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► To cite this version:

Thibault Liard, Ismaïla Balogoun, Swann Marx, Franck Plestan. Boundary sliding mode control of a system of linear hyperbolic equations: a Lyapunov approach. *Automatica*, In press, 135, 10.1016/j.automatica.2021.109964 . hal-03084995

HAL Id: hal-03084995

<https://hal.science/hal-03084995v1>

Submitted on 21 Dec 2020

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Boundary sliding mode control of a system of linear hyperbolic equations: a Lyapunov approach

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Abstract

This paper introduces a new sliding mode methodology for a system of two hyperbolic equations. The main difference with the existing literature is the definition of the sliding variable, given here by the gradient of a Lyapunov functional. We state and prove an existence theorem and a global asymptotic stability result. The efficiency of our feedback-law is illustrated by some numerical simulations relying on implicit schemes.

Key words: Sliding mode control, hyperbolic equations, robust control, stabilization of PDEs, nonlinear systems.

1 Introduction

This paper is concerned with the generalization of the sliding mode control (for short SMC) strategy for a class of partial differential equations (for short PDEs) subject to an unknown boundary disturbance. To be more precise, our focus will be on a system of two hyperbolic systems controlled from the boundary (see [9] for a review on this class of system).

Stabilization of systems of linear transport equations of one dimension have been considered for many years. This is surely because this kind of systems models many physical phenomena. If [9] provides a good overview of the actual research lines concerning this topic of stabilization, it is worth mentioning some important articles in that direction. In [14], the local stabilization of a 2×2 nonlinear hyperbolic system is considered, where some distributed couplings are allowed. The feedback-law is built thanks to the *backstepping* method (see e.g., [22] for an overview on this topic). In [10], hyperbolic systems with some non-local terms are studied. More precisely, the design method relies on a Fredholm transform

instead of a more usual Volterra one. Generalizations to the case of systems of n linear hyperbolic equations have been considered in [17] or [21], to cite only few of the numerous papers dealing with such a topic. Let us also mention some recent papers about optimal time of stabilization [5,13] or about delay-robust stabilization [4,6].

For many years, the SMC strategy has been proved to be efficient for the robust control of nonlinear systems described with ordinary differential equations (for short ODEs). Most of the classical techniques for ODEs are exposed in [32,35]. Roughly speaking, this technique is decomposed into two steps: firstly, one selects a *sliding variable* such that the trajectories reach a *sliding surface*, where the global asymptotic stability is ensured; secondly, one designs a state feedback-law so that the sliding variable reaches the surface in finite-time on which the disturbance is rejected.

To the best of our knowledge, robust control for PDEs has been investigated for only few years. Most of the existing papers (see e.g., [34] or [26]) use PI controllers. In contrast with the sliding mode strategy, this imposes to consider disturbances which do not depend on time. In [15,16], a feedback-law constructed using the backstepping method is proposed to reject periodic disturbances. This design, based on the internal model approach, allows to reject disturbances which might not match with the control, at the price of firstly assuming the disturbances to be periodic, and of secondly knowing the related frequencies. It is in contrast with the sliding mode approach, which only requires to know the bounds of the perturbations. However, the sliding mode approach

* This research was conducted in the framework of the regional programme "Atlantc 2020, Research, Education and Innovation in Pays de la Loire", supported by the French Region Pays de la Loire and the European Regional Development Fund.

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needs the disturbances to match with the control.

The generalization of the SMC procedure to the PDE case is not new. However, many questions remain open. We believe that the main challenge is the choice of the sliding variable. Indeed, since one faces with an infinite-dimensional system, there exist many ways to select it. In [20], [33] [30], a backstepping strategy is used in order to design a sliding mode controller. Note that an active disturbance rejection control¹ (ADRC for short) strategy is also followed in [20] and [18], which is a different approach from the SMC one. In [33], the use of the backstepping method imposes to measure the full-state, and therefore the boundaries, where are located the control and the disturbance. Let us also mention [7,8,29], where a distributed SMC is considered. In the latter case, the sliding variable is the full state itself.

In this paper, a different strategy is followed. The sliding variable is derived from the gradient of some well-known Lyapunov functional in the hyperbolic context [9, Section 2.1.2]. We thus assume that we measure this quantity, which does not imply the knowledge of the boundaries where are located the control and the disturbance. Moreover, in contrast with the usual backstepping strategy, the bound on the derivative of the disturbance does not need to be known. Furthermore, the global asymptotic analysis is provided in some L^p space, which is not common in the literature. Indeed, most of the existing works give their stability results in the Hilbertian space L^2 . Let us mention however some recent works in that direction: [12], [3] or [11]. We also emphasize on the fact that our approach is adjustable to more general SMC methods such as adaptive approaches. Adaptive techniques [31,37] are interesting, due to the fact that reduced knowledge of the model is required (bounds of perturbations are not required). To that end, we illustrate numerically the efficiency of the adaptive SMC taken from [31] in our context.

It is also important to note that our approach could be extended to more general systems. Indeed, as soon as one has a Lyapunov function, the sliding variable and its related sliding control can be derived from it in a systematic way. As a consequence, one may consider the case of $n \times n$ linear hyperbolic equations with space varying velocities [9, Section 2.3] or distributed coupling terms as in [14]. In the latter ones, the length of the domain has to be sufficiently small as explained in [9, Section 5.6]. Systems of nonlinear hyperbolic equations may be also considered, but the results would become local. However, in

¹ ADRC method proposes to build an observer allowing the estimation of the disturbance. This estimation is then added in the feedback-law and allows to reject the perturbation. An interesting aspect of this strategy relies on the fact that the disturbance does not need to be matched with the control, i.e. it does not need to be located at the same place than the control.

order to avoid any further technicalities, and in order to ease the reading of our paper, we chose to focus on the case of a system of 2×2 linear hyperbolic equations.

This paper is organized as follows. Section 2 presents the system of linear hyperbolic equations, the sliding mode method and introduces the notion of solutions that will be used all along the paper. Section 3 gathers the main results of the paper, namely an existence theorem and a global asymptotic result. Section 4 is devoted to the proof of the main theorems. Section 5 illustrates via numerical simulations the efficiency of our sliding mode control. Finally, Section 6 collects some remarks and introduces some future research lines to be followed.

Notation. The set of non-negative real numbers is denoted in this paper by \mathbb{R}_+ . When a function f only depends on the time variable t (resp. on the space variable x), its derivative is denoted by \dot{f} (resp. f'). Given any subset of \mathbb{R} denoted by Ω (\mathbb{R}_+ or an interval, for instance), $L^p(\Omega; \mathbb{R}^n)$ denotes the set of (Lebesgue) measurable functions f_1, \dots, f_n such that, for $i = \{1, \dots, n\}$, $\int_{\Omega} |f_i(x)|^p dx < +\infty$ when $p \neq +\infty$ and such that $\sup \text{ess}_{x \in \Omega} |f_i(x)| < +\infty$ when $p = +\infty$. The associated norms are, for $p \neq +\infty$, $\|(f_1, \dots, f_n)\|_{L^p(\Omega; \mathbb{R}^n)}^p := \int_{\Omega} |f_1(x)|^p dx + \dots + \int_{\Omega} |f_n(x)|^p dx$ and, for $p = +\infty$, $\|f_1, \dots, f_n\|_{L^\infty(\Omega; \mathbb{R}^n)} := \sup \text{ess}_{x \in [0, L]} |f_1(x)| + \dots + |f_n(x)|$. For any $p \in [1, \infty]$, the Sobolev space $W^{1,p}(\Omega; \mathbb{R}^n)$ is defined by the set $\{(f_1, \dots, f_n) \in L^p(\Omega; \mathbb{R}^n) \mid (f'_1, \dots, f'_n) \in L^p(\Omega; \mathbb{R}^n)\}$. When $n = 1$, we simplify the notation and use $L^p(\Omega)$ or $W^{1,p}(\Omega)$. Given an interval Ω of \mathbb{R} , the set $C_c^\infty(\Omega)$ denotes the set of infinitely differentiable functions with compact support. All along the paper, we will say *a.e.* to denote properties satisfied almost everywhere, i.e. properties satisfied everywhere except in a set of Lebesgue measure equal to 0.

Acknowledgement: We would like to thank Vincent Andrieu, Yacine Chitour and Vincent Perrollaz for fruitful discussions, which improve significantly the results of the paper.

2 Problem statement

2.1 Boundary linear Hyperbolic System

Let $\lambda_1 > 0$, $\lambda_2 > 0$ and $k_2 \in \mathbb{R}$. We consider the following linear hyperbolic system

$$\begin{cases} \partial_t R_1(t, x) + \lambda_1 \partial_x R_1(t, x) = 0, \\ \partial_t R_2(t, x) - \lambda_2 \partial_x R_2(t, x) = 0, \\ R_1(t, 0) = u(t) + d(t), \\ R_2(t, L) = k_2 R_1(t, L), \\ R_1(0, x) = R_1^0(x), R_2(0, x) = R_2^0(x), \end{cases} \quad (1)$$

where $d(\cdot) \in L^\infty(\mathbb{R})$ is an unknown disturbance and u is a control function. When the system (1) is undisturbed ($d = 0$), it is nowadays well-known that the control function

$$u(t) := k_1 R_2(t, 0),$$

allows to stabilize the system when $|k_1 k_2| < 1$, see [9, Theorem 2.4.]. The proof, for the L^2 -topology, relies on the Lyapunov functional

$$V_2(t) = \int_0^L \left(\frac{p_1}{\lambda_1} R_1^2(t, x) e^{\frac{-\nu x}{\lambda_1}} + \frac{p_2}{\lambda_2} R_2^2(t, x) e^{\frac{\nu x}{\lambda_2}} \right) dx. \quad (2)$$

As classical in the SMC literature, our goal is to find a state feedback control u which allows to reject the disturbance and to globally asymptotically stabilize the system around the equilibrium point $(0, 0)$ in the functional space $L^p(0, L; \mathbb{R}^2)$. As mentioned in the introduction, one seeks a sliding surface (i.e., a subspace of the state space) where some desired global asymptotic stability properties are satisfied.

In our case, our aim is to find a sliding surface on which (1) becomes

$$\begin{cases} \partial_t R_1(t, x) + \lambda_1 \partial_x R_1(t, x) = 0, \\ \partial_t R_2(t, x) - \lambda_2 \partial_x R_2(t, x) = 0, \\ R_1(t, 0) = k_1 R_2(t, 0), \\ R_2(t, L) = k_2 R_1(t, L), \end{cases} \quad (3)$$

in finite time $t_0 > 0$ with k_1 chosen such that $|k_1 k_2| < 1$. We know from [9, Theorem 2.1 and 2.2] that $(0, 0)$ is exponentially stable for (3). The next section will provide a definition of this sliding surface (and its related sliding variable) together with the sliding mode controller.

Remark 1 (Generalization to $n \times n$ systems) *Our approach could be extended to systems in the form*

$$\partial_t \mathbf{R}(t, x) + \Lambda \partial_x \mathbf{R}(t, x) = 0, \quad (t, x) \in \mathbb{R}_+ \times [0, L], \quad (4)$$

where $\mathbf{R}(t, x) \in \mathbb{R}^n$ and Λ is a diagonal matrix defined by $\Lambda := \text{diag}(\lambda_1, \dots, \lambda_m, -\lambda_{m+1}, \dots, -\lambda_n)$ with $\lambda_i > 0$ for any $i \in \{1, \dots, n\}$. Associated to these velocities, one defines

$$\mathbf{R}^+(t, x) := \begin{pmatrix} R_1(t, x) \\ \vdots \\ R_m(t, x) \end{pmatrix}, \quad \mathbf{R}^-(t, x) := \begin{pmatrix} R_{m+1}(t, x) \\ \vdots \\ R_n(t, x) \end{pmatrix} \quad (5)$$

In this case, one may consider

$$\mathbf{R}^+(t, 0) = \mathbf{B}(u(t) + d(t)), \quad \mathbf{R}^-(t, L) = \mathbf{K}_2 \mathbf{R}^+(t, L) \quad (6)$$

with u and d scalar valued functions, and $\mathbf{B} \in \mathbb{R}^{m \times 1}$ and $\mathbf{K}_2 \in \mathbb{R}^{(n-m) \times m}$. As in (1), u denotes the control and d the disturbance.

On the sliding surface that one aims at reaching, the system becomes

$$\begin{cases} \partial_t \mathbf{R}(t, x) + \Lambda \partial_x \mathbf{R}(t, x) = 0, \quad (t, x) \in \mathbb{R}_+ \times [0, L], \\ \mathbf{R}^+(t, 0) = \mathbf{B} \mathbf{K}_1 \mathbf{R}^-(t, 0), \quad t \in \mathbb{R}_+, \\ \mathbf{R}^-(t, L) = \mathbf{K}_2 \mathbf{R}^+(t, L), \quad t \in \mathbb{R}_+, \\ \mathbf{R}(0, x) = \mathbf{R}_0(x), \end{cases} \quad (7)$$

with $\mathbf{K}_1 \in \mathbb{R}^{1 \times (n-m)}$ such that the pair $(\mathbf{B} \mathbf{K}_1, \mathbf{K}_2)$ satisfies the condition given in [9, Theorem 3.2.]. This imposes to assume some stabilizability condition on the pair $(\mathbf{K}_1, \mathbf{B})$.

2.2 Sliding surface

Let $p \in [1, \infty]$. We introduce the following sliding surface Σ defined as follows. For any $t > 0$

$$\Sigma := \{ (R_1(t, \cdot), R_2(t, \cdot)) \in L^p(0, L)^2 \mid S(t) = 0 \},$$

with $S : \mathbb{R}_+ \rightarrow \mathbb{R}$ defined by

$$S(t) := \int_0^L \left(\frac{p_1}{\lambda_1} R_1(t, x) e^{\frac{-\nu x}{\lambda_1}} + \frac{p_2}{\lambda_2} R_2(t, x) e^{\frac{\nu x}{\lambda_2}} \right) dx. \quad (8)$$

We will sometimes write $S(t)$ by $S(R_1(t, \cdot), R_2(t, \cdot))$ to lighten the statements. This function represents the sliding variable mentioned earlier. As explained in [28], this sliding variable represents the gradient of the Lyapunov function (2). We consider the set-valued sliding mode controller u defined by, for a.e $t \geq 0$,

$$u(t) = k_1 R_2(t, 0) - K \text{sign}(S(t)), \quad (9)$$

where k_1 is defined later, S is introduced in (8), the gain $K > \|d\|_{L^\infty(\mathbb{R}_+)}$ and the set-valued function sign is defined by

$$\text{sign}(z) = \begin{cases} -1 & \text{if } z < 0, \\ [-1, 1] & \text{if } z = 0, \\ 1 & \text{if } z > 0. \end{cases}$$

From (9), the system (1) is rewritten as

$$\begin{cases} \partial_t R_1(t, x) + \lambda_1 \partial_x R_1(t, x) = 0, \\ \partial_t R_2(t, x) - \lambda_2 \partial_x R_2(t, x) = 0, \\ R_1(t, 0) = k_1 R_2(t, 0) - K \text{sign}(S(t)) + d(t), \\ R_2(t, L) = k_2 R_1(t, L), \\ R_1(0, x) = R_1^0(x), \quad R_2(0, x) = R_2^0(x). \end{cases} \quad (10)$$

For a.e $t \geq 0$, we can deduce, after some formal integration by parts, that

$$\begin{aligned} \dot{S}(t) &= -\nu S(t) - [p_1 e^{-\frac{\nu L}{\lambda_1}} - p_2 k_2 e^{\frac{\nu L}{\lambda_2}}] R_1(t, L) \\ &\quad + [p_1 k_1 - p_2] R_2(t, 0) - K p_1 \text{sign}(S(t)) + p_1 d(t). \end{aligned}$$

One picks $p_2 = k_1 = \frac{e^{-\nu(\frac{L}{\lambda_1} + \frac{L}{\lambda_2})}}{k_2}$ and $p_1 = 1$. Then, we deduce that

$$\dot{S}(t) = -\nu S(t) - K \text{sign}(S(t)) + d(t).$$

Hence, since $K > \|d\|_{L^\infty(\mathbb{R}_+)}$, for all t sufficiently large, the solution (R_1, R_2) belongs to the sliding surface Σ and $S(t) = \dot{S}(t) = 0$. This implies that the latter equation reduces to be $K \text{sign}(S(t)) + d(t) = 0$. Going back to $R_1(t, 0)$ in (1), one therefore obtains that $R_1(t, 0) = u(t) + d(t) = k_1 R_2(t, 0)$. This shows (formally) that, on the sliding surface Σ , (1) is equal to (3).

Remark 2 (Space-varying velocities) When the velocities λ_1 and λ_2 are space-varying, we may consider the following sliding variable defined by

$$S(t) := \int_0^L (q_1(x) R_1(t, x) + q_2(x) R_2(t, x)) dx,$$

with

$$\begin{aligned} q_1(x) &= \frac{p_1}{\lambda_1(x)} e^{-\left(\int_0^x \frac{\mu}{\lambda_1(y)} dy\right)}, \\ q_2(x) &= \frac{p_2}{\lambda_2(x)} e^{\left(\int_0^x \frac{\mu}{\lambda_2(y)} dy\right)}. \end{aligned}$$

The positive constants p_1 and p_2 are chosen in an appropriate way.

Remark 3 (SMC for $n \times n$ systems (4)) At the light of Remark 1, the sliding variable would become for the system (4):

$$\begin{aligned} S(t) &:= \sum_{i=1}^m \frac{p_i}{\lambda_i} \int_0^L e^{-\frac{\mu}{\lambda_i} x} R_i(t, x) dx \\ &\quad + \sum_{i=m+1}^n \frac{p_i}{\lambda_i} \int_0^L e^{\frac{\mu}{\lambda_i} x} R_i(t, x) dx, \end{aligned}$$

with the positive constants p_i and μ chosen in an appropriate way.

2.3 Weak solution to the Cauchy problem

This subsection is devoted to the introduction of some notions of solutions. As usual in the PDE context, one needs to consider a weak notion of solutions, i.e. solutions admitting discontinuities in their domain. One can

expect such phenomena since the initial data and the control under consideration in this paper are naturally discontinuous.

Definition 1 Let $p \in [1, \infty]$ and $R_1^0, R_2^0 \in L^p(0, L)$. We say that a map $(R_1, R_2) : [0, \infty) \times (0, L) \rightarrow \mathbb{R}^2$ is a solution of the Cauchy problem (10) if there exists $u \in L^p(\mathbb{R})$ such that

- (1) If $p \neq \infty$, $(R_1, R_2) \in C^0([0, \infty); L^p((0, L); \mathbb{R}^2))$. If $p = \infty$, $(R_1, R_2) \in C^0([0, \infty); L^1((0, L); \mathbb{R}^2))$.
- (2) For every $T > 0$, for every test functions $(\varphi_1, \varphi_2) \in C^1([0, T] \times [0, L]; \mathbb{R}^2)$

$$\lambda_1 \varphi_1(t, L) - k_2 \lambda_2 \varphi_2(t, L) = 0, \quad (11)$$

we have

$$\begin{aligned} &\int_0^L (\varphi_1(T, x) R_1(T, x) + \varphi_2(T, x) R_2(T, x)) dx \\ &- \int_0^L (\varphi_1(0, x) R_1^0(x) + \varphi_2(0, x) R_2^0(x)) dx \\ &- \int_0^T \lambda_1 \varphi_1(t, 0) (u(t) + d(t)) - \lambda_2 \varphi_2(t, 0) R_2(t, 0) dt \\ &- \int_0^L \int_0^T (\partial_t \varphi_1(t, x) + \lambda_1 \partial_x \varphi_1(t, x)) R_1(t, x) \\ &\quad + (\partial_t \varphi_2(t, x) - \lambda_2 \partial_x \varphi_2(t, x)) R_2(t, x) dt dx = 0. \end{aligned} \quad (12)$$

- (3) For a.e $t \geq 0$,

$$R_1(t, 0) \in k_1 R_2(t, 0) - K \text{sign}(S(t)) + d(t),$$

with S given in (8).

Remark 4 They may exist multiple measurable selection functions u (see e.g., [36] for a survey on such a notion) such that (R_1, R_2) satisfies (1), (2) and (3) of Definition 1. This leads to the possible existence of multiple weak solutions of (10).

3 Main results

We are now in position to state our two main results, namely: an existence theorem and a robust stabilization theorem.

Theorem 1 (Existence) Let $p \in [1, \infty]$ and $R_1^0, R_2^0 \in L^p(0, L)$. The system (10) admits at least one weak solution (R_1, R_2) . Moreover, if $p = \infty$ then $R_1, R_2 \in L^\infty((0, \infty) \times (0, L))$.

Remark 5 (Uniqueness) We may prove the uniqueness of the solution in this case. However, when considering for instance an adaptive controller, uniqueness fails to be true. For the sake of generality, we prove the robust stabilization for any possible weak solutions.

Theorem 2 (Stabilization) Let $p \in [1, \infty]$ and $R_1^0, R_2^0 \in L^p(0, L)$ and $K - \|d\|_{L^\infty(\mathbb{R}_+)} > 0$. There exist positive constants C, μ such that, for any $t \in [0, \infty)$,

$$\begin{aligned} & \| (R_1(t, \cdot), R_2(t, \cdot))^T \|_{L^p((0, L); \mathbb{R}^2)} \leq \\ & C e^{-\mu t} \| (R_1^0(\cdot), R_2^0(\cdot))^T \|_{L^p((0, L); \mathbb{R}^2)}, \end{aligned}$$

for any weak solution (R_1, R_2) of (10).

Remark 6 We want to emphasize on the fact that our framework is quite general. We believe that it allows to use more sophisticated tools from the sliding mode literature. One may for instance consider adaptive SMC as in [31] or super-twisting controller as in [23]. However, to make easier the reading of the paper, we do not want to enter in such details. In Section 5, we illustrate the efficiency of the algorithm proposed in [31] on the hyperbolic case.

4 Proof of Theorems 1 and 2

4.1 Existence of weak solutions

We consider the following ODE

$$\begin{cases} \dot{\gamma}(t) \in -\nu\gamma(t) - K \text{sign}(\gamma(t)) + d(t), & t \in \mathbb{R}_+, \\ \gamma(0) = \gamma_0, \end{cases} \quad (13)$$

where $\nu > 0, d \in L^\infty(\mathbb{R}_+), K > \|d\|_{L^\infty(\mathbb{R}_+)}$ and $\gamma_0 \in \mathbb{R}$. The ODE is understood in the sense of Filippov [19]. A Filippov solution of (13) is an absolutely continuous map that satisfies (13) for almost all $t \geq 0$. In [1, Example 2.19], the existence of Filippov solutions for (13) is proved.

Lemma 1 The ODE (13) admits a Filippov solution. Moreover, there exists $t_0 > 0$ such that, for any Filippov solution γ of (13),

$$\gamma \in W^{1, \infty}(\mathbb{R}) \text{ and } \gamma(t) = 0, \forall t \geq t_0.$$

Remark 7 When $K > \|d\|_{L^\infty(\mathbb{R}_+)}$, there exists a unique Filippov solution and $t_0 < \frac{|\gamma(0)|}{K - \|d\|_{L^\infty(\mathbb{R}_+)}}$.

Let γ be a Filippov solution of (13) with initial condition $\gamma(0) = S(R_1^0, R_2^0)$. We consider the following PDE

$$\begin{cases} \partial_t Y_1(t, x) + \lambda_1 \partial_x Y_1(t, x) = 0, \\ \partial_t Y_2(t, x) - \lambda_2 \partial_x Y_2(t, x) = 0, \\ Y_1(t, 0) = k_1 Y_2(t, 0) + \dot{\gamma}(t) + \nu \gamma(t), \\ Y_2(t, L) = k_2 Y_1(t, L), \\ Y_1(0, x) = R_1^0(x), Y_2(0, x) = R_2^0(x). \end{cases} \quad (14)$$

From Appendix A, the PDE (14) admits a unique weak solutions (Y_1, Y_2) constructed using the method of characteristics, see [9, Section 2.1]. In particular, (Y_1, Y_2) satisfies the points (1) and (2) of Definition 1 with $u(t) = -d(t) + k_1 Y_2(t, 0) + \dot{\gamma}(t) + \nu \gamma(t)$.

We now prove that the weak solution (Y_1, Y_2) of (14) satisfies Definition 1 (3). To that end, we introduce the function

$$\sigma(t) = S(Y_1(t, \cdot), Y_2(t, \cdot)),$$

where S is defined in (8) and (Y_1, Y_2) the weak solution of (14). From Appendix B, for any $T > 0$, σ is a Carathéodory solution of

$$\begin{cases} \dot{\sigma}(t) = -\nu\sigma(t) + \dot{\gamma}(t) + \nu\gamma(t), & t \in [0, T], \\ \sigma(0) = S(R_1^0, R_2^0). \end{cases} \quad (15)$$

A Carathéodory solution of (15) is an absolutely continuous map that satisfies (15) for almost every t . Moreover, from (15) and Lemma 1, $\sigma \in W^{1, \infty}(\mathbb{R})$. We introduce $g \in W^{1, 1}(\mathbb{R}) \cap W^{1, \infty}(\mathbb{R})$ defined by $g(t) = \sigma(t) - \gamma(t)$. From (13) and (15) with $\gamma(0) = S(R_1^0, R_2^0)$, g is solution of

$$\begin{cases} \dot{g}(t) = -\nu g(t) \\ g(0) = 0 \end{cases} \quad (16)$$

Thus, for any $t \in \mathbb{R}$, $g(t) = 0$. By definition of g , we deduce that for any $t \in \mathbb{R}$, $\sigma(t) = \gamma(t)$. Therefore, we have

$$\begin{aligned} Y_1(t, 0) &= k_1 Y_2(t, 0) + \dot{\gamma}(t) + \nu \gamma(t) \\ &\in k_1 Y_2(t, 0) - K \text{sign}(\sigma(t)) + d(t) \end{aligned}$$

Thus, (Y_1, Y_2) satisfies the point (3) of Definition 1. We conclude that, for any Filippov solution γ of (13) with initial condition $\gamma(0) = S(R_1^0, R_2^0)$, the associated weak solution $(Y_1(\cdot), Y_2(\cdot))$ of (14) is a weak solution of (10).

Remark 8 If the ODE (13) admits multiple Filippov solutions, then the PDE (10) has multiple weak solutions constructed from the coupled PDE-ODE (14)-(13).

4.2 Robust stabilization

Let (R_1, R_2) a weak solution of (10). From Definition 1, there exists $u \in L^p(\mathbb{R})$ such that $R_1(t, 0) = u(t) + d(t) \in k_1 R_2(t, 0) - K \text{sign}(S(t)) + d(t)$, whence the existence of $v \in L^p(\mathbb{R})$ such that

$$u(t) = k_1 R_2(t, 0) + v(t),$$

with $v(t) \in -K \text{sign}(S(t))$. From Appendix B which holds replacing (B.1) by $f(t) = v(t) + d(t)$, for any $T > 0$,

S defined in (8) is a Carathéodory solution of

$$\begin{cases} \dot{S}(t) = -\nu S(t) + v(t) + d(t), & t \in [0, T], \\ S(0) = S(R_1^0, R_2^0). \end{cases} \quad (17)$$

Since $v(\cdot) \in -K \text{sign}(S(t))$, then S is a Filippov solution of (13) with initial condition $S(R_1^0, R_2^0)$. From Lemma 1, $S \in W^{1,\infty}(\mathbb{R})$ and there exists a finite time t_0 (independent of v) such that

$$\dot{S}(t) + \nu S(t) = 0 \text{ for any } t > t_0.$$

Thus, from (17), for any $t > t_0$, $v(t) + d(t) = 0$.

We conclude that there exists a finite time t_0 such that for any weak solution (R_1, R_2) of (10), for any $t > t_0$, (R_1, R_2) is a weak solution of

$$\begin{cases} \partial_t R_1(t, x) + \lambda_1 \partial_x R_1(t, x) = 0, \\ \partial_t R_2(t, x) - \lambda_2 \partial_x R_2(t, x) = 0, \\ R_1(t, 0) = k_1 R_2(t, 0), \\ R_2(t, L) = k_2 R_1(t, L), \end{cases} \quad (18)$$

From Appendix C, the system (18) is exponentially stable in L^p -norm, that is to say there exist $C, \mu > 0$ such that

$$\begin{aligned} & \|(R_1(t, \cdot), R_2(t, \cdot))^T\|_{L^p((0,L), \mathbb{R}^2)} \\ & \leq C e^{-\mu(t-t_0)} \|(R_1(t_0, \cdot), R_2(t_0, \cdot))^T\|_{L^p((0,L), \mathbb{R}^2)}. \end{aligned} \quad (19)$$

Moreover, according to Theorem 1 and Appendix C, there exists $C > 0$ such that $\forall t \in [0, t_0]$,

$$\begin{aligned} & \|(R_1(t, \cdot), R_2(t, \cdot))^T\|_{L^p((0,L), \mathbb{R}^2)} \leq \\ & C \|(R_1^0(\cdot), R_2^0(\cdot))^T\|_{L^p((0,L), \mathbb{R}^2)}, \end{aligned} \quad (20)$$

which concludes the proof of Theorem 2

Remark 9 Since S is a Filippov solution of (13) with initial condition $\gamma(0) = S(R_1^0, R_2^0)$, a weak solution of (10) is a weak solution of (14). As a consequence, the set of weak solutions of (10) is the set

$$\{(Y_1^\gamma, Y_2^\gamma) \mid \gamma \text{ Filippov solution of (13)}\}$$

where (Y_1^γ, Y_2^γ) is the unique weak solution of (14) associated to γ .

5 Numerical illustration

5.1 Conventional SMC

Numerical scheme: the approximate solution (R_1^n, R_2^n) of (10) at time $t_n = n\Delta t$ is constructed using an upwind

scheme as described in [24, Section 10.7]. The ghost-cell boundary conditions [25, Section 7] are computed as follows

$$\begin{aligned} R_1^n(t, 0) &= k_1 \bar{R}_2^n(t, 0) - K \text{sign}_v(S(\bar{R}_1^n(t), \bar{R}_2^n(t))) + d(t), \\ R_2^n(t, L) &= k_2 \bar{R}_1^n(t, L), \end{aligned}$$

where S is defined in (8), $(\bar{R}_1^n, \bar{R}_2^n)$ are constructed using an upwind scheme from (R_1^{n-1}, R_2^{n-1}) and sign_v is the Yoshida approximation of sign defined as

$$\text{sign}_v(x) = \begin{cases} \text{sign}(x) & \text{if } |x| > v, \\ \frac{x}{v} & \text{if } |x| \leq v. \end{cases}$$

Note that the construction of the sliding mode control is inspired from [2, Section 5.1], where the sign function is approximated with the projection of this function on the interval $[-K, K]$, which corresponds indeed to a saturation. As a consequence, the chattering phenomenon is reduced, see Figure 1.

Simulations: let $L = 3$, $\lambda_1 = 2$, $\lambda_2 = -1$, $k_2 = -1$, $\nu = 0.1$, $K = 2$ and $v = 0.02$. The time-space step variation $(\Delta t, \Delta x) = (0.0037, 0.0150)$ satisfies the CFL condition $\max(|\lambda_1|, |\lambda_2|) \leq \Delta x / \Delta t$. We consider the following disturbance $d(t) = \sin(t)$. The associated approximate solutions $(R_1^n(t, \cdot), R_2^n(t, \cdot))$ of (10) at time $t = 0$ and at $t = 50$ are plotted with respect to $x \in [0, L]$, see Figure 1, 2, 3, 4.

Figure 1 shows that the sliding surface is reached in finite time. In Figures 2 and 3, the robust stabilization of R_1^n and R_2^n , respectively, is illustrated. Figure 4 shows the behavior of the controller. After a certain amount of time, one can note that the controller is almost equal to the disturbance $-d$, which is a classical behavior in the SMC literature. This shows that one can reconstruct the perturbation from a certain time thanks to the feedback control law defined in (9).

5.2 Adaptive SMC

If $\|d\|_{L^\infty(\mathbb{R}_+)}$ is an unknown parameter, we use the first adaptive sliding mode control law in [31]. As a consequence, the sliding mode control u in (9) is replaced by

$$u(t) = k_1 R_2(t, 0) - K(t) \text{sign}(S(t)) \quad (21)$$

where the adaptive gain $K(\cdot)$ is defined as follows. Let $t \geq 0$ and Δt the time step variation, if $|S(t)| > 4K(t)\Delta t$ then

$$\begin{cases} \dot{K}(t) = \bar{K}_1 |S(t)|, \\ K(0) = \bar{K}_0, \end{cases}$$

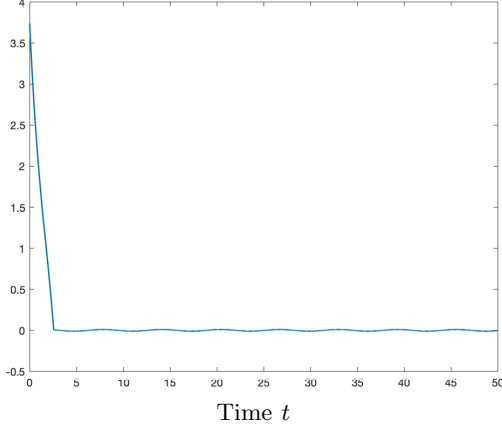


Fig. 1. Plotting of the approximate sliding surface $S(R_1^n(t, \cdot), R_2^n(t, \cdot))$ with respect to the time t where $(R_1^n(t, \cdot), R_2^n(t, \cdot))$ is an approximate solution of (10) at time t and $d(t) = \sin(t)$

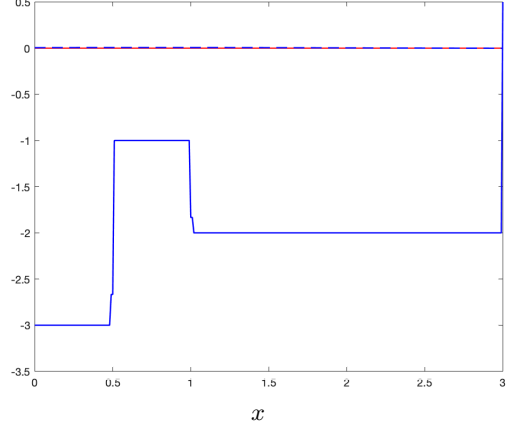


Fig. 3. Plotting of $R_2^n(t, \cdot)$ at time $t = 0$ (—) and $t = 50$ (- -) with respect to $x \in [0, L]$ with $d(t) = \sin(t)$. The equilibrium point 0 is plotted in red straight line.

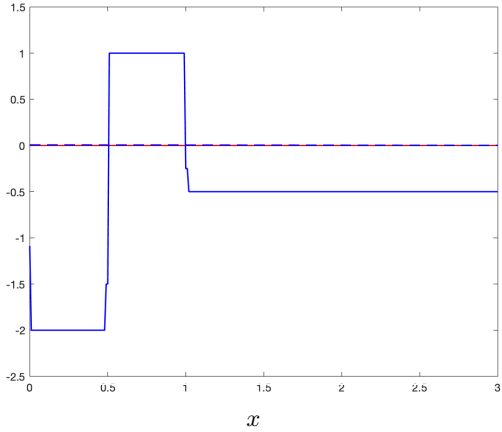


Fig. 2. Plotting of $R_1^n(t, \cdot)$ at time $t = 0$ (—) and $t = 50$ (- -) with respect to $x \in [0, L]$ with $d(t) = \sin(t)$. The equilibrium point 0 is plotted in red straight line.

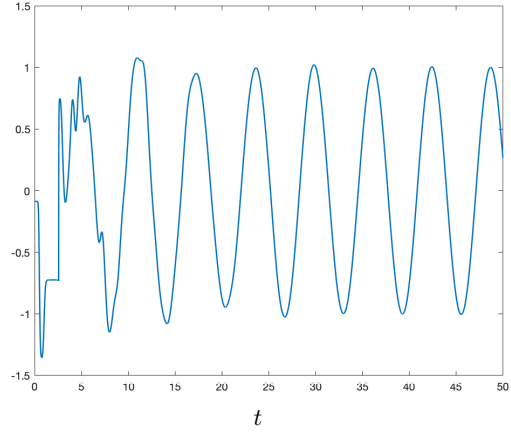


Fig. 4. Plotting of the sliding mode control u defined in (9) with respect to the time t .

with \bar{K}_0, \bar{K}_1 two positive constants and S defined in (8). If $|S(t)| \leq 4K(t)\Delta t$ then

$$K(t) = \bar{K}_2(t)|\eta| + \bar{K}_3.$$

Above, $\bar{K}_2(\cdot)$ is a piecewise constant function with discontinuous points $(t_i)_i$ defined by $S(t_i - \Delta t) > 4K(t_i - \Delta t)\Delta t$ and $S(t_i) \leq 4K(t_i)\Delta t$. For any $t \in [t_i, t_{i+1})$, $\bar{K}_2(t) = K(t_i)$. Note that t_i is on-line updated. The function η is solution of

$$\begin{cases} \tau \dot{\eta}(t) + \eta(t) = \text{sign}(S(t)), \\ \eta(0) = 0, \end{cases}$$

with $\tau > 0$ and \bar{K}_3 is a positive constant.

Numerical scheme: We choose $\bar{K}_0 = 2, \bar{K}_1 = 1, \tau = 0.1, \bar{K}_3 = 1$ and $d(t) = \sin(t)$. The other parameters are the same as in Section 5.1.

Figure 5 shows that the sliding surface is reached in finite-time. Figures 6 and 7 show that R_1^n and R_2^n are stabilized around the equilibrium point $(0, 0)$, meaning that our method is quite efficient. In Figure 8, the behavior of the adaptive gain K is given. One can see that it behaves almost like the function $t \mapsto \sin(t)$, which is the disturbance to be rejected. Finally, in Figure 9, the behavior of the controller u given in (21) is illustrated. Note that the amplitude of the adaptive control in Figure 9 is larger than the one in Figure 4 since the bound of the disturbance is unknown. From a certain time, it behaves like in the case where the control is not time-varying.

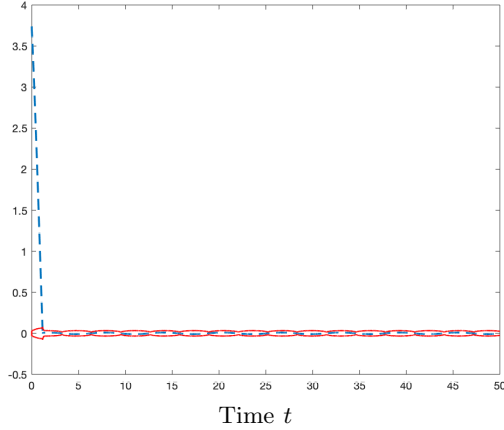


Fig. 5. Plotting of the approximate sliding surface $S(R_1^n(t, \cdot), R_2^n(t, \cdot))$ (---) and $t \mapsto 4K(t)\Delta t$ (—) with respect to the time t where $(R_1^n(t, \cdot), R_2^n(t, \cdot))$ is an approximate solution of (1) with u defined in (21) and $d(t) = \sin(t)$

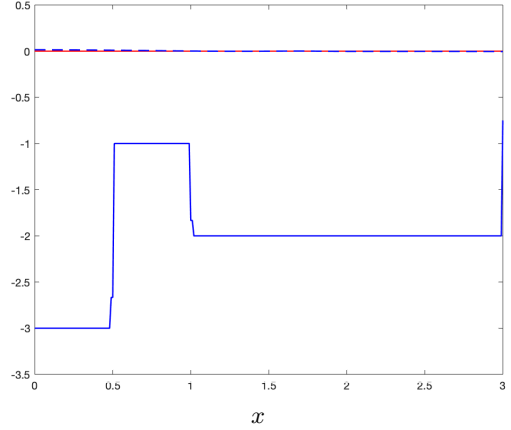


Fig. 7. Plotting of $R_2^n(t, \cdot)$ at time $t = 0$ (—) and $t = 50$ (---) with respect to the time t with u defined in (21) and $d(t) = \sin(t)$. The equilibrium point 0 is plotted in red straight line.

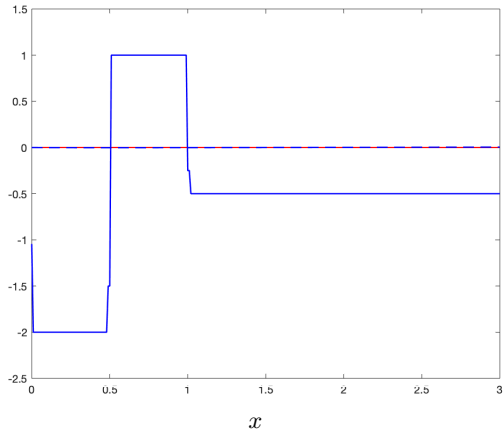


Fig. 6. Plotting of $R_1^n(t, \cdot)$ at time $t = 0$ (—) and $t = 50$ (---) with respect to the time t with u defined in (21) and $d(t) = \sin(t)$. The equilibrium point 0 is plotted in red straight line.

6 Conclusion

In this paper, we proposed a new approach for the sliding mode control applied to a specific class of PDEs, namely a system of two transport equations. It is a Lyapunov approach, since the sliding variable is based on the gradient of the classical Lyapunov function given in [9]. We have proved existence of the closed-loop system and have shown that the robust stabilization holds.

Many further research lines remain to be followed. It is important to note that our approach is a *state-feedback* approach, and one may instead consider an *output-feedback* approach, which would require to introduce a notion of sliding mode observer. It might be also interesting to investigate the case of systems described with

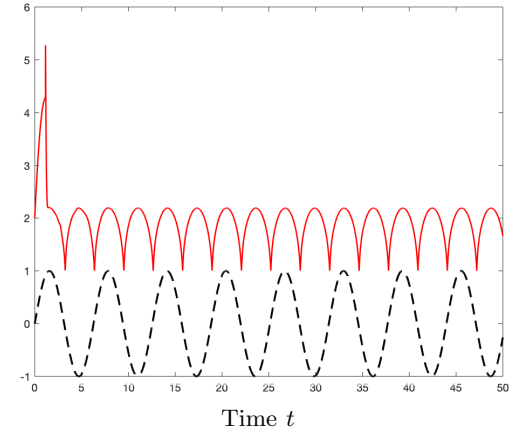


Fig. 8. Plotting of the adaptive gain K (—) and the disturbance d (---) with respect to the time t .

operators, as it has been done in [27], in the case of saturated feedback-laws. Finally, we may also consider sliding mode of higher order, as it has been done in [23].

A Notion of solutions

Let $p \in [1, \infty]$, $R_1^0, R_2^0 \in L^p(0, L)$ and γ be a Filippov solution of (13) with initial condition $\gamma(0) = S(R_1^0, R_2^0)$. We consider the linear hyperbolic system (14).

Definition 2 (Y_1, Y_2) is a weak solution of (14) if the points (1) and (2) of Definition 1 is satisfied replacing $u(\cdot)$ by $-d(\cdot) + \dot{\gamma}(\cdot) + \nu\gamma(\cdot)$.

Lemma 2 There exists a unique weak solution (Y_1, Y_2) of (14). Moreover, if $p = \infty$ then $Y_1, Y_2 \in L^\infty((0, \infty) \times (0, L))$.

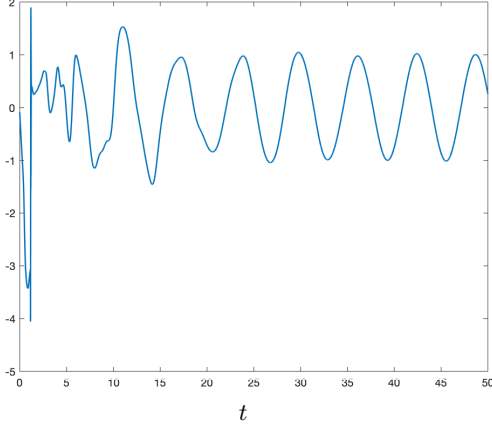


Fig. 9. Plotting of the adaptive sliding mode control u defined in (21) with respect to the time t .

Using the method of characteristics (see [9, Section 2.1]), we give an explicit weak solution (Y_1, Y_2) of (14). We introduce the function f defined by

$$f(t) = \dot{\gamma}(t) + \nu\gamma(t).$$

Let $m_1 \in \mathbb{Z}$ such that

$$m_1 \left(\frac{L}{\lambda_1} + \frac{L}{\lambda_2} \right) \leq t - \frac{x}{\lambda_1} < (m_1 + 1) \left(\frac{L}{\lambda_1} + \frac{L}{\lambda_2} \right). \quad (\text{A.1})$$

We define Y_1 for a.e. (t, x) as follows. If $m_1 < 0$ then

$$Y_1(t, x) = R_1^0(-\lambda_1 t + x)$$

We now assume that $m_1 \geq 0$ and we define

$$t_i := t - i \left(\frac{L}{\lambda_1} + \frac{L}{\lambda_2} \right) - \frac{x}{\lambda_1}, \quad i \in \mathbb{N} \quad (\text{A.2})$$

If $m_1 \left(\frac{L}{\lambda_1} + \frac{L}{\lambda_2} \right) \leq t - \frac{x}{\lambda_1} < m_1 \left(\frac{L}{\lambda_1} + \frac{L}{\lambda_2} \right) + \frac{L}{\lambda_2}$ then

$$Y_1(t, x) = (k_1 k_2)^{m_1} k_1 R_2^0(\lambda_2 t_{m_1}) + \sum_{i=0}^{m_1} (k_1 k_2)^i f(t_i). \quad (\text{A.3})$$

If $m_1 \left(\frac{L}{\lambda_1} + \frac{L}{\lambda_2} \right) + \frac{L}{\lambda_2} \leq t - \frac{x}{\lambda_1} < (m_1 + 1) \left(\frac{L}{\lambda_1} + \frac{L}{\lambda_2} \right)$ then

$$Y_1(t, x) = (k_1 k_2)^{m_1+1} R_1^0 \left(-\lambda_1 \left(t_{m_1} - \frac{L}{\lambda_2} \right) + L \right) + \sum_{i=0}^{m_1} (k_1 k_2)^i f(t_i). \quad (\text{A.4})$$

Let $m_2 \in \mathbb{Z}$ such that

$$m_2 \left(\frac{L}{\lambda_1} + \frac{L}{\lambda_2} \right) \leq t - \frac{L-x}{\lambda_2} < (m_2 + 1) \left(\frac{L}{\lambda_1} + \frac{L}{\lambda_2} \right).$$

We define Y_2 for a.e. (t, x) as follows. If $m_2 < 0$ then

$$Y_2(t, x) = R_2^0(\lambda_2 t + x).$$

We now assume that $m_2 \geq 0$ and we define $\bar{t}_i := t - i \left(\frac{L}{\lambda_1} + \frac{L}{\lambda_2} \right) - \frac{L-x}{\lambda_2}$.

If $m_2 \left(\frac{L}{\lambda_1} + \frac{L}{\lambda_2} \right) \leq t - \frac{L-x}{\lambda_2} < m_2 \left(\frac{L}{\lambda_1} + \frac{L}{\lambda_2} \right) + \frac{L}{\lambda_1}$ then two different cases occur. If $m_2 = 0$, $Y_2(t, x) = k_2 R_1^0(-\lambda_1 \bar{t}_0 + L)$. Otherwise,

$$Y_2(t, x) = (k_1 k_2)^{m_2} k_2 R_1^0(-\lambda_1 \bar{t}_{m_2} + L) + \sum_{i=0}^{m_2-1} (k_1 k_2)^i k_2 f(\bar{t}_i - \frac{L}{\lambda_1}). \quad (\text{A.5})$$

If $m_2 \left(\frac{L}{\lambda_1} + \frac{L}{\lambda_2} \right) + \frac{L}{\lambda_1} \leq t - \frac{L-x}{\lambda_2} < (m_2 + 1) \left(\frac{L}{\lambda_1} + \frac{L}{\lambda_2} \right)$ then

$$Y_2(t, x) = (k_1 k_2)^{m_2+1} R_2^0 \left(\lambda_2 \left(\bar{t}_{m_2} - \frac{L}{\lambda_1} \right) \right) + \sum_{i=0}^{m_2} (k_1 k_2)^i k_2 f(\bar{t}_i - \frac{L}{\lambda_1}). \quad (\text{A.6})$$

An illustration of the construction of Y_1 with $m_1 = 0$ and $\frac{L}{\lambda_2} \leq t - \frac{x}{\lambda_1} < \left(\frac{L}{\lambda_1} + \frac{L}{\lambda_2} \right)$ in Figure A.1.

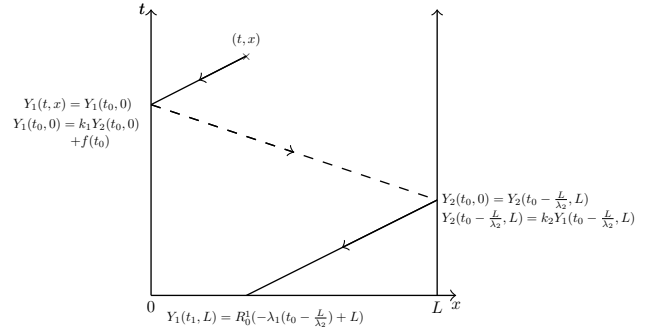


Fig. A.1. Construction of $Y_1(t, x)$ using the method of characteristics with $m_1 = 0$ and $\frac{L}{\lambda_2} \leq t - \frac{x}{\lambda_1}$.

Let us now prove the uniqueness. Let (Y_1^1, Y_2^1) and (Y_1^2, Y_2^2) two weak solutions of (14). Then, the couple (Y_1, Y_2) defined by $Y_1 = Y_1^2 - Y_1^1$ and $Y_2 = Y_2^2 - Y_2^1$ is a solution of (14) with $(R_1^0, R_2^0) = (0, 0)$ and $\dot{\gamma} + \nu\gamma = 0$. Let $1 \leq p < \infty$ and $(\psi_1^T, \psi_2^T) \in W^{1,p}([0, L]; \mathbb{R}^2)$ such that

$$\begin{aligned} k_1 \lambda_1 \psi_1^T(0) - \lambda_2 \psi_2^T(0) &= 0, \\ \lambda_1 \psi_1^T(L) - k_2 \lambda_2 \psi_2^T(L) &= 0. \end{aligned} \quad (\text{A.7})$$

Using the method of characteristics as in the proof of Lemma 2, we state the existence of

$$\psi_1, \psi_2 \in C^1([0, T]; L^p(0, L)) \cap C^0([0, T]; W^{1,p}(0, L)), \quad (\text{A.8})$$

satisfying

$$\begin{cases} \partial_t \psi_1(t, x) + \lambda_1 \partial_x \psi_1(t, x) = 0, \\ \partial_t \psi_2(t, x) - \lambda_2 \partial_x \psi_2(t, x) = 0, \\ k_1 \lambda_1 \psi_1(t, 0) - \lambda_2 \psi_2(t, 0) = 0, \\ \lambda_1 \psi_1(t, L) - k_2 \lambda_2 \psi_2(t, L) = 0, \\ \psi_1(T, x) = \psi_1^T(x) \\ \psi_2(T, x) = \psi_2^T(x). \end{cases} \quad (\text{A.9})$$

By simple density argument, (12) holds if we replace (φ_1, φ_2) by (ψ_1, ψ_2) satisfying (A.8) and (A.9). So, in (12) replacing (R_1, R_2) by (Y_1, Y_2) and $u(t)$ by $k_1 Y_2(t, 0) - d(t)$, we deduce that

$$\int_0^L (\psi_1^T(x) Y_1(T, x) + \psi_2^T(x) Y_2(T, x)) = 0. \quad (\text{A.10})$$

Since the set of $(\psi_1^T, \psi_2^T) \in W^{1,p}((0, L); \mathbb{R}^2)$ satisfying (A.7) is dense in $L^p((0, L); \mathbb{R}^2)$ for any $p \neq \infty$, it follows from (A.10) that

$$(Y_1(T, x), Y_2(T, x)) = (0, 0)$$

for all $T > 0$ and for all $x \in [0, L]$. This achieves the proof of uniqueness of solution when $1 \leq p < \infty$. If $p = \infty$, the proof is similar as before choosing $\psi_1^T, \psi_2^T \in W^{1,1}(0, L)$.

B Conservation of mass

Let $\sigma : t \rightarrow S(Y_1(t, \cdot), Y_2(t, \cdot))$ with S defined in (8) and (Y_1, Y_2) a weak solution of (14). We consider γ a Filippov solution of (13) with initial condition $\gamma(0) = S(R_1^0, R_2^0)$ and we introduce the function f defined by

$$f(t) = \dot{\gamma}(t) + \nu \gamma(t) \quad (\text{B.1})$$

Lemma 3 *For any $T > 0$, σ is a Carathéodry solution of*

$$\begin{cases} \dot{\sigma}(t) = -\nu \sigma(t) + f(t), & t \in [0, T], \\ \sigma(0) = S(R_1^0, R_2^0). \end{cases}$$

Proof 1 *Using the change of variable*

$$z_1(t, x) = \frac{1}{\lambda_1} e^{-\frac{\nu x}{\lambda_1}} Y_1(t, x), \quad z_2(t, x) = \frac{k_1}{\lambda_2} e^{\frac{\nu x}{\lambda_2}} Y_2(t, x), \quad (\text{B.2})$$

(z_1, z_2) is a weak solution of

$$\begin{cases} \partial_t z_1 + \lambda_1 \partial_x z_1 + \nu z_1 = 0, & (t, x) \in [0, T] \times [0, L], \\ \partial_t z_2 - \lambda_2 \partial_x z_2 + \nu z_2 = 0, & (t, x) \in [0, T] \times [0, L], \\ z_1(t, 0) = \frac{1}{\lambda_1} (\lambda_2 z_2(t, 0) + f(t)) & t \in [0, T], \\ z_2(t, L) = \frac{\lambda_1}{\lambda_2} z_1(t, L) & t \in [0, T], \\ z_1(0, x) = \frac{1}{\lambda_1} e^{-\frac{\nu x}{\lambda_1}} R_1^0(x), & x \in [0, L], \\ z_2(0, x) = \frac{k_1}{\lambda_2} e^{\frac{\nu x}{\lambda_2}} R_2^0(x), & x \in [0, L], \end{cases} \quad (\text{B.3})$$

That is to say, (z_1, z_2) satisfies the point (1) of Definition 1. Moreover, the point (2) of Definition 1 is replaced by, for any test function $\varphi^1, \varphi^2 \in C_c^1([0, T] \times]0, L[)$, we have

$$\begin{aligned} & \iint_{(0, T) \times (0, L)} z_1 (\varphi_t^1 + \lambda_1 \varphi_x^1 + \nu \varphi^1) dx dt \\ & + \iint_{(0, T) \times (0, L)} z_2 (\varphi_t^2 - \lambda_2 \varphi_x^2 + \nu \varphi^2) dx dt = 0 \end{aligned} \quad (\text{B.4})$$

Moreover, the boundary conditions of (B.3) are satisfied almost everywhere. A standard density argument shows that the equation (B.4) is still admissible if φ_1, φ_2 are just Lipschitz functions. Let $\bar{t} > 0$ and $\theta > 0$, we define

$$\varphi_\theta^i(t, x) := \psi_\theta^i(t) \phi_\theta^i(x),$$

with $i \in \{1, 2\}$ and

$$\psi_\theta^i(t) = \begin{cases} \frac{t}{\theta} & \text{if } t \leq \theta, \\ 1 & \text{if } \theta \leq t \leq \bar{t} - \theta, \\ \frac{\bar{t} - t}{\theta} & \text{if } \bar{t} - \theta \leq t \leq \bar{t}, \end{cases} \quad (\text{B.5})$$

and

$$\phi_\theta^i(x) = \begin{cases} \frac{x}{\theta} & \text{if } x \leq \theta, \\ 1 & \text{if } \theta \leq x \leq L - \theta, \\ \frac{L - x}{\theta} & \text{if } L - \theta \leq x \leq L, \end{cases} \quad (\text{B.6})$$

When $\theta \rightarrow 0$ in (B.4), we have

$$\begin{aligned} & \int_0^L z_1(\bar{t}, x) dx - \int_0^L z_1(0, x) dx + \lambda_1 \int_0^{\bar{t}} z_1(t, L) dt \\ & - \lambda_1 \int_0^{\bar{t}} z_1(t, 0) dt + \nu \iint_{(0, \bar{t}) \times (0, L)} z_1(t, x) dx dt \\ & + \int_0^L z_2(\bar{t}, x) dx - \int_0^L z_2(0, x) dx - \lambda_2 \int_0^{\bar{t}} z_2(t, L) dt \\ & + \lambda_2 \int_0^{\bar{t}} z_2(t, 0) dt + \nu \iint_{(0, \bar{t}) \times (0, L)} z_2(t, x) dx dt = 0. \end{aligned}$$

Using (B.2), (B.3) and (8) we have for any $\bar{t} \in (0, T)$,

$$S(\bar{t}) = S(0) - \nu \int_0^{\bar{t}} S(u) du + \int_0^{\bar{t}} f(u) du \quad (\text{B.7})$$

which conclude the proof of Lemma 3.

C Exponential stability

The proof of exponential stability is divided into two parts: first we prove the Lyapunov stability of the system (14) over the time interval $[0, t_0]$ and then we prove the exponential stability of the system (18). Let us start by proving the first part.

From Appendix A, (14) admits a unique weak solution constructed using the method of characteristics and given by (A.3), (A.4), (A.5) and (A.6). Assuming that when $p \neq \infty$. From (A.3) and (A.4) and $k_1 k_2 < 1$, we deduce that there exists a positive constant C such that for all $t \in [0, t_0]$ we have

$$\begin{aligned} \|Y_1(t, \cdot)\|_{L^p((0,L),\mathbf{R})}^p &\leq C \|R_1^0(\cdot), R_2^0(\cdot)\|_{L^p((0,L),\mathbf{R}^2)}^p \\ &+ C \int_0^L \left| \sum_{i=0}^{m_1(t,x)} f(t_i(t,x)) \right|^p dx \end{aligned} \quad (\text{C.1})$$

where $m_1(t, x)$ and $t_i(t, x)$ are defined in Appendix A. For the sake of clarity, the dependence of m_1 and t_i with respect to t and x is emphasized in (C.1). From (A.1), for any $t \geq 0$, the map $x \mapsto m_1(t, x)$ is a decreasing function and $0 \leq m_1(t, 0) - m_1(t, L) \leq 1$. Thus, two different cases occur. In the first case, there exists $m_0 \in \mathbf{N}$ such that for every $x \in [0, L]$, $m_1(t, x) = m_0$ then using the change of variable $s = t_i$ with t_i defined in Appendix A and $t \leq t_0$, there exists $C > 0$ such that

$$\begin{aligned} \|Y_1(t, \cdot)\|_{L^p((0,L),\mathbf{R})}^p &\leq C \|R_1^0(\cdot), R_2^0(\cdot)\|_{L^p((0,L),\mathbf{R}^2)}^p \\ &+ C \|f\|_{L^p((0,t_0),\mathbf{R})}^p \end{aligned} \quad (\text{C.2})$$

In the second case, there exist $m_0 \in \mathbf{N}$ and $x_0 \in [0, L]$ such that $m_1(t, x) = m_0$ for any $0 \leq x \leq x_0$ and $m_1(t, x) = m_0 - 1$ for any $x_0 < x \leq L$. Then,

$$\begin{aligned} \int_0^L \left| \sum_{i=0}^{m_1(t,x)} f(t_i(t,x)) \right|^p dx &= \int_0^{x_0} \left| \sum_{i=0}^{m_0} f(t_i(t,x)) \right|^p dx \\ &+ \int_{x_0}^L \left| \sum_{i=0}^{m_0-1} f(t_i(t,x)) \right|^p dx \end{aligned} \quad (\text{C.3})$$

Using the change of variable $s = t_i$ with t_i defined in Appendix A for each right-hand term of (C.3) and $t \leq t_0$, we deduce that there exists $C > 0$ such that (C.2) holds.

On the other hand, we have

$$\|f\|_{L^\infty(0,t_0)}^p = \|\dot{S}(t) + \nu S(t)\|_{L^\infty(0,t_0)}^p = \|v + d\|_{L^\infty(0,t_0)}^p. \quad (\text{C.4})$$

This implies that $\|f\|_{L^\infty(0,t_0)}^p \leq (2K)^p$. Now, using $K > \|d\|_{L^\infty(\mathbf{R}_+)}$ and Remark 7, we know that

$t_0 < \frac{|S(0)|}{K - \|d\|_{L^\infty(\mathbf{R}_+)}}$. Therefore, there exists $C > 0$ such that

$$\|f\|_{L^p((0,t_0),\mathbf{R})}^p \leq C |S(0)|^p. \quad (\text{C.5})$$

By continuous injection of $L^p((0, L)$ in $L^1((0, L)$, there exists $C > 0$ such that

$$\begin{aligned} |S(0)| &\leq \|R_1^0(\cdot), R_2^0(\cdot)\|_{L^1((0,L),\mathbf{R}^2)} \\ &\leq C \|R_1^0(\cdot), R_2^0(\cdot)\|_{L^p((0,L),\mathbf{R}^2)} \end{aligned} \quad (\text{C.6})$$

From (C.2), (C.5) and (C.6), we can conclude that, when $1 \leq p < \infty$, there exists a positive constant C such that for all $t \in [0, t_0]$ we have

$$\|Y_1(t, \cdot)\|_{L^p((0,L),\mathbf{R})} \leq C \|R_1^0(\cdot), R_2^0(\cdot)\|_{L^p((0,L),\mathbf{R}^2)}. \quad (\text{C.7})$$

Similarly, we prove that there exists a positive constant C such that for all $t \in [0, t_0]$ we have

$$\|Y_2(t, \cdot)\|_{L^p((0,L),\mathbf{R})} \leq C \|R_1^0(\cdot), R_2^0(\cdot)\|_{L^p((0,L),\mathbf{R}^2)}. \quad (\text{C.8})$$

Now, when $p = \infty$, according to (A.3), (A.4), (A.5), (A.6), (C.4) and (C.6) we obtain also (C.7) and (C.8), which conclude the proof of the first part.

Now, we provide a global asymptotic stability result related to the linear system (18). From Appendix A with $f(\cdot) := \dot{\gamma}(\cdot) + \nu\gamma(\cdot) = 0$, (18) admits a unique weak solution constructed using the method of characteristics. In particular, the weak solution R_1, R_2 of (18) is written as (A.3), (A.5), (A.4) and (A.6) replacing f by 0 and (R_1^0, R_2^0) by $(R_1(t_0, \cdot), R_2(t_0, \cdot))$. Thus, the inequality (19) follows from $k_1 k_2 = e^{-\nu(\frac{L}{\lambda_1} + \frac{L}{\lambda_2})} < 1$ (see also [9, Theorem 2.1]).

For the sake of completeness, we give the proof when $p \in [2, \infty)$ using the following Lyapunov function.

$$V(t) = \int_0^L \left[\frac{a}{\lambda_1} e^{-\frac{\nu x}{\lambda_1}} |R_1(t, x)|^p + \frac{b}{\lambda_2} e^{-\frac{\nu x}{\lambda_2}} |R_2(t, x)|^p \right] dx \quad (\text{C.9})$$

with $a > 0$ and $b > 0$. We will consider classical solutions and deduce the result in the space $L^p(0, L)$ by a standard density argument.

The time derivative of V along the classical solutions to

(18) is

$$\begin{aligned}
\dot{V}(t) &= \frac{ap}{\lambda_1} \int_0^L \partial_t R_1(t, x) e^{-\frac{\nu x}{\lambda_1}} R_1(t, x) |R_1(t, x)|^{p-2} dx \\
&\quad + \frac{bp}{\lambda_2} \int_0^L \partial_t R_2(t, x) e^{\frac{\nu x}{\lambda_2}} R_2(t, x) |R_2(t, x)|^{p-2} dx \\
&= -ap \int_0^L \partial_x R_1(t, x) e^{-\frac{\nu x}{\lambda_1}} R_1(t, x) |R_1(t, x)|^{p-2} dx \\
&\quad + bp \int_0^L \partial_x R_2(t, x) e^{\frac{\nu x}{\lambda_2}} R_2(t, x) |R_2(t, x)|^{p-2} dx \\
&= \frac{-ap}{2} \left(e^{-\frac{\nu L}{\lambda_1}} |R_1(t, L)|^p - |R_1(t, 0)|^p \right) \\
&\quad - \frac{ap\nu}{2\lambda_1} \int_0^L e^{-\frac{\nu x}{\lambda_1}} |R_1(t, x)|^p dx \\
&\quad + \frac{ap(p-2)}{2} \int_0^L \partial_x R_1(t, x), e^{-\frac{\nu x}{\lambda_1}} R_1(t, x) |R_1(t, x)|^{p-2} \\
&\quad + \frac{bp}{2} \left(e^{\frac{\nu L}{\lambda_2}} |R_2(t, L)|^p - |R_2(t, 0)|^p \right) \\
&\quad - \frac{bp\nu}{2\lambda_2} \int_0^L e^{\frac{\nu x}{\lambda_2}} |R_2(t, x)|^p dx \\
&\quad - \frac{bp(p-2)}{2} \int_0^L \partial_x R_2(t, x), e^{\frac{\nu x}{\lambda_2}} R_2(t, x) |R_2(t, x)|^{p-2} \\
&= -\nu V(t) - a \left(e^{-\frac{\nu L}{\lambda_1}} |R_1(t, L)|^p - |R_1(t, 0)|^p \right) \\
&\quad + b \left(e^{\frac{\nu L}{\lambda_2}} |R_2(t, L)|^p - |R_2(t, 0)|^p \right)
\end{aligned}$$

Using the boundary condition of (18) we have, for all $t \geq t_0$

$$\begin{aligned}
\dot{V}(t) &= -\nu V(t) - a \left(e^{-\frac{\nu L}{\lambda_1}} |R_1(t, L)|^p - k_1^p |R_2(t, 0)|^p \right) \\
&\quad + b \left(e^{\frac{\nu L}{\lambda_2}} k_2^p |R_1(t, L)|^p - |R_2(t, 0)|^p \right) \\
&= -\nu V(t) - |R_2(t, 0)|^p (-ak_1^p + b) \\
&\quad - |R_1(t, L)|^p (-be^{\frac{\nu L}{\lambda_2}} k_2^p + ae^{-\frac{\nu L}{\lambda_1}}) \tag{C.10}
\end{aligned}$$

Since $p \in [2, \infty)$, then

$$e^{\nu(\frac{L}{\lambda_1} + \frac{L}{\lambda_2})} k_1^p k_2^p = e^{(1-p)\nu(\frac{L}{\lambda_1} + \frac{L}{\lambda_2})} < 1,$$

thus, we can select a and b such that

$$e^{-\nu(\frac{L}{\lambda_1} + \frac{L}{\lambda_2})} k_2^p < \frac{a}{b} < \frac{1}{k_1^p}$$

which implies that

$$-be^{\frac{\nu L}{\lambda_2}} k_2 + ae^{-\frac{\nu L}{\lambda_1}} > 0 \text{ and } -ak_1 + b > 0.$$

Hence, we see that

$$\dot{V}(t) \leq -\nu V(t) \quad \forall t \geq t_0. \tag{C.11}$$

along the trajectories of the system (18) which are of class C^1 . By density, this inequality also holds in the sense of distribution, i.e. for every solution to (18) in $C^0([0, \infty); L^p(0, L))$.

On the other hand, there exists $\gamma > 0$ such that

$$\begin{aligned}
&\frac{1}{\gamma} \|(R_1(t, \cdot), R_2(t, \cdot))\|_{L^p((0, L); \mathbb{R}^2)}^p \\
&\leq V(t) \leq \gamma \|(R_1(t, \cdot), R_2(t, \cdot))\|_{L^p((0, L); \mathbb{R}^2)}^p. \tag{C.12}
\end{aligned}$$

Then, we have

$$\begin{aligned}
&\|(R_1(t, \cdot), R_2(t, \cdot))\|_{L^p((0, L); \mathbb{R}^2)} \\
&\leq \gamma e^{-\nu \frac{t}{p}} \|(R_1(t_0, \cdot), R_2(t_0, \cdot))\|_{L^p((0, L); \mathbb{R}^2)} \quad \forall t \geq t_0. \tag{C.13}
\end{aligned}$$

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