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A mixed variational formulation of dynamic viscoelastic problems with adhesion and friction

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Dedicated to the memory of Professor Nicolaie Dan Cristescu

Keywords

Dynamic viscoelastic problems, pointwise contact conditions, adhesion, slip depending friction, mixed variational formulation.

Abstract

The aim of this work is to study a class of dynamic contact problems coupling adhesion and friction between two viscoelastic bodies with nonlinear viscosity operators. The boundary conditions concern relaxed unilateral contact, pointwise friction and adhesion including possible recoverable behavior. A mixed variational formulation of these problems is given as a five-field evolution implicit equation coupled with a differential inclusion describing the evolution of the intensity of adhesion. Based on several estimates and a classical fixed point theorem for multivalued functions, the existence of a strong variational solution is proved.

1 Introduction

This paper deals with the analysis of a class of nonsmooth dynamic contact problems which describe various surface interactions between two viscoelastic bodies with nonlinear viscosity operators. These interactions include some relaxed unilateral contact, friction with slip depending coefficient of friction, and complex adhesion conditions.

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The quasistatic elastic problems with unilateral contact and local Coulomb friction have been studied in [1, 32, 33] and extensions by taking into account the adhesion, described by the intensity of adhesion introduced in [18, 19], were investigated in [31, 12], see also [34] and references therein.

Dynamic frictional contact problems with normal compliance laws have been studied in [24, 20, 4] and (non)local friction laws were considered in [21, 22, 16, 6, 13], for viscoelastic bodies. Dynamic frictionless problems with adhesion have been studied in [5, 23, 36] and dynamic viscoelastic problems coupling unilateral contact, recoverable adhesion and nonlocal friction have been analyzed in [14, 7].

Using the hemivariational inequalities theory, several nonsmooth quasistatic and dynamic contact problems were investigated, see [26, 27, 28, 29] and references therein. An elastic contact problem with relaxed unilateral conditions and pointwise Coulomb friction in the static case was studied in [30] and the extension to an elastic quasistatic contact problem was investigated in [10]. The corresponding viscoelastic dynamic case was analyzed in [7, 8, 9, 11] for different contact conditions.

Based on a new mixed variational formulation, given as a five-field evolution implicit equation coupled with a differential inclusion that describes the evolution of the intensity of adhesion, this work generalizes and extends the results presented in [7, 8, 9].

The results presented in this paper extend some of those proved in [11] in the more general case of rate-depending contact interactions, but with a limited adhesion (there is no intensity of adhesion), a linear viscosity operator and under stronger regularity assumptions.

The approach described in this paper enables to consider more general constitutive laws, as, for example, the ones characterizing some elastoviscoplastic materials investigated in [15].

The paper is organized as follows. In Section 2 the classical formulation of the dynamic contact problem is presented. In Section 3 two mixed variational formulations and some auxiliary results are given. Section 4 is devoted to the existence of a strong variational solution which is proved for an equivalent fixed point problem for a multi-valued function.

2 Classical formulation

We consider two viscoelastic bodies that occupy the reference domains Ω^α of the three-dimensional Euclidean point space \mathcal{E} with Lipschitz boundaries denoted by $\Gamma^\alpha = \partial\Omega^\alpha$, $\alpha = 1, 2$. Let Γ_U^α , Γ_F^α and Γ_C^α denote three open disjoint sufficiently smooth parts of Γ^α such that $\Gamma^\alpha = \bar{\Gamma}_U^\alpha \cup \bar{\Gamma}_F^\alpha \cup \bar{\Gamma}_C^\alpha$ and, to

simplify the estimates, $\text{meas}(\Gamma_U^\alpha) > 0$, $\alpha = 1, 2$.

We assume the small deformation hypothesis and we use Cartesian coordinates representations with the summation convention for $i, j, k, l = 1, 2, 3$. To simplify the presentation of the functional framework, we shall continue to use the vector and tensor notations for some inclusions.

Let $\mathbf{y}^\alpha(\mathbf{x}^\alpha, t)$ denote the position at time $t \in [0, T]$, where $0 < T < +\infty$, of the material point \mathbf{x}^α represented by the coordinates $x^\alpha = (x_1^\alpha, x_2^\alpha, x_3^\alpha)$ in the reference configuration Ω^α , and $\mathbf{u}^\alpha(\mathbf{x}^\alpha, t) = \mathbf{y}^\alpha(\mathbf{x}^\alpha, t) - \mathbf{x}^\alpha$ denote the displacement vector of \mathbf{x}^α at time t , with the components $u^\alpha = (u_1^\alpha, u_2^\alpha, u_3^\alpha)$.

Let $\boldsymbol{\varepsilon}^\alpha$, with the components $\varepsilon^\alpha = (\varepsilon_{ij}(u^\alpha))$, and $\boldsymbol{\sigma}^\alpha$, with the components $\sigma^\alpha = (\sigma_{ij}^\alpha)$, be the infinitesimal strain tensor and the stress tensor, respectively, corresponding to Ω^α , $\alpha = 1, 2$.

Denote by $(\mathbb{S}, \cdot, \|\cdot\|_{\mathbb{S}})$ the space of symmetric second-order tensors with its inner product and the associated norm.

Let $\mathbf{A}^\alpha : \Omega^\alpha \times \mathbb{S} \rightarrow \mathbb{S}$ be the linear elasticity tensor corresponding to Ω^α and denote by $A^\alpha = (A_{ijkl}^\alpha)$ its components satisfying the following classical symmetry and ellipticity conditions: $A_{ijkl}^\alpha = A_{jikl}^\alpha = A_{klij}^\alpha \in L^\infty(\Omega^\alpha)$, $\forall i, j, k, l = 1, 2, 3$, $\exists k_1^\alpha > 0$ such that $A_{ijkl}^\alpha \tau_{ij} \tau_{kl} \geq k_1^\alpha \tau_{ij} \tau_{ij}$ $\forall \boldsymbol{\tau} = (\tau_{ij}) \in \mathbb{R}^9$ verifying $\tau_{ij} = \tau_{ji} \forall i, j = 1, 2, 3$, $\alpha = 1, 2$. Thus, for all $\boldsymbol{\tau} \in \mathbb{S}$ with components $\tau = (\tau_{ij})$, $\mathbf{A}^\alpha(\mathbf{x}^\alpha, \boldsymbol{\tau})$ has the components $A^\alpha(x^\alpha)_{ijkl} \tau_{ij} \forall k, l = 1, 2, 3$, $\alpha = 1, 2$, and \mathbf{A}^α satisfies the following conditions for $\alpha = 1, 2$:

$$(\mathbf{A}^\alpha(\mathbf{x}^\alpha, \boldsymbol{\tau})) \cdot \boldsymbol{\tau} \geq k_1^\alpha \|\boldsymbol{\tau}\|_{\mathbb{S}}^2 \quad \text{a.e. } \mathbf{x}^\alpha \in \Omega^\alpha, \forall \boldsymbol{\tau} \in \mathbb{S},$$

$$\exists k_2^\alpha > 0 \text{ such that}$$

$$\|\mathbf{A}^\alpha(\mathbf{x}^\alpha, \boldsymbol{\tau})\|_{\mathbb{S}} \leq k_2^\alpha \|\boldsymbol{\tau}\|_{\mathbb{S}} \quad \text{a.e. } \mathbf{x}^\alpha \in \Omega^\alpha, \forall \boldsymbol{\tau} \in \mathbb{S}.$$

Let $\mathbf{B}^\alpha : \Omega^\alpha \times \mathbb{S} \rightarrow \mathbb{S}$ denote the nonlinear viscosity tensor corresponding to Ω^α , satisfying the following conditions for $\alpha = 1, 2$:

$$\exists k_{3,4}^\alpha > 0 \text{ such that a.e. } \mathbf{x}^\alpha \in \Omega^\alpha, \forall \boldsymbol{\tau}_{1,2} \in \mathbb{S},$$

$$(\mathbf{B}^\alpha(\mathbf{x}^\alpha, \boldsymbol{\tau}_1) - \mathbf{B}^\alpha(\mathbf{x}^\alpha, \boldsymbol{\tau}_2)) \cdot (\boldsymbol{\tau}_1 - \boldsymbol{\tau}_2) \geq k_3^\alpha \|\boldsymbol{\tau}_1 - \boldsymbol{\tau}_2\|_{\mathbb{S}}^2,$$

$$\|\mathbf{B}^\alpha(\mathbf{x}^\alpha, \boldsymbol{\tau}_1) - \mathbf{B}^\alpha(\mathbf{x}^\alpha, \boldsymbol{\tau}_2)\|_{\mathbb{S}} \leq k_4^\alpha \|\boldsymbol{\tau}_1 - \boldsymbol{\tau}_2\|_{\mathbb{S}},$$

$$\mathbf{B}^\alpha(\cdot, \mathbf{0}) = \mathbf{0},$$

$$\forall \boldsymbol{\tau} \in \mathbb{S}, \mathbf{B}^\alpha(\cdot, \boldsymbol{\tau}) \text{ is measurable on } \Omega^\alpha.$$

Assume that the displacements $\mathbf{u}^\alpha = \mathbf{0}$ on $\Gamma_U^\alpha \times (0, T)$, $\alpha = 1, 2$, and that the densities of both bodies are equal to 1. Let $\mathbf{f}_1 = (\mathbf{f}_1^1, \mathbf{f}_1^2)$, $\mathbf{f}_2 = (\mathbf{f}_2^1, \mathbf{f}_2^2)$ be the given densities of body forces in $\Omega^1 \cup \Omega^2$ and of tractions on

$\Gamma_F^1 \cup \Gamma_F^2$, respectively. Let $\mathbf{u}_0 = (\mathbf{u}_0^1, \mathbf{u}_0^2)$, $\mathbf{u}_1 = (\mathbf{u}_1^1, \mathbf{u}_1^2)$ denote the initial displacements and velocities of the bodies, respectively.

Suppose that the solids can be in contact between the potential contact surfaces Γ_C^1 and Γ_C^2 which are parametrized by two C^1 functions, φ^1, φ^2 , defined on an open and bounded subset Ξ of \mathbb{R}^2 , such that $\varphi^1(\xi) - \varphi^2(\xi) \geq 0 \forall \xi \in \Xi$ and each Γ_C^α is the graph of φ^α on Ξ that is $\Gamma_C^\alpha = \{(\xi, \varphi^\alpha(\xi)) \in \mathbb{R}^3; \xi \in \Xi\}$, $\alpha = 1, 2$, see, e.g., [3]. Define an initial normalized gap between the two contact surfaces by

$$g_0(\xi) = \frac{\varphi^1(\xi) - \varphi^2(\xi)}{\sqrt{1 + |\nabla\varphi^1(\xi)|^2}} \quad \forall \xi \in \Xi.$$

Let \mathbf{n}^α denote the unit outward normal vector to Γ^α , $\alpha = 1, 2$.

We introduce the following notations for the normal and tangential components of a displacement field \mathbf{v}^α , of the relative displacement corresponding to $\mathbf{v} := (\mathbf{v}^1, \mathbf{v}^2)$ and of the stress vector $\boldsymbol{\sigma}^\alpha \mathbf{n}^\alpha$ on Γ_C^α , $\alpha = 1, 2$, respectively:

$$\begin{aligned} \mathbf{v}^\alpha(\xi, t) &:= \mathbf{v}^\alpha(\xi, \varphi^\alpha(\xi), t), \quad v_N^\alpha(\xi, t) := \mathbf{v}^\alpha(\xi, t) \cdot \mathbf{n}^\alpha(\xi), \\ v_N(\xi, t) &:= v_N^1(\xi, t) + v_N^2(\xi, t), \\ [v_N](\xi, t) &:= v_N(\xi, t) - g_0(\xi), \\ \mathbf{v}_T^\alpha(\xi, t) &:= \mathbf{v}^\alpha(\xi, t) - v_N^\alpha(\xi, t) \mathbf{n}^\alpha(\xi), \\ \mathbf{v}_T(\xi, t) &:= \mathbf{v}_T^1(\xi, t) - \mathbf{v}_T^2(\xi, t), \\ \sigma_N^\alpha(\xi, t) &:= (\boldsymbol{\sigma}^\alpha(\xi, t) \mathbf{n}^\alpha(\xi)) \cdot \mathbf{n}^\alpha(\xi), \\ \sigma_T^\alpha(\xi, t) &:= \boldsymbol{\sigma}^\alpha(\xi, t) \mathbf{n}^\alpha(\xi) - \sigma_N^\alpha(\xi, t) \mathbf{n}^\alpha(\xi), \end{aligned}$$

for all $\xi \in \Xi$ and for all $t \in [0, T]$, where we denoted the inner product of two vectors by " \cdot ".

In Ξ , we consider an internal state variable β (see [18, 19]) that represents the intensity of adhesion: $\beta = 1$ means that the adhesion is total, $\beta = 0$ means that there is no adhesion and $0 < \beta < 1$ is the case of partial adhesion. We assume that the evolution of β is governed, for all $t \in (0, T)$, by the inclusion $\dot{\beta} \in \psi([u_N], \beta)$ in Ξ , where ψ is a given constitutive set-valued mapping. Denote by β_0 the initial intensity of adhesion.

Let $\underline{\kappa}, \bar{\kappa} : \mathbb{R}^2 \rightarrow \mathbb{R}$ be two mappings with $\underline{\kappa}$ lower semicontinuous and $\bar{\kappa}$ upper semicontinuous, satisfying the following conditions for all $s \in \mathbb{R}^2$:

$$\underline{\kappa}(s) \leq \bar{\kappa}(s) \quad \text{and} \quad 0 \notin (\underline{\kappa}(s), \bar{\kappa}(s)), \quad (2.1)$$

$$\exists r_0 \geq 0 \quad \text{such that} \quad \max(|\underline{\kappa}(s)|, |\bar{\kappa}(s)|) \leq r_0, \quad (2.2)$$

Let $\mu : \Xi \times \mathbb{R}^3 \rightarrow \mathbb{R}_+$ be the sliding velocity dependent coefficient of friction and assume that μ is a bounded function such that for a.e. $\xi \in \Xi$

$\mu(\xi, \cdot)$ is Lipschitz continuous with the Lipschitz constant independent of ξ , and for every $\mathbf{v} \in \mathbb{R}^3$ $\mu(\cdot, \mathbf{v})$ is measurable. Define a truncation operator $\vartheta = \vartheta_{l_0}$ by $\vartheta : \mathbb{R} \rightarrow \mathbb{R}$, $\vartheta(s) = -l_0$ if $s \leq -l_0$, $\vartheta(s) = s$ if $|s| < l_0$ and $\vartheta(s) = l_0$ if $s \geq l_0$, where $l_0 > 0$ is a given characteristic length, see, e.g., [31, 36].

We choose the following state variables: the infinitesimal strain tensor $(\boldsymbol{\varepsilon}^1, \boldsymbol{\varepsilon}^2) = (\boldsymbol{\varepsilon}(\mathbf{u}^1), \boldsymbol{\varepsilon}(\mathbf{u}^2))$ in $\Omega^1 \cup \Omega^2$, the normal relative displacement $[u_N] = u_N^1 + u_N^2 - g_0$, the tangential relative displacement $[\mathbf{u}_T] = \mathbf{u}_T^1 - \mathbf{u}_T^2$, and the intensity of adhesion β in Ξ .

Consider the following dynamic viscoelastic contact problem coupling adhesion and Coulomb friction.

Problem P_c : Find $\mathbf{u} = (\mathbf{u}^1, \mathbf{u}^2)$ and β such that $\mathbf{u}(0) = \mathbf{u}_0$, $\dot{\mathbf{u}}(0) = \mathbf{u}_1$, $\beta(0) = \beta_0$ in Ξ and, for all $t \in (0, T)$,

$$\begin{aligned} \ddot{\mathbf{u}}^\alpha - \operatorname{div} \boldsymbol{\sigma}^\alpha(\mathbf{u}^\alpha, \dot{\mathbf{u}}^\alpha) &= \mathbf{f}_1^\alpha \quad \text{in } \Omega^\alpha, \\ \boldsymbol{\sigma}^\alpha(\mathbf{u}^\alpha, \dot{\mathbf{u}}^\alpha) &= \mathbf{A}^\alpha \boldsymbol{\varepsilon}(\mathbf{u}^\alpha) + \mathbf{B}^\alpha \boldsymbol{\varepsilon}(\dot{\mathbf{u}}^\alpha) \quad \text{in } \Omega^\alpha, \\ \mathbf{u}^\alpha &= \mathbf{0} \quad \text{on } \Gamma_U^\alpha, \quad \boldsymbol{\sigma}^\alpha \mathbf{n}^\alpha = \mathbf{f}_2^\alpha \quad \text{on } \Gamma_F^\alpha, \quad \alpha = 1, 2, \\ \boldsymbol{\sigma}^1 \mathbf{n}^1 + \boldsymbol{\sigma}^2 \mathbf{n}^2 &= \mathbf{0} \quad \text{in } \Xi, \\ \underline{\kappa}([u_N], \beta) \leq \sigma_N \leq \bar{\kappa}([u_N], \beta) &\quad \text{in } \Xi, \\ |\boldsymbol{\sigma}_T| \leq \mu(\dot{\mathbf{u}}_T) |\sigma_N| &\quad \text{in } \Xi \quad \text{and} \\ \dot{\mathbf{u}}_T \neq \mathbf{0} \Rightarrow \boldsymbol{\sigma}_T &= -\mu(\dot{\mathbf{u}}_T) |\sigma_N| \frac{\dot{\mathbf{u}}_T}{|\dot{\mathbf{u}}_T|}, \\ \beta \in [0, 1] \quad \text{and} \quad \dot{\beta} &\in \psi(\vartheta([u_N]), \beta) \quad \text{in } \Xi, \end{aligned}$$

where, for all $(\mathbf{x}^\alpha, t) \in \Omega^\alpha \times (0, T)$,

$$\begin{aligned} \mathbf{A}^\alpha \boldsymbol{\varepsilon}(\mathbf{u}^\alpha)(\mathbf{x}^\alpha, t) &= \mathbf{A}^\alpha(\mathbf{x}^\alpha, \boldsymbol{\varepsilon}(\mathbf{u}^\alpha(\mathbf{x}^\alpha, t))), \\ \mathbf{B}^\alpha \boldsymbol{\varepsilon}(\dot{\mathbf{u}}^\alpha)(\mathbf{x}^\alpha, t) &= \mathbf{B}^\alpha(\mathbf{x}^\alpha, \boldsymbol{\varepsilon}(\dot{\mathbf{u}}^\alpha(\mathbf{x}^\alpha, t))), \\ \boldsymbol{\sigma}^\alpha &= \boldsymbol{\sigma}^\alpha(\mathbf{u}^\alpha, \dot{\mathbf{u}}^\alpha), \quad \alpha = 1, 2, \quad \sigma_N := \sigma_N^1, \quad \boldsymbol{\sigma}_T := \boldsymbol{\sigma}_T^1. \end{aligned}$$

The nonlinear constitutive law represents a generalization of the classical Kelvin-Voigt law.

Different choices for $\underline{\kappa}$, $\bar{\kappa}$ and ψ give various contact and friction conditions, including irreversible or recoverable (healing) adhesion, see [31, 12, 14, 9, 7].

We also remark that in this model friction and adhesion are strongly coupled, as sliding friction can occur even under tensile loads [25], but the case of the solely compressive loads can be easily considered.

3 Mixed variational formulations and approximation results

We adopt the following notations:

$$\begin{aligned}
\mathbf{H}^s(\Omega^\alpha) &:= H^s(\Omega^\alpha; \mathbb{R}^3), \quad \alpha = 1, 2, \\
\mathbf{H}^s &:= \mathbf{H}^s(\Omega^1) \times \mathbf{H}^s(\Omega^2) \quad \forall s \in \mathbb{R}, \\
\mathbf{V}^\alpha &= \{\mathbf{v}^\alpha \in \mathbf{H}^1(\Omega^\alpha); \mathbf{v}^\alpha = \mathbf{0} \text{ a.e. on } \Gamma_U^\alpha\}, \quad \alpha = 1, 2, \\
\mathbf{V} &:= \mathbf{V}^1 \times \mathbf{V}^2, \quad \mathbf{H} := \mathbf{H}^0 = L^2(\Omega^1; \mathbb{R}^3) \times L^2(\Omega^2; \mathbb{R}^3).
\end{aligned}$$

$(\mathbf{H}, |\cdot|)$ and $(\mathbf{V}, \|\cdot\|)$ are Hilbert spaces with the associated inner products denoted by (\cdot, \cdot) and by $\langle \cdot, \cdot \rangle$, respectively, $\mathbf{V} \subset \mathbf{H} \subset \mathbf{V}'$ with the inclusion mapping of \mathbf{V} into \mathbf{H} continuous and densely defined, where $(\mathbf{V}', \|\cdot\|_{\mathbf{V}'})$ is the dual of \mathbf{V} and \mathbf{H} is identified with its own dual. Let $\langle \cdot, \cdot \rangle_{\mathbf{V}', \mathbf{V}}$ denote the duality pairing between \mathbf{V}' and \mathbf{V} .

Define $\Xi_T = \Xi \times (0, T)$, the closed convex cones $L_+^2(\Xi)$, $L_+^2(\Xi_T)$ and the closed convex set $L_{[0,1]}^2(\Xi)$ as follows:

$$\begin{aligned}
L_+^2(\Xi) &:= \{\delta \in L^2(\Xi); \delta \geq 0 \text{ a.e. in } \Xi\}, \\
L_+^2(\Xi_T) &:= \{\eta \in L^2(\Xi_T); \eta \geq 0 \text{ a.e. in } \Xi_T\}, \\
L_{[0,1]}^2(\Xi) &:= \{\delta \in L^2(\Xi); \delta \in [0, 1] \text{ a.e. in } \Xi\}.
\end{aligned}$$

Let $\mathcal{A}: \mathbf{V} \rightarrow \mathbf{V}'$, $\mathcal{B}: \mathbf{V} \rightarrow \mathbf{V}'$ be two operators defined by

$$\begin{aligned}
\langle \mathcal{A}\mathbf{v}, \mathbf{w} \rangle_{\mathbf{V}', \mathbf{V}} &= \sum_{\alpha=1,2} \int_{\Omega^\alpha} (\mathbf{A}^\alpha \boldsymbol{\varepsilon}(\mathbf{v}^\alpha)) \cdot \boldsymbol{\varepsilon}(\mathbf{w}^\alpha) dx, \\
\langle \mathcal{B}\mathbf{v}, \mathbf{w} \rangle_{\mathbf{V}', \mathbf{V}} &= \sum_{\alpha=1,2} \int_{\Omega^\alpha} (\mathbf{B}^\alpha \boldsymbol{\varepsilon}(\mathbf{v}^\alpha)) \cdot \boldsymbol{\varepsilon}(\mathbf{w}^\alpha) dx \\
\forall \mathbf{v} &= (\mathbf{v}^1, \mathbf{v}^2), \quad \mathbf{w} = (\mathbf{w}^1, \mathbf{w}^2) \in \mathbf{V}.
\end{aligned}$$

Under the above assumptions on \mathbf{A}^α , \mathbf{B}^α , $\alpha = 1, 2$, it follows that there exist $M_{\mathcal{A}}$, $M_{\mathcal{B}}$, such that, for all $\mathbf{v}, \mathbf{w} \in \mathbf{V}$,

$$\|\mathcal{A}\mathbf{v}\|_{\mathbf{V}'} \leq M_{\mathcal{A}} \|\mathbf{v}\|_{\mathbf{V}}, \quad \|\mathcal{B}\mathbf{v} - \mathcal{B}\mathbf{w}\|_{\mathbf{V}'} \leq M_{\mathcal{B}} \|\mathbf{v} - \mathbf{w}\|_{\mathbf{V}}. \quad (3.1)$$

As $\text{meas}(\Gamma_U^\alpha) > 0$, by using Korn's inequality it also follows that there exist $m_{\mathcal{A}}$, $m_{\mathcal{B}} > 0$ such that, for all $\mathbf{v}, \mathbf{w} \in \mathbf{V}$,

$$\langle \mathcal{A}\mathbf{v}, \mathbf{v} \rangle_{\mathbf{V}', \mathbf{V}} \geq m_{\mathcal{A}} \|\mathbf{v}\|_{\mathbf{V}}^2, \quad (3.2)$$

$$\langle \mathcal{B}\mathbf{v} - \mathcal{B}\mathbf{w}, \mathbf{v} - \mathbf{w} \rangle_{\mathbf{V}', \mathbf{V}} \geq m_{\mathcal{B}} \|\mathbf{v} - \mathbf{w}\|_{\mathbf{V}}^2. \quad (3.3)$$

Assume $\mathbf{u}_0 \in \mathbf{V}$, $\mathbf{u}_1 \in \mathbf{H}$, $g_0 \in L^2_+(\Xi)$, $\mathbf{f}_1^\alpha \in L^2(0, T; L^2(\Omega^\alpha; \mathbb{R}^3))$, $\mathbf{f}_2^\alpha \in L^2(0, T; L^2(\Gamma_F^\alpha; \mathbb{R}^3))$, $\alpha = 1, 2$, and define the mapping $\mathbf{f} \in L^2(0, T; \mathbf{V}')$ by

$$\begin{aligned} \langle \mathbf{f}, \mathbf{v} \rangle_{\mathbf{V}', \mathbf{V}} &= \sum_{\alpha=1,2} \int_{\Omega^\alpha} \mathbf{f}_1^\alpha \cdot \mathbf{v}^\alpha dx + \sum_{\alpha=1,2} \int_{\Gamma_F^\alpha} \mathbf{f}_2^\alpha \cdot \mathbf{v}^\alpha ds \\ \forall \mathbf{v} = (\mathbf{v}^1, \mathbf{v}^2) \in \mathbf{V}, \quad \text{a.e. } t \in [0, T]. \end{aligned}$$

Assume also the following initial conditions: $\beta_0 \in L^2_{[0,1]}(\Xi)$, $[u_{0N}] \leq 0$, and $\bar{\kappa}([u_{0N}], \beta_0) = 0$ a.e. in Ξ .

For every $\zeta = (\zeta_1, \zeta_2) \in L^2(0, T; (L^2(\Xi))^2) = (L^2(\Xi_T))^2$, define the following nonempty, closed, and convex sets:

$$\begin{aligned} \Lambda^0(\zeta_1, \zeta_2) &= \{ \eta \in L^2(\Xi_T); \underline{\kappa} \circ (\zeta_1, \zeta_2) \leq \eta \\ &\leq \bar{\kappa} \circ (\zeta_1, \zeta_2) \text{ a.e. in } \Xi_T \}, \\ \Lambda^0_+(\zeta_1, \zeta_2) &= \{ \eta \in L^2_+(\Xi_T); \underline{\kappa}_+ \circ (\zeta_1, \zeta_2) \leq \eta \\ &\leq \bar{\kappa}_+ \circ (\zeta_1, \zeta_2) \text{ a.e. in } \Xi_T \}, \\ \Lambda^0_-(\zeta_1, \zeta_2) &= \{ \eta \in L^2_-(\Xi_T); \bar{\kappa}_- \circ (\zeta_1, \zeta_2) \leq \eta \\ &\leq \underline{\kappa}_- \circ (\zeta_1, \zeta_2) \text{ a.e. in } \Xi_T \}, \end{aligned}$$

where, for each $r \in \mathbb{R}$, $r_+ := \max(0, r)$ and $r_- := \max(0, -r)$ denote the positive and negative parts, respectively. Also, for every $\mathbf{w} \in W^{1,2}(0, T; \mathbf{V})$, $v \in L^2(\Xi_T)$, define the following nonempty and closed sets:

$$\begin{aligned} \Lambda^1(\mathbf{w}, v) &= \{ (\eta, \boldsymbol{\varsigma}) \in L^2(\Xi_T) \times (L^2(\Xi_T))^3; \\ &\eta \in \Lambda^0([w_N], v), |\boldsymbol{\varsigma}| \leq \mu(\dot{\mathbf{w}}_T) |\eta|, \\ &\boldsymbol{\varsigma} \cdot \dot{\mathbf{w}}_T + \mu(\dot{\mathbf{w}}_T) |\eta| |\dot{\mathbf{w}}_T| = 0 \text{ a.e. in } \Xi_T \}, \\ \Lambda^2(\mathbf{w}, v) &= \{ (\eta, \boldsymbol{\varsigma}) \in L^2(\Xi_T) \times (L^2(\Xi_T))^3; \\ &\eta_+ \in \Lambda^0_+([w_N], v), \eta_- \in \Lambda^0_-([w_N], v), \\ &|\boldsymbol{\varsigma}| \leq \mu(\dot{\mathbf{w}}_T) (\eta_+ + \eta_-), \\ &\boldsymbol{\varsigma} \cdot \dot{\mathbf{w}}_T + \mu(\dot{\mathbf{w}}_T) (\eta_+ + \eta_-) |\dot{\mathbf{w}}_T| = 0 \text{ a.e. in } \Xi_T \}, \\ \Lambda^3(\mathbf{w}, v) &= \{ (\eta_1, \eta_2, \boldsymbol{\varsigma}) \in (L^2(\Xi_T))^5; \eta_1 \in \Lambda^0_+([w_N], v), \\ &\eta_2 \in \Lambda^0_-([w_N], v), |\boldsymbol{\varsigma}| \leq \mu(\dot{\mathbf{w}}_T) (\eta_1 + \eta_2), \\ &\boldsymbol{\varsigma} \cdot \dot{\mathbf{w}}_T + \mu(\dot{\mathbf{w}}_T) (\eta_1 + \eta_2) |\dot{\mathbf{w}}_T| = 0 \text{ a.e. in } \Xi_T \}. \end{aligned}$$

The relations $\text{meas}(\Xi) < \infty$ and (2.2) imply that for all $\zeta = (\zeta_1, \zeta_2) \in L^2(\Xi_T)$ the sets $\Lambda^0(\zeta), \Lambda^0_+(\zeta)$ and $\Lambda^0_-(\zeta)$ are bounded in norm in $L^2(0, T; L^2(\Xi)) = L^2(\Xi_T)$

by $R_0 = r_0(\text{meas}(\Xi))^{1/2}T$ and are bounded in norm in $L^\infty(0, T; L^\infty(\Xi))$ by r_0 .

As the coefficient of friction μ is a bounded function, it follows that for all $\mathbf{w} \in W^{1,2}(0, T; \mathbf{V})$, $v \in L^2(\Xi_T)$ the sets $\Lambda^1(\mathbf{w}, v)$, $\Lambda^2(\mathbf{w}, v)$, and $\Lambda^3(\mathbf{w}, v)$ are bounded in norm. Thus, there exists $R_1 > 0$ such that $\Lambda^3(\mathbf{w}, v) \subset D_0 \times D_1$ for all $\mathbf{w} \in W^{1,2}(0, T; \mathbf{V})$, $v \in L^2(\Xi_T)$, where $D_0 = \{(\eta_1, \eta_2) \in (L^2(\Xi_T))^2; \|\eta_1\|_{L^2(\Xi_T)} \leq R_0, \|\eta_2\|_{L^2(\Xi_T)} \leq R_0\}$ and $D_1 = \{\boldsymbol{\varsigma} \in (L^2(\Xi_T))^3; \|\boldsymbol{\varsigma}\|_{(L^2(\Xi_T))^3} \leq R_1\}$.

A first variational formulation of the problem P_c is the following.

Problem P_v^1 : Find $\mathbf{u} \in C^1([0, T]; \mathbf{H}) \cap W^{1,2}(0, T; \mathbf{V}) \cap W^{2,2}(0, T; \mathbf{V}')$, $\lambda \in L^2(\Xi_T)$, $\boldsymbol{\gamma} \in (L^2(\Xi_T))^3$, $\beta \in W^{1,\infty}(0, T; L^\infty(\Xi))$, such that $(\lambda, \boldsymbol{\gamma}) \in \Lambda^1(\mathbf{u}, \beta)$, $\beta(t) \in L^2_{[0,1]}(\Xi)$ for all $t \in (0, T)$, $\mathbf{u}(0) = \mathbf{u}_0$, $\dot{\mathbf{u}}(0) = \mathbf{u}_1$, $\beta(0) = \beta_0$, and for almost all $t \in (0, T)$

$$\begin{aligned} & \langle \ddot{\mathbf{u}}, \mathbf{v} \rangle_{\mathbf{V}', \mathbf{V}} + \langle \mathbf{A}\mathbf{u}, \mathbf{v} \rangle_{\mathbf{V}', \mathbf{V}} + \langle \mathbf{B}\dot{\mathbf{u}}, \mathbf{v} \rangle_{\mathbf{V}', \mathbf{V}} \\ & - (\lambda, v_N)_{L^2(\Xi)} - (\boldsymbol{\gamma}, \mathbf{v}_T)_{(L^2(\Xi))^3} = \langle \mathbf{f}, \mathbf{v} \rangle_{\mathbf{V}', \mathbf{V}} \quad \forall \mathbf{v} \in \mathbf{V}, \\ & \dot{\beta} \in \psi(\vartheta([u_N]), \beta) \quad \text{a.e. in } \Xi_T, \end{aligned} \quad (3.4)$$

where $(\cdot, \cdot)_{L^2(\Xi)}$ and $(\cdot, \cdot)_{(L^2(\Xi))^3}$ denote the inner products of the corresponding spaces.

The formal equivalence between the variational problem P_v^1 and the classical problem P_c can be proved as usual by Green's formula, where the Lagrange multipliers λ , $\boldsymbol{\gamma}$ satisfy the relations $\lambda = \sigma_N$, $\boldsymbol{\gamma} = \boldsymbol{\sigma}_T$.

The sets $\Lambda^0(\zeta_1, \zeta_2)$, $\Lambda^0_+(\zeta_1, \zeta_2)$ and $\Lambda^0_-(\zeta_1, \zeta_2)$ have the following useful properties, see [9].

Lemma 3.1. *Let $(\zeta_1, \zeta_2) \in (L^2(\Xi))^2$ and $(\eta_1, \eta_2) \in \Lambda^0_+(\zeta_1, \zeta_2) \times \Lambda^0_-(\zeta_1, \zeta_2)$. Then $\eta_1\eta_2 = 0$ a.e. in Ξ_T and there exists $\eta \in \Lambda^0(\zeta_1, \zeta_2)$ such that $\eta_+ = \eta_1$, $\eta_- = \eta_2$ a.e. in Ξ_T .*

Since $\lambda \in \Lambda^0([u_N], \beta)$ if and only if $(\lambda_+, \lambda_-) \in \Lambda^0_+([u_N], \beta) \times \Lambda^0_-([u_N], \beta)$, the previous lemma enables to consider the following variational problem P_v^2 , which has the same solutions \mathbf{u} , $\boldsymbol{\gamma}$, β as the problem P_v^1 and the solutions λ_1, λ_2 satisfy the relation $\lambda = \lambda_1 - \lambda_2$, where λ is a solution of P_v^1 .

Problem P_v^2 : Find $\mathbf{u} \in C^1([0, T]; \mathbf{H}) \cap W^{1,2}(0, T; \mathbf{V}) \cap W^{2,2}(0, T; \mathbf{V}')$, $(\lambda_1, \lambda_2) \in (L^2(\Xi_T))^2$, $\boldsymbol{\gamma} \in (L^2(\Xi_T))^3$, $\beta \in W^{1,\infty}(0, T; L^\infty(\Xi))$, such that $(\lambda_1, \lambda_2, \boldsymbol{\gamma}) \in \Lambda^3(\mathbf{u}, \beta)$, $\beta(t) \in L^2_{[0,1]}(\Xi)$ for all $t \in (0, T)$, $\mathbf{u}(0) = \mathbf{u}_0$,

$\dot{\mathbf{u}}(0) = \mathbf{u}_1$, $\beta(0) = \beta_0$, and for almost all $t \in (0, T)$

$$\begin{aligned} & \langle \ddot{\mathbf{u}}, \mathbf{v} \rangle_{\mathbf{V}', \mathbf{V}} + \langle \mathcal{A}\mathbf{u}, \mathbf{v} \rangle_{\mathbf{V}', \mathbf{V}} + \langle \mathcal{B}\dot{\mathbf{u}}, \mathbf{v} \rangle_{\mathbf{V}', \mathbf{V}} \\ & - (\lambda_1 - \lambda_2, v_N)_{L^2(\Xi)} - (\boldsymbol{\gamma}, \mathbf{v}_T)_{(L^2(\Xi))^3} \\ & = \langle \mathbf{f}, \mathbf{v} \rangle_{\mathbf{V}', \mathbf{V}} \quad \forall \mathbf{v} \in \mathbf{V}, \end{aligned} \quad (3.5)$$

$$\dot{\beta} \in \psi(\vartheta([u_N]), \beta) \quad \text{a.e. in } \Xi_T, \quad (3.6)$$

Assume that the set-valued mapping $\psi : (L^2(\Xi_T))^2 \rightarrow 2^{L^2(\Xi_T)}$ verifies the following properties: for each $\mathbf{u} \in W^{1,2}(0, T; \mathbf{V})$ there exists a unique solution $\beta_{\mathbf{u}} \in W^{1,\infty}(0, T; L^\infty(\Xi))$ of the inclusion (3.6), such that $\beta(0) = \beta_0$, $\beta(t) \in L^2_{[0,1]}(\Xi)$ for all $t \in (0, T)$, and if $\beta_{\mathbf{u}_1}, \beta_{\mathbf{u}_2}$ are the solutions of (3.6) corresponding to $\mathbf{u}_1, \mathbf{u}_2 \in W^{1,2}(0, T; \mathbf{V})$, respectively, with the same initial condition β_0 , then the following estimate holds for all $t \in [0, T]$:

$$\|\beta_{\mathbf{u}_1}(t) - \beta_{\mathbf{u}_2}(t)\|_{L^2(\Xi)}^2 \leq C_0 \int_0^t \|\mathbf{u}_1(s) - \mathbf{u}_2(s)\|^2 ds, \quad (3.7)$$

where C_0 is a positive constant independent of $\mathbf{u}_1, \mathbf{u}_2, \beta_{\mathbf{u}_1}, \beta_{\mathbf{u}_2}$.

Several choices of ψ , as subdifferentials or single-valued mappings satisfying the previous properties, have been considered, see, e.g., [31, 34, 14, 7].

The existence of strong solutions to problem P_v^2 will be established by using some auxiliary results and an equivalent fixed point problem.

Similar arguments to those used to prove Theorem 5.2 in [36], now applied to the product space $\mathbf{V} = \mathbf{V}^1 \times \mathbf{V}^2$ and to the simple case without adhesion, enable to obtain the following existence and uniqueness result for the strong solution, in the \mathbf{V}' sense, of the following intermediate problem.

Lemma 3.2. *For each $(\eta_1, \eta_2) \in (L^2(\Xi_T))^2$, $\boldsymbol{\varsigma} \in (L^2(\Xi_T))^3$, there exists a unique function $\mathbf{u} = \mathbf{u}_{(\eta_1, \eta_2, \boldsymbol{\varsigma})} \in C^1([0, T]; \mathbf{H}) \cap W^{1,2}(0, T; \mathbf{V}) \cap W^{2,2}(0, T; \mathbf{V}')$, such that $\mathbf{u}(0) = \mathbf{u}_0$, $\dot{\mathbf{u}}(0) = \mathbf{u}_1$, and for almost all $t \in (0, T)$*

$$\begin{aligned} & \langle \ddot{\mathbf{u}}, \mathbf{v} \rangle_{\mathbf{V}', \mathbf{V}} + \langle \mathcal{A}\mathbf{u}, \mathbf{v} \rangle_{\mathbf{V}', \mathbf{V}} + \langle \mathcal{B}\dot{\mathbf{u}}, \mathbf{v} \rangle_{\mathbf{V}', \mathbf{V}} \\ & - (\eta_1 - \eta_2, v_N)_{L^2(\Xi)} - (\boldsymbol{\varsigma}, \mathbf{v}_T)_{(L^2(\Xi))^3} \\ & = \langle \mathbf{f}, \mathbf{v} \rangle_{\mathbf{V}', \mathbf{V}} \quad \forall \mathbf{v} \in \mathbf{V}, \end{aligned} \quad (3.8)$$

where $\ddot{\mathbf{u}}$ is the second order strong derivative of \mathbf{u} considered in \mathbf{V}' .

For the involved abstract mathematical result, see, e.g., Theorem 4.10 in [2].

Lemma 3.3. Let $\mathbf{u}_{(\eta_1, \eta_2, \varsigma_1)}$, $\mathbf{u}_{(\delta_1, \delta_2, \varsigma_2)}$ be the solutions of variational equation (3.8) corresponding to (η_1, η_2) , $(\delta_1, \delta_2) \in (L^2(\Xi_T))^2$, $\varsigma_1, \varsigma_2 \in (L^2(\Xi_T))^3$, respectively.

Then there exists a constant $C_1 > 0$, independent of (η_1, η_2) , (δ_1, δ_2) , and ς_1, ς_2 , such that for all $t \in [0, T]$

$$\begin{aligned}
& |\dot{\mathbf{u}}_{(\eta_1, \eta_2, \varsigma_1)}(t) - \dot{\mathbf{u}}_{(\delta_1, \delta_2, \varsigma_2)}(t)|^2 + \|\mathbf{u}_{(\eta_1, \eta_2, \varsigma_1)}(t) - \mathbf{u}_{(\delta_1, \delta_2, \varsigma_2)}(t)\|^2 \\
& \quad + \int_0^t \|\dot{\mathbf{u}}_{(\eta_1, \eta_2, \varsigma_1)} - \dot{\mathbf{u}}_{(\delta_1, \delta_2, \varsigma_2)}\|^2 d\tau \\
& \leq C_1 \int_0^t \{(\eta_1 - \eta_2 - \delta_1 + \delta_2, \dot{\mathbf{u}}_{(\eta_1, \eta_2, \varsigma_1)N} - \dot{\mathbf{u}}_{(\delta_1, \delta_2, \varsigma_2)N})_{L^2(\Xi)} \\
& \quad + (\varsigma_1 - \varsigma_2, \dot{\mathbf{u}}_{(\eta_1, \eta_2, \varsigma_1)T} - \dot{\mathbf{u}}_{(\delta_1, \delta_2, \varsigma_2)T})_{(L^2(\Xi))^3}\} d\tau. \tag{3.9}
\end{aligned}$$

Proof. Let $(\eta_1, \eta_2), (\delta_1, \delta_2) \in (L^2(\Xi_T))^2$ and $\varsigma_1, \varsigma_2 \in (L^2(\Xi_T))^3$ with $\mathbf{u}_1 := \mathbf{u}_{(\eta_1, \eta_2, \varsigma_1)}$, $\mathbf{u}_2 := \mathbf{u}_{(\delta_1, \delta_2, \varsigma_2)}$ the corresponding solutions of (3.8) which exist according to Lemma 3.2. Taking in each equation $\mathbf{v} = \dot{\mathbf{u}}_1 - \dot{\mathbf{u}}_2$, for a.e. $\tau \in (0, T)$ it follows that

$$\begin{aligned}
& \langle \ddot{\mathbf{u}}_1 - \ddot{\mathbf{u}}_2, \dot{\mathbf{u}}_1 - \dot{\mathbf{u}}_2 \rangle_{\mathbf{V}', \mathbf{V}} + \langle \mathcal{A}\mathbf{u}_1 - \mathcal{A}\mathbf{u}_2, \dot{\mathbf{u}}_1 - \dot{\mathbf{u}}_2 \rangle_{\mathbf{V}', \mathbf{V}} \\
& \quad + \langle \mathcal{B}\dot{\mathbf{u}}_1 - \mathcal{B}\dot{\mathbf{u}}_2, \dot{\mathbf{u}}_1 - \dot{\mathbf{u}}_2 \rangle_{\mathbf{V}', \mathbf{V}} \\
& = (\eta_1 - \eta_2 - \delta_1 + \delta_2, \dot{\mathbf{u}}_{1N} - \dot{\mathbf{u}}_{2N})_{L^2(\Xi)} \\
& \quad + (\varsigma_1 - \varsigma_2, \dot{\mathbf{u}}_{1T} - \dot{\mathbf{u}}_{2T})_{(L^2(\Xi))^3}.
\end{aligned}$$

Since the solutions $\mathbf{u}_1, \mathbf{u}_2$ belong to $\mathbf{u} \in C^1([0, T]; \mathbf{H}) \cap W^{1,2}(0, T; \mathbf{V}) \cap W^{2,2}(0, T; \mathbf{V}')$ and verify the same initial conditions, by integrating over $(0, t)$ it follows that for all $t \in [0, T]$

$$\begin{aligned}
& \frac{1}{2} |\dot{\mathbf{u}}_1(t) - \dot{\mathbf{u}}_2(t)|^2 + \frac{1}{2} \langle \mathcal{A}(\mathbf{u}_1 - \mathbf{u}_2), \mathbf{u}_1 - \mathbf{u}_2 \rangle_{\mathbf{V}', \mathbf{V}} \\
& \quad + \int_0^t \langle \mathcal{B}\dot{\mathbf{u}}_1 - \mathcal{B}\dot{\mathbf{u}}_2, \dot{\mathbf{u}}_1 - \dot{\mathbf{u}}_2 \rangle_{\mathbf{V}', \mathbf{V}} d\tau \\
& = \int_0^t \{(\eta_1 - \eta_2 - \delta_1 + \delta_2, \dot{\mathbf{u}}_{1N} - \dot{\mathbf{u}}_{2N})_{L^2(\Xi)}\} d\tau \\
& \quad + \int_0^t \{(\varsigma_1 - \varsigma_2, \dot{\mathbf{u}}_{1T} - \dot{\mathbf{u}}_{2T})_{(L^2(\Xi))^3}\} d\tau.
\end{aligned}$$

By (3.2) and (3.3), the estimate (3.9) follows. \square

Lemma 3.4. *Under the assumptions of Sections 2 and 3, for every $(\eta_1, \eta_2) \in (L^2_+(\Xi_T))^2$ and every $\boldsymbol{\varsigma} \in (L^2(\Xi_T))^3$, let $(\eta_1^n, \eta_2^n)_n$ be a sequence in $(L^2_+(\Xi_T))^2$ and $(\boldsymbol{\varsigma}^n)_n$ be a sequence in $(L^2(\Xi_T))^3$ such that $\eta_1^n \rightharpoonup \eta_1$, $\eta_2^n \rightharpoonup \eta_2$ in $L^2(\Xi_T)$, and $\boldsymbol{\varsigma}^n \rightharpoonup \boldsymbol{\varsigma}$ in $(L^2(\Xi_T))^3$. Let $\mathbf{u}_{(\eta_1^n, \eta_2^n, \boldsymbol{\varsigma}^n)}$ be the solution of (3.8) corresponding to $(\eta_1^n, \eta_2^n, \boldsymbol{\varsigma}^n)$ according to Lemma 3.2, for every $n \in \mathbb{N}$. Then $(\mathbf{u}_{(\eta_1^n, \eta_2^n, \boldsymbol{\varsigma}^n)})_n$ is strongly convergent in $C^1([0, T]; \mathbf{H}) \cap W^{1,2}(0, T; \mathbf{V})$ to the solution $\mathbf{u} = \mathbf{u}_{(\eta_1, \eta_2, \boldsymbol{\varsigma})}$ of (3.8) corresponding to $(\eta_1, \eta_2, \boldsymbol{\varsigma})$ according to Lemma 3.2.*

Proof. We adopt the following notations:

$$\mathbf{u}_n := \mathbf{u}_{(\eta_1^n, \eta_2^n, \boldsymbol{\varsigma}^n)}, \quad u_{nN} := u_{(\eta_1^n, \eta_2^n, \boldsymbol{\varsigma}^n)N},$$

so that we have for almost all $t \in (0, T)$

$$\begin{aligned} & \langle \ddot{\mathbf{u}}_n, \mathbf{v} \rangle_{\mathbf{V}', \mathbf{V}} + \langle \mathcal{A} \mathbf{u}_n, \mathbf{v} \rangle_{\mathbf{V}', \mathbf{V}} + \langle \mathcal{B} \dot{\mathbf{u}}_n, \mathbf{v} \rangle_{\mathbf{V}', \mathbf{V}} \\ & - (\eta_1^n - \eta_2^n, v_N)_{L^2(\Xi)} - (\boldsymbol{\varsigma}^n, \mathbf{v}_T)_{(L^2(\Xi))^3} \\ & = \langle \mathbf{f}, \mathbf{v} \rangle_{\mathbf{V}', \mathbf{V}} \quad \forall \mathbf{v} \in \mathbf{V}. \end{aligned} \quad (3.10)$$

By Lemma 3.3, for all $n \in \mathbb{N}$ and $t \in (0, T]$ we have

$$\begin{aligned} & \frac{1}{2} |\dot{\mathbf{u}}_n(t)|^2 + \frac{1}{2} \|\mathbf{u}_n(t)\|^2 + \frac{1}{2} \int_0^t \|\dot{\mathbf{u}}_n\|^2 d\tau \\ & \leq |\dot{\mathbf{u}}_n(t) - \dot{\mathbf{u}}(t)|^2 + \|\mathbf{u}_n(t) - \mathbf{u}(t)\|^2 + \int_0^t \|\dot{\mathbf{u}}_n - \dot{\mathbf{u}}\|^2 d\tau \\ & \quad + |\dot{\mathbf{u}}(t)|^2 + \|\mathbf{u}(t)\|^2 + \int_0^t \|\dot{\mathbf{u}}\|^2 d\tau \\ & \leq C_1 \int_0^t \{(\eta_1^n - \eta_2^n - \eta_1 + \eta_2, \dot{u}_{nN} - \dot{u}_N)_{L^2(\Xi)} \\ & \quad + (\boldsymbol{\varsigma}^n - \boldsymbol{\varsigma}, \dot{\mathbf{u}}_{nT} - \dot{\mathbf{u}}_T)_{(L^2(\Xi))^3}\} d\tau + C_2, \end{aligned}$$

where C_2 is a positive constant dependent only on \mathbf{u} .

Since the sequences $(\eta_1^n, \eta_2^n)_n$, $(\boldsymbol{\varsigma}^n)_n$ are bounded in $(L^2(\Xi_T))^2$, $(L^2(\Xi_T))^3$, respectively, by Young's inequality it follows that there exists a positive constant C_3 , depending only on \mathbf{u} , C_1 , C_2 , the bounds of $(\eta_1^n, \eta_2^n)_n$ and $(\boldsymbol{\varsigma}^n)_n$, such that the following estimates hold for all $n \in \mathbb{N}$:

$$|\dot{\mathbf{u}}_n(t)| \leq C_3, \quad \|\mathbf{u}_n(t)\| \leq C_3 \quad \forall t \in [0, T], \quad \|\dot{\mathbf{u}}_n\|_{L^2(0, T; \mathbf{V})} \leq C_3. \quad (3.11)$$

For all $\mathbf{v} \in L^2(0, T; \mathbf{V})$, by (3.10) we have

$$\begin{aligned} & \int_0^T \langle \ddot{\mathbf{u}}_n, \mathbf{v} \rangle_{\mathbf{V}', \mathbf{V}} dt + \int_0^T \langle \mathcal{A}\mathbf{u}_n, \mathbf{v} \rangle_{\mathbf{V}', \mathbf{V}} dt \\ & + \int_0^T \langle \mathcal{B}\dot{\mathbf{u}}_n, \mathbf{v} \rangle_{\mathbf{V}', \mathbf{V}} dt - \int_0^T (\eta_1^n - \eta_2^n, v_N)_{L^2(\Xi)} dt \\ & - \int_0^T (\boldsymbol{\varsigma}^n, \mathbf{v}_T)_{(L^2(\Xi))^3} dt = \int_0^T \langle \mathbf{f}, \mathbf{v} \rangle_{\mathbf{V}', \mathbf{V}} dt. \end{aligned}$$

This relation and the estimates (3.1), (3.11) imply that there exists a positive constant C_4 , depending only on C_3 , $M_{\mathcal{A}}$, and $M_{\mathcal{B}}$, such that

$$\forall n \in \mathbb{N}, \quad \|\ddot{\mathbf{u}}_n\|_{L^2(0, T; \mathbf{V}')} \leq C_4. \quad (3.12)$$

From (3.11), (3.12), it follows that there exist a subsequence $(\mathbf{u}_{n_k})_k$ and $\tilde{\mathbf{u}}$ such that

$$\begin{aligned} \dot{\mathbf{u}}_{n_k} &\rightharpoonup^* \dot{\tilde{\mathbf{u}}} \text{ in } L^\infty(0, T; \mathbf{H}), \quad \mathbf{u}_{n_k} \rightharpoonup^* \tilde{\mathbf{u}} \text{ in } L^\infty(0, T; \mathbf{V}), \\ \dot{\mathbf{u}}_{n_k} &\rightharpoonup \dot{\tilde{\mathbf{u}}} \text{ in } L^2(0, T; \mathbf{V}), \quad \ddot{\mathbf{u}}_{n_k} \rightharpoonup \ddot{\tilde{\mathbf{u}}} \text{ in } L^2(0, T; \mathbf{V}'). \end{aligned}$$

Since $\mathbf{V} \subset \mathbf{H}^\iota \subset \mathbf{H} \subset \mathbf{V}'$ with compact embedding from \mathbf{V} into \mathbf{H}^ι , according to a classical compactness result, see, e.g., [35], it follows that

$$\dot{\mathbf{u}}_{n_k} \rightarrow \dot{\tilde{\mathbf{u}}} \text{ in } L^2(0, T; \mathbf{H}^\iota),$$

where $1 > \iota > \frac{1}{2}$, so that, by the trace theorem,

$$\dot{\mathbf{u}}_{n_k} \rightarrow \dot{\tilde{\mathbf{u}}} \text{ in } L^2(0, T; (L^2(\Xi))^3) = (L^2(\Xi_T))^3. \quad (3.13)$$

By Lemma 3.3, for all $k \in \mathbb{N}$ and $t \in (0, T]$ we have

$$\begin{aligned} & |\dot{\mathbf{u}}_{n_k}(t) - \dot{\tilde{\mathbf{u}}}(t)|^2 + \|\mathbf{u}_{n_k}(t) - \mathbf{u}(t)\|^2 + \int_0^t \|\dot{\mathbf{u}}_{n_k} - \dot{\tilde{\mathbf{u}}}\|^2 d\tau \\ & \leq C_1 \int_0^t \{(\eta_1^{n_k} - \eta_2^{n_k} - \eta_1 + \eta_2, \dot{u}_{n_k N} - \dot{u}_N)_{L^2(\Xi)} \\ & \quad + (\boldsymbol{\varsigma}^{n_k} - \boldsymbol{\varsigma}, \dot{\mathbf{u}}_{n_k T} - \dot{\mathbf{u}}_T)_{(L^2(\Xi))^3}\} d\tau. \end{aligned}$$

Using the weak convergence properties of $(\eta_1^n)_n$, $(\eta_2^n)_n$, $(\boldsymbol{\varsigma}^n)_n$, and the strong convergence property (3.13), we can pass to limits in the right hand side of the previous estimates for $(\mathbf{u}_{n_k})_k$ and so we obtain that $\mathbf{u} = \tilde{\mathbf{u}}$ and

$$\mathbf{u}_{n_k} \rightarrow \mathbf{u} \text{ in } C^1([0, T]; \mathbf{H}) \cap W^{1,2}(0, T; \mathbf{V}). \quad (3.14)$$

As for every subsequence of $(\mathbf{u}_n)_n$, by using the same arguments as above that enabled to obtain the relation (3.14), one can always find a (sub)subsequence converging to \mathbf{u} in $C^1([0, T]; \mathbf{H}) \cap W^{1,2}(0, T; \mathbf{V})$, it follows that

$$\mathbf{u}_{(\eta_1^n, \eta_2^n, \varsigma^n)} \rightarrow \mathbf{u}_{(\eta_1, \eta_2, \varsigma)} \text{ in } C^1([0, T]; \mathbf{H}) \cap W^{1,2}(0, T; \mathbf{V}). \quad (3.15)$$

□

4 An equivalent fixed point problem and existence result

Let $\Phi : (L_+^2(\Xi_T))^2 \times (L^2(\Xi_T))^3 \rightarrow 2^{(L_+^2(\Xi_T))^2 \times (L^2(\Xi_T))^3} \setminus \{\emptyset\}$ be the set-valued mapping defined by

$$\begin{aligned} \forall (\eta_1, \eta_2, \varsigma) \in (L_+^2(\Xi_T))^2 \times (L^2(\Xi_T))^3 \\ \Phi(\eta_1, \eta_2, \varsigma) = \Lambda^3(\mathbf{u}_{(\eta_1, \eta_2, \varsigma)}, \beta_{\mathbf{u}_{(\eta_1, \eta_2, \varsigma)}}) \end{aligned} \quad (4.1)$$

where $\mathbf{u}_{(\eta_1, \eta_2, \varsigma)}$ is the solution of the variational equation (3.8) which corresponds to $(\eta_1, \eta_2, \varsigma)$ according to Lemma 3.2 and $\beta_{\mathbf{u}_{(\eta_1, \eta_2, \varsigma)}}$ is the solution of (3.6) corresponding to $\mathbf{u} = \mathbf{u}_{(\eta_1, \eta_2, \varsigma)}$.

Since $(\lambda_1, \lambda_2, \gamma)$ is a fixed point of Φ , i.e. $(\lambda_1, \lambda_2, \gamma) \in \Phi(\lambda_1, \lambda_2, \gamma)$, if and only if $(\mathbf{u}_{(\lambda_1, \lambda_2, \gamma)}, \lambda_1, \lambda_2, \gamma, \beta_{\mathbf{u}_{(\lambda_1, \lambda_2, \gamma)}})$ is a solution of the Problem P_v^2 , we consider the problem which consists in finding a fixed point of the set-valued mapping Φ , called also multivalued function or multifunction.

The existence of a fixed point of the multifunction Φ will be proved by using a corollary of the Ky Fan's fixed point theorem [17], proved in [30] in the particular case of a reflexive Banach space. We recall this result for the reader's convenience.

Definition 4.1. Let Y be a reflexive Banach space, D a weakly closed set in Y , and $F : D \rightarrow 2^Y \setminus \{\emptyset\}$ be a set-valued mapping. F is called sequentially weakly upper semicontinuous if $z_n \rightharpoonup z$, $y_n \in F(z_n)$ and $y_n \rightharpoonup y$ imply $y \in F(z)$.

Proposition 4.1. ([30]) Let Y be a reflexive Banach space, D a convex, closed and bounded set in Y , and $F : D \rightarrow 2^D \setminus \{\emptyset\}$ a sequentially weakly upper semicontinuous set-valued mapping such that $F(z)$ is convex for every $z \in D$. Then F has a fixed point.

Theorem 4.2. Under the assumptions of Sections 2 and 3, there exists $(\lambda_1, \lambda_2, \gamma) \in (L_+^2(\Xi_T))^2 \times (L^2(\Xi_T))^3$ such that $(\lambda_1, \lambda_2, \gamma) \in \Phi(\lambda_1, \lambda_2, \gamma)$. For

each fixed point $(\lambda_1, \lambda_2, \gamma)$ of the multifunction Φ , $(\mathbf{u}_{(\lambda_1, \lambda_2, \gamma)}, \lambda, \gamma, \beta_{\mathbf{u}_{(\lambda_1, \lambda_2, \gamma)}})$, where $\lambda = \lambda_1 - \lambda_2$, is a solution of the Problem P_v^1 and $(\mathbf{u}_{(\lambda_1, \lambda_2, \gamma)}, \lambda_1, \lambda_2, \gamma, \beta_{\mathbf{u}_{(\lambda_1, \lambda_2, \gamma)}})$ is a solution of the Problem P_v^2 .

Proof. We apply Proposition 4.1 to $Y = (L^2(\Xi_T))^5$, $F = \Phi$ and $D = [(L^2_+(\Xi_T))^2 \cap D_0] \times [(L^2(\Xi_T))^3 \cap D_1]$.

The set $D \subset (L^2(\Xi_T))^5$ is clearly convex, closed, and bounded.

For all $\mathbf{w} \in W^{1,2}(0, T; \mathbf{V})$, $v \in L^2(\Xi_T)$, the set $\Lambda^3(\mathbf{w}, v)$ is nonempty, closed, and convex, so that $\Phi(\eta_1, \eta_2, \varsigma)$ is a nonempty, closed, and convex subset of D for every $(\eta_1, \eta_2, \varsigma) \in D$.

In order to prove that the multifunction Φ is sequentially weakly upper semicontinuous, let $(\eta_1^n, \eta_2^n, \varsigma^n) \in D$, $(\delta_1^n, \delta_2^n, \varpi^n) \in \Phi(\eta_1^n, \eta_2^n, \varsigma^n) \forall n \in \mathbb{N}$, $(\eta_1^n, \eta_2^n, \varsigma^n) \rightharpoonup (\eta_1, \eta_2, \varsigma)$, $(\delta_1^n, \delta_2^n, \varpi^n) \rightharpoonup (\delta_1, \delta_2, \varpi)$ and let us verify that $(\delta_1, \delta_2, \varpi) \in \Phi(\eta_1, \eta_2, \varsigma)$.

We adopt the following notations:

$$\mathbf{u}_n := \mathbf{u}_{(\eta_1^n, \eta_2^n, \varsigma^n)}, \quad u_{nN} := u_{(\eta_1^n, \eta_2^n, \varsigma^n)N},$$

$$\mathbf{u}_\eta := \mathbf{u}_{(\eta_1, \eta_2, \varsigma)}, \quad u_{\eta N} := u_{(\eta_1, \eta_2, \varsigma)N},$$

$$\beta_n := \beta_{\mathbf{u}_{(\eta_1^n, \eta_2^n, \varsigma^n)}}, \quad \beta_\eta := \beta_{\mathbf{u}_{(\eta_1, \eta_2, \varsigma)}},$$

where $\beta_{\mathbf{u}_{(\eta_1^n, \eta_2^n, \varsigma^n)}}$ is the solution of (3.6) corresponding to $\mathbf{u}_{(\eta_1^n, \eta_2^n, \varsigma^n)}$ and $\beta_{\mathbf{u}_{(\eta_1, \eta_2, \varsigma)}}$ is the solution of (3.6) corresponding to $\mathbf{u}_{(\eta_1, \eta_2, \varsigma)}$.

By Lemma 3.4, we have

$$\mathbf{u}_n \rightarrow \mathbf{u}_\eta \quad \text{in } C^1([0, T]; \mathbf{H}) \cap W^{1,2}(0, T; \mathbf{V}), \quad (4.2)$$

which implies

$$\mathbf{u}_n \rightarrow \mathbf{u}_\eta, \quad \dot{\mathbf{u}}_n \rightarrow \dot{\mathbf{u}}_\eta \quad \text{in } (L^2(\Xi_T))^3, \quad (4.3)$$

and, by (3.7),

$$\beta_n(t) \rightarrow \beta_\eta(t) \quad \text{in } L^2(\Xi) \quad \text{for all } t \in [0, T]. \quad (4.4)$$

Now, by Lemma 3.1, if $(\delta_1^n, \delta_2^n, \varpi^n) \in \Phi(\eta_1^n, \eta_2^n, \varsigma^n) = \Lambda^3(\mathbf{u}_{(\eta_1^n, \eta_2^n, \varsigma^n)}, \beta_{\mathbf{u}_{(\eta_1^n, \eta_2^n, \varsigma^n)}})$ for all $n \in \mathbb{N}$, then

$$\underline{\kappa}([u_{nN}], \beta_n) \leq \delta_1^n - \delta_2^n \leq \bar{\kappa}([u_{nN}], \beta_n) \quad \text{a.e. in } \Xi_T, \quad (4.5)$$

$$|\varpi^n| \leq \mu(\dot{\mathbf{u}}_{nT}) (\delta_1^n + \delta_2^n) \quad \text{a.e. in } \Xi_T, \quad (4.6)$$

$$\varpi^n \cdot \dot{\mathbf{u}}_{nT} + \mu(\dot{\mathbf{u}}_{nT}) (\delta_1^n + \delta_2^n) |\dot{\mathbf{u}}_{nT}| = 0 \quad \text{a.e. in } \Xi_T, \quad (4.7)$$

for all $n \in \mathbb{N}$.

First, the relations (4.5) are equivalent to

$$\int_{\omega} \underline{\kappa}([u_{nN}], \beta_n) \leq \int_{\omega} (\delta_1^n - \delta_2^n) \leq \int_{\omega} \bar{\kappa}([u_{nN}], \beta_n), \quad (4.8)$$

for every measurable subset $\omega \subset \Xi_T$ and for all $n \in \mathbb{N}$.

By (4.3) and a converse of Lebesgue's dominated convergence theorem, it follows that there exists a subsequence of $(\mathbf{u}_n)_n$, denoted by $(\mathbf{u}_{n_k})_k$, such that

$$[u_{n_k N}] \rightarrow [u_{\eta N}], \quad \dot{\mathbf{u}}_{n_k T} \rightarrow \dot{\mathbf{u}}_{\eta T} \quad \text{a.e. in } \Xi_T. \quad (4.9)$$

The first convergence property in (4.9) enables to pass to limits in (4.8) with respect to n_k according to Fatou's lemma, by using (4.4), the semi-continuity of $\underline{\kappa}$ and $\bar{\kappa}$, the relation (2.2), and the convergence property

$$\int_{\omega} (\delta_1^n - \delta_2^n) \rightarrow \int_{\omega} (\delta_1 - \delta_2).$$

Thus, we obtain

$$\int_{\omega} \underline{\kappa}([u_{\eta N}], \beta_{\eta}) \leq \int_{\omega} (\delta_1 - \delta_2) \leq \int_{\omega} \bar{\kappa}([u_{\eta N}], \beta_{\eta}),$$

for every measurable subset $\omega \subset \Xi_T$, which implies

$$\underline{\kappa}([u_{\eta N}], \beta_{\eta}) \leq \delta_1 - \delta_2 \leq \bar{\kappa}([u_{\eta N}], \beta_{\eta}) \quad \text{a.e. in } \Xi_T. \quad (4.10)$$

Second, the relation (4.6) is equivalent to

$$\int_{\omega} |\varpi^n| \leq \int_{\omega} \mu(\dot{\mathbf{u}}_{nT}) (\delta_1^n + \delta_2^n),$$

for every measurable subset $\omega \subset \Xi_T$ and for all $n \in \mathbb{N}$. As $\mu(\xi, \cdot)$ is Lipschitz continuous with the Lipschitz constant independent of ξ , by using the second convergence property in (4.9) and passing to limits with respect to n_k we obtain

$$\begin{aligned} \int_{\omega} |\varpi| &\leq \liminf \int_{\omega} |\varpi^{n_k}| \leq \lim \int_{\omega} \mu(\dot{\mathbf{u}}_{n_k T}) (\delta_1^{n_k} + \delta_2^{n_k}) \\ &= \int_{\omega} \mu(\dot{\mathbf{u}}_{\eta T}) (\delta_1 + \delta_2). \end{aligned}$$

Thus

$$\int_{\omega} |\varpi| \leq \int_{\omega} \mu(\dot{\mathbf{u}}_{\eta T}) (\delta_1 + \delta_2),$$

for every measurable subset $\omega \subset \Xi_T$, which implies

$$|\varpi| \leq \mu(\dot{\mathbf{u}}_{\eta T}) (\delta_1 + \delta_2) \text{ a.e. in } \Xi_T. \quad (4.11)$$

Finally, consider the relation (4.7) which is equivalent to

$$\int_{\omega} \varpi^n \cdot \dot{\mathbf{u}}_{nT} + \int_{\omega} \mu(\dot{\mathbf{u}}_{nT}) (\delta_1^n + \delta_2^n) |\dot{\mathbf{u}}_{nT}| = 0, \quad (4.12)$$

for every measurable subset $\omega \subset \Xi_T$ and for all $n \in \mathbb{N}$. By the second convergence property in (4.9), we have

$$\mu(\dot{\mathbf{u}}_{n_k T}) |\dot{\mathbf{u}}_{n_k T}| \rightarrow \mu(\dot{\mathbf{u}}_{\eta T}) |\dot{\mathbf{u}}_{\eta T}| \text{ in } L^1(\Xi_T), \quad (4.13)$$

and, by the relations (2.2), (4.5),

$$\delta_1^n + \delta_2^n \rightharpoonup^* \delta_1 + \delta_2 \text{ in } L^\infty(\Xi_T). \quad (4.14)$$

Passing to limits in (4.12) by using (4.13) and (4.14), we obtain

$$\int_{\omega} \varpi \cdot \dot{\mathbf{u}}_{\eta T} + \int_{\omega} \mu(\dot{\mathbf{u}}_{\eta T}) (\delta_1 + \delta_2) |\dot{\mathbf{u}}_{\eta T}| = 0,$$

for every measurable subset $\omega \subset \Xi_T$, which implies

$$\varpi \cdot \dot{\mathbf{u}}_{\eta T} + \mu(\dot{\mathbf{u}}_{\eta T}) (\delta_1 + \delta_2) |\dot{\mathbf{u}}_{\eta T}| = 0. \quad (4.15)$$

From (4.10), (4.11), (4.15), it follows that $(\delta_1, \delta_2, \varpi) \in \Phi(\eta_1, \eta_2, \varsigma)$, so that we may apply the Proposition 4.1 to conclude the proof. \square

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