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► **To cite this version:**

Susmit Bagchi. Topological Analysis of Fibrations in Multidimensional (\mathbb{C} , \mathbb{R}) Space. *Symmetry*, 2020, 12, 10.3390/sym12122049 . hal-03085806

HAL Id: hal-03085806

<https://hal.science/hal-03085806v1>

Submitted on 22 Dec 2020

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Article

Topological Analysis of Fibrations in Multidimensional (C, R) Space

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Received: 15 November 2020; Accepted: 9 December 2020; Published: 10 December 2020



Abstract: A holomorphically fibred space generates locally trivial bundles with positive dimensional fibers. This paper proposes two varieties of fibrations (compact and non-compact) in the non-uniformly scalable quasinormed topological (C, R) space admitting cylindrically symmetric continuous functions. The projective base space is dense, containing a complex plane, and the corresponding surjective fiber projection on the base space can be fixed at any point on real subspace. The contact category fibers support multiple oriented singularities of piecewise continuous functions within the topological space. A composite algebraic operation comprised of continuous linear translation and arithmetic addition generates an associative magma in the non-compact fiber space. The finite translation is continuous on complex planar subspace under non-compact projection. Interestingly, the associative magma resists transforming into a monoid due to the non-commutativity of composite algebraic operation. However, an additive group algebraic structure can be admitted in the fiber space if the fibration is a non-compact variety. Moreover, the projection on base space supports additive group structure, if and only if the planar base space passes through the real origin of the topological (C, R) space. The topological analysis shows that outward deformation retraction is not admissible within the dense topological fiber space. The comparative analysis of the proposed fiber space with respect to Minkowski space and Seifert fiber space illustrates that the group algebraic structures in each fiber spaces are of different varieties. The proposed topological fiber bundles are rigid, preserving sigma-sections as compared to the fiber bundles on manifolds.

Keywords: topological spaces; fiber bundles; group; projection; norm

MSC: 54F65; 55R15; 55R65

1. Introduction

The Minkowski space is a *four-dimensional* topological vector space over reals (i.e., *four-manifold* admitting Poincare symmetry group of isometries) with applications in physical and mathematical sciences [1,2]. In general, the Minkowski space is not well behaved if the corresponding Euclidean topological space is considered to be a locally homogeneous space [3]. The reason is that the Minkowski topological space gets decomposed into two locally homogeneous Euclidean subspaces, where the two topological subspaces are separated in nature. Note that the finest topology in *n-dimensional* Minkowski space is Zeeman topology, which is separable, Hausdorff, locally non-compact, and also non-Lindeloff in nature. Moreover, the Zeeman topology generates a first countable topological space. On the other hand, the $n > 1$ dimensional Minkowski space equipped with *t*-topology (M^t) is not completely Euclidean in nature [4]. The topological space on M^t is first countable, where the compactification and continuity of a function can be maintained through the Zeno sequences. In view of general topology, the 4D Minkowski space equipped with *s*-topology is not a normal topological space, and it is a non-compact Hausdorff space. The axial rotations of a Minkowski space generate various geometric hypersurfaces

in space. For example, the three types of helicoidal hypersurfaces are generated by axial rotation of 4-dimensional Minkowski space [5].

In this paper, the construction of topological fiber space in a non-uniformly scaled quasinormed (C, R) space and the corresponding topological, as well as algebraic analysis of fibration varieties are presented. First, the brief descriptions about the related concepts, such as topological fiber spaces and immersion of manifolds are presented in Sections 1.1 and 1.2, respectively. Next, the motivation for this work and contributions made in this paper are explained in Section 1.3. In this paper, the symbols R , C and Z represent sets of extended real numbers, complex numbers, and integers, respectively.

1.1. Topological Fiber Spaces

The topological fiber bundles over a sphere exhibit a set of interesting topological properties if the respective fiber space is Euclidean. It is shown that if M is a closed and compact manifold maintaining Hausdorff topological property, then the function space X containing functions $m : M \times R \rightarrow R$ without interchanging the ends in R is a contractible space [6]. In view of algebraic topology, the structure G_n represents a monoid of homotopy self-equivalences of $(n - 1)$ -spheres denoted by S^{n-1} [6]. Note that in this case, the topological space is in the compact-open category.

However, from the viewpoint of differential geometry, the structure of a fiber space can be considered to be rigid under specific conditions, such as orientations [7,8]. Specifically, a Seifert fiber space is profinitely rigid if it is an oriented variety [8]. Otherwise, the Seifert fiber space is based on three-manifold, which is classified depending upon a set of invariants, and it incorporates rigidity of infinite π_1 with a hyperbolic two-orbifold base of fibration [9]. Similarly, the Haken orientable hyperbolic three-manifold is an irreducible variety, and it supports the rigidity of fibration [10].

1.2. Manifolds and Immersions

In view of geometric topology, the immersion of surfaces into the null submanifolds in 3D offers various interesting observations. If the structure (M, g) is a Lorentzian manifold under tensor h with $i : N \rightarrow M$ immersion function, then (N, h) is a degenerative null submanifold [11]. Interestingly, the Lorentzian manifolds are not metrizable, preserving isometry in reference to the space of immersion. It is known that the immersion of a hypersurface S into a Euclidean space with normal vector field V_N is self-adjoint in the presence of a suitable shape operator. In general, a Lorentz space can be finitely covered by a circle bundle if it is a compact space [12]. Note that topologically the spaces with Lorentzian geometry (such as a torus bundle) are locally Hausdorff, and the corresponding manifold is not a globally Hausdorff topological space. The immersion of a k -dimensional manifold in a $k + l$ -dimensional Euclidean space is given by $f_i : M_k \rightarrow E^{k+l}$ induces the one-to-one map in tangent space on the manifold [13]. The immersion space is considered to be a regular topological space, and the manifold is a connected topological space with orientation.

1.3. Motivation and Contributions

The fiber bundles are geometric as well as topological objects, which can be simulated and visualized in computer models. The computer visualization of fiber bundles as geometric objects has opened up a wide array of applications in various domains of physical sciences as well as computational sciences [14]. In the topological spaces of fiber bundles, the determination of equivalence between fiber bundles is a challenging task. The fiber bundle space reduction theorem indicates that two topological fiber bundle spaces are equivalent if and only if the corresponding Ehresmann bundles have cross-sections over common base space [15]. Interestingly, if we consider subspace $X_0 \subset X$ in the corresponding topological base space, then the cross-sections of an automorphic bundle within the subspace form an algebraic group structure. This paper proposes the construction and analysis of fiber space in the non-uniformly scalable multidimensional topological (C, R) space [16]. One of the interesting aspects of multidimensional topological (C, R) space is that the space is quasinormed, admitting cylindrically symmetric continuous functions, and does not always preserve compactness

under topological projections. Hence, the interesting and motivating questions are: what are the possible varieties of fibrations in such space, and is it possible to establish any algebraic structures within the respective fiber space? Moreover, what are the topological properties of the resulting fiber space within the quasinormed multidimensional (C, R) space if the holomorphic condition of complex subspace is relaxed? These questions are addressed in this paper. The presented analysis considers algebraic as well as topological standpoints as required. The elements of functional analysis are employed whenever necessary.

The main contributions made in this paper can be summarized as follows. The topological (C, R) space is a non-uniformly scalable and quasinormed space, where the cylindrical open sets form the topological basis. The proposed fibrations within the space can be constructed in two varieties, such as compact fibration and non-compact fibration. The fiber space is considered to be dense, and it can admit the concept of a special category of fibers called contact fibers. The fiber space is equipped with finite linear translation operation. The resulting fiber space in the topological (C, R) space supports the expansion and orientations of multiple singularities of a piecewise continuous function on contact fibers. It is shown that a composite algebraic operation comprised of linear translation, and arithmetic addition prepares an associative magma in the non-compact fiber space. The associative magma space is commutative under linear translation within the magma space of fibers, and it resists the formation of a monoid under composite algebraic operation. However, an additive group algebraic structure can be admitted in the fiber space and in projective base space under specific conditions. Interestingly, the proposed fiber space does not support outward deformation retraction in a dense subspace.

The rest of the paper is organized as follows. Section 2 presents a set of preliminary concepts enhancing the completeness of the paper. The definitions related to topological fiber spaces and fibrations are presented in Section 3. The algebraic and topological properties are presented in Section 4. Section 5 illustrates the concepts of expansion and singularities in the proposed topological fiber space. A detailed comparative analysis of this work with respect to other contemporary works in the domain is presented in Section 6. Finally, Section 7 concludes the paper.

2. Preliminary Concepts

Let $v = (x_0, x_1, \dots, x_{n-1})$ and $w = (y_0, y_1, \dots, y_{n-1})$ be two vectors in a n -dimensional vector space represented as $(V_n, +, \cdot)$. The Lorentzian inner product between v, w is given as $L(v, w) = -x_0y_0 + \sum_{i=1}^{n-1} x_iy_i$. The real vector space in R^n endowed with symmetric Lorentzian inner product with non-degenerate bi-linear form is called a n -dimensional Minkowski space M_n . In the Minkowski space, the linear operator $T : M_n \rightarrow M_n$ preserves $L(\cdot, \cdot)$ because it maintains the condition represented as $\forall v \in M_n, L(v, v) = L(T(v), T(v))$ generating a Lorentz group structure. The Euclidean e -topology in M_n is formulated by the topological basis elements given as a set of neighborhoods $B = \{N(x, \varepsilon) : x \in M_n, \varepsilon > 0\}$. Note that s -topology on M_n is strictly finer than the e -topology. Moreover, in the Minkowski space M_n , the geometry of the space cone in the Euclidean topology maintains $\sigma - e - open$ criteria [3].

The Steenrod formulation of fiber bundles is represented as an algebraic structure $F = \{B, X, p, Y, G, V_j, \varphi_j\}$, where B is the bundle space, X is a topological base space, $p : B \rightarrow X$ is a projection function, Y is a topological space called fiber, and G is a bundle group [15]. In the formulation, the elements V_j, φ_j are coordinate neighborhoods and the corresponding coordinate functions, respectively. The coordinate functions must satisfy a condition which is given by $\varphi_j : V_j \times Y \rightarrow p^{-1}(V_j)$. Note that a four-dimensional Minkowski space is essentially a real vector space preserving Hausdorff topological property [17].

3. Topological Fiber Space and Fibrations

In this section, a set of definitions related to fibrations and the resulting generation of fiber spaces in a quasinormed topological (C, R) space is formulated. A point x_p in the quasinormed topological (C, R) space (X, τ_X) is represented as $x_p = (z_p, r_p)$, and the origin of (X, τ_X) is denoted as $x_0 = (z_0, 0)$, where z_0 is the Gauss origin. The corresponding topological projections are given by $\pi_R : X \rightarrow R$ and $\pi_C : X \rightarrow C$ on the real subspace and complex subspace, respectively. For the simplicity of algebraic representation $\{z_p\} \times I \equiv (z_p, I)$ is used to denote a sectional subspace within the topological fiber space, where I is an interval (either open or closed). In this paper, A° and \bar{A} represent the interior and closure of an arbitrary set A such that $\bar{A} = A^\circ \cup \partial A$. Moreover, if A is homeomorphic to B then it is denoted as $\text{hom}(A, B)$. First, we define the projective base of a fiber space in the topological (C, R) space.

3.1. Projective Base in (C, R) Tangent Space

Let $A_m \in \tau_X$ be an open cylindrical subspace in the topological (C, R) space (X, τ_X) such that $A_m = D(z_m, \varepsilon > 0) \times I$, where $(z_m, r) \in C \times \{r\}$ and $I \subseteq R$ is open. Suppose we determine $\{\alpha, \varepsilon\} \subset R^+$ maintaining the ordering relation given by $0 < \alpha < \varepsilon$. The projection of corresponding tangent subspace onto topological base space at $r \in R$ is defined as:

$$\begin{aligned} \forall D(z_n, \alpha) \subset D(z_m, \varepsilon), B_n &= \partial D(z_n, \alpha) \times I, \\ \forall z_p \in \partial D(z_n, \alpha), \forall r \in R, \pi_r : B_n &\rightarrow (\partial D(z_n, \alpha) \times \{r\}), \\ \pi_r((z_p, I)) &= (z_p, r). \end{aligned} \quad (1)$$

Note that the projection $\pi_r(\cdot)$ is surjective in nature onto the dense base space $D(z_n, \varepsilon) \times \{r\}$, where the $\text{hom}(\partial D(z_n, \alpha), S^1)$ condition is maintained for every $z_n \in C$. Moreover, it preserves the standard notion of projection in the tangent space of a topological manifold (M, τ_M) . Next, the definition of fiber bundles in the tangent space considering a base space is formulated.

3.2. Topological Fiber Bundle in (C, R) Space

As stated earlier, let $A_m = D(z_m, \varepsilon) \times I$ be a subspace in the topological (C, R) space for $\varepsilon > 0$. The topological fiber bundle in the corresponding tangent space $B_m = \partial D(z_m, \varepsilon) \times I$ is given by:

$$\begin{aligned} F_m &= \{(z_p, I) : z_p \in \partial D(z_m, \varepsilon)\}, \\ \forall E_p \subseteq \partial D(z_m, \varepsilon) \times \{r_p\}, \sigma : E_p &\rightarrow F_m, \\ (\pi_r \circ \sigma)(x_p \in E_p) &= (z_p, r). \end{aligned} \quad (2)$$

A fiber $(\sigma \circ \pi_r)((z_p, I)) = \mu_{p \times I} \equiv (z_p, I)$ where $(z_p, I) \in F_m$ is called a compact fiber, if and only if $I \subset R \setminus \{-\infty, +\infty\}$ and $I = I^\circ \cup \partial I$. Otherwise, it is a result of non-compact fibration in the topological (C, R) space.

Remark 1. The topological fiber bundle F_m in (C, R) space is more rigid as compared to the fiber bundle in (M, τ_M) because F_m preserves the σ -sections within the fiber bundles in the corresponding tangent (C, R) space, where the topological base space is a dense set. The σ -sections in the topological manifold (M, τ_M) may not be preserved in every case within the fiber bundle of (M, τ_M) . The reason is that a set of σ -sections of (M, τ_M) is constructed separately independent of projections onto topological base space.

3.3. Fiber Space

The fiber space in a topological (C, R) space (X, τ_X) is formulated based on the sets of fiber bundles in the dense subspaces. Let us consider $A_m = D(z_m, \varepsilon) \times I$ be a dense subspace in topological (C, R) space for $\varepsilon > 0$. The corresponding fiber space Γ_m in A_m considering $0 \leq \varepsilon < k < +\infty$ is defined as:

$$\Gamma_m = \bigcup_{\forall z_a \in D(z_m, \varepsilon)} F_a = \bigcup_{\forall \varepsilon \in [0, k]} F_m. \quad (3)$$

In the above definition F_a is a local fiber bundle of corresponding topological subspace $B_a = \partial D(z_a, \varepsilon) \times I$ in (X, τ_X) . The entire fiber space of topological (C, R) space can be generated from the sets of local fiber bundles as given below:

$$\Gamma_{CR} = \bigcup_{\forall z_m \in C} \Gamma_m. \quad (4)$$

Note that at $\varepsilon = 0$ in the topological subspace, A_m the subspace $\mu_m \times I$ is also a fiber at $z_m \in C$. Earlier it is mentioned that there are two varieties of fibrations, which can be admitted into the topological (C, R) space. However, it is interesting to observe that in any case, the fiber space in (X, τ_X) is not compact, which is presented in the following proposition.

Proposition 1. *Every fiber space Γ_m in (X, τ_X) is not compact.*

Proof. Let (X, τ_X) be a topological (C, R) space and $F_a \subset \Gamma_m$ be a topological fiber bundle in the topological subspace $A_a \subset A_m$. The topological projection $\pi_R : X \rightarrow R$ prepares a real subspace in (X, τ_X) . We will prove the proposition by considering two separate cases. In the first case, suppose $I \subset R \setminus \{-\infty, +\infty\}$ such that $I = I^0 \cup \partial I$ preparing a compact fibration in the topological space. However, recall that as $A_a = D(z_a, \varepsilon > 0) \times I$, thus, the projection $\pi_C(v \in A_a) \subset C$ is an open subspace. Otherwise, if $I = R$ in A_a then the fibration $F_a \subset \Gamma_m$ is a non-compact variety. As a result, the projective subspace $\pi_R(v \in A_a) = R$ is an open set. Hence, in every case $F_a \subset \Gamma_m$ is not compact and as a result $\Gamma_m = \bigcup_{\forall z_a \in D(z_m, \varepsilon)} F_a$ is also not compact in (X, τ_X) . \square

Remark 2. *The above proposition further indicates that the fiber space Γ_{CR} in (X, τ_X) is not a compact space.*

In general, the Seifert fiber space on an oriented manifold maintains profinite rigidity [8]. However, in the case of a holomorphic map $f : M \rightarrow N$ between two complex manifolds M and N , the rigidity of fibration allows sending a single fiber to another fiber, as long as the complex manifolds, are individually connected spaces [18]. In this paper, the topological (C, R) space is a connected space, and the associated fiber space supports finite translation of fibers within the space.

3.4. Translation in Fiber Space

Let $\Gamma_m \subset \Gamma_{CR}$ be a fiber subspace in the topological (C, R) space (X, τ_X) . The topological translation of a fiber subspace in Γ_m is defined as:

$$\begin{aligned} \exists \Gamma_p : \Gamma_m \subset \Gamma_p \subseteq \Gamma_{CR}, \\ T : \Gamma_{CR} \rightarrow \Gamma_{CR}, \\ T(\mu_a \times I) = \mu_{Ta} \times I = T(z_a, I), \\ \forall \mu_a \times I \in \Gamma_m, \mu_{Ta} \times I \in \Gamma_p. \end{aligned} \quad (5)$$

Remark 3. Note that, the translation $T : \Gamma_{CR} \rightarrow \Gamma_{CR}$ is continuous in C under projection $\pi_C : X \rightarrow C$ within the open subspaces of fibers. However, the translation can be transformed into a strictly closed and convergent variety if the following restrictions are imposed on it: $\Gamma_p = \Gamma_p^o \cup \partial\Gamma_p$ and $\Gamma_p \subset \Gamma_{CR}$.

The properties of finite translations within the topological fiber space depend on the Baire categorization of a subspace (i.e., dense or meager) and compactness of the corresponding projective subspace. The following proposition presents such observation.

Proposition 2. If $\Gamma_p \subset \Gamma_{CR}$ is dense and the topological projection on subspace $\pi_R \circ \sigma(x_m \in X)$ is not compact, then $T : (\Gamma_m \subset \Gamma_{CR}) \rightarrow (\Gamma_p \subset \Gamma_{CR})$ is a continuous and finite fiber space translation.

Proof. Let Γ_m, Γ_p be two fiber subspaces in (X, τ_X) such that the translation is given as $T : (\Gamma_m \subset \Gamma_{CR}) \rightarrow (\Gamma_p \subset \Gamma_{CR})$. Note that $\Gamma_m^o \subset \overline{\Gamma_p}$ in the topological space and as a result $T : \Gamma_{CR} \rightarrow \Gamma_{CR}$ is continuous in all monotone classes of projective subspaces $\pi_C(\Gamma_m) \subset \pi_C(\Gamma_p)$ in (X, τ_X) . If the topological projection $\pi_R \circ \sigma(x_m \in X)$ is not compact, then the corresponding fibration is a non-compact variety resulting in the generation of fiber $\mu_{m \times I} \equiv (z_m, I = R)$. Suppose fibration is non-compact everywhere in the topological space (X, τ_X) . Thus $\forall \mu_{a \times I} \in \Gamma_m, T(\mu_{a \times I}) = \mu_{T a \times I}$ is maintained in the subspaces such that $\mu_{T a \times I} \in \Gamma_p$ and $\mu_{T a \times I} = T(z_a, I)$, where $T(z_a) \in C$ and $T(I = R) = I$. If $\Gamma_p \subset \Gamma_{CR}$ is dense, then $\overline{\Gamma_p} \subset \Gamma_{CR}$ is a compact subspace of fibers. As a result, it is true that $\forall \mu_{a \times I} \in \Gamma_m, 0 \leq \|\pi_C(\mu_{T a \times I}) - z_a\|_2 < +\infty$ considering the inclusion of translation invariant fiber, which is given by $\exists \mu_{a \times I} \in \Gamma_m, T(\mu_{a \times I}) = \mu_{a \times I}$. Hence, the continuous translation of fiber subspace $T : \Gamma_{CR} \rightarrow \Gamma_{CR}$ in (X, τ_X) is finite if $\Gamma_p \subset \Gamma_{CR}$ is dense and the topological projection $\pi_R \circ \sigma(x_m \in X)$ on real subspace is not compact. \square

The proposed fibration in a topological (C, R) space is equipped with a special category of fiber called contact category, as defined below. A contact category fiber admits multiple oriented singularities of a function within the fibered topological space under specific conditions.

3.5. Contact Category Fiber

Let $\Gamma_m \subset \Gamma_{CR}, \Gamma_p \subset \Gamma_{CR}$ be two fiber subspaces in the topological (C, R) space (X, τ_X) . A fiber $\mu_{s \times I}$ is defined to be in the contact fiber category if the following conditions are maintained by it:

$$\begin{aligned} \Gamma_m \cap \Gamma_p &= \{\mu_{s \times I}\}, \\ \forall r \in R, \pi_R \circ \sigma((z_s, r)) &= R. \end{aligned} \quad (6)$$

Note that the restriction for a contact fiber to be maintained is given as, $z_s \in C \setminus \{-\infty, +\infty\}$. This restriction is required to retain the finiteness of projection of fiber space into the topological base space of the contact fiber category. Once the contact fiber is defined, we can formulate the oriented singularities of a function in the topological space.

3.6. Oriented Singularities of Function

Let $f : [a \in R, b \in R] \rightarrow (X, \tau_X)$ be a continuous function in the topological (C, R) space. The function $f(\cdot)$ is defined to have positively oriented singularity in the space if the following conditions are maintained by it on a fiber.

$$\begin{aligned} I &\subseteq R \setminus \{-\infty, +\infty\}, \exists \mu_{f \times I} \subset X, \\ \exists c \in (a, b) : \forall x_{f_r} \in f([a, b] \setminus \{c\}), x_{f_r} &= (z_f, r \in I), \\ \lim_{t \rightarrow c} f(t) &= (z_f, r), r \rightarrow +\infty. \end{aligned} \quad (7)$$

Remark 4. It is straightforward to verify that the negatively oriented singularity of a continuous function can be defined by following a similar concept, as stated above, with a change in orientation. Moreover, a planar and symmetric variety of the function with multiple oriented singularities can be formulated if the following restrictions are maintained by it:

$$\begin{aligned}
 \Gamma_m \cap \Gamma_p &= \{\mu_{s \times I}\}, \\
 D(z_m, \varepsilon) \cap D(z_p, \varepsilon) &= \{z_s\}, \exists r \in R \setminus \{-\infty, +\infty\}, \\
 f([a, c]) \in D(z_m, \varepsilon) \times \{r\}, f((c, b]) &\in D(z_p, \varepsilon) \times \{r\}, \\
 \lim_{t \rightarrow c^+} f(t) &= (z_s, r), r \rightarrow +\infty, \\
 \lim_{t \rightarrow c^-} f(t) &= (z_s, r), r \rightarrow -\infty.
 \end{aligned}
 \tag{8}$$

It is important to note that a planar and symmetric function with multiple oriented singularities in the fiber space of topological (C, R) space is a discontinuous variety (i.e., the function is piecewise continuous). Furthermore, the fiber admitting multiple singularities of a planar as well as symmetric function is in the contact fiber category, and the corresponding fibration is non-compact type.

4. Algebraic and Topological Properties

In this section, the algebraic and topological properties of fiber space in the quasinormed topological (C, R) space are presented. First, we show that the fiber space Γ_{CR} is an associative magma under the algebraic composite operation $(+T)$ as given below:

$$\begin{aligned}
 \forall \mu_{a \times I}, \mu_{b \times I} \in \Gamma_{CR}, \\
 \mu_{a \times I} (+T) \mu_{b \times I} &= \mu_{(a+Tb) \times I}
 \end{aligned}
 \tag{9}$$

The above representation indicates that $(+T)$ is a composite algebraic operation, which is comprised of functional translation and arithmetic addition within the topological space. Note that the linearity of translation operation is considered to be maintained for generality. It is interesting to observe that the algebraic structure $(\Gamma_{CR}, (+T))$ constructs an associative magma in the topological space (X, τ_X) if and only if the translation is invariant to first-order and the fibration is non-compact type.

Theorem 1. The non-compact fiber space $(\Gamma_{CR}, (+T))$ is an associative magma if and only if $T : \Gamma_{CR} \rightarrow \Gamma_{CR}$ is linear and $\forall n \geq 2, T^n = T, n \in Z^+$.

Proof. Let $(\Gamma_{CR}, (+T))$ be a fiber space associated with the algebraic operation in the topological space (X, τ_X) . If $T : \Gamma_{CR} \rightarrow \Gamma_{CR}$ is linear and the fibration is non-compact type, then a σ - section preserves the following condition involving corresponding topological base space, $\forall \mu_{a \times I} \in \Gamma_{CR}, (\pi_R \circ \sigma)(x_a \in X) = R$. Moreover, due to the linearity of translation of fibers in the space it can be verified that $\forall \{k_1, k_2\} \subset R, T(k_1 \mu_{a \times I} + k_2 \mu_{b \times I}) = k_1 T(\mu_{a \times I}) + k_2 T(\mu_{b \times I})$, where $\{\mu_{a \times I}, \mu_{b \times I}\} \subset \Gamma_{CR}$. Recall that $\forall \mu_{a \times I} \in \Gamma_{CR}, T(\mu_{a \times I}) = \mu_{Ta \times I}$ condition is maintained in the corresponding fiber space. Thus we can derive the following equations considering the algebraic operation $(+T)$:

$$\begin{aligned}
 \{\mu_{a \times I}, \mu_{b \times I}, \mu_{c \times I}\} &\subset \Gamma_{CR}, \\
 (\mu_{a \times I} (+T) \mu_{b \times I}) (+T) \mu_{c \times I} &= \mu_{(a+Tb+Tc) \times I} \in \Gamma_{CR}, \\
 \mu_{a \times I} (+T) (\mu_{b \times I} (+T) \mu_{c \times I}) &= \mu_{(a+T(b+Tc)) \times I} \in \Gamma_{CR}.
 \end{aligned}
 \tag{10}$$

This indicates that the fiber space is associative if $T^2 = T$ in $(\Gamma_{CR}, (+T))$. Moreover, if $T^2 = T$ then $\forall n \in Z^+, T^n = T^{n-2}(T^2) = T^{n-1} = \dots = T$ and it is closed in $(\Gamma_{CR}, (+T))$. Hence, the fiber space $(\Gamma_{CR}, (+T))$ is an associative magma under the non-compact fibration in the topological space. \square

Corollary 1. *It is important to note that the fiber space $(\Gamma_{CR}, (+T))$ is not a monoid as no fixed fiber (i.e., original fiber) can be identified within the space. This indicates that it is a relatively rigid structure because even if the invariance condition of translation is further relaxed at the origin of (X, τ_X) so that $\exists x_0 \in X, T(\mu_{0 \times I}) = \mu_{0 \times I} = T(z_0, I)$ then also the algebraic structure $(\Gamma_{CR}, (+T))$ is not transformed into a monoid, where z_0 is Gauss origin. The main reason is that the algebraic operation $(+T)$ is not commutative in nature.*

Interestingly, although the fiber space $(\Gamma_{CR}, (+T))$ is an associative magma under non-commutative algebraic operation $(+T)$; however, an additional linear translation of the magma of fiber space is a commutative space. This observation is presented in Theorem 2.

Theorem 2. *In $(\Gamma_{CR}, (+T))$ associative magma of fiber space, the translation $T : (\Gamma_{CR}, (+T)) \rightarrow (\Gamma_{CR}, (+T))$ admits commutativity in magma under T .*

Proof. Let $\{\mu_{a \times I}, \mu_{b \times I}\} \subset \Gamma_{CR}$ be a subset of fibers in associative magma $(\Gamma_{CR}, (+T))$. According to the principle of commutativity, the following equations can be derived in the magma of fiber space:

$$\begin{aligned}\mu_{a \times I} (+T) \mu_{b \times I} &= \mu_{(a+Tb) \times I}, \\ \mu_{b \times I} (+T) \mu_{a \times I} &= \mu_{(b+Ta) \times I}.\end{aligned}\quad (11)$$

Thus, further translation in the associative magma of fiber space leads to the following conclusions considering $T^2 = T$.

$$\begin{aligned}T(\mu_{a \times I} (+T) \mu_{b \times I}) &= \mu_{(Ta+Tb) \times I}, \\ T(\mu_{b \times I} (+T) \mu_{a \times I}) &= \mu_{(Tb+Ta) \times I}.\end{aligned}\quad (12)$$

However, the algebraic structure $(C, +)$ is commutative in nature, indicating that $T(z_b) + T(z_a) = T(z_a) + T(z_b)$. This leads to the following algebraic identity:

$$T(\mu_{a \times I} (+T) \mu_{b \times I}) = T(\mu_{b \times I} (+T) \mu_{a \times I}). \quad (13)$$

Hence, the function $T : (\Gamma_{CR}, (+T)) \rightarrow (\Gamma_{CR}, (+T))$ admits commutativity within the associative magma of fiber space in (X, τ_X) . \square

From the algebraic standpoint, the above theorem illustrates that the linear translation in the associative magma of fiber space admits commutativity within the space. However, the space does not support a group structure under such given conditions. On the other hand, if the fibration is a non-compact variety in topological (C, R) space, then the algebraic structure $(\Gamma_{CR}, +)$ in the respective fiber space attains a group structure as presented in the next theorem.

Theorem 3. *The structure $(\Gamma_{CR}, +)$ is a fiber group in (X, τ_X) if and only if $\forall \mu_{a \times I} \in \Gamma_{CR}, \forall r_a \in R$ the fibration maintains $(\pi_R \circ \sigma)(x_a \in X) = R$.*

Proof. Let Γ_{CR} be a fiber space in the topological (C, R) space (X, τ_X) . Suppose we consider a subspace $M_\mu = \{\mu_{a \times I}, \mu_{b \times I}, \mu_{c \times I}\} \subset \Gamma_{CR}$ such that the projection on base space is $M_v = \{\pi_r(v \in M_\mu)\}$ and the corresponding topological projection maintains $(\pi_R \circ \sigma)(y \in M_v) = R$. This leads to the following result maintaining commutativity under non-compact fibration.

$$\begin{aligned}I &= R, z_a + z_b = z_{a+b} \in C, \\ \mu_{a \times I} + \mu_{b \times I} &= \mu_{(a+b) \times I} = (z_{a+b}, I), \\ (\mu_{a \times I} + \mu_{b \times I}) + \mu_{c \times I} &= \mu_{a \times I} + (\mu_{b \times I} + \mu_{c \times I}).\end{aligned}\quad (14)$$

Moreover, at the origin x_0 of (X, τ_X) the fiber $\mu_{0 \times I}$ is an identity fiber (i.e., original fiber) because, $\forall \mu_{a \times I} \in \Gamma_{CR}, \mu_{a \times I} + \mu_{0 \times I} = \mu_{0 \times I} + \mu_{a \times I} = \mu_{a \times I}$. Furthermore, if we consider fibers such that

$\forall \mu_{a \times I} \in \Gamma_{CR}, \exists \mu_{-a \times I} \in \Gamma_{CR}$ maintaining $\mu_{a \times I} + \mu_{-a \times I} = \mu_{-a \times I} + \mu_{a \times I} = \mu_{0 \times I}$ then $\mu_{-a \times I}$ is the inverse of $\mu_{a \times I}$. Hence, the algebraic structure $(\Gamma_{CR}, +)$ is a fiber group if the fibration is a non-compact variety. \square

Lemma 1. *In the topological (C, R) space (X, τ_X) , the algebraic structure $(\pi_r(\Gamma_{CR}), +)$ is a group, if and only if $r = 0$.*

Proof. The proof is relatively straightforward. Suppose we select an arbitrary $r \in R \setminus \{0\}$ to prepare a topological base space for projection. In this case, the $\{x_0\} \cap \pi_r(\Gamma_{CR}) = \phi$ condition is satisfied within the projective space of fibers. This leads to the conclusion that $\forall z_a, z_b \in (\pi_C \circ \pi_r)(\Gamma_{CR}), z_a + z_b \neq z_a \neq z_b$ if $z_a \neq z_b \neq z_0$ and in addition the following condition is maintained, $\{(\pi_R \circ \pi_r)(v \in \Gamma_{CR})\} \cap \{\pi_R(x_0)\} = \phi$. Thus, the algebraic structure $(\pi_r(\Gamma_{CR}), +)$ is not a group because $\forall x_a, x_b \in \pi_r(\Gamma_{CR}), x_a + x_b \neq x_a \neq x_b$ indicating that the additive identity element does not exist in $(\pi_r(\Gamma_{CR}), +)$ if $r \neq 0$. Hence, the algebraic structure $(\pi_r(\Gamma_{CR}), +)$ is a group, if and only if $r = 0$. \square

Remark 5. *If $\psi : \Gamma_{CR} \rightarrow \Gamma_{CR}$ is a function in fiber space such that $\psi(\mu_{a \times I}) = \mu_{a \times \psi(I)}$ with $(\pi_R \circ \sigma)(x_a \in X) = R$, then it can be observed that $\psi(\mu_{a \times I}) \cong \mu_{a \times \psi(I)}$, if and only if $\mu_{a \times \psi(I)} \cong \mu_{a \times I}$. Hence, the function $\psi(\cdot)$ is a 2π -radian rotation of a fiber maintaining symmetry property. Moreover, it preserves the identity in a σ -section as $(\psi \circ (\sigma \circ \pi_r)) = (\sigma \circ \pi_r)$ due to symmetry.*

Theorem 4. *In the fiber space Γ_{CR} , there is no deformation retract $\eta_D : D(z_m, \varepsilon) \times I \rightarrow \partial D(z_m, \varepsilon) \times I$, where $\varepsilon > 0, \varepsilon \in R^+$.*

Proof. Let Γ_{CR} be a fiber space in the topological (C, R) space (X, τ_X) . If we consider $\varepsilon > 0, \varepsilon \in R^+$ then $D(z_m, \varepsilon) \times I$ is an open subspace in (X, τ_X) . Note that in the corresponding subspace $\text{hom}(\partial D(z_m, \varepsilon), S^1)$ condition is maintained. According to the deformation retraction theorem, there can be no retraction from convex $D(z_m, \varepsilon)$ to $\partial D(z_m, \varepsilon)$ maintaining continuity of retraction function within the topological space. As a result, there can be no deformation retract function $\eta_D : D(z_m, \varepsilon) \times I \rightarrow \partial D(z_m, \varepsilon) \times I$ admissible within (X, τ_X) . \square

Remark 6. *Although $\eta_D : D(z_m, \varepsilon) \times I \rightarrow \partial D(z_m, \varepsilon) \times I$ outward deformation retract is not admissible in (X, τ_X) ; however, the inward deformation retract given by $\eta_E : D(z_m, \varepsilon) \times I \rightarrow D(z_m, \alpha) \times I$ can be admitted, if and only if $\alpha < \varepsilon$ and $\alpha > 0, \alpha \in R^+$.*

5. Expansion and Singularity

The fiber space Γ_{CR} in a quasinormed topological (C, R) space (X, τ_X) supports the continuous and uniform expansion of a subspace Γ_m under the translation function $T : \Gamma_{CR} \rightarrow \Gamma_{CR}$. First, we define the uniformity of T in Γ_{CR} as given below, considering the Euler representation of a point $z_m \in C$:

$$\begin{aligned} z_m &= \alpha_m e^{i\theta_m}, \\ \forall z_m \in C, \mu_{Tm \times I} &= (\alpha_m^T e^{i\theta_m}, I). \end{aligned} \quad (15)$$

If we equip the fiber subspace $\Gamma_m \subset \Gamma_{CR}$ with uniform $T : \Gamma_{CR} \rightarrow \Gamma_{CR}$ then the algebraic structure (Γ_m, T) admits continuous and uniform expansion in (X, τ_X) , within the monotone class of fiber space. In other words, the fiber subspace $\Gamma_m \subset \Gamma_{CR}$ admits continuous and uniform expansion within the fiber space if it maintains the following properties:

$$\begin{aligned} \forall \Gamma_m \subset \Gamma_{CR}, \exists \Gamma_{Tm} \subset \Gamma_{CR}, \Gamma_m \subset \Gamma_{Tm}, \\ \forall \mu_{m \times I} \in \Gamma_m, n > 1, T^n(\mu_{m \times I}) \in \Gamma_{Tm}, \\ T^n \neq T. \end{aligned} \quad (16)$$

Note that it indicates Γ_{T_m} is dense in Γ_{CR} because $\Gamma_{T_m} \cup \partial\Gamma_{T_m} = \overline{\Gamma_{T_m}}$ within (X, τ_X) . Moreover, the continuous and uniform expansion of a fiber subspace is finite within a compact subspace. However, this restriction may not be valid in the entire fiber space Γ_{CR} within (X, τ_X) , where $X \in \tau_X$ is open.

The contact category fiber in (X, τ_X) admits multiple singularities of a function $f : [0, 1] \rightarrow (X, \tau_X)$. In this case, the function is piecewise continuous in (X, τ_X) . Let $\Gamma_m, \Gamma_p \subset \Gamma_{CR}$ be two fiber subspaces such that $\Gamma_m \cap \Gamma_p = \{\mu_{s \times I}\}$ is a fiber in the contact fiber category. Suppose $a \in (0, 1)$ such that $\forall x \in [0, a], f(x) \in D(z_m, \varepsilon) \times I$ and $\forall x \in (a, 1], f(x) \in D(z_p, \varepsilon) \times I$ for some $\varepsilon > 0$. Consider that fibration is a non-compact variety, and as a result the $(\pi_R \circ \sigma)(f(x)) = R$ condition is maintained under projection. If $\lim_{x \rightarrow a^+} f(x) \rightarrow (z_s, +\infty)$ and $\lim_{x \rightarrow a^-} f(x) \rightarrow (z_s, -\infty)$ conditions are maintained by the function, then $f : [0, 1] \rightarrow (\Gamma_m \cup \Gamma_p)$ admits multiple singularities in the fiber space. Furthermore, the proposed structure does not exclude the possibility of the existence of oriented singularity within the space under the additional requirement. If $\lim_{x \rightarrow a^-} f(x) \cong \lim_{x \rightarrow a^+} f(x)$ equivalence relation is maintained by the limiting values of the function, then $\lim_{x \rightarrow a^\pm} f(x) \rightarrow (z_s, +\infty)$ is a strictly positively oriented singularity of the function on the respective contact fiber in (X, τ_X) .

6. Comparative Analysis

There are varieties in the construction of fiber bundles and associated singularities based on various parameters, such as the connectedness of topological spaces, locality of homeomorphism, and different types of projection maps. The comparison of various properties associated with different varieties of fiber spaces is summarized in Table 1.

Table 1. Summary of comparative analysis of various spaces and fibrations.

Space/Fibration	Geometric Property	Topological Decomposed Subspaces	Local Compactness	Group Structure
Seifert fiber space	three-Manifold	S^1 —fiber space	Yes	Fundamental group with unique normal subgroup
Minkowski space	four-Manifold	3D real, 1D real spaces	Yes	Lorentz group in fiber sub-bundle
Complex fiber bundle	Holomorphic	nD normal complex space ($n \geq 2$)	Yes	Lie group
(C, R) space	Quasinormed	2D complex, 1D real spaces	Yes	Two additive group varieties in fiber space

In the case of Seifert fiber space, the fiber bundles are S^1 —bundles of a three-manifold M_3 [7,8]. The Seifert fiber space admits local compactness along with the fundamental group $\pi_1(M_3)$. The fundamental group $\pi_1(M_3)$ includes a unique normal subgroup structure. However, the fiber space of topological (C, R) space is a set of elements given by $\{\mu_{a \times I} : (z_a \in C, I \subseteq R) \subset X\}$. As a result, the fiber space in topological (C, R) space admits oriented singularity and two respective varieties of fibrations. Moreover, unlike Seifert space, the topological (C, R) space is a quasinormed space with non-uniform scaling of points in space. The fiber space of topological (C, R) space forms an additive group algebraic structure under certain conditions. Furthermore, the composite algebraic operation comprised of arithmetic addition and functional translation forms an associative magma in the fiber space of topological (C, R) space.

The generalization of singular fibration considers that the topological space (X, τ_X) is connected in nature [19]. Specifically, if $A_1 \cup A_2 = X$ then the singular fibration in (X, τ_X) maintains the condition that $A_1 \cap A_2 \neq \phi$. Moreover, the function $h : (B \subset A_1) \rightarrow A_2$ is a homeomorphism, and for each component, $E \subset A_2 \setminus h(B)$, the projection map is given by $h \setminus h^{-1}(E)$ for a fiber bundle, where the base space is $E \subset X$ [19]. The similarity of singularities between the topological (C, R) space and the generalized singular fibration is that in both cases, the topological space is a connected type. However, the main distinction between the two constructions is that the function of admitting multiple singularities in a topological (C, R) space does not require the existence of local homeomorphism within the topological

space. There are varieties of constructions to preserve compactness. In the case of a generalized singular fibration, the functions $f : A_1 \rightarrow A_2$ and $f^{-1} : A_2 \rightarrow A_1$ maintain compactness [19]. On the contrary, the compactness in a topological (C, R) space is not always preserved. For example, in the topological (C, R) space, $\pi_C(B \subset X)$ is compactible but $\pi_R(B \subset X)$ is not compactible in nature.

The fibration and fiber bundles in Minkowski space consider four-manifold (M_4) structure [20,21]. Interestingly the fibration in Minkowski space is possible if the space is Hausdorff and connected topological space, which is locally Euclidean on M_4 . The fibration in topological (C, R) space considers similar topological properties (i.e., space is connected T_2 space). However, the fibration in Minkowski space is *four-dimensional*, whereas the fibration in topological (C, R) space is three dimensional in nature. Furthermore, the topological (C, R) space contains a complex subspace under projection, and the Minkowski M_4 is a completely real manifold. Note that the Minkowski space and topological (C, R) space both support the formation of sub-bundles within the respective fiber spaces. The fiber space of topological (C, R) space admits two varieties of additive group algebraic structures under projections. However, the Minkowski fiber space admits a sub-bundle with Lorentz group structure [20]. Interestingly, the holomorphic and principal fiber bundle of a compact complex manifold supports a complex Lie group structure [22]. The topological decomposition indicates that Minkowski space can be completely decomposed into two components, such as 3D real Euclidean subspace and 1D real Euclidean subspace [21]. On the contrary, the topological (C, R) space is a quasinormed space supporting non-uniform scaling, and it can be topologically decomposed into two components, such as 2D complex subspace and 1D real subspace.

7. Conclusions

A fiber space is a topological space where it locally behaves as a product space, but globally the space has a different topological structure. The analytical properties of fibrations and resulting fiber spaces vary depending upon the constructions as well as the structures of topological spaces. The multidimensional topological (C, R) space is a non-uniformly scaled quasinormed space admitting a topological group under composite algebraic operations. The fibrations in multidimensional topological (C, R) space have two varieties in view of compactness. The non-compact fiber space forms associative magma under composite algebraic operation comprised of linear translation and addition within the complex subspace. In this case, the translation is invariant to first order. However, it resists the formation of a monoid structure under composite algebraic operation, and the magma space supports commutativity under linear translation operation within the space of fibers. The fiber space forms an additive group structure if the fibration is a non-compact variety. The projective base space admits an additive group structure if and only if the base space contains the origin of real subspace. The fiber subspace is dense, and it does not allow outward deformation retraction. The fiber space of multidimensional topological (C, R) space allows expansion under uniform translation and preserves multiple oriented singularities of a piecewise continuous function on contact fibers. The structures of Minkowski space and multidimensional topological (C, R) space have different dimensions and partial similarities. The comparative analysis of fiber spaces of multidimensional topological (C, R) space, Minkowski space, and Seifert fiber space illustrates that the supporting group algebraic structures are different for each space. Moreover, the topologically decomposed subspaces have different dimensions as well as properties in each space. Finally, this is to note that the proposed constructions of topological fibrations may find applications in mathematical sciences (i.e., manifold immersion and surface classifications) and in physical sciences (i.e., supersymmetry, topological string theory, cosmology, and analyzing symmetry fibration in biological networks).

Author Contributions: The author (S.B., Department of Aerospace and Software Engineering (Informatics), Gyeongsang National University, Jinju, ROK) is the sole and single author of this paper, and the paper contains his own contributions. The author has read and agreed to the published version of the manuscript.

Funding: The research is funded by Gyeongsang National University, Jinju, Korea.

Acknowledgments: The author would like to thank anonymous reviewers and editors for their valuable comments and suggestions.

Conflicts of Interest: The author declares no conflict of interest.

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