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Port Hamiltonian systems with moving interface: the two-phase Stefan problem

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Abstract: In this paper we propose a port Hamiltonian formulation of the two-phase Stefan problem. This problem describes the evolution of a pure material which is at two different states of matter: liquid and solid. This is a parabolic distributed parameter system which possesses a sharp moving interface. The two-phase Stefan problem results from the interconnection of two heat conduction equations at the interface position. The interface dynamics is governed by an ordinary differential equation. The port Hamiltonian formulation relies on the introduction of color functions which identify the liquid and the solid states. The contribution of this paper concerns the formulation of the two-phase Stefan problem as a sharp interface port Hamiltonian system. A thermodynamic-based modeling approach is considered to provide a physical insight on the proposed model.

Keywords: Boundary control systems, Stefan problem, Port Hamiltonian systems, Moving interface systems

1. INTRODUCTION

The two-phase Stefan problem describes the dynamical behaviour of two phases in a pure medium (Visintin, 1996). This can be used to model melting/solidification or vaporization/condensation processes for example. The interface refers to the narrow region which spatially separates the two sub-domains. The two-phase Stefan model proposes an explicit tracking of the interface. Indeed, its dynamics are governed by an ordinary differential equation, resulting from thermodynamic assumptions. The two sub-domains divided by the moving interface are governed by parabolic equations (heat conduction equations), which are partial differential equations. The two sub-domains dynamics are interconnected at the interface through boundary conditions and constitutive relations. A review and a study of the problem can be found in (Visintin, 1996), for example.

The one-phase Stefan problem recently gains attention, in the control community, with its application on the selective laser sintering process (Koga et al., 2019; de Andrade et al., 2019), where the boundary backstepping control was investigated. The estimation of the Arctic ice thickness and temperature profile is proposed, with an observer, based on a Stefan-like model (Koga and Krstic, 2020a). The energy

shaping boundary control problem of the two-phase Stefan problem was also proposed in (Koga and Krstic, 2020b).

In this paper we are considering a structured modeling approach for the two-phase Stefan problem. The boundary control port Hamiltonian formulation (Duindam et al., 2009) is a natural framework for interconnected systems as boundary port variables are defined with respect to the Hamiltonian storage function (van der Schaft and Maschke, 2002). For example to model systems governed by conservation laws (van der Schaft and Maschke, 2002) such as Timoshenko beams (Macchelli and Melchiorri, 2004), vibro acoustic tubes (Trenchant et al., 2015), shallow water equations (Hamroun et al., 2010), plasmas in Tokamaks (Vu et al., 2016), adsorption columns (Baaiu et al., 2008), etc. The port Hamiltonian formulation of sharp moving boundary in 1D, described by a system of two conservation laws was addressed by (Diagne and Maschke, 2013). Our contribution extends this result to the case of dissipative port Hamiltonian systems, illustrated with the interconnection of heat conduction equations.

The paper is structured as follows. In Section 2 we present the two-phase Stefan problem with a thermodynamic based modeling approach. This justifies the interface model. In Section 3 we are presenting the necessary background for the port Hamiltonian representation. The main contribution is given in Section 4, following the approach proposed in (Diagne and Maschke, 2013), where the port Hamiltonian representation of the two-phase Stefan

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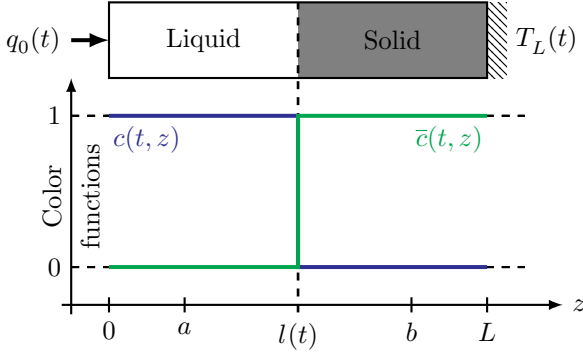


Fig. 1. Schematic of the two-phase 1D Stefan problem (top), and the typical shape of color functions (bottom)

problem is given and discussed. Section 5 concludes the paper.

2. THE TWO-PHASE STEFAN PROBLEM

The two-phase Stefan model governs the dynamics of phase changes, *i.e.* of melting (from solid to liquid state) or conversely of solidification, in a pure medium. Consider the 1D two-phase Stefan problem, depicted at Figure 1, defined in the fixed domain $\Omega = [0, L] \subset \mathbb{R}$. The liquid and solid states are defined in the time-varying spatial (complementary on Ω) sub-domains Ω_l and Ω_s , respectively. The interface position, labelled $l(t) \in]0, L[$, is point in Ω , and represents the sharp (with a null thickness) transition between the liquid and the solid state of matter. This interface possesses its own dynamics and is governed by an ordinary differential equation. Consider the subscript $i \in \{l, s\}$ to denote either the liquid or solid states of matter. The two-phase Stefan model considered here is subject to the following three assumptions (Visintin, 1996).

Assumption 1. The liquid density and the solid density are constants and equals. Therefore a phase density $\rho_i(t, z) = \rho_i \in \mathbb{R}$ is taken as a constant parameter.

Assumption 2. We assume that there exists only one interface in the system such that only liquid is present in the liquid phase Ω_l and conversely for the solid phase Ω_s . Thus the following relations hold:

$$T_l(t, z) \geq T_m, \quad \forall z \in \Omega_l, \forall t > 0, \quad (1)$$

and

$$T_s(t, z) \geq T_m, \quad \forall z \in \Omega_s, \forall t > 0, \quad (2)$$

where $T_i \in L^2(\Omega_i, \mathbb{R})$ and $T_m > 0$ denote the temperature profile in the sub-domain Ω_i and the pure material melting temperature, respectively.

Assumption 3. The interface is assumed at the thermodynamic equilibrium. Therefore, at the interface position $l(t) \in]0, L[$ the temperature is equal to the melting temperature T_m :

$$T_l(t, l(t)) = T_s(t, l(t)) = T_m, \quad (3)$$

for all $t > 0$.

We have privileged a thermodynamic presentation of the two-phase Stefan problem to facilitate the structured port Hamiltonian derivation of the model. Furthermore, this

will help in the physical interpretation of the interface dynamics.

2.1 Balance equations

Within each sub-domain Ω_i , the internal energy balance equations are defined as:

$$\frac{\partial \rho_i u_i}{\partial t}(t, z) + \frac{\partial}{\partial z} (J_{q,i}(t, z)) = 0, \quad \forall z \in \Omega_i, \forall t > 0, \quad (4)$$

where $u_i(t, z) \in L^2(\Omega_i, \mathbb{R})$ and $J_{q,i}(t, z) \in H^1(\Omega_i, \mathbb{R})$ are the internal energy density profile and the heat flux density in the sub-domains Ω_i , respectively. The fundamental law of thermodynamics applies (de Groot and Mazur, 1984), and Gibbs's law reads:

$$\delta u_i = T_i \delta s_i, \quad (5)$$

from which one derives the entropy balance equations as:

$$\frac{\partial \rho_i s_i}{\partial t}(t, z) + \frac{\partial}{\partial z} (J_{s,i}(t, z)) = \sigma_i(t, z), \quad (6)$$

$z \in \Omega_i, \forall t > 0$, where the entropy flux $J_{s,i} \in H^1(\Omega_i, \mathbb{R})$ and the source of irreversible entropy production $\sigma_{s,i} \in L^2(\Omega_i, \mathbb{R})$ are given by:

$$J_{s,i}(t, z) = \tau_i(t, z) J_{q,i}(t, z), \quad (7)$$

and

$$\sigma_{s,i}(t, z) = J_{q,i}(t, z) \frac{\partial \tau_i}{\partial z}(t, z) \geq 0, \quad (8)$$

$\forall z \in \Omega_i, \forall t > 0$, respectively.

2.2 Closure equations

The heat flux in Ω_i is defined by Fourier's law, as:

$$J_{q,i}(t, z) = -\lambda_i(t, z) \frac{\partial T_i}{\partial z}(t, z), \quad \forall z \in \Omega_i, \quad (9)$$

where $\lambda_i(t, z) \in L^2(\Omega_i, \mathbb{R})$, for all $z \in \Omega_i$, denotes the positive heat conduction coefficient. If one uses the entropy as a thermodynamic potential, the thermodynamic driving force F'_i is given by:

$$F'_i(t, z) = \frac{\partial}{\partial z} \left(\frac{ds_i}{du_i} \right) = \frac{\partial \tau_i}{\partial z}(t, z), \quad \forall z \in \Omega_i, \quad (10)$$

where τ_i denotes the inverse of the temperature T_i . Then the Fourier's law is equivalent to:

$$J_{q,i} = \lambda_i(t, z) T_i^2(t, z) F'_i(t, z), \quad \forall z \in \Omega_i. \quad (11)$$

2.3 Boundary conditions

Following the two-phase Stefan problem illustrated at Figure 1, we have mixed boundary conditions. Consider Neumann boundary condition at position $z = 0$:

$$J_{q,l}(t, 0) = q_0(t), \quad (12)$$

and Dirichlet boundary condition at position $z = L$, such that:

$$T_s(t, L) = T_L(t). \quad (13)$$

Both the heat flux $q_0(t)$ and the temperature $T_L(t)$ are considered as boundary input variables.

Remark 1. The boundary heat flux $q_0(t)$ must be positive to respect Assumption 2. Furthermore, the boundary temperature $T_L(t)$ must be lower than the melting temperature T_m to guaranty the presence of only solid state of matter in the solid phase Ω_s (Koga and Krstic, 2020b).

2.4 Interface dynamics

The interface is governed by the following ordinary differential equation:

$$\frac{dl}{dt}(t) = \frac{J_{q,s}(t, l(t)) - J_{q,l}(t, l(t))}{\rho_s u_s(t, l(t)) - \rho_l u_l(t, l(t))}, \quad (14)$$

where the liquid and the solid heat fluxes are taken as the right and left limits at position $z = l(t)$, respectively.

Remark 2. The model employed here is similar to the model proposed in (Visintin, 1996) since the denominator of equation (14) denotes the variation interface internal energy variation. For constant pressure and volume, this is equivalent to the local enthalpy variation, that is the latent heat.

Remark 3. The total internal energy, over a fixed domain containing the moving interface $[a, b] \in \Omega$, such that $0 < a < l(t) < b < L$, is given by the integral:

$$U_{\text{tot}} = \int_a^{l(t)} \rho_l u_l(t, z) dz + \int_{l(t)}^b \rho_s u_s(t, z) dz. \quad (15)$$

Thus, the total internal energy balance equation is given by:

$$\frac{dU_{\text{tot}}}{dt}(t) = -[J_{q,l}(t, z)]_a^{l(t)} - [J_{q,s}(t, z)]_{l(t)}^b \quad (16)$$

$$+ \frac{dl}{dt}(t) [\rho_l u_l(t, l(t)) - \rho_s u_s(t, l(t))], \quad (17)$$

where we have applied Leibniz integral theorem, and injected the internal energy balance equations (4). The interface dynamics (14) is obtained by cancelling the terms associated to the interface in equation (16). Then, one obtains the following total internal energy balance equation:

$$\frac{dU_{\text{tot}}}{dt}(t) = J_{q,l}(t, a) - J_{q,s}(t, b). \quad (18)$$

Remark 4. Similarly, the total entropy balance equation of a sub-domain $[a, b]$ which includes the interface $l(t)$ can be computed to identify the entropic contribution of the interface. Applying the same derivation rules and using the entropy balance equation 6, the total entropy defined as:

$$S_{\text{tot}} = \int_a^{l(t)} \rho_l s_l(t, z) dz + \int_{l(t)}^b \rho_s s_s(t, z) dz, \quad (19)$$

provides the following total entropy balance equation:

$$\frac{dS_{\text{tot}}}{dt} = J_{s,l}(t, a) - J_{s,s}(t, b) + \int_a^{l(t)} \sigma_1(t, z) dz + \int_{l(t)}^b \sigma_s(t, z) dz \quad (20)$$

$$+ [J_{q,s}(t, l(t)) - J_{q,l}(t, l(t))] (\tau_m - \alpha).$$

The term α in equation (20) is identified as:

$$\alpha = \frac{\rho_s s_s(t, l(t)) - \rho_l s_l(t, l(t))}{\rho_s u_s(t, l(t)) - \rho_l u_l(t, l(t))} = \frac{\delta \rho s(t, l(t))}{\delta \rho u(t, l(t))} = \tau_m, \quad (21)$$

where we have used Gibb's relation (5). Then the total entropy balance equation (20) reduces to:

$$\frac{dS_{\text{tot}}}{dt} = J_{s,l}(t, a) - J_{s,s}(t, b) + \int_a^{l(t)} \sigma_1(t, z) dz + \int_{l(t)}^b \sigma_s(t, z) dz. \quad (22)$$

The interface dynamics does not contribute to the generation of entropy through an irreversible entropy source term in equation (22). This is consistent with Assumption 3 since the interface is at equilibrium.

To summarize, the two-phase Stefan problem is then governed by the internal energy balance equations (4) (or alternatively by the entropy balance equations (6)), coupled to an ordinary differential equation governing the interface dynamics (14). The system is completed by the boundary conditions (12), (3) and (13). Finally, consider the initialization data $T_{l,0}(z) = T_l(0, z)$, $T_{s,0}(z) = T_s(0, z)$ and $l_0 = l(0)$, respecting the boundary conditions and Assumptions 2–3.

Remark 5. The steady state solutions of the proposed problem was studied for the case of only Neumann boundary conditions in (Cannon and Primicerio, 1971a) and only Dirichlet boundary conditions in (Cannon and Primicerio, 1971b).

3. BACKGROUND AND PROBLEM STATEMENTS

Let us recall in this section the port Hamiltonian formulation of distributed parameter systems (van der Schaft and Maschke, 2002; Duindam et al., 2009).

3.1 Stokes–Dirac structures

Definition 1. (Courant (1990)). Consider two real vector spaces, \mathcal{F} the space of flow variables and \mathcal{E} the space of effort variables, together with a pairing, that is, a bilinear product $\mathcal{F} \times \mathcal{E} \rightarrow \mathbb{R}$, $(f, e) \mapsto \langle e, f \rangle$ which introduces the symmetric bilinear form $\langle\langle, \rangle\rangle$ on the bond space $\mathcal{B} = \mathcal{F} \times \mathcal{E}$ of conjugated power variables $(e, f) \in \mathcal{B}$ as:

$$\langle\langle (f_1, e_1), (f_2, e_2) \rangle\rangle = \langle e_1, f_2 \rangle + \langle e_2, f_1 \rangle, \quad (23)$$

where $(f_i, e_i) \in \mathcal{B}$, $i \in \{1, 2\}$. A Dirac structure is a linear subspace $\mathcal{D} \subset \mathcal{B}$ which is isotropic and co-isotropic that is satisfied $\mathcal{D} = \mathcal{D}^\perp$, with \perp denoting the orthogonal complement with respect to the bilinear form.

Proposition 1 (van der Schaft and Maschke (2002))

The linear subspace of the bond space $\mathcal{B} = \mathcal{E} \times \mathcal{F}$, product of the space of flow variables \mathcal{F} and effort variables \mathcal{E} where $\mathcal{F} = \mathcal{E} = L^2((a, b), \mathbb{R}^2) \times \mathbb{R}^2$ defined by:

$$\mathcal{D} = \left\{ \left(\begin{pmatrix} f_1 \\ f_2 \\ f_\partial \end{pmatrix}, \begin{pmatrix} e_1 \\ e_2 \\ e_\partial \end{pmatrix} \right) \in \mathcal{F} \times \mathcal{E} \text{ s.t. } \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \in H^1((a, b), \mathbb{R}^2)^2, \right. \\ \left. \begin{pmatrix} f_1 \\ f_\partial \end{pmatrix} = \mathcal{J} \begin{pmatrix} e_1 \\ e_\partial \end{pmatrix} \text{ and } \begin{pmatrix} f_\partial \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_\partial \end{pmatrix} \Big|_{a,b} \right\}, \quad (24)$$

is a Dirac structure, called a Stokes–Dirac structure, with respect to the pairing

$$\left\langle \begin{pmatrix} f_1 \\ f_2 \\ f_\partial \end{pmatrix}, \begin{pmatrix} e_1 \\ e_2 \\ e_\partial \end{pmatrix} \right\rangle = \int_a^b (f_1 e_1 + f_2 e_2) dz + e_\partial^\top \Sigma e_\partial \quad (25)$$

with $\Sigma = \text{diag}(-1, 1)$.

The Stokes–Dirac structure is suited for the class of systems defined by two conservation laws (van der Schaft and Maschke, 2002). As highlighted in (Duindam et al., 2009), this class of Stokes–Dirac structure is adapted also to the class of parabolic systems, for example the heat conduction equation (4).

3.2 Dissipative port Hamiltonian systems

To illustrates the concept of dissipative port Hamiltonian systems we consider the example of heat conduction in

the sub-domain Ω_i . Following (Duindam et al., 2009) we have the following port Hamiltonian representation of the internal energy balance equation (4) coupled with the thermodynamic force (10):

$$\begin{pmatrix} \frac{\partial \rho_i u_i}{\partial t} \\ -F_i \end{pmatrix} = \begin{pmatrix} 0 & -\frac{\partial}{\partial z} \\ -\frac{\partial}{\partial z} & 0 \end{pmatrix} \begin{pmatrix} \frac{ds_i}{du_i} \\ J_{q,i} \end{pmatrix}, \quad z \in \Omega_i \quad (26)$$

with the boundary port variables:

$$\begin{pmatrix} f_{\partial,i} \\ e_{\partial,i} \end{pmatrix} \Big|_{\partial\Omega_i} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{ds_i}{du_i} \\ J_{q,i} \end{pmatrix} \Big|_{\partial\Omega_i}, \quad (27)$$

where $\partial\Omega_i$ denotes the boundary of the interval Ω_i , and the closure relation (11).

3.3 Problem statement

The contribution of this paper is to propose a port Hamiltonian formulation of the 1D Stefan problem which includes both the sub-systems (liquid and solid) with the interface dynamics. This work follows the previous development proposed in (Diagne and Maschke, 2013) where we extend the result with an parabolic system that is heat conduction. Firstly, we will extend the variables of the sub-domains Ω_i in the complete domain Ω with the help of color functions. They are Heaviside functions indicating the locations of the liquid and solid states. Color functions are distributed parameter governed by transport partial differential equations. Secondly, we will prove that the augmented system defines a Stokes–Dirac structure and can be expressed as a port Hamiltonian system. In the sequel of the paper, and to introduce more generality on the methodology we assume that the interface is subject to the following balance equation:

$$J_{q,l}(t, l(t)) + J_{q,s}(t, l(t)) + e_I = 0, \quad (28)$$

and the continuity equation

$$\tau_l(t, l(t)) = \tau_s(t, l(t)) = f_I = \tau_m, \quad (29)$$

where the pair of variables (e_I, f_I) represents a source/sink of power associated to the interface dynamics.

4. PORT HAMILTONIAN SYSTEMS WITH MOVING INTERFACE

In this section we present the port Hamiltonian formulation of a dissipative system by following the formulation of moving interface systems derived in (Diagne and Maschke, 2013). The extension to this class of system is the main contribution of the paper. We formulate the Stokes–Dirac structure on an arbitrary interval $[a, b]$ containing the moving interface: $0 < a < l(t) < b < L$, as depicted at Figure 1.

4.1 Color functions

Consider the color functions (Diagne and Maschke, 2013)) $(t, z) \mapsto c(t, z)$ defined as:

$$c(t, z) = \begin{cases} 1, & 0 < z < l(t), \\ 0, & l(t) < z < L, \end{cases} \quad (30)$$

and its complementary functional $(t, z) \mapsto \bar{c}(t, z)$, defined as:

$$\bar{c}(t, z) = \begin{cases} 0, & 0 < z < l(t), \\ 1, & l(t) < z < L. \end{cases} \quad (31)$$

Color functions are depicted at Figure 1 for the Stefan problem. These color functions are governed by the following partial differential equations, convection transport equations:

$$\frac{\partial c}{\partial t}(t, z) = -\dot{l}(t) \frac{\partial c}{\partial z}(t, z), \quad (32)$$

and

$$\frac{\partial \bar{c}}{\partial t}(t, z) = -\dot{l}(t) \frac{\partial \bar{c}}{\partial z}(t, z). \quad (33)$$

Considering the color functions (30) and (31), one has to define the extended states and flux to reformulate the governing balance equations (4) and (6). Note that in equations (32) and (33), the interface velocity $\dot{l}(t)$ is an input source term.

4.2 New state equation and thermodynamic forces

Consider $\rho u(t, z)$ to be the internal energy variable defined on the total spatial domain $[0, L] \ni z$, for any $t > 0$. When this internal energy profile is pre-multiplied by the color functions c or \bar{c} , one recovers the internal energy in one or the other sub-domain:

$$c(t, z) \rho u(t, z) = \rho_l u_l(t, z), \quad \forall z \in [0, l(t)[, \quad (34)$$

and

$$\bar{c}(t, z) \rho u(t, z) = \rho_s u_s(t, z), \quad \forall z \in]l(t), L], \quad (35)$$

respectively. Similarly, we define the heat flux on the complete spatial domain, such that $J_q(t, z)$, for all $z \in [0, L]$ and $t > 0$, which is solution on the sub-domains to:

$$c(t, z) J_q(t, z) = J_{q,l}(t, z), \quad \forall z \in [0, l(t)[, \quad (36)$$

and

$$\bar{c}(t, z) J_q(t, z) = J_{q,s}(t, z), \quad \forall z \in]l(t), L]. \quad (37)$$

In a fixed frame $[a, b]$, which includes the interface $l(t)$, i.e. $0 < a < l(t) < b < L$, for all $t > 0$, the total internal energy balance equation is given by:

$$\begin{aligned} \frac{dU_{\text{tot}}}{dt} &= \frac{d}{dt} \int_a^b \rho u(t, z) dz, \\ &= \frac{d}{dt} \int_a^{l(t)} \rho u(t, z) dz + \frac{d}{dt} \int_{l(t)}^b \rho u(t, z) dz, \\ &= \int_a^{l(t)} \frac{\partial}{\partial t} [c(t, z) \rho_l u_l(t, z)] dz + \int_{l(t)}^b \frac{\partial}{\partial t} [\bar{c}(t, z) \rho_s u_s(t, z)] dz \\ &\quad + \dot{l}(t) [c(t, l(t)) \rho_l u_l(t, l(t)) - \bar{c}(t, l(t)) \rho_s u_s(t, l(t))], \\ &= \int_a^b (\mathbf{d}_c J_q(u, c, \bar{c}) - e_I) dz \\ &\quad + \dot{l}(t) [c(t, l(t)) \rho_l u_l(t, l(t)) - \bar{c}(t, l(t)) \rho_s u_s(t, l(t))], \end{aligned}$$

where \mathbf{d}_c is the following nonlinear differential operator:

$$\mathbf{d}_c = - \left[\frac{\partial c}{\partial z} \cdot + \frac{\partial \bar{c}}{\partial z} \cdot \right]. \quad (38)$$

In the internal energy balance equation the interface input source term e_I results from the local conservation law (28). Thus the local version of the internal energy balance

equation, in a fixed domain which include the moving interface is:

$$\frac{\partial \rho u}{\partial t}(t, z) = \mathbf{d}_c J_q(u, c, \bar{c}) - e_I + \dot{l}(t) \left[cu \frac{\partial c}{\partial z} + \bar{c} u \frac{\partial \bar{c}}{\partial z} \right]. \quad (39)$$

Following the same approach carried out with the internal energy balance equation, one defines the thermodynamic driving force $F'(u, c, \bar{c})$ such that:

$$c(t, z)F'(t, z) = F'_1(t, z), \quad \forall z \in \Omega_1, \forall t > 0, \quad (40)$$

and

$$\bar{c}(t, z)F'(t, z) = F'_s(t, z), \quad \forall z \in \Omega_s, \forall t > 0. \quad (41)$$

Consider the following definition of the dual operator of \mathbf{d}_c .

Proposition 2 (Dagne and Maschke (2013))

The dual operator of \mathbf{d}_c , labelled \mathbf{d}_c^* , is equal to:

$$\mathbf{d}_c^* = -\mathbf{d}_c + \left[\frac{\partial c}{\partial z}(t, z) + \frac{\partial \bar{c}}{\partial z}(t, z) \right]. \quad (42)$$

Thus the thermodynamic force F' can be expressed in terms of the dual operator \mathbf{d}_c^* such that:

$$-F' = -\mathbf{d}_c^* \tau(t, z), \quad (43)$$

where the inverse temperature $\tau(t, z)$ defined in the domain Ω is defined with the color functions, as for the other variables $F'(t, z)$, $J_q(t, z)$ and $\rho u(t, z)$. This enables the port Hamiltonian representation of the two-phase systems.

4.3 Port Hamiltonian formulation

In a domain $[a, b]$ which includes the interface, i.e. $0 \leq a < l(t) < b \leq L$, the systems is described by the internal energy balance equation (39), the thermodynamic driving force (43), and the color functions dynamics (32) and (33). Consider the vector of flux $\tilde{f} \in \mathcal{F} = L^2(\Omega, \mathbb{R})^4$ as:

$$\tilde{f} = \left(\frac{\partial \rho u}{\partial t}, -F', \frac{\partial c}{\partial t}, \frac{\partial \bar{c}}{\partial t} \right)^\top, \quad (44)$$

and the vector of effort $\tilde{e} \in \mathcal{E} = H^1(\Omega, \mathbb{R})^4$ defined as:

$$\tilde{e} = (\tau, J_q, cu, \bar{c}u)^\top. \quad (45)$$

Then one identifies the following state equation:

$$\tilde{f} = \mathcal{J}_c \tilde{e} + I e_I + \dot{l}(t) \mathcal{G}(u, c, \bar{c}), \quad (46)$$

where the interconnection operator is:

$$\mathcal{J}_c = \begin{pmatrix} 0 & \mathbf{d}_c & 0 \\ -\mathbf{d}_c^* & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (47)$$

The input map $I = \mathbb{R}^4$ associated to the interface input term e_I is defined as:

$$I = (-1, 0, 0, 0)^\top. \quad (48)$$

Finally, the input operator associated to the interface velocity $\dot{l}(t)$ is:

$$\mathcal{G}(u, c, \bar{c}) = \begin{pmatrix} cu & \bar{c}u \\ 0 & 0 \\ -1 & 0 \\ 0 & -1 \end{pmatrix} \frac{\partial}{\partial z} \begin{pmatrix} c \\ \bar{c} \end{pmatrix}. \quad (49)$$

The output e_l , conjugated to the interface velocity $\dot{l}(t)$, is defined as:

$$e_l = \int_a^b \tilde{e}^\top \mathcal{G}(u, c, \bar{c}) dz, \quad (50)$$

which can be reformulated as the pairing:

$$e_l = \langle \mathcal{G}^\top(u, c, \bar{c}) |, \tilde{e} \rangle. \quad (51)$$

Proposition 3

Consider the bond space $\mathcal{B} = \mathcal{F} \times \mathcal{E}$ with $\mathcal{F} = \mathcal{E} = L^2((a, b), \mathbb{R})^5 \times \mathbb{R} \times \mathbb{R}^5$, the flow variables (44), the effort variables (45), the differential operator (47), the moving interface of velocity $\dot{l}(t)$, and its conjugated variable (51), the continuity relation (29), the balance equation (28) and the input matrix (48). The following linear subspace $\mathcal{D}_{MI} \subset \mathcal{B}$:

$$\mathcal{D}_{MI} = \left\{ \begin{pmatrix} \tilde{f} \\ f_I \\ -e_l \end{pmatrix} = \begin{pmatrix} \mathcal{J}_c & I & \mathcal{G}(x, c, \bar{c}) \\ -I^\top & 0 & 0 \\ -\langle \mathcal{G}^\top(x, c, \bar{c}) | & 0 & 0 \end{pmatrix} \begin{pmatrix} \tilde{e} \\ e_I \\ \dot{l}(t) \end{pmatrix} \right. \\ \left. \text{and } \begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ (c + \bar{c}) & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}_{(a,b)} \right\} \quad (52)$$

endowed with the pairing:

$$\left\langle \begin{pmatrix} \tilde{f} \\ f_I \\ e_l \end{pmatrix}, \begin{pmatrix} \tilde{e} \\ e_I \\ \dot{l}(t) \end{pmatrix} \right\rangle = \int_a^b \tilde{e}^\top \tilde{f} dz + \int_a^b e_I^\top f_I dz + e_\partial^\top \Sigma f_\partial - e_l \dot{l}(t) \quad (53)$$

with $\Sigma = \text{diag}(-1, 1)$, defines a Dirac structure.

Proof 1. The proof follows the development of Dagne and Maschke (2013), where classical arguments are used. The differences are in the definitions of the input vector I and in the input operator \mathcal{G} .

With the definition of this Stokes-Dirac structure \mathcal{D}_{MI} , one defines the port Hamiltonian system of a parabolic moving interface system as follows.

Proposition 4

The augmented Hamiltonian system with the conjugated interface flow variable and conjugated variable to the the interface velocity may be defined as a boundary port Hamiltonian system with respect to the Dirac structure \mathcal{D}_{MI} by:

$$\left(\begin{pmatrix} \tilde{f} \\ f_I \\ e_l \end{pmatrix}, \begin{pmatrix} \tilde{e} \\ e_I \\ \dot{l}(t) \end{pmatrix} \right) \in \mathcal{D}_{MI}, \quad (54)$$

where the pair of port variables (f_I, e_I) are associated to the interface inputs, the pair of port variables (\dot{l}, e_l) are associated with the velocity of the interface and the pair of port variables (f_∂, e_∂) are associated with the boundary of the spatial domain $[0, L]$.

5. CONCLUSION

We have proposed a structured representation of the two-phase Stefan problem. This representation is based on the coupling of two parabolic systems by a moving interface. Firstly, we have presented the two-phase Stefan problem with a thermodynamic point of view which enables us to derive the interface dynamics. Secondly, we have augmented the system with color functions. Finally we have presented the port Hamiltonian representation of the two-phase Stefan problem whose interface's dynamics are included within the structure. The development of the port Hamiltonian model follows the same approach considered

in (Diagne and Maschke, 2013), where we have extended the result to the interconnection of parabolic systems at the moving interface. At the interface, the temperature profile is continuous and the heat flux follows a balance equation.

Outgoing work concerns the development of a boundary controller for the presented sharp moving interface port Hamiltonian system. The control problem being at stabilizing in closed loop the interface position $l(t)$ at a desired position l_d when $t \mapsto +\infty$. An energy shaping control or a passivity based control methods are privileged. A second perspective aims at representing the two-phase Stefan problem as a phase field boundary control system, following the work (Fabrizio, 2008), The port Hamiltonian representation should be investigated toward the comparison of the two modeling approaches with a boundary control perspective.

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