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# Port Hamiltonian systems with moving interface: a phase field approach

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**Abstract:** In this paper, we give a formulation of distributed parameter systems with a moving diffuse interface using the Port Hamiltonian formalism. For this purpose, we suggest to use the phase field modeling approach. In the first part we recall the phase field models, in particular the Cahn–Hilliard and Allen–Cahn equations, and show that they may be expressed in terms of a dissipative Hamiltonian system. In the second part we show how this Hamiltonian model may be extended to a Boundary Port Hamiltonian System and illustrate the construction on the example of crystallization.

*Keywords:* Boundary control systems, Port Hamiltonian systems, Phase fields, Solidification

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## 1. INTRODUCTION

In this paper we consider the modeling problem of distributed parameter systems with internal moving interface. The interface is the narrow region which separates two spatial regions of different material states (solid/liquid, liquid/gas, solid/solid, etc). One way to model such systems is to consider *sharp interfaces* and to divide the spatial domain into sub-systems. Each sub-systems being governed by a set of conservation laws interconnected at the interfaces through boundary conditions resulting from flux conditions and constitutive relations. This implies an explicit tracking of the interface position, both in space and time. This representation of multi-phase distributed parameter systems increases the complexity of modeling, analysis, and numerical discretization schemes (Godlewski and Raviart, 2004; Ambroso et al., 2008; Boutin et al., 2008).

An alternative approach is to consider *diffuse interfaces* where phases are defined by continuous variables in space and time. This is the phase field approach. The dynamics of phase variable are governed by partial differential equations derived from thermodynamics potentials. Most models falling into the scope of phase field systems are then described by Cahn–Hilliard (Cahn and Hilliard, 1958) or Allen–Cahn (Allen and Cahn, 1979) equations which describe conservative and non-conservative processes, respectively. Sharp interface models can be recovered from phase field ones by considering infinitely small interface

thickness (Elder et al., 2001). See the monography (Emmerich, 2003) or the review articles (Emmerich, 2008; Moe-lans et al., 2008) (and the references therein) for more details on the development, the analysis and the applications of this modeling tool. Allen–Cahn and Cahn–Hilliard equations are employed as boundary control models. Chehab et al. (2017) used a 1–D Allen–Cahn equation to model the interface formation in a lithium electric battery and to control the number of interface through a Neumann boundary control variable. Chen (1993) proposed a boundary optimal control law for the Allen–Cahn equation.

The boundary control port Hamiltonian formulation of infinite-dimensional systems is convenient to model systems governed by conservation laws (van der Schaft and Maschke, 2002) such as Timoshenko beams (Macchelli and Melchiorri, 2004), vibro acoustic tubes (Trenchant et al., 2015), shallow water equations (Hamroun et al., 2010), plasmas in Tokamaks (Vu et al., 2016), adsorption columns (Baaij et al., 2008), etc. The boundary control port Hamiltonian formulation (Duindam et al., 2009) is a natural framework for interconnected systems as boundary port variables are defined with respect to the Hamiltonian storage function. The port Hamiltonian formulation of sharp moving boundary in 1–D systems was addressed by Diagne and Maschke (2013). In this work we define phase field systems within the boundary control port Hamiltonian framework. Toward this end we introduce phase fields models. We define the potential of the interface and the conservative and non-conservatives modeling approaches. The port Hamiltonian representations for phase field systems are obtained by prolongation of the systems on their jet spaces. This approach was already introduced in various examples Maschke and van der Schaft (2005,

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2013); Schberl and Siuka (2014). This is then illustrated with a solidification process (Elder et al., 2001).

The paper is structured as follows. In Section 2 the phase field modeling is introduced. The definition of Stokes–Dirac structures are recalled in Section 3. The main contributions are presented in sections 4 and 5 where boundary port Hamiltonian formulations of a non-conserved and a conserved phase field models are stated, respectively. In Section 6 a solidification example illustrates the contribution. Section 7 concludes the paper.

## 2. A VERY SHORT INTRODUCTION TO PHASE FIELD MODELS

Interfaces are implicitly represented through a continuous time-varying distributed parameter named the phase field variable. This parameter possesses its own dynamics and physical properties. Hence, a phase field model is defined by two elements. Firstly a functional representing a thermodynamical potential such as the energy or the entropy (or any other Legendre transformations). Secondly a state equation governing the phase field dynamics. Two phenomenological behaviors are distinguished depending on the phase field conservative properties. The state equations are conservation laws or gradient systems. For an overview of phase field models see the review articles (Emmerich, 2008; Kobayashi, 2010) or the monography (Emmerich, 2003).

Consider  $z$  to be the space variable defined in the spatial domain  $\Omega \subset \mathbb{R}^3$  with boundary  $\partial\Omega \subset \mathbb{R}^2$  and  $t \in [0, +\infty)$  to be the time variable. The conserved phase field variable  $c(t, z) \in [0, +\infty) \times \Omega \rightarrow [0, 1]$  is a scalar distributed parameter. Values 0 and 1 enable the representation of two phases. Intermediate values indicate the proximity with the dissipative interface. We have deliberately chosen the interval  $[0, 1]$  for simplicity but any interval  $[a, b]$  for any  $a > b \in \mathbb{R}$  is adequate to represent two phases. Non-conserved phase field variables are labeled  $\phi(t, z) \in [0, +\infty) \times \Omega \rightarrow [0, 1]$ . Phase fields are initialized at time  $t = 0$  as  $\phi(0, z) = \phi_0(z) \in C^\infty(\Omega, [0, 1])$  and  $c(0, z) = c_0(z) \in C^\infty(\Omega, [0, 1])$  for all  $z \in \Omega$ .

### 2.1 The Landau–Ginzburg functional

The cornerstone of phase field modeling relies on the definition of a functional representing indifferently its entropy density, its energy density or other thermodynamical potentials. This function is a thermodynamic potential where a phase field variable is added to the thermodynamic variables present in the system (Emmerich, 2003). Consider any, conserved or non-conserved, phase field variable  $x$ . The Landau–Ginzburg model for binary interfaces is defined with the functional  $\mathcal{G} : C^\infty(\Omega) \rightarrow \mathbb{R}$  (Emmerich, 2003):

$$\mathcal{G}(x) = \int_{\Omega} g(x) + \frac{1}{2} \kappa_x(z) (\text{grad } x)^2 dV, \quad (1)$$

where  $\kappa_x(z) \in C^\infty(\Omega, \mathbb{R}^+)$  represents the non-negative gradient coefficient and is related to the interface surface tension and width (Kobayashi, 2010). The functional (1) is the sum of a quadratic term that represents the cost for inhomogeneities and  $g(x) \in C^\infty([0, 1], \mathbb{R})$  is an analytic

potential function that generates the interface dynamics. This function exhibits two minima in the interval  $[0, 1]$  enabling the phase dynamics (Emmerich, 2003).

The dynamics of either a non-conserved phase field variable or a conserved one are generated by variations of the potential function (1).

### 2.2 Non-conserved phase field variable

A non-conserved phase field variable,  $\phi(t, z)$ , is governed by a gradient equation of the following form:

$$\frac{\partial \phi}{\partial t}(t, z) = -\Gamma_\phi(z) \frac{\delta \mathcal{G}}{\delta \phi}, \quad (2)$$

where  $\Gamma_\phi(z) \in C^\infty(\Omega, \mathbb{R}^+)$  represents the isotropic interface mobility and  $\delta \mathcal{G} / \delta \phi$  denotes the functional derivative of the functional  $\mathcal{G}$  with respect to the phase field variable  $\phi(t, z)$ . The variational differentiation of the functional (1) is given by:

$$\frac{\delta \mathcal{G}}{\delta \phi}(\phi) = \frac{\partial g}{\partial \phi}(\phi) - \text{div} \left( \kappa_\phi(z) \text{grad } \phi(t, z) \right). \quad (3)$$

According to the functional derivative (3), the phase field state equation (2) reads as follows:

$$\frac{\partial \phi}{\partial t} = -\Gamma_\phi \left( -\text{div} \left( \kappa_\phi \text{grad } \phi \right) + \frac{\partial g}{\partial \phi}(\phi) \right), \quad (4)$$

where time and space dependences on the state variable and coefficients are omitted for sake of clarity. Equation (4) is named the Allen–Cahn equation after Allen and Cahn (1979). To define a Cauchy problem we include initial conditions at time  $t = 0$  with  $\phi(0, z) = \phi_0(z)$  for all  $z \in \Omega$ , and boundary conditions, evaluated on  $\partial\Omega$ , that can be Dirichlet, Neumann or Danckwerts (Nauman and He, 2001).

*Remark 1.* The Allen–Cahn equation (4) is a resistive diffusion equation (Duindam et al., 2009), *e.g.* a heat diffusion equation, and can be formulated as a Hamiltonian system. Nevertheless one has to use a different Hamiltonian function than the potential (1). Consider us consider the following Hamiltonian function:

$$\mathcal{H}(\phi) = \int_{\Omega} \frac{1}{2} \phi^2 dV, \quad (5)$$

and the conservation law

$$\frac{\partial \phi}{\partial t} + \text{div } j_\phi = 0, \quad (6)$$

where  $j_\phi$  denotes the non-conserved flux given by:

$$j_\phi = -\Gamma_\phi \text{grad } \frac{\delta \mathcal{H}}{\delta \phi}. \quad (7)$$

Then the Hamiltonian representation of the Allen–Cahn equation (4) is given by:

$$\begin{pmatrix} \frac{\partial \phi}{\partial t} + \frac{\partial g}{\partial \phi}(\phi) \\ F_\phi \end{pmatrix} = \begin{pmatrix} 0 & -\text{div} \\ -\text{grad} & 0 \end{pmatrix} \begin{pmatrix} \frac{\delta \mathcal{H}}{\delta \phi} \\ j_\phi \end{pmatrix}. \quad (8)$$

The Hamiltonian function (5) is not physically based as the potential function (1). therefore the contribution due to the density potential function  $g(\phi)$  is not included as an effort term but arises in the left hand side of equation (8) as a flux. This implies the definition of non-physical port variables and motivates the formulation of an extended port Hamiltonian representation of the Allen–Cahn equation (4).

### 2.3 Conserved phase field variable

Dynamics of a conserved phase field variable  $c(t, z) \in [0, 1]$ , are governed by a balance equation of the following form:

$$\frac{\partial c}{\partial t}(t, z) + \operatorname{div} j_c(t, z) = 0, \quad (9)$$

where the phase field flux  $j_c(t, z) \in C^\infty(\Omega, \mathbb{R}^3)$  is closed by the following linear transport relation:

$$j_c = -\Gamma_c(z) \operatorname{grad} \left( \frac{\delta G}{\delta c}(c) \right), \quad (10)$$

where  $\Gamma_c(z) \in C^\infty(\Omega, \mathbb{R}^+)$  represents the phase field transport coefficient. After injecting the phase field flux (10) into the conservation law (9), with the variational derivative (3), one finds the Cahn–Hilliard equation, derived by Cahn and Hilliard (1958):

$$\frac{\partial c}{\partial t} = \operatorname{div} \left( \Gamma_c \operatorname{grad} \left( -\operatorname{div}(\kappa_c \operatorname{grad} c) + \frac{\partial g}{\partial c} \right) \right), \quad (11)$$

where time and space dependences on the state variable and coefficients are omitted. To define a Cauchy problem we add initial conditions  $c(0, z) = c_0(z)$  for all  $z \in \Omega$  and boundary conditions on  $\partial\Omega$  to the phase field equation (11). Those are chosen in coherence with the considered application but one can consider Dirichlet, Neuman, or Danckwerts boundary conditions (Nauman and He, 2001).

*Remark 2.* Similarly to the Allen–Cahn equation, the Cahn–Hilliard equation (11) can be represented as a Hamiltonian system. Indeed we have a conservation law (9) which can be formulated with the same operator as in equation (8):

$$\left( \frac{\partial c}{\partial t} \right) = \begin{pmatrix} 0 & -\operatorname{div} \\ -\operatorname{grad} & 0 \end{pmatrix} \begin{pmatrix} \delta \mathcal{G} \\ \delta c \end{pmatrix}, \quad (12)$$

with the thermodynamic force  $F_c$  is defined as:

$$F_c = -\operatorname{grad} \frac{\delta \mathcal{G}}{\delta c}. \quad (13)$$

One should not the presence of differential operators in the functional derivative (3). This motivates the formulation of an alternative Hamiltonian formulation of the Cahn–Hilliard equation (11) where the extended representation includes all differential terms in the structure.

Both conserved and non-conserved phase field variables can be present in the same dynamical system. This will be illustrated with a solidification process in Section 6.

### 3. PORT HAMILTONIAN SYSTEMS AND STOKES–DIRAC STRUCTURES

The port Hamiltonian formulation of infinite-dimensional systems relies on the definition of a Stokes–Dirac structure (van der Schaft and Maschke, 2002). Hereby we recall the definition needed for the structured representation of the Cahn–Hilliard (11) and the Allen–Cahn (4) equations.

Consider an  $n$ -dimensional space  $\mathcal{F} \subset \mathbb{R}$  and  $\mathcal{E} = \mathcal{F}'$  its dual, *i.e.* the space of linear operator  $e : \mathcal{F} \rightarrow \mathbb{R}$ . The elements of  $f \in \mathcal{F}$  and of  $e \in \mathcal{E}$  are called flows and efforts, respectively. They are boundary port variables whose combinations represent the flowing power inside the system. The pair of boundary port variables  $(e^\partial, f^\partial)$  are

defined in the boundary spaces  $\mathcal{E}^\partial$  and  $\mathcal{F}^\partial$ , respectively. Power are defined with the dual product between  $e$  and  $f$  as  $\langle e, f \rangle = e(f)$ . The space of power variables is given by:

$$\mathcal{B} = \{(f, f^\partial, e, e^\partial) \in \mathcal{F} \times \mathcal{F}^\partial \times \mathcal{E} \times \mathcal{E}^\partial\}, \quad (14)$$

such that the duality pairing between elements of  $\mathcal{B}$  is defined as:

$$\langle\langle (f_1, f_1^\partial, e_1, e_1^\partial), (f_2, f_2^\partial, e_2, e_2^\partial) \rangle\rangle = \langle e_1, f_2 \rangle + \langle e_2, f_1 \rangle + \langle e_1^\partial, f_2^\partial \rangle_{\partial\Omega} + \langle e_2^\partial, f_1^\partial \rangle_{\partial\Omega}. \quad (15)$$

*Definition 1*

Le Gorrec et al. (2005) A Stokes–Dirac structure  $\mathcal{D}$  on the bond space  $\mathcal{B}$  is a subspace of  $\mathcal{B}$  which is maximally isotropic with respect to the canonical symmetrical pairing (14), *i.e.*

$$\mathcal{D} = \mathcal{D}^\perp, \quad (16)$$

where  $\mathcal{D}^\perp$  denotes the orthogonal subspace of  $\mathcal{D}$  with respect to the pairing (14).

Port Hamiltonian systems defined by the state variable  $x(t) \in \mathcal{F}$ , the potential function  $\mathcal{H}(x(t)) \in \mathcal{E}$  with boundary port variables  $\zeta(t) \in \mathcal{Z}$ . Then the port Hamiltonian system is defined by the Stokes–Dirac structure

$$\left( -\dot{x}(t), f^\partial, \frac{\delta \mathcal{H}}{\delta x}(x(t)), e^\partial \right) \in \mathcal{D}. \quad (17)$$

Dirac structures are defined for various physical applications. For details and properties concerning their composition one can refers to (Duindam et al., 2009). The case of a system with two conservations laws is discussed in (Kotyczka et al., 2018). Dirac structures are also defined on Hilbert spaces (Kurula et al., 2010).

### 4. PORT HAMILTONIAN FORMULATION OF NON-CONSERVED PHASE FIELDS

Consider the model of a non-conserved phase field variable  $\phi(t, z) \in C^\infty(\Omega, [0, 1])$  defined with the storage function (1) and the gradient equation (2). As a partial differential equation this model is known as the Allen–Cahn equation (4). To emphasize the linear structure behind the phase field model we augment the system. This is a modeling technique already known for port–Hamiltonian systems. For example for the vibrating string, one uses its strain and momentum instead of the string position and velocity (see example 4.3 in Maschke and van der Schaft (2005)), for the Timoshenko beam (Schberl and Siuka, 2014) or for the Boussinesq equations (Maschke and van der Schaft, 2013). The state space representation (4) is augmented with the new state  $\psi$ :

$$\begin{pmatrix} \phi \\ \psi \end{pmatrix} \triangleq \begin{pmatrix} \phi \\ \operatorname{grad} \phi \end{pmatrix} \in C^\infty(\Omega, [0, 1]) \times C^\infty(\Omega, \mathbb{R}^3), \quad (18)$$

Hence the potential (1) is re-written as:

$$\bar{\mathcal{G}}(\phi, \psi) = \int_\Omega \frac{\kappa_\phi}{2} \psi^2 + g(\phi) dV. \quad (19)$$

Furthermore we introduce the new distributed parameter  $E_\phi(t, z) \in C^\infty(\Omega, \mathbb{R})$  given by:

$$E_\phi(t, z) = \Gamma_\phi(z) F_\phi(t, z), \quad (20)$$

where  $F_\phi(t, z) \in C^\infty(\Omega, \mathbb{R})$  denotes the variational derivative of the total free potential function (3) expressed in terms of the new pair of variables (18) as:

$$F_\phi = \frac{\delta \bar{\mathcal{G}}}{\delta \phi} - \operatorname{div} \left( \frac{\delta \bar{\mathcal{G}}}{\delta \psi} \right). \quad (21)$$

The gradient equation (2) is now:

$$\frac{\partial \phi}{\partial t}(t, z) = -E_\phi(t, z). \quad (22)$$

The time variation of the second variable  $\psi(t, z) \in C^\infty(\Omega, \mathbb{R}^3)$  is given by

$$\frac{\partial \psi}{\partial t}(t, z) = \frac{\partial}{\partial t} \text{grad } \phi(t, z). \quad (23)$$

The gradient operator and the time derivative commute since  $\phi(t, z) \in C^\infty(\Omega, [0, 1])$ , and plugging equation (22) in (23) one obtains:

$$\frac{\partial \psi}{\partial t}(t, z) = -\text{grad } E_\phi(t, z). \quad (24)$$

The balance equation of the potential function (19) is given by

$$\frac{d\bar{\mathcal{G}}}{dt} = \int_{\Omega} \left[ \frac{\delta \bar{\mathcal{G}}}{\delta \phi}, \frac{\delta \bar{\mathcal{G}}}{\delta \psi} \right] \left[ \frac{\partial \phi}{\partial t}, \frac{\partial \psi}{\partial t} \right] dV, \quad (25)$$

where we plug in the state equations (22) and (24) to obtain:

$$\frac{d\bar{\mathcal{G}}}{dt} = - \int_{\Omega} \left( \frac{\delta \bar{\mathcal{G}}}{\delta \phi} E_\phi + \frac{\delta \bar{\mathcal{G}}}{\delta \psi} \text{grad } E_\phi \right) dV. \quad (26)$$

After applying an integration by part one identifies a divergence term:

$$\frac{d\bar{\mathcal{G}}}{dt} = - \int_{\Omega} \left( \frac{\delta \bar{\mathcal{G}}}{\delta \phi} E_\phi - E_\phi \text{div} \frac{\delta \bar{\mathcal{G}}}{\delta \psi} \right) dV - \int_{\Omega} \text{div} \left( \frac{\delta \bar{\mathcal{G}}}{\delta \psi} E_\phi \right) dV.$$

Using definition (21) and applying Stokes' theorem on the first and on the second term on the right hand side of the equation above, respectively, one obtains:

$$\frac{d\bar{\mathcal{G}}}{dt} = - \int_{\Omega} E_\phi F_\phi dV - \int_{\partial\Omega} \vec{n} \frac{\delta \bar{\mathcal{G}}}{\delta \psi} E_\phi dS, \quad (27)$$

where  $\vec{n}$  denotes the outward unit normal vector acting on the boundary  $\partial\Omega$ . One identifies a distributed dissipative term, due to the interface diffusion, in the first integral term in equation (27). The second term in the right hand side of equation (27) denotes the potential variation due to the exchanges through the boundary  $\partial\Omega$ .

With the state equations (22) and (24), and the closure relation (20), one identifies the following implicit structured representation:

$$f_\phi = \mathcal{J}_\phi e_\phi, \quad (28)$$

where the flow variable  $f_\phi$  and the effort variables  $e_\phi$  are

$$f_\phi^\top = \left( \frac{\partial \phi}{\partial t}, \frac{\partial \psi}{\partial t}, F_\phi \right)^\top \in \mathcal{F}_\phi, \quad (29)$$

where  $\mathcal{F}_\phi = C^\infty(\Omega, \mathbb{R}) \times C^\infty(\Omega, \mathbb{R}^3)^2$  and

$$e_\phi^\top = \left( \frac{\delta \bar{\mathcal{G}}}{\delta \phi}, \frac{\delta \bar{\mathcal{G}}}{\delta \psi}, E_\phi \right)^\top \in \mathcal{E}_\phi, \quad (30)$$

where  $\mathcal{E}_\phi = C^\infty(\Omega, \mathbb{R}) \times C^\infty(\Omega, \mathbb{R}^3) \times C^\infty(\Omega, \mathbb{R})$ , respectively. Furthermore the linear operator  $\mathcal{J}_\phi$  takes the following form:

$$\mathcal{J}_\phi = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & -\text{grad} \\ 1 & -\text{div} & 0 \end{pmatrix}. \quad (31)$$

With the functional (27) we identify the following pair of boundary port variables

$$\begin{pmatrix} f_\phi^\phi \\ e_\phi^\phi \end{pmatrix} = \mathcal{W}_\phi e_\phi|_{\partial\Omega} = \begin{pmatrix} -\vec{n} \frac{\delta \bar{\mathcal{G}}}{\delta \psi} \\ E_\phi \end{pmatrix}, \quad (32)$$

where the boundary operator  $\mathcal{W}_\phi$  is defined as:

$$\mathcal{W}_\phi = \begin{pmatrix} 0 & -\vec{n} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (33)$$

Boundary port variables  $f_\phi^\phi$  and  $e_\phi^\phi$  are defined in the linear spaces of boundary flows and efforts  $\mathcal{F}_\phi^\partial = C^\infty(\partial\Omega, \mathbb{R}^3)$  and  $\mathcal{E}_\phi^\partial = C^\infty(\partial\Omega, \mathbb{R})$ , respectively. Finally, the space of power variables is the Cartesian product of the bulk and boundary efforts and flow variables:

$$\mathcal{B}_\phi = \{(f_\phi, f_\phi^\partial, e_\phi, e_\phi^\partial) \in \mathcal{F}_\phi \times \mathcal{F}_\phi^\partial \times \mathcal{E}_\phi \times \mathcal{E}_\phi^\partial\}. \quad (34)$$

The duality pairing between elements of  $\mathcal{B}_\phi$  is defined as:

$$\langle\langle (f_\phi^1, f_\phi^{1\partial}, e_\phi^1, e_\phi^{1\partial}), (f_\phi^2, f_\phi^{2\partial}, e_\phi^2, e_\phi^{2\partial}) \rangle\rangle = \langle e_\phi^1, f_\phi^2 \rangle + \langle e_\phi^2, f_\phi^1 \rangle + \langle e_\phi^{1\partial}, f_\phi^{2\partial} \rangle_{\partial\Omega} + \langle e_\phi^{2\partial}, f_\phi^{1\partial} \rangle_{\partial\Omega}. \quad (35)$$

Let us formally show that the non-conserved phase field model possesses a Stokes-Dirac structure.

*Proposition 1*

Consider the space of power variables  $\mathcal{B}_\phi$ , the bilinear product  $\ll \cdot, \cdot \gg$ , and the linear operator  $\mathcal{J}_\phi$  defined in (34), (35), and (31), respectively. Then the following linear subspace  $\mathcal{D}_\phi \subset \mathcal{B}_\phi$ :

$$\mathcal{D}_\phi = \left\{ \begin{pmatrix} \begin{pmatrix} f_\phi^\phi \\ f_\phi^\partial \end{pmatrix}, \begin{pmatrix} e_\phi^\phi \\ e_\phi^\partial \end{pmatrix} \end{pmatrix} \in \mathcal{B}_\phi \text{ s.t. } \begin{matrix} f_\phi = \mathcal{J}_\phi e_\phi, \begin{pmatrix} f_\phi^\phi \\ f_\phi^\partial \end{pmatrix} = \mathcal{W}_\phi e_\phi|_{\partial\Omega} \end{matrix} \right\}, \quad (36)$$

is a Stokes-Dirac structure.

*Proof 1.* The linear operator  $\mathcal{J}_\phi$  verifies

$$\langle\langle (f_\phi^1, f_\phi^{1\partial}, e_\phi^1, e_\phi^{1\partial}), (f_\phi^2, f_\phi^{2\partial}, e_\phi^2, e_\phi^{2\partial}) \rangle\rangle = 0, \quad (37)$$

for null boundary conditions. Thus  $\mathcal{J}_\phi$  is a skew-symmetric operator. Following (van der Schaft and Maschke, 2002) one shows that the subspace  $\mathcal{D}_\phi$  is a Dirac structure, *i.e.*  $\mathcal{D}_\phi = \mathcal{D}_\phi^\perp$  where  $\mathcal{D}_\phi^\perp$  denotes the orthogonal complement with respect to the bilinear form (35).

*Remark 3.* Integrations over the domain  $\Omega$  and its boundary  $\partial\Omega$ , *e.g.* the potential balance equation (27), are independent on the interface position. This is a major difference with the traditional way of modeling moving interface systems where the domain of integration moves with the interface (Diagne and Maschke, 2013). Hence the Stokes-Dirac structure (36), and by extension the port Hamiltonian system is defined independently of the interface position.

To summarize, the structured model of a non-conserved phase field is defined by the states  $(\phi, \psi)$ , with the Hamiltonian function (19) and the Stokes-Dirac structure:

$$\left( \frac{\partial \phi}{\partial t}, \frac{\partial \psi}{\partial t}, F_\phi, f_\phi^\partial, \frac{\delta \bar{\mathcal{G}}}{\delta \phi}, \frac{\delta \bar{\mathcal{G}}}{\delta \psi}, E_\phi, e_\phi^\partial \right) \in \mathcal{D}_\phi \quad (38)$$

where  $F_\phi$  and  $E_\phi$  are closed by the algebraic equation (20). Boundary conditions  $f_\phi^\partial$  and  $e_\phi^\partial$  are defined by the relation (32). They physically represent the gradient of the phase field variable passing through the boundary  $\partial\Omega$  and the functional derivative of the potential function.

## 5. PORT HAMILTONIAN FORMULATION OF CONSERVED PHASE FIELDS

Consider now the case of conserved phase field variables as introduced in Section 2.3. The same methodology as for the non-conserved case is followed. Let us introduce the augmented state variables:

$$\begin{pmatrix} c \\ \gamma \end{pmatrix} \triangleq \begin{pmatrix} c \\ \nabla c \end{pmatrix} \in C^\infty(\Omega, \mathbb{R}) \times C^\infty(\Omega, \mathbb{R}^3), \quad (39)$$

such that the potential (1) is now given by:

$$\bar{\mathcal{G}}(c, \gamma) = \int_{\Omega} g(c) + \frac{\kappa_c}{2} \gamma^2 dV. \quad (40)$$

The phase field variable  $c$  is governed by the following balance equation (9):

$$\frac{\partial c}{\partial t} = -\operatorname{div} j_c, \quad (41)$$

and the state equation associated to the variable  $\gamma$  is given by:

$$\frac{\partial \gamma}{\partial t} = -\operatorname{grad}(\operatorname{div} j_c), \quad (42)$$

where we have used the definition of  $\gamma$ , and the balance equation (41). The gradient operator and the time derivative commute since  $c(t, z) \in C^\infty(\Omega, [0, 1])$ . The phase field flux  $j_c \in C^\infty(\Omega, \mathbb{R}^3)$  is closed by a linear transport model (10),

$$j_c = \Gamma_c F_c, \quad (43)$$

where  $F_c \in C^\infty(\Omega, \mathbb{R}^3)$  denotes the phase field thermodynamic force:

$$F_c = -\operatorname{grad} \left( \frac{\delta \bar{\mathcal{G}}}{\delta c} - \operatorname{div} \frac{\delta \bar{\mathcal{G}}}{\delta \gamma} \right). \quad (44)$$

*Remark 4.* With the structure representation of conserved phase field variables the constitutive relation (43) is an algebraic equation. The differential part of the closure equation is gathered in the thermodynamic force (44) which will be included in the Stokes–Dirac Structure.

The balance equation of the potential function (40) enable us to identify the boundary port variables and is given by:

$$\frac{d\bar{\mathcal{G}}}{dt}(c, \gamma) = \int_{\Omega} \left[ \frac{\delta \bar{\mathcal{G}}}{\delta c}, \frac{\delta \bar{\mathcal{G}}}{\delta \gamma} \right] \begin{bmatrix} \frac{\partial c}{\partial t} \\ \frac{\partial \gamma}{\partial t} \end{bmatrix} dV, \quad (45)$$

where we plug in the state equations to obtain:

$$\frac{d\bar{\mathcal{G}}}{dt} = - \int_{\Omega} \left( \frac{\delta \bar{\mathcal{G}}}{\delta c} \operatorname{div} j_c + \frac{\delta \bar{\mathcal{G}}}{\delta \gamma} \operatorname{grad} \operatorname{div} j_c \right) dV. \quad (46)$$

Applying one and two integrations by part to the first and the second term in the right hand side of equation (46), respectively, gives:

$$\begin{aligned} \frac{d\bar{\mathcal{G}}}{dt} &= - \int_{\partial\Omega} \vec{n} \left( \frac{\delta \bar{\mathcal{G}}}{\delta c} j_c + \frac{\delta \bar{\mathcal{G}}}{\delta \gamma} \operatorname{div} j_c - j_c \operatorname{div} \frac{\delta \bar{\mathcal{G}}}{\delta \gamma} \right) dS \\ &+ \int_{\Omega} j_c \operatorname{grad} \frac{\delta \bar{\mathcal{G}}}{\delta c} dV - \int_{\Omega} j_c \operatorname{grad} \left( \operatorname{div} \frac{\delta \bar{\mathcal{G}}}{\delta \gamma} \right) dV \end{aligned} \quad (47)$$

where we identify the force (44) to finally deduce:

$$\begin{aligned} \frac{d\bar{\mathcal{G}}}{dt} &= - \int_{\partial\Omega} \vec{n} \left( -j_c \operatorname{div} \frac{\delta \bar{\mathcal{G}}}{\delta \gamma} + \frac{\delta \bar{\mathcal{G}}}{\delta \gamma} \operatorname{div} j_c + \frac{\delta \bar{\mathcal{G}}}{\delta c} j_c \right) dS \\ &- \int_{\Omega} j_c F_c dV. \end{aligned} \quad (48)$$

The balance equation (41), the state equation (42) and the closure relation (43) are gathered in the following unique equation:

$$f_c = \mathcal{J}_c e_c. \quad (49)$$

Let us define the linear operator  $\mathcal{J}_c$ :

$$\mathcal{J}_c = \begin{pmatrix} 0 & 0 & -\operatorname{div}(\cdot) \\ 0 & 0 & -\operatorname{grad}(\operatorname{div}(\cdot)) \\ -\operatorname{grad}(\cdot) & \operatorname{grad}(\operatorname{div}(\cdot)) & 0 \end{pmatrix}. \quad (50)$$

The vector of flows  $f_c$  and effort  $e_c$  variables are defined as:

$$f_c^\top = \left( \frac{\partial c}{\partial t}, \frac{\partial \gamma}{\partial t}, F_c \right)^\top \in \mathcal{F}^c, \quad (51)$$

with  $\mathcal{F}^c = C^\infty(\Omega, \mathbb{R}) \times C^\infty(\Omega, \mathbb{R}^3)^2$ , and

$$e_c^\top = \left( \frac{\delta \bar{\mathcal{G}}}{\delta c}, \frac{\delta \bar{\mathcal{G}}}{\delta \gamma}, j_c \right)^\top \in \mathcal{E}_c, \quad (52)$$

with  $\mathcal{E}_c = C^\infty(\Omega, \mathbb{R}) \times C^\infty(\Omega, \mathbb{R}^3)^2$ , respectively. One identifies in the balance equation (48) the following boundary port variables:

$$\begin{pmatrix} f_c^\partial \\ e_c^\partial \end{pmatrix} = \mathcal{W}_c e_c|_{\partial\Omega}, \quad (53)$$

where the boundary operator  $\mathcal{W}_c$  is set as follow:

$$\mathcal{W}_c = \begin{pmatrix} 0 & \operatorname{div} & 0 \\ 0 & 0 & -\operatorname{div} \\ -1 & 0 & 0 \\ 0 & 0 & \vec{n} \\ 0 & \vec{n} & 0 \\ 0 & 0 & \vec{n} \end{pmatrix}. \quad (54)$$

where  $\mathcal{E}_c^\partial \in C^\infty(\partial\Omega, \mathbb{R}^3)^2 \times C^\infty(\partial\Omega, \mathbb{R})$  and  $\mathcal{F}_c^\partial \in C^\infty(\partial\Omega, \mathbb{R}^3)^3$ .

The space of power variables is given by the:

$$\mathcal{B}_c = \{(f_c, f_c^\partial, e_c, e_c^\partial) \in \mathcal{F}_c \times \mathcal{F}_c^\partial \times \mathcal{E}_c \times \mathcal{E}_c^\partial\}. \quad (55)$$

The duality pairing between elements of  $\mathcal{B}_c$  is defined as:

$$\begin{aligned} \langle \langle (f_c^1, f_c^{1\partial}, e_c^1, e_c^{1\partial}), (f_c^2, f_c^{2\partial}, e_c^2, e_c^{2\partial}) \rangle \rangle &= \\ \langle e_c^1, f_c^2 \rangle + \langle e_c^2, f_c^1 \rangle + \langle e_c^{1\partial}, f_c^{2\partial} \rangle_{\partial\Omega} + \langle e_c^{2\partial}, f_c^{1\partial} \rangle_{\partial\Omega}. \end{aligned} \quad (56)$$

Let us show that the conserved phase field model possesses a Stokes–Dirac structure.

*Proposition 2*

Consider the space of power variables  $\mathcal{B}_c$ , the bilinear product  $\ll \cdot, \cdot \gg$ , and the linear operator  $\mathcal{J}_c$  defined in (55), (56), and (50), respectively. The following linear subspace  $\mathcal{D}_c \subset \mathcal{B}_c$ :

$$\mathcal{D}_c = \left\{ \begin{pmatrix} \begin{pmatrix} f_c \\ f_c^\partial \end{pmatrix}, \begin{pmatrix} e_c \\ e_c^\partial \end{pmatrix} \end{pmatrix} \in \mathcal{B}_c \text{ s. t. } \begin{aligned} f_c &= \mathcal{J}_c e_c, \\ \begin{pmatrix} f_c^\partial \\ e_c^\partial \end{pmatrix} &= \mathcal{W}_c e_c|_{\partial\Omega} \end{aligned} \right\} \quad (57)$$

is a Stokes–Dirac structure.

*Proof 2.* Firstly let us show the skew-symmetry of the second order linear operator  $\mathcal{J}_c$  defined at equation (50). Therefore we compute  $\langle e^1, f^2 \rangle$  and  $\langle e^2, f^1 \rangle$  such that:

$$\begin{aligned}
\langle e^1, f^2 \rangle &= \int_{\Omega} (e_1^2 \operatorname{div}(e_3^1) + e_3^2 \operatorname{grad}(e_1^1)) dV \\
&+ \int_{\Omega} (e_2^2 \operatorname{grad}(\operatorname{div} e_3^1) - e_3^2 \operatorname{grad}(\operatorname{div} e_2^1)) dV \\
&+ \int_{\partial\Omega} \vec{n} (e_3^1 \operatorname{div}(e_2^2) - e_2^1 \operatorname{div}(e_3^2) - e_3^1 e_2^1) dS \\
&+ \int_{\partial\Omega} \vec{n} (e_3^2 \operatorname{div}(e_2^1) - e_2^2 \operatorname{div}(e_3^1) - e_3^2 e_1^1) dS,
\end{aligned} \tag{58}$$

and

$$\begin{aligned}
\langle e^2, f^1 \rangle &= - \int_{\Omega} (e_3^2 \operatorname{grad}(e_1^1) + e_1^2 \operatorname{div}(e_3^1)) dV \\
&+ \int_{\Omega} (e_3^2 \operatorname{grad}(\operatorname{div} e_2^1) - e_2^2 \operatorname{grad}(\operatorname{div} e_3^1)) dV,
\end{aligned} \tag{59}$$

holds. Then we sum the two power products where only the boundary terms remains

$$\begin{aligned}
\langle e^1, f^2 \rangle + \langle e^2, f^1 \rangle &= \\
&+ \int_{\partial\Omega} \vec{n} (e_3^1 \operatorname{div}(e_2^2) - e_2^1 \operatorname{div}(e_3^2) - e_3^1 e_2^1) dS \\
&+ \int_{\partial\Omega} \vec{n} (e_3^2 \operatorname{div}(e_2^1) - e_2^2 \operatorname{div}(e_3^1) - e_3^2 e_1^1) dS,
\end{aligned} \tag{60}$$

that is:

$$\langle e^1, f^2 \rangle + \langle e^2, f^1 \rangle = -\langle e_c^{1\partial}, f_c^{2\partial} \rangle_{\partial\Omega} - \langle e_c^{2\partial}, f_c^{1\partial} \rangle_{\partial\Omega}. \tag{61}$$

The power product (60) cancels out with zero boundary conditions, thus the linear operator  $\mathcal{J}_c$  is skew-symmetric.

Secondly, following (van der Schaft and Maschke, 2002), one can prove that the subspace  $\mathcal{D}_c$  is a Dirac structure, i.e.  $\mathcal{D}_c = \mathcal{D}_c^\perp$  where  $\mathcal{D}_c^\perp$  denotes the orthogonal complement with respect to the bilinear form (56). The first step consists of verifying  $\mathcal{D}_c \subset \mathcal{D}_c^\perp$ . The second step aims at showing that  $\mathcal{D}_c^\perp \subset \mathcal{D}_c$ .

To summarize the structured model of a conserved phase field is defined by the state variables  $(c, \gamma)$ , the Hamiltonian function (1), and the following Stokes-Dirac structure:

$$\left( \frac{\partial c}{\partial t}, \frac{\partial \gamma}{\partial t}, F_c, f_c^\partial, \frac{\delta \bar{\mathcal{G}}}{\delta c}, \frac{\delta \bar{\mathcal{G}}}{\delta \gamma}, j_c, e_c^\partial \right) \in \mathcal{D}_c, \tag{62}$$

where  $F_c$  and  $j_c$  are closed by equation (43). Boundary terms are defined at equation (53).

## 6. EXAMPLE OF A SOLIDIFICATION PROCESS

Let us consider the solidification process proposed in Elder et al. (2001), where the spatial domain is labeled  $\Omega \subset \mathbb{R}^3$  with its boundary  $\partial\Omega \subset \mathbb{R}^2$ . This binary concentration solidification process is achieved at uniform and constant temperature denoted  $T$ . This system possesses two phases: a solid phase and a liquid one. The interface between the two states of matter is represented by the non-conserved phase field variable  $\phi \in C^\infty(\Omega, [0, 1])$ . The binary solution, which solidifies, is represented by the conserved phase field  $c \in C^\infty(\Omega, [0, 1])$ . This later represents the mass or mole fraction of one of the two species present in the system, where the second species admits a concentration equal to  $1 - c$ . The potential used to drive the system's dynamics, here the Gibb's free energy is given by the integral

$$\mathcal{G}(\phi, c) = \int_{\Omega} G(\phi, c) dz, \tag{63}$$

with the density function:

$$G(\phi, c) = \frac{1}{2} \kappa_\phi (\operatorname{grad} \phi)^2 + \frac{1}{2} \kappa_c (\operatorname{grad} c)^2 + g(\phi, c). \tag{64}$$

The local bulk potential density  $g(\phi, c)$  is defined as:

$$g(\phi, c) = -a \frac{T}{T_m} h(c) + k(\phi, c), \tag{65}$$

where the first term is function of the conserved field  $c$  and is pre-multiplied to the scalar coefficient  $a \in \mathbb{R}$  and the ratio between the uniform temperature  $T$  and the melting temperature  $T_m$ . The second term on the right hand side of equation (65) denotes the free entropy associated to the diffusive interface. The function  $h(c)$  defines the entropy of a random mixing for binary solutions, and is defined as:

$$h(c) = -Rn [c \ln c + (1 - c) \ln(1 - c)], \tag{66}$$

where  $R$  is the ideal gas constant, and  $n$  denotes the total moles number in the system. The mixing function  $k(\phi, c)$  is given by:

$$k(\phi, c) = \left( \alpha \Delta T - \beta \left( c - \frac{1}{2} \right)^2 \right) \Phi(\phi) - \frac{1}{2} \phi^2 + \frac{1}{4} \phi^4, \tag{67}$$

where  $\Phi(\phi) = 2\phi - \frac{4}{3}\phi^3 + \frac{2}{5}\phi^5$ ,  $\Delta T = (T - T_m)/T_m$  denotes the normalized temperature, and parameters  $\alpha$ ,  $\beta$  and  $u \in \mathbb{R}$  are empiric scalar values. With the potential function (63) one associates the Allen-Cahn equation (4) and the Cahn-Hilliard equation (11). Both phase field variables are initialized at time  $t = 0$  as  $c(0, z) = c_0(z) \in C^\infty(\Omega)$  and  $\phi(0, z) = \phi_0(z) \in C^\infty(\Omega)$ . The port Hamiltonian formulation naturally follows from the development of Sections 4 and 5 where one considers the augmented state variable:

$$(\phi, \psi, c, \gamma) \triangleq (\phi, \operatorname{grad} \psi, c, \operatorname{grad} c), \tag{68}$$

with the re-formulated potential (63):

$$\bar{\mathcal{G}}(\phi, \psi, c, \gamma) = \int_{\Omega} \frac{\kappa_\phi}{2} \psi^2 + \frac{\kappa_c}{2} \gamma^2 + g(\phi, c). \tag{69}$$

Then the system under its structured representation is expressed as

$$f_s = \mathcal{J}_s e_s, \tag{70}$$

where the vectors of flow and effort variables are defined as:

$$f_s^\top = \left( \frac{\partial \phi}{\partial t}, \frac{\partial \psi}{\partial t}, F_\phi, \frac{\partial c}{\partial t}, \frac{\partial \gamma}{\partial t}, F_c \right)^\top \tag{71}$$

and

$$e_s^\top = \left( \frac{\delta \bar{\mathcal{G}}}{\delta \phi}, \frac{\delta \bar{\mathcal{G}}}{\delta \psi}, E_\phi, \frac{\delta \bar{\mathcal{G}}}{\delta c}, \frac{\delta \bar{\mathcal{G}}}{\delta \gamma}, j_c \right)^\top, \tag{72}$$

respectively. The skew-symmetric operator  $\mathcal{J}$  is diagonal

$$\mathcal{J}_s = \begin{pmatrix} \mathcal{J}_\phi & 0 \\ 0 & \mathcal{J}_c \end{pmatrix}. \tag{73}$$

From Proposition 1 and 2 it follows that the solidification example is a port Hamiltonian system with the boundary port variables defined for each sub-systems, see equations (32) and (53).

*Remark 5.* The conserved and non-conserved dynamics are not coupled through the structure, represented by the interconnection operator  $\mathcal{J}_s$  in the dynamical equation (70). The coupling is intrinsically defined in the potential function (65). Thus there is no shared (boundary) port variables between the conserved and non-conserved sub-systems. This is where the phase field modeling approach

differs from sharp interface models (Diagne and Maschke, 2013).

## 7. CONCLUSION

In this contribution we have introduced the formulation of phase field models as boundary control port Hamiltonian systems. The originality of this work resides in the presence of an underlying structure within the thermodynamic fluxes. The idea is to extend the system's state on their jet spaces (Maschke and van der Schaft, 2005) such that the port Hamiltonian systems of phase field models are defined along side algebraic constitutive relations. The conserved and non-conserved structured representation can be applied to multi-phases problems with moving interfaces. In Section 6 the problem of solidification was addressed.

Outgoing work include the extension to multiple-phases field models (Boyer et al., 2010). Examples such as the evaporator in heating pumps (Rasmussen and Alleyne, 2006), spinodal decomposition (Nauman and He, 2001) or the behavior of lithium batteries (Chehab et al., 2017) are investigated for the design of boundary control laws. An early lumping approach is investigated with the use of structure preserving discretization method (Cardoso-Ribeiro et al., 2019).

## REFERENCES

- Allen, S.M. and Cahn, J.W. (1979). A microscopic theory for antiphase boundary motion and its application to antiphase domain coarsening. *Acta Metallurgica*, 27(6), 1085–1095.
- Ambroso, A., Chalons, C., Coquel, F., Godlewski, E., Lagoutiere, F., Raviart, P.A., and Seguin, N. (2008). A relaxation method for the coupling of systems of conservation laws. In *Hyperbolic Problems: Theory, Numerics, Applications*, 947–954. Springer.
- Baaiu, A., Couenne, F., Le Gorrec, Y., Lefvre, L., and Tayakout, M. (2008). Port based modelling of a multi-scale adsorption column. *Mathematical and Computer Modelling of Dynamical Systems*, 14(3), 195–211.
- Boutin, B., Coquel, F., and Godlewski, E. (2008). Dafermos regularization for interface coupling of conservation laws. In *Hyperbolic problems: theory, numerics, applications*, 567–575. Springer.
- Boyer, F., Lapuerta, C., Minjeaud, S., Piar, B., and Quintard, M. (2010). Cahn–Hilliard/Navier–Stokes model for the simulation of three-phase flows. *Transport in Porous Media*, 82(3), 463–483.
- Cahn, J.W. and Hilliard, J.E. (1958). Free energy of a nonuniform system. I. Interfacial free energy. *The Journal of chemical physics*, 28(2), 258–267.
- Cardoso-Ribeiro, F.L., Matignon, D., and Lefvre, L. (2019). A partitioned finite element method for power-preserving discretization of open systems of conservation laws.
- Chehab, J.P., Franco, A., and Mammeri, Y. (2017). Boundary control of the number of interfaces for the one-dimensional Allen–Cahn equation.
- Chen, Z. (1993). Optimal boundary controls for a phase field model. *IMA Journal of Mathematical Control and Information*, 10(2), 157–176.
- Diagne, M. and Maschke, B. (2013). Port Hamiltonian formulation of a system of two conservation laws with a moving interface. *European Journal of Control*, 19(6), 495–504. Lagrangian and Hamiltonian Methods for Modelling and Control.
- Duindam, V., Macchelli, A., Stramigioli, S., and Bruyninckx, H. (2009). *Modeling and Control of Complex Physical Systems: The Port–Hamiltonian Approach*. Springer, Berlin Heidelberg.
- Elder, K., Grant, M., Provatas, N., and Kosterlitz, J. (2001). Sharp interface limits of phase-field models. *Phys. Rev. E*, 64, 021604.
- Emmerich, H. (2003). *The diffuse interface approach in materials science: thermodynamic concepts and applications of phase-field models*, volume 73. Springer Science & Business Media.
- Emmerich, H. (2008). Advances of and by phase-field modelling in condensed-matter physics. *Advances in Physics*, 57(1), 1–87.
- Godlewski, E. and Raviart, P.A. (2004). The numerical interface coupling of nonlinear hyperbolic systems of conservation laws: I. the scalar case. *Numerische Mathematik*, 97(1), 81–130.
- Hamroun, B., Dimofte, A., Lefèvre, L., and Mendes, E. (2010). Control by interconnection and energy-shaping methods of port Hamiltonian models. Application to the shallow water equations. *European Journal of Control*, 16(5), 545–563.
- Kobayashi, R. (2010). A brief introduction to phase field method. *AIP Conference Proceedings*, 1270(1), 282–291.
- Kotyczka, P., Maschke, B., and Lefvre, L. (2018). Weak form of Stokes–Dirac structures and geometric discretization of port–Hamiltonian systems. *Journal of Computational Physics*, 361, 442–476.
- Kurula, M., Zwart, H., van der Schaft, A., and Behrndt, J. (2010). Dirac structures and their composition on Hilbert spaces. *Journal of Mathematical Analysis and Applications*, 372(2), 402–422.
- Le Gorrec, Y., Zwart, H., and Maschke, B. (2005). Dirac structures and boundary control systems associated with skew-symmetric differential operators. *SIAM J. Control and Optimization*, 44, 1864–1892. doi: 10.1137/040611677.
- Macchelli, A. and Melchiorri, C. (2004). Modeling and Control of the Timoshenko Beam. The Distributed Port–Hamiltonian Approach. *SIAM Journal on Control and Optimization*, 43(2), 743–767.
- Maschke, B. and van der Schaft, A. (2005). Compositional Modelling of Distributed-Parameter Systems. In *Advanced topics in control systems theory*, 115–154. Springer.
- Maschke, B. and van der Schaft, A. (2013). On alternative Poisson brackets for fluid dynamical systems and their extension to Stokes–Dirac structures. *IFAC Proceedings Volumes*, 46(26), 109 – 114. 1st IFAC Workshop on Control of Systems Governed by Partial Differential Equations.
- Moelans, N., Blanpain, B., and Wollants, P. (2008). An introduction to phase-field modeling of microstructure evolution. *Calphad*, 32(2), 268–294.
- Nauman, E. and He, D. (2001). Nonlinear diffusion and phase separation. *Chemical Engineering Science*, 56(6), 1999–2018.



- Rasmussen, B.P. and Alleyne, A.G. (2006). Dynamic modeling and advanced control of air conditioning and refrigeration systems. Technical report, Air Conditioning and Refrigeration Center. College of Engineering. University of Illinois at Urbana-Champaign.
- Schberl, M. and Siuka, A. (2014). Jet bundle formulation of infinite-dimensional port-Hamiltonian systems using differential operators. *Automatica*, 50(2), 607 – 613.
- Trenchant, V., Fares, Y., Ramírez, H., and Le Gorrec, Y. (2015). A port-Hamiltonian formulation of a 2D boundary controlled acoustic system. *IFAC-PapersOnLine*, 48(13), 235 – 240. 5th IFAC Workshop on Lagrangian and Hamiltonian Methods for Nonlinear Control LHMNC 2015.
- van der Schaft, A. and Maschke, B. (2002). Hamiltonian formulation of distributed-parameter systems with boundary energy flow. *Journal of Geometry and Physics*, 42(12), 166–194.
- Vu, N., Lefèvre, L., and Maschke, B. (2016). A structured control model for the thermo-magneto-hydrodynamics of plasmas in Tokamaks. *Mathematical and Computer Modelling of Dynamical Systems*, 22(3), 181–206.