_{ij} \quad \text{so} \quad K_{tt} = N^i N^j K_{ij} \quad K_{ti} = N^\mu K_{\mu i} = N^\mu K_{i\mu} = K_{it}. \quad (6)$$

The N -independent covariant speed $D_0 g_{ij}$ of a leaf along its normal is

$$D_0 g_{ij} \equiv 2N K_{ij} = \partial_0 g_{ij} - N^k \partial_k g_{ij} - g_{kj} \partial_i N^k - g_{ki} \partial_j N^k. \quad (7)$$

It is in fact interesting to introduce the leaf covariant rate of evolution a_{ij} of the speed K_{ij} ,

$$a_{ij} \equiv N \text{Lie}_{\mathbf{N}}(D_0 g_{ij}) = (\partial_0 - N^k \partial_k) D_0 g_{ij} - 2 D_0 g_{k(i} \partial_{j)} N^k. \quad (8)$$

This acceleration a_{ij} of the leaf along its normal \mathbf{N} is the part of the d dimensional Riemann tensor $R_{\nu\rho\sigma}^\mu$ that is covariant in the leaf at constant t and contains the term $\partial_0^2 g_{ij}$ but no derivative of the lapse function N . This tensor is a useful leaf covariant entity to study the various non relativistic approximation of general relativity.

4 Defining the classical and quantum evolution

We will define the gravitational physical degrees of freedom of the gravity field as the unimodular part of g_{ij} , denoted as \bar{g}_{ij} with $\det \bar{g}_{ij} = 1$. One has the relation $\bar{g}_{ij} = \frac{g_{ij}}{(\det g_{ij})^{\frac{1}{d-1}}}$. When \bar{g}_{ij} is defined modulo internal reparametrization, it counts the right number of degrees of freedom of gravity quanta in each leaf, that is, $\frac{d(d-1)}{2} - 1 - (d-1) = \frac{d(d-3)}{2}$. For the special case $d=2$, one rather expresses the 2 dimensional metric by its Beltrami differential and conformal factor and the relevant field of $2d$ gravity is its Beltrami differential [21]. Note that for $d=3$ one has $\frac{d(d-3)}{2} = 0$ and $3d$ gravity is indeed topological. The conformal classes of its $2d$ spatial leaves are then represented by their moduli, whose finite ranges can be restricted to fundamental domains.

A first question is to better understand the classical t evolution of the leaves Σ_t defined modulo local dilatations through their \bar{g}_{ij} dependance. [2] did so by using extensively the foliation concept.

A second question is to define the expectation values of functionals of these physical degrees of freedom. It requires a plausible definition of the amplitude of probability

$$\mathcal{A}(\Sigma_I \rightarrow \Sigma_F) \quad \text{for} \quad t_I \rightarrow t_F \quad (9)$$

of going from a given leaf Σ_I at time t_I to another given leaf Σ_F at time t_F . The definition of $\mathcal{A}(\Sigma_I \rightarrow \Sigma_F)$ will follow from a suitably gauge fixed BRST invariant path integral formula that we will introduce for observables.

A third question is how to realistically compute such averages of functionals of the \bar{g}_{ij} 's once they have been defined as the observables. But then, one expects to hit all known local quantum field theory difficulties. In particular, one has those coming from the sign changes of the d dimensional Euclidean Einstein action when performing the summation over all metrics in the path integral. Then stochastic quantization might be useful [7].

What is wonderful for both first and second questions is that the use of the ADM parametrization of the metric tensor $g_{\mu\nu}$ changes the conceptual visualisation of the space $\{x^\mu\}$, as an ordered accumulations of leaves rather than a disorganised ensemble of points. From a physical point of view, the foliation picture is such that, within the same system of coordinates, one can heuristically change the foliation by local small dilatations of neighbouring leaves along their normal and by local small shifts without changing the physics, while maintaining the $d-1$ reparametrization invariance $\text{Diff}_\Sigma(\mathcal{M})$ in each leaf.

This property is often referred to by the sentence, “the choice of slicing of space-time does not matter”. One of its illustrations is that one can solve the Cauchy problem of the gravity theory by adjusting at will the lapse and shift functions in each infinitesimal step. The role of a sophisticated enough BRST symmetry is to impose this property directly at the quantum level, while allowing the definition of a consistent local gauge fixed action.

4.1 Classical action and the positivity property of its “kinetic energy” term

Both the speed $D_0 g_{ij}$ and the acceleration a_{ij} are covariant tensors in each leaf with respect to $d-1$ dimensional diffeomorphisms with t -independent parameters $\xi^i(x)$, denoted as $\text{Diff}_\Sigma(\mathcal{M})$. N and N^i are respectively a scalar and a vector for such diffeomorphisms. Using the ADM field variables one has for the cosmological term $\sqrt{|g|} \equiv \sqrt{-\det g_{\mu\nu}} = N \sqrt{\det g_{ij}}$ and for the Einstein-Hilbert action

$$\int d^d x R^d(g_{\mu\nu}) = \int \sqrt{\det g_{ij}} d^{d-1} x dt \left(\frac{g^{ijkl}}{N} D_0 g_{ij} D_0 g_{kl} + N R^{d-1}(g_{ij}) \right) \quad (10)$$

Here, $g^{ijkl} = \frac{1}{2}(g^{ik}g^{jl} + g^{jk}g^{il}) - \lambda g^{ij}g^{kl}$ is the (so-called DeWitt) leaf metric 4-tensor over the space of Euclidean $d-1$ dimensional metrics. The constant λ is to be computed by the decomposition $R^d = \frac{1}{N^2} D_0 g_{ij} g^{ijkl} D_0 g_{kl} +$

$R^{d-1}(g_{ij})$. g^{ijkl} has $\frac{d(d-1)}{2}$ diagonal elements with signature $(1 - \lambda(d-1), 1, \dots, 1)$. This signature is non Euclidean for all values of $d > 2$ (although g_{ij} is Euclidean). Indeed the computation shows that $1 - \lambda(d-1) < 0$ generically. The non positivity of the kinetic term can be seen by decomposing g^{ijkl} in traceless and trace parts

$$g^{ijkl} = (g^{ijkl})^T + \frac{1 - \lambda(d-1)}{d-1} g^{ij} g^{kl}. \quad (11)$$

Here the λ independent traceless part of g^{ijkl} is

$$(g^{ijkl})^T = \frac{1}{2} (g^{ik} g^{jl} + g^{jk} g^{il}) - \frac{1}{d-1} g^{ij} g^{kl}. \quad (12)$$

Besides, $K_{ij}^T = K_{ij} - \frac{1}{d-1} g_{ij} K$ is traceless where $K \equiv g^{ij} K_{ij}$. One has thus

$$g^{ijkl} K_{ij} K_{kl} = K_{ij}^T K^{Tij} + \frac{1 - \lambda(d-1)}{d-1} K^2 \quad \text{where } K^{Tij} \equiv g^{ik} g^{jl} K_{kl}^T. \quad (13)$$

With $K_{ij} = \frac{1}{2N} D_0 g_{ij}$ and $1 - \lambda(d-1) < 0$, Eq. (13) implies that the traceless and trace parts of the “kinetic energy term” $g^{ijkl} D_0 g_{ij} D_0 g_{kl}$ have opposite signs. One motivation for the unimodular gauge choice we will shortly impose in a BRST invariant way is then quite clear. The gauge choice $\sqrt{\det g_{ij}} = 1$ implies $g_{ij} = \bar{g}_{ij}$ and $K|_{g_{ij}=\bar{g}_{ij}} = N g^{ij} D_0(\bar{g}_{ij}) = 0$ since the trace of any given variation of an unimodular metric vanishes. So,

$$\frac{g^{ijkl}}{N} D_0 g_{ij} D_0 g_{kl}|_{g_{ij}=\bar{g}_{ij}} = \frac{1}{N} D_0 \bar{g}_{ij} D_0 \bar{g}^{ij} > 0 \quad \text{where } D_0 \bar{g}^{ij} \equiv \bar{g}^{ik} \bar{g}^{jl} D_0 \bar{g}_{kl}. \quad (14)$$

The “kinetic energy term” $\frac{1}{N} D_0 g_{ij} g^{ijkl} D_0 g_{kl}$ is thus a sum of positive terms in each leaf expressed in the unimodular gauge. It must be noted that the negative direction in the space of deformations of spatial metrics is associated with overall volume deformations. The choice of the unimodular gauge is quite suggestive from this point of view, as it implies that all deformations of the metrics be purely traceless. As it is well known the lapse and shift N and N^i have no genuine dynamics since (10) has no dependance in their time derivatives, so this picture remains true for all possible choices of N and N^i .[‡]

4.2 Using the leaf covariant tensors in classical gravity

$D_0 g_{ij}$ defines the classical evolution of the leaves. Both $D_0 g_{ij}$ and α_{ij} are independent on N . This sustains the importance of the foliation language in gravity, for a step by step evolution of leafs, with an arbitrarily given choice of N and N^i at each step, while remaining in the same system of coordinates x^μ . Suggestively, a local change of the lapse can be compensated by a local dilation of g_{ij} , so a local change of N is of no relevance for the conformal classes of leaf internal $d-1$ dimensional metrics g_{ij} . Changing the shift vector is also classically irrelevant for the evolution of these conformal classes. In fact, the independence of the physical evolution of the gravity field under changes of the initial lapse and shift functions that is explicit in the work of York has been widely used in the context of numerical gravity. It allows to possibly optimize the stability of numerical computations since a blind simulation implies more systematic errors in the enforcement of gravitational constraints.

4.3 Using the leaf covariant tensors for a proper definition of the path integral

Classically, the boundaries of the Cauchy problem of Einstein equations can be chosen as the metric $g_{ij}^I(x^k, t_I)$ and extrinsic curvature $K_{ij}^I(x^k, t_I)$ (basically the initial speed of the leaf) of an initial spacelike leaf Σ_I . The classical limit suggests using $\phi = \log \sqrt{\det g_{ij}(x^k, t)}$ and the $\frac{d(d-1)}{2} - 1$ independent components of the unimodular $\bar{g}_{ij}(x^k, t)$ as the leaf field variables rather than the $\frac{d(d-1)}{2}$ components of g_{ij} 's. In each leaf of the

[‡]The positivity of the ADM kinetic energy the unimodular gauge is an important point, but more is needed to prove that the standard conformal mode instability in Euclidean quantum gravity can be cured. Indeed, although the elimination of the modes that comes with the wrong sign is granted in the unimodular gauge, this does not resolve entirely the problem since the gauge-invariant scalar mode in a York-like decomposition still has the wrong sign [2]. A careful analysis of possible compensations from the ghost dependence of BRST invariant action is needed to help solving this question. The relation between this question and the BRST framework developed here will be the subject of another publication.

evolution, one has the internal diffeomorphism symmetry $\text{Diff}_\Sigma(\mathcal{M})$ $x^i \rightarrow x'^i(x)$, whose infinitesimal transformations are represented by the Lie derivative Lie_ξ^Σ along a t -independent vector field $\xi^i(x^j)$. The differential operator Lie_ξ^Σ acting on leaf tensors only involves the partial derivatives ∂_i 's and the g_{ij} 's. $N^i = g^{it}$ and N transform as a $d-1$ vector and a $d-1$ scalar under $\text{Diff}_\Sigma(\mathcal{M})$. The transformation laws of g_{ij}, N^i, N under the complete reparametrization symmetry in \mathcal{M}_d is defined from $s^d g_{\mu\nu} = \text{Lie}_{\xi^\rho} g_{\mu\nu}$ and the chain rule applied to the relations (3). Their parameters and ghosts are $\xi^i(x^j) \rightarrow \xi^\mu(x^j, t) = (\xi^0(x^j, t), \xi^i(x^j, t))$. The action of Lie_ξ^Σ is thus completed in \mathcal{M}_d by terms involving $\xi^t \equiv \xi^0$ and transformations under ∂_0 . Explicit formula will be shortly displayed.

We can now tentatively define the path integral that computes the amplitude of probability $\mathcal{A}(\Sigma_I \rightarrow \Sigma_F)$ introduced in (9) of going from a given leaf Σ_I at time t_I to another given leaf Σ_F at time t_F . The boundaries are thus defined as both $d-1$ dimensional metrics of initial and final leaves. The guidelines are as follows.

At the quantum level, $\mathcal{A}(\Sigma_I \rightarrow \Sigma_F)$ is non zero when the path in the metric space $\{g_{ij}\}$ that connects Σ_I to Σ_F is not a solution of the equation of motion, although it should concentrate around it when $\hbar \rightarrow 0$.

The leaf evolution can be decomposed into N steps, each one becoming an infinitesimal one in the limit $N \rightarrow \infty$, with the following composition formula, to be read from right to left

$$\mathcal{A}(\Sigma_I \rightarrow \Sigma_F) = \int \int \dots \int [dg_{ij}^{(n-1)}][dg_{ij}^{(n-2)}] \dots [dg_{ij}^{(1)}] \mathcal{A}(\Sigma_{n-1} \rightarrow \Sigma_F) \mathcal{A}(\Sigma_{n-2} \rightarrow \Sigma_{n-1}) \dots \mathcal{A}(\Sigma_I \rightarrow \Sigma_1). \quad (15)$$

This formula must remain consistently true for $N \rightarrow \infty$, meaning that each intermediary leaves become infinitesimally near its right and left neighbours.

The goal is to provide a consistent definition of the gravitational observables. One expects

$$\begin{aligned} \langle \mathcal{O}(\bar{g}_{ij}) \rangle &\equiv \int [dg_{kl}][dN][dN^k][d(\text{ghost})][d(\text{antighost})][d(\text{Lagrange multiplier})] \\ &\mathcal{O}(\bar{g}_{ij}) \exp \int -\frac{i}{\hbar} S^{BRST}[\bar{g}_{kl}, \text{ghost}, \text{antighost}, \text{Lagrange multiplier}]. \end{aligned} \quad (16)$$

In this definition, S^{BRST} corresponds to a BRST invariant action with a gauge fixing of the zero modes of the Einstein action, obtained by adding a BRST exact term to the classical gravity action. We will shortly define what are all ghosts and antighosts in(16). In the foliation picture, the BRST symmetry in (16) involves at least the d -vector ghost $\xi^\mu(x^j, t) \equiv (\xi^0, \xi^i)$. Other ghosts are needed in the unimodular gauge. S^{BRST} must be such that all observable mean values $\langle \mathcal{O}(\bar{g}_{ij}) \rangle$ are covariant under the full d -space coordinates transformations $x^\mu \rightarrow x^\mu + \epsilon^\mu$. Once (16) will be made precise, and if proven consistent, we expect it will define $\mathcal{A}(\Sigma_I \rightarrow \Sigma_F)$.

4.4 BRST approach

The equations of motion of the Einstein action in the ADM variables $\int \sqrt{\det g_{ij}} d^{d-1}x dt (\frac{1}{N} g^{ijkl} D_0 g_{ij} D_0 g_{kl} + NR^{d-1}(g_{ij}))$ show that there is no classical dynamics for the lapse N and shift N^i at the classical level. The path integral must sum over all possible evolutions of leafs identified by the field variable g_{ij} modulo a gauge fixing for the reparametrization symmetry in the leaf, between g_{ij}^I at initial time $t = t_I$ and g_{ij}^F at final time $t = t_F$. Among all path connecting g_{ij}^I and g_{ij}^F , only a single one satisfies the Einstein equation, with a determined value of the extrinsic tensor $K_{ij}^I(x, t_I)$ at the initial time.

The expected consistency is from the reasonable assumption that the configuration space of $d-1$ dimensional metrics over which one performs a path integration between g_{ij}^I and g_{ij}^F is a complete one for each value of t . Indeed the set of all metrics $g_{ij}(x^k)$ defined modulo $d-1$ dimensional reparametrization symmetry describes all possible $d-1$ dimensional manifolds Σ that may occur at any given intermediate value of the time t . It is in fact our freedom to postulate the definition of the path integral over all possible intermediate leafs that may occur for the quantum evolution between Σ_I and Σ_F . But, once it is stated, the consistency of the definition must be checked. In particular, one must ensure the compatibility of the path integral definition with everything we know to be true in the classical limit as expressed in [2]. Since one sums over all intermediary $d-1$ dimensional metrics in each of the steps $1, 2, \dots, n-2, n-1$, one must ensure that one can reinitialize the lapse at each step, meaning for instance that one can impose

$$N_{n-1} = N_{n-2} \dots = N_1 = N_I. \quad (17)$$

This requirement will be proven to be possible by a BRST invariant gauge-fixing procedure. Appendix A shows that the BRST invariant gauge fixing we will do for the shift field N^i is analogous to what one does for gauge fixing the temporal component of a gauge field with a regularized Coulomb gauge condition $A_0 = \alpha \partial_i A_i$.

Let us begin by rewriting a bit more precisely the path integral proposed formula (16):

$$\begin{aligned} \langle \mathcal{O}(\bar{g}_{ij}) \rangle = & \int [dg_{kl}] [dN] [dN^k] [d(\text{ghost})] [d(\text{antighost})] [d(\text{Lagrange multiplier})] \mathcal{O}(\bar{g}_{ij}) \\ & \exp \int -\frac{i}{\hbar} \int_{t_I, g_{kl}^I}^{t_F, g_{kl}^F} d^{d-1} x dt \sqrt{\det g_{kl}} \left[\frac{1}{N} g^{klmn} D_0 g_{kl} D_0 g_{mn} + N R^{d-1}(g_{kl}) \right. \\ & \left. + \text{BRST exact term (ghost, antighost, Lagrange multiplier, } g_{kl}, N, N^k) \right] \end{aligned} \quad (18)$$

The Einstein action in the second line is invariant under the full \mathcal{M}_d reparametrization symmetry $sg_{\mu\nu} = \text{Lie}_{\xi^\rho} g_{\mu\nu}$, with $g_{\mu\nu} = (\bar{g}_{ij}, \sqrt{\det g_{ij}}, N, N^i)$. Therefore, the ghost sector includes at minima d -space reparametrization anticommuting ghosts and antighosts d -vector $\xi^\mu = (\xi^i, \xi^0)$ and $\bar{\xi}^\mu = (\bar{\xi}^i, \bar{\xi}^0)$ with Lagrange multipliers $b^\mu = (b^i, b^0)$, $s\xi^\mu = \xi^\nu \partial_\nu \xi^\mu$, $s\bar{\xi}^\mu = b^\mu$ and $sb^\mu = 0$. The i and 0 components of those fields are respectively scalar and vectors with respect to $\text{Diff}_\Sigma(\mathcal{M})$. Other ghosts and Lagrange multipliers will be introduced as trivial BRST quartets as in [8] to enforce the unimodular gauge both in Σ_t and in \mathcal{M}_d , because the transverse and longitudinal parts of the vector ghosts and antighosts must be distinguished in this case.

The BRST symmetry operation s is thus the nilpotent differential operator that one can associate to the \mathcal{M}_d reparametrization symmetry. Finding a suitable gauge fixing must be adapted to the foliation scheme, meaning choosing relevant gauge functions possibly allowed by the addition of a new trivial BRST field quartet. The action of s on the leaf variables is obtained by using the relations (3). An equivariance of s with respect to each leaf internal symmetry $\text{Diff}_\Sigma(\mathcal{M})$ must appear. It is explicit by the leaf identification of g_{ij} as an internal leaf $d-1$ dimensional metric, of N^i and of ξ^i as leaf $d-1$ vectors, and N and ξ^0 as leaf scalars.

The unimodular leaf gauge choice is classically well defined with the following gauge functions

$$N = 1 \quad \sqrt{-g} = 1 \quad \partial^i \bar{g}_{ij} = 0. \quad (19)$$

This generalizes [8] within the ADM parametrization. Imposing $N = 1$ and $\sqrt{-g} = 1$ is as imposing $N = 1$ and $\det g_{ij} = 1$ since $g = -N^2 \det g_{ij}$, and thus $g_{ij} = \bar{g}_{ij}$. We will shortly improve (19) and discuss the gauge-fixing of the shift N^i .

One could refine the gauge fixing of the lapse function, $N = 1 \rightarrow N = f(x^i, t)$, where the background function f is chosen at will, with $sf = 0$, for instance $f(x) = N(x^i, t = t_I)$. Because the f dependence would be through a BRST-exact term, there should be no f dependence for the expectation values of physical observables. For simplicity we will stick to the choice $f(x^i, t) = 1$.

4.5 BRST spectrum for the ADM parametrization unimodular gauge fixing

The unimodularity gauge condition for the metrics implies that one makes a distinction between the longitudinal and transverse parts of the reparametrization vector ghosts and antighosts ξ and $\bar{\xi}^i$ in the leaf. Having such a decomposition of vector ghosts modulo longitudinal parts in a BRST invariant way implies the necessity of ghosts of ghosts [8]. The longitudinal and transverse components of the Lagrange multiplier auxiliary field b^i must be also separated. Altogether, one needs introducing a trivial BRST quartet as in [8]

$$L^{(00)}, \eta^{(10)}, \bar{\eta}^{(01)}, b^{(11)}, \quad (20)$$

which completes the ordinary BRST system for $g_{\mu\nu}, \xi^\mu, \bar{\xi}^\mu, b^\mu$. Having such a quartet actually corrects the wrong statement that the $d+1$ conditions $\partial_i \bar{g}^{ij} = 0$, $g = 1$ and $N = 1$ might imply an over-gauge fixing. The gauge function $\partial^i \bar{g}_{ij} = 0$ in (19) will be smeared by imposing instead $\bar{g}_{ij} \partial_k \bar{g}^{kj} = -\partial_i L - \alpha N_i$ and summing over possible configurations of L , in the spirit of [8]. α is some real parameter. In analogy with the case of the Coulomb gauge in the Yang-Mills theory (Appendix A), having $\alpha = 0$ is a singular gauge choice beyond the classical level. One thus generalizes the leaf reparametrization gauge function in (19) by

$$\bar{g}_{ij} \partial_k \bar{g}^{kj} = 0 \quad \rightarrow \quad \bar{g}_{ij} \partial_k \bar{g}^{kj} + \partial_i L + \alpha N_i = 0. \quad (21)$$

The scalar bosonic fields L, b and the fermionic fields $\eta, \bar{\eta}$ will count altogether for zero=1+1-1-1 degrees of freedom in the BRST unitary relations provided their dynamics is governed by an s -exact action with invertible propagators. Having available the extra unphysical fields (20) turns out to be what one needs to get a Lagrangian with invertible propagators and no zero modes in a BRST invariant gauge fixing for the unimodular gauge.

The following diagram displays suggestively all necessary ghosts, antighosts and Lagrange multipliers[§]

$$\begin{array}{ccc}
& \bar{g}_{ij}, \sqrt{\det g_{ij}}, N, L^{(00)}, N^i & \\
\swarrow & & \searrow \\
\xi^{i(10)}, \xi^{0(10)}, \eta^{(10)} & & \bar{\xi}_i^{(01)}, \bar{\xi}_i 0^{(01)}, \bar{\eta}^{(01)} \\
& \swarrow & \searrow \\
& b^{i(11)}, b^{0(11)}, b^{(11)} & \\
1 & 0 & -1
\end{array} \quad . \quad (22)$$

The numbers $-1, 0, 1$ in the bottom line indicate the net ghost number of fields that are aligned vertically above each number. The BRST transformations are

$$\begin{aligned}
s g_{\mu\nu} &= \text{Lie}_\xi g_{\mu\nu} = \xi^\rho \partial_\rho g_{\mu\nu} + \bar{g}_{\rho\mu} \partial_\nu \xi^\rho + \bar{g}_{\rho\nu} \partial_\mu \xi^\rho \\
s \xi^\mu &= \text{Lie}_\xi \xi^\mu = \xi^\nu \partial_\nu \xi^\mu \\
s \bar{\xi}^\mu &= b^\mu & s b^\mu &= 0 \\
s L &= \eta & s \eta &= 0 \\
s \bar{\eta} &= b & s b &= 0.
\end{aligned} \quad (23)$$

The operations $d, s, i_\xi, \text{Lie}_\xi = [i_\xi, d]$ build a system of nilpotent graded differential operators acting with $[s, \partial_\mu] \equiv 0, \{s, d\} \equiv 0$ and $s^2 = 0$ on all fields[¶]. Both last lines in Eqs. (23) identify $L, \eta, \bar{\eta}, b$ as the elements of a BRST exact quartet. The extra commuting scalar b in (20) is a scalar Lagrange multiplier with ghost number 0 and both anticommuting scalars $\eta, \bar{\eta}$ are odd Lagrange multipliers with ghost numbers -1 and 1 .

One has $s\sqrt{-g} = \partial_\mu(\sqrt{-g}\xi^\mu)$ and the s transformation of the unimodular $\hat{g}_{\mu\nu} = g_{\mu\nu}(\det g_{\mu\nu})^{-\frac{1}{d}}$ is traceless,

$$s\hat{g}_{\mu\nu} = \hat{g}_{\rho\mu} \partial_\nu \xi^\rho + \hat{g}_{\rho\nu} \partial_\mu \xi^\rho - \frac{2}{d} \hat{g}_{\mu\nu} \partial_\rho \xi^\rho. \quad (24)$$

The nilpotent BRST s transformations of g^{ij}, g^{0i} and g^{00} determine those of N, N^i and $\sqrt{\det g_{ij}}$ according to

$$\begin{aligned}
sN &= \text{Lie}_{\xi^k} N + \xi^0 \partial_0 N + N(\partial_0 - N^i \partial_i) \xi^0 \\
sN_i &= \text{Lie}_{\xi^k} N_i + g_{ij} \partial_0 \xi^j + \xi^0 \partial_0 N_i + (N_i \partial_0 + (N^k N_k - N^2) \partial_i) \xi^0 \\
s\sqrt{\det g_{ij}} &= \partial_k(\sqrt{\det g_{ij}} \xi^k) + \left(\frac{1}{N} - \sqrt{\det g_{ij}}\right) \partial_0 \xi^0 + \sqrt{\det g_{ij}} (N^k \partial_k \xi^0 - \frac{1}{N} \xi^0 \partial_0 N)
\end{aligned} \quad (25)$$

where $\text{Lie}_{\xi^k} N = \xi^k \partial_k N$ and $\text{Lie}_{\xi^k} N_i = \xi^k \partial_k N_i + N_k \partial_i \xi^k$. $s\bar{g}_{ij}$ is traceless and lapse independent, with

$$s\bar{g}_{ij} = (\xi^k \partial_k + \xi^0 \partial_0) \bar{g}_{ij} + \bar{g}_{ik} \partial_j \xi^k + \bar{g}_{jk} \partial_i \xi^k + N_i \partial_j \xi^0 + N_j \partial_i \xi^0 - \frac{2}{d-1} \bar{g}_{ij} (\partial_k \xi^k + N^k \partial_k \xi^0). \quad (26)$$

4.6 The BRST invariant action in the leaf unimodular gauge

We can now express the s -exact BRST invariant gauge fixed action in the refined leaf unimodular gauge:

$$I_{BRST}^{Unimodular} = \int d^d x \sqrt{\det g_{ij}} \left[\frac{1}{N} g^{klmn} D_0 g_{ij} D_0 g_{kl} + N R^{d-1}(g_{ij}) + s \left(\xi^0 (N-1) + \bar{\xi}^i (\bar{g}_{ij} \partial_k \bar{g}^{kj} + \alpha N_i + \partial_i L) + \bar{\eta} (\sqrt{-g} - 1) \right) \right]$$

[§]The notation $\phi^{g,g'}$ means that the field $\phi^{g,g'}$ carries ghost number g and antighost number g' for a total net ghost number $G = g - g'$. $\phi^{g,g'}$ is a boson if G is even and a fermion if G is odd. We often skip these ghost and antighost indices in the formula.

[¶]The convenient graded operation $\hat{s} = s - L_\xi$ is nilpotent in the absence of local supersymmetry because in this case $s\xi = \text{Lie}_\xi \xi$. In supergravity, one has also $s^2 = 0$ with a complete system of auxiliary fields but then $\hat{s}^2 = i_\Phi \neq 0$ where $\Phi^\mu = \chi \gamma^\mu \chi$ is the vector field quadratic in the commuting supersymmetry ghost χ [27].

$$\begin{aligned}
&= \int dt d^{d-1}x \left[\sqrt{\det g_{ij}} \left(\frac{1}{N} g^{klmn} D_0 g_{ij} D_0 g_{kl} + N R^{d-1}(g_{ij}) \right) + b^0(N-1) + b(\sqrt{-g}-1) + b^i(\bar{g}_{ij} \partial_k \bar{g}^{kj} + \alpha N_i + \partial_i L) \right. \\
&\quad \left. - \bar{\xi}^0 sN - \bar{\xi}^i (s(\bar{g}_{ij} \partial_k \bar{g}^{kj} + \alpha s N_i) + \partial_i \eta) - \bar{\eta} \partial_\mu \sqrt{-g} \xi^\mu \right]. \tag{27}
\end{aligned}$$

Eliminating b^0 and b by their algebraic equations of motion imposes $\sqrt{-g} = 1, N = 1 \rightarrow g_{ij} = \bar{g}_{ij}$. Thus

$$\begin{aligned}
I_{BRST}^{Unimodular} \sim \int dt d^{d-1}x \left[D_0 \bar{g}_{ij} D_0 \bar{g}^{ij} + R^{d-1}(\bar{g}_{ij}) + b^i(\bar{g}_{ij} \partial_k \bar{g}^{kj} + \alpha N_i + \partial_i L) \right. \\
\left. - \bar{\xi}^0 sN - \bar{\xi}^i (s(\bar{g}_{ij} \partial_k \bar{g}^{kj} + \alpha N_i) + \partial_i \eta) + \bar{\eta} \partial_\mu \xi^\mu \right]. \tag{28}
\end{aligned}$$

Eq. (14) has reduced $D_0 g_{ij} g^{ijkl} D_0 g_{kl} |_{g_{mn} = \bar{g}_{mn}}$ to the positive kinetic term $D_0 \bar{g}_{ij} D_0 \bar{g}^{ij} = g^{ik} g^{jl} D_0 \bar{g}_{ij} D_0 \bar{g}_{kl} > 0$ as a benefit of using the unimodular gauge. One must check all propagators are invertible.

Consider the bosonic sector. The $d-1$ components of the Lagrange multiplier b^i gauge fix the transversality of the quadratic approximation of $R^{d-1}(\bar{g}_{ij})$ in function of \bar{g}_{ij} . This gauge fixing involves mixings with L and N^i as seen in (28). The term $\partial_i L$ is justified because the eigenvalues \bar{g}_{ij} depend on only $d-2$ independent combinations of the \bar{g}_{ij} 's due to $\det \bar{g}_{ij} = 1$. The b^i propagation is thus between the $d-2$ longitudinal degrees of freedom in \bar{g}_{ij} plus the additional one in $\partial_i L$ and the $d-1$ degrees of freedom in b^i .

For $\alpha = 0$, the propagation of the shift vector N^i is left undetermined, in analogy with the time dependence of the temporal component of a gauge field that is left undetermined in the Coulomb gauge.

For $\alpha \neq 0$, the elimination of b^i gives a delta function $\delta(\alpha N_i + \bar{g}_{ij} \partial_k \bar{g}^{kj} + \partial_i L)$ in the measure whose argument is linear in N^i . One can then integrate over all possibilities over the lapse N^i , meaning that N^i is gauge fixed at the zero of the delta function. The gauge fixed dependence in N^i is through the extrinsic curvature $K_{ij}(\bar{g}_{kl}, N^k)$ gauge-fixed at $N = 1$, namely through $K_{ij}|_{N=1} = D_0 \bar{g}_{ij} = \partial_0 \bar{g}_{ij} - N^{(k)} \partial_k \bar{g}_{ij} - \bar{g}_{ki} \partial_j N^{(k)} - \bar{g}_{kj} \partial_i N^{(k)}$ where $N_i = -\frac{1}{\alpha}(\bar{g}_{ij} \partial_k \bar{g}^{kj} + \partial_i L)$. One gets a propagator for the longitudinal part of \bar{g}_{ij} with a quadratic behaviour in the time derivative of \bar{g}_{ij} and a quartic behaviour in its space derivatives. This propagator is analogous to that of A_i in the Yang-Mills regularized Coulomb gauge choice $\alpha A_0 + \partial_i A_i = 0$, Appendix A ^{||}. Thus, $\alpha \neq 0$ provides an unambiguous unimodular gauge fixing of each leaf.

Consider now the fermionic sector and check that the system of the ghost propagators is also invertible. It is sufficient to investigate the field quadratic approximation of the second line in Eq. (28). In this approximation, $\bar{\xi}^0 sN \sim \bar{\xi}^0 \partial_0 \xi^0$, $\bar{\xi}^i \partial_i sL + \bar{\eta} s \sqrt{-g} \sim \eta \partial_i \bar{\xi}^i + \bar{\eta} \partial_\mu \xi^\mu$ and $\bar{\xi}^i s(\bar{g}_{ij} \partial_k \bar{g}^{kj} + \alpha N_i) \sim \bar{\xi}^i [(\bar{g}_{ij} \bar{\Delta} + \frac{d-3}{d-1} \partial_i \partial_j) \xi^j + \alpha(\partial_i \xi^0 + \bar{g}_{ij} \partial_0 \xi^j)]$. Thus the quadratic approximation of the ghost action is

$$\begin{aligned}
&- \int dt d^{d-1}x \left[\bar{\xi}^0 sN + \bar{\xi}^i (s(\bar{g}_{ij} \partial_k \bar{g}^{kj} + \alpha N_i) + \partial_i L) + \bar{\eta} \partial_\mu \xi^\mu \right] \\
&\sim - \int dt d^{d-1}x \left[\bar{\xi}^i \left(\bar{g}_{ij} (\bar{\Delta} + \alpha \partial_0) + \frac{d-3}{d-1} \partial_i \partial_j \right) \xi^i - \eta' \partial_i \bar{\xi}^i + \bar{\eta} \partial_i \xi^i + \bar{\xi}^{0'} \partial_0 \xi^0 \right]. \tag{29}
\end{aligned}$$

with $\eta' = \eta + \alpha \xi^0$ and $\bar{\xi}^{0'} = \bar{\xi}^0 + \bar{\eta}$. All propagators between (ξ^i, ξ^0, η) and $(\bar{\xi}^i, \bar{\xi}^0, \bar{\eta})$ are invertible. The gauge fixing that we have constructed is thus well defined perturbatively. In the absence of gravitational anomaly, the Ward identities that control all interactions are enforced by the BRST invariance and ensure the gauge invariance of the observables and the unitarity order by order in perturbation theory. Eventually the class of α -unimodular gauges introduced by the above discussion combine both its physicality and narrow contact with the classical properties of the theory as well as its consistency.

5 Conclusion

Given the classical gravity action $I[g_{ij}, N, N^i] = \int_{t_I, g_{ij}^I}^{t_F, g_{ij}^F} d^d x L^{Einstein}(g_{ij}, N, N^i)$, the expression (16) of the gravity observables has been made explicit in Eq. (28). The path integral is over all possible internal metrics of

^{||}See [20] for a detailed study of the regularized Coulomb gauge in the Yang-Mills theory.

leaf trajectories between two boundary leaves at times t_I and t_F . The class of α -unimodular gauges determines a positive kinetic energy term $D_0 \bar{g}_{ij} D_0 \bar{g}^{ij}$ in a path integral formulation and the conformal factor decouples. We have discussed the differences between both choices $\alpha \neq 0$ or $\alpha = 0$. The propagation of the shift vector N^i is formally analogous to that of the temporal component of a gauge field in a regularized Coulomb gauge.

The case $\alpha \neq 0$ gives unambiguous field propagators both in \mathcal{M}_d and in each leaf Σ_{d-1} . The comparison with the gauge fixing in the regularized Coulomb gauge of Yang–Mills fields enlightens some aspects of the gravity compactification into lower dimensional gravity and Yang–Mills theories, with shift components g_{M0} identified as time components A_0 of a gauge field with spatial components A_k made of corresponding g_{Mk} components.

Eq.(28) is a BRST invariant path integral definition that involves the $\frac{d(d-1)}{2} - 1$ off shell components \bar{g}_{ij} of the unimodular parts of the $d - 1$ dimensional internal metric g_{ij} as the genuinely propagating components of the quantum gravity field $g_{\mu\nu}$. It identifies ab initio the $\frac{d(d-3)}{2}$ physical quanta of gravity as the physical degrees of freedom of the quantum theory. The BRST invariance of the unimodular gauge fixing implies Ward identities for the observables $\langle \mathcal{O}(\bar{g}_{ij}) \rangle$'s. They must ensure that a physical quantum propagation with unitary properties occurs only for the unimodular parts of the internal leaf metrics (for $d \geq 4$). The unimodular leaf gauge can be viewed as a physical choice because the classical limit of the corresponding quantum theory is then automatically compatible with the classical properties found in [2]. Unphysical fields propagate in this gauge, but, owing to the BRST symmetry, their contributions compensate each other in the physical amplitudes. The validity of this reasoning at the non perturbative level is a question worth further investigation.

One must note that the lapse function has been gauge fixed in a BRST invariant way with $N(x^i, t) = 1$, but it can also be fixed more generally to any possible fixed positive function $N(x^i, t) = f(x^i, t)$, still in a BRST invariant way. This generalization might be useful to perhaps solve more conveniently certain problems. Let us conclude by emphasising that the physical equivalence between Weyl related internal metrics of each leaf is quite subtle and goes beyond topics of the genuine covariance under reparametrization symmetry. One might be tempted to consider this property as a defining concept of the gauge invariance of general relativity.

A rather technical and relevant comment is that practical calculations of quantum gravity often use the background field method. Both subjects are not the subject of this paper, but they are very interesting. One might thus attempt to further investigate the possibility of defining the background field method within the ADM formulation, maybe with the help of some of the advances presented in this work. In fact, since the ADM fields N, N^i, g_{ij} undergo the whole repametrization transformations in a one to one correspondence with those of $g_{\mu\nu}$, using them within the background field formalism seems a priori doable, even if they get split in Weyl invariant and non Weyl invariant components, as it is done systematically in the unimodular gauge. On the other hand, one must solve the non trivial issue of whether or not one has the necessity of introducing background fields for both the lapse and shift fields, which have no relevant physical propagation, as well as for the conformal factor. The latter questions are non trivial and deserve further studies.

Acknowledgements: I am most grateful to John Iliopoulos for many discussions on the subject of this work.

A Regularized Coulomb gauge

A.1 Yang–Mills

Consider the abelian curvature $F_{ij} = \partial_i A_j - \partial_j A_i$ and $F_{0i} = \partial_0 A_i - \partial_i A_0$ of the gauge field $A_\mu = (A_i, A_0)$. $F_{\mu\nu}^2 = F_{ij}^2 - F_{0i}^2$ is gauge invariant under $sA_\mu = \partial_\mu c$. With the gauge choice $\alpha A_0 = \partial_i A_i$, one has

$$\begin{aligned} F_{\mu\nu}^2|_{A_0=\alpha^{-1}\partial_i A_i} &= -A_i \square A_i + (\partial_i A_i)^2 + \alpha^{-2} (\partial_i A_i) \Delta (\partial_i A_i) + \text{boundary term} \\ &= -A_i \left(\delta_{ij} - \frac{\partial_i \partial_j}{\Delta} \right) \square A_j - \partial_i A_i \frac{\partial_0^2 - \alpha^{-2} \Delta^2}{\Delta} \partial_i A_i + \text{boundary term} \end{aligned} \quad (30)$$

where $\Delta = \partial^i \partial_i$. The propagator of A_i stemming from this gauge fixing of $F_{\mu\nu}^2$ is

$$\langle A_i, A_j \rangle = - \left(\delta_{ij} - \frac{\partial_i \partial_j}{\Delta} \right) \frac{1}{\square} + \frac{\partial_i \partial_j}{\Delta} \frac{1}{\partial_0^2 - \alpha^{-2} \Delta^2} \quad (31)$$

A ghost propagation must occur to compensate for that of unphysical modes implied by the longitudinal part in (31). It is defined by the Faddeev–Popov ghost propagator one gets from the enforcement of a BRST

invariance, by adding the term $\bar{c}s(\alpha A_0 + \partial_i A^i) = \bar{c}(\alpha\partial_0 + \Delta)c$ to the action (30). This ghost propagator is

$$\langle \bar{c}, c \rangle = \frac{1}{\alpha\partial_0 + \Delta}. \quad (32)$$

Beyond technical difficulties, the regularised Coulomb gauge is well-defined from (31), and, perturbatively, can be shown to yield the same physical results as the Lorentz gauge $\partial_\mu A_\mu = 0$ (or, more subtly, $\alpha\partial_0 A_0 + \partial_i A^i = 0$) [20].

The propagator (31) is to be compared with that of the pure Coulomb gauge condition $\partial_i A_i = 0$ for which

$$F_{\mu\nu}^2|_{\partial_i A_i=0} = -A_i \square A_i + A_0 \Delta A_0 + \text{boundary term}. \quad (33)$$

The gauge field propagator implied by (33) suffers from the pure time dependent zero modes of A_0 . The limiting case $\alpha = 0$ is generally ambiguous because of contact terms, except for questions where one can consistently set $A_0 = 0$ everywhere through the computations.

For Yang–Mills, the possibility of practical perturbative computations comes from the following “improved” successions of gauge functions: the “physical” but ambiguous gauge $\partial_i A_i = 0 \rightarrow$ the non ambiguous gauge $\partial_i A_i = \alpha A_0, (\alpha \neq 0), \rightarrow$ the non ambiguous renormalisable but not so easily usable gauge $\partial_i A_i = \alpha\partial_0 A_0 \rightarrow$ the non ambiguous and more easily renormalisable gauge with $\alpha = 1, \partial_i A_i = \partial_0 A_0$, as it is Lorentz invariant.

A.2 A simpler but less physical gauge–fixing for gravity inspired from the Yang–Mills Coulomb gauge

Consider the following gravity simpler gauge-fixing in the ADM parametrisation, where no separation between the conformal and non conformal part of both $g_{\mu\nu}$ and g_{ij} is made. The BRST invariant action for the gauge functions $N^i + \alpha\partial_j g^{ij} = 0$ and $N - \gamma\sqrt{\det g_{ij}} = 0$, where $\alpha \neq 0 \gamma \neq 0$ are a pair of gauge parameters, is

$$\begin{aligned} & \int d^d x \left[\sqrt{-g} R^d + s(\bar{\xi}^i (g_{ij} \partial_k g^{jk} + \alpha N_i)) + s(\bar{\xi}^0 (N - \gamma\sqrt{\det g_{ij}})) \right] \\ &= \int d^d x \left[\sqrt{-g} R^d + b_i (\partial_j g^{ji} + \alpha N^i) + b^0 (N - \gamma\sqrt{\det g_{ij}}) - \bar{\xi}^i s(g_{ij} \partial_k g^{jk} + \alpha N_i) - \bar{\xi}^0 (sN - \gamma s\sqrt{\det g_{ij}}) \right] \\ &\sim \int d^{d-1} x dt \left[\frac{1}{\gamma} g^{ijkl} D_0 g_{ij} D_0 g_{kl} |_{N^i = -\alpha \partial_j g^{ij}} + \gamma \det g_{ij} R^{d-1}(g_{ij}) - \bar{\xi}^i s(g_{ij} \partial_k g^{jk} + \alpha N_i) - \bar{\xi}^0 (sN - \gamma s\sqrt{\det g_{ij}}) \right]. \end{aligned}$$

For $\gamma \neq 0$, the ghost part of this BRST invariant action has an invertible quadratic approximation, namely, $-\bar{\xi}^i s(g_{ij} \partial_k g^{jk} + \alpha N_i) - \bar{\xi}^0 (sN - \gamma\sqrt{\det g_{ij}}) \sim -\bar{\xi}^i (\Delta + \alpha\partial_0) \delta_{ij} + \partial_i \partial_j \xi^j - \alpha \bar{\xi}^i \partial_i \xi^0 - \bar{\xi}^0 (\partial_0 \xi^0 - \gamma \partial_i \xi^i)$. The quartet (20) is unneeded in such classes of α, γ gauges. In these perfectly admissible but non-unimodular gauges, the conformal factor propagates with a negative contribution to the kinetic energy (13). The reason for that is the contribution of the opposite sign of the trace part of the metric tensor g^{ijkl} over the space of $d - 1$ dimensional Euclidean metrics when the internal leaf metric is not gauge fixed in an unimodular way. The relevance of the Weyl invariance for defining the observables is not as explicit in the classes of α, γ gauges than it is in the unimodular gauges.

References

- [1] R. L. Arnowitt, S. Deser and C. W. Misner, *The dynamics of general relativity*, Gen. Rel. Grav. **40** (2008) 1997–2027.
- [2] J. W. York, Jr., *Role of conformal three-geometry in the dynamics of gravitation*, Phys. Rev. Lett. **28** (1972) 1082–1085; *Conformally invariant orthogonal decomposition of symmetric tensors on Riemannian manifolds and the initial value problem of general relativity*, J. Math. Phys. **14** (1973) 456–464.
- [3] J. M. Thomas, *Conformal correspondence of Riemann spaces*, Proc. Nat. Acad. Sci. **11** (1925) 257–259.
- [4] J. M. Thomas, *Conformal invariants*, Proc. Nat. Acad. Sci. **12** (1926) 389–393.

- [5] O. Veblen and J. M. Thomas, *Projective invariants of affine geometry of paths*, Ann. Math. **27** (1926) 279–296.
- [6] A. Einstein, *Die Grundlage der Allgemeinen Relativitätstheorie*, Annalen der Physik, **354** 769-822, 1916.
- [7] L. Baulieu, L. Ciambelli, and S. Wu, *Weyl Symmetry in Stochastic Quantum Gravity*, Class. Quant. Grav. **37** (2020) 4, [hep-th]1909.11478.
- [8] L. Baulieu, *Unimodular Gauge in Perturbative Gravity and Supergravity*, Phys.Lett. **B808** (2020), [hep-th] 2004.05950.
- [9] M. Henneaux and C. Teitelboim, *The Cosmological Constant As A Canonical Variable* Phys.Lett. B **143** (1984) 415-420; *The Cosmological Constant and General Covariance* Phys.Lett. B **222** (1989) 195-199.
- [10] E. Alvarez, *The Weight of Matter*, arXiv:1204.6162.
- [11] A. Padilla, I. D. Saltas, *A Note on Classical and Quantum Unimodular gravity*, arXiv:1409.3573.
- [12] S. Upadhyay, M. Oksanen, R. Bufalo, *BRST Quantization of Unimodular Gravity*, Braz. J. Phys. **47** (2017), arXiv:1510.00188.
- [13] E. Álvarez, S. González-Martin, C. P. Martin, *A note on the Gauge Symmetries of Unimodular Gravity*, arXiv:1604.07263.
- [14] I. Oda, *Fake Conformal Symmetry in Unimodular Gravity*, Phys. Rev. **D94** 044032 (2016), arXiv:1606.01571.
- [15] I. Oda, *Classical Weyl Transverse Gravity*, arXiv:1610.05441.
- [16] R. de Leon Ardon, N. Ohta, R. Percacci, *The Path Integral of Unimodular Gravity*, arXiv:1710.02457.
- [17] S. Nagy, A. Padilla, I. Zavala, *The Super-Stuckelberg procedure and dS in Pure Supergravity*, arXiv:1910.14349.
- [18] J. Anero, C. P. Martin, R. Santos-Garcia, *Off-shell unimodular $N=1$, $d=4$ supergravity*, JHEP (2020) 145, arXiv:1911.04160, *A note on unimodular $N=1, d=4$ AdS supergravity*, arXiv:2001.05365.
- [19] P. Jirouek, A. Vikman, *New Weyl-invariant vector-tensor theory for the cosmological constant* JCAP **04** (2019) 00, e-Print: 1811.09547 [gr-qc] <https://inspirehep.net/literature/1704721>; K. Hammer, P. Jirouek, A. Vikman *Axionic cosmological constant* e-Print: 2001.03169 [gr-qc], <https://inspirehep.net/literature/1774715>.
- [20] L. Baulieu and D. Zwanziger *Renormalizable noncovariant gauges and Coulomb gauge limit*, Nucl. Phys. cal B **548** (1999) 527-562, hep-th/9807024 [hep-th].
- [21] L. Baulieu, C. Becchi and R. Stora, *On the Covariant quantization of the free bosonic string*, Phys. Lett. **B180** (1986) 55–60; L. Baulieu and M. P. Bellon, *Beltrami parametrization and string theory*, Phys. Lett. **B196** (1987) 142–150; L. Baulieu, M. P. Bellon and R. Grimm, *Left-right asymmetric conformal anomalies*, Phys. Lett. **B228** (1989) 325–331.
- [22] C. Kiefer and B. Nikolic, *Conformal and Weyl-Einstein gravity: classical geometrodynamics*, Phys. Rev. **D95** (2017) 084018, [gr-qc] 1702.04973.
- [23] E.ourgoulhon and J. L. Jaramillo, *A 3+1 perspective on null hypersurfaces and isolated horizons*, Phys. Rep. **423** (2006) 159–294, [gr-qc] 0503113.
E.ourgoulhon, *3+1 formalism and bases of numerical relativity*, [gr-qc] 0703035.
- [24] M. Blau, *Lectures Notes in General Relativity*, <http://www.blau.itp.unibe.ch/newlecturesGR.pdf>.
- [25] B. S. DeWitt, *Quantum theory of gravity: the canonical theory*, Phys. Rev. **160** (1967) 1113–1148.
- [26] D. Giulini and C. Kiefer, *Wheeler-DeWitt metric and the attractivity of gravity*, Phys. Lett. **A193** (1994) 21–24, [gr-qc] 9405040.
- [27] L. Baulieu and M. Bellon *p -Forms and Supergravity: Gauge Symmetries in Curved Space*, Nucl.Phys. **B 266** (1986) 75-124.