



HAL
open science

A density theorem for the difference Galois groups of regular singular Mahler equations

Marina Poulet

► **To cite this version:**

Marina Poulet. A density theorem for the difference Galois groups of regular singular Mahler equations. 2020. hal-03084197v1

HAL Id: hal-03084197

<https://hal.science/hal-03084197v1>

Preprint submitted on 20 Dec 2020 (v1), last revised 7 Mar 2022 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

A DENSITY THEOREM FOR THE DIFFERENCE GALOIS GROUPS OF REGULAR SINGULAR MAHLER EQUATIONS

MARINA POULET

ABSTRACT. The difference Galois theory of Mahler equations is an active research area. The present paper aims at developing the analytic aspects of this theory. We first attach a pair of connection matrices to any regular singular Mahler equation. We then show that these connection matrices can be used to produce a Zariski-dense subgroup of the difference Galois group of any regular singular Mahler equation.

CONTENTS

Introduction	3
Notations	4
Part I. Preliminary results	5
1 Difference equations, difference systems and difference modules	5
1.1 Generalities	5
1.1.a The category of difference modules	5
1.1.b Difference systems and equations	6
1.1.c The category of difference systems and the category of difference mod- ules are equivalent	6
1.2 The category of Mahler systems with different base fields	7
1.2.a The category $\mathcal{E}(K_{p^\infty})$	7
1.2.b The category \mathcal{E}	7
2 Regular singular Mahler systems	8
2.1 Neighbourhood of the point 0	8
2.2 Neighbourhood of the point ∞	10
2.3 Neighbourhood of the point 1	11
3 Meromorphic functions on (a part of) the universal cover of \mathbb{C}^*	13
Part II. The categories of Fuchsian systems and regular singular systems at 0, 1 and ∞	14
4 The category \mathcal{E}_{rs} of regular singular systems at 0, 1 and ∞	14
5 The category \mathcal{E}_f of Fuchsian systems at 0, 1 and ∞	14

Date: December 20, 2020.

2010 Mathematics Subject Classification. 39A06, 12H10, 11R45.

Key words and phrases. Mahler equations, difference Galois theory, density theorem.

6	The categories \mathcal{E}_{rs} and \mathcal{E}_f are equivalent	14
Part III. Local categories and local groupoids		16
7	Preliminary results	16
7.1	On morphisms between constant systems	16
7.2	Finite dimensional complex representations of \mathbb{Z}	17
8	Localization at 0	17
8.1	The categories $\mathcal{E}_{rs}^{(0)}$ and $\mathcal{P}^{(0)}$	17
8.2	Local Galois groupoid at 0	18
9	Localization at ∞	18
9.1	The categories $\mathcal{E}_{rs}^{(\infty)}$ and $\mathcal{P}^{(\infty)}$	18
9.2	Local Galois groupoid at ∞	18
10	Localization at 1	19
10.1	The categories $\mathcal{E}_{rs}^{(1)}$ and $\mathcal{P}^{(1)}$	19
10.2	Local Galois groupoid at 1	20
Part IV. Global categories		21
11	The category \mathcal{E}_{rs} of regular singular systems at 0, 1 and ∞	21
12	The category \mathcal{C} of connections	21
13	The category \mathcal{S} of solutions	22
14	Equivalence between the categories	23
15	Structure of neutral Tannakian category over \mathbb{C}	24
Part V. Mahler analogue of the density theorem of Schlesinger		25
16	The Galois groupoid of the category \mathcal{C}	25
17	Construction of isomorphisms of tensor functors	26
17.1	The category $\mathcal{C}_{E_0, E_\infty}$	26
17.2	Construction of isomorphisms of tensor functors.	29
18	Analogue of the density theorem of Schlesinger for Mahler equations.	30
18.1	Density criteria.	30
18.2	Density theorem in the regular singular case	31
18.3	Density theorem in the regular case.	34
References		34

Introduction

A Mahler equation of order $n \in \mathbb{N}^*$ is a functional equation of the form

$$(1) \quad a_n(z)f(z^{p^n}) + a_{n-1}(z)f(z^{p^{n-1}}) + \cdots + a_0(z)f(z) = 0$$

with $a_0(z), \dots, a_n(z) \in \mathbb{C}(z)$, with $a_0(z)a_n(z) \neq 0$ and with p an integer greater than or equal to 2. The solutions of these equations are called Mahler functions. The study of these functions began with the work of Mahler on the algebraic relations between their special values in [Mah29], [Mah30a], [Mah30b]. This work has been a source of inspiration for many authors e.g. [LP77], [Lox84], [Nis96], [Pel11], [BCZ14], [AF16], [CDDM16], [AB17], [Fer19]. Philippon in [Phi15], then Adamczewski and Faverjon in [AF15], showed that the algebraic relations between special values of Mahler functions come from algebraic relations between the functions; hence the importance of the study of algebraic relations between Mahler functions. Moreover, there is an increased interest in these functions because of their connection with automata theory (see for instance [Cob68], [MF80], [AS92], [Bec94] for more details).

The algebraic relations between Mahler functions are reflected by certain difference Galois groups. This is one of the reasons why the difference Galois theoretic aspects of the theory of Mahler equations has been the subject of many recent works. For more details on this difference Galois theory and applications see for instance [vdPS97], [DHR15], [Phi15], [Roq18]. These articles focus on the algebraic aspects of the difference Galois theory of Mahler equations. The analytic aspects have not yet been studied so much. The aim of the present paper is to start to fill this gap: our main objective is to introduce for any Mahler equation a Zariski-dense subgroup of analytic nature of its difference Galois group.

Our result is inspired by a celebrated density theorem due to Schlesinger for differential equations (see [Sch95]) and by its generalizations to (q -)difference equations discovered by Etingof and developed further by Duval, Ramis, Sauloy, Singer, van der Put. More precisely, the density theorem of Schlesinger ensures that the monodromy of a differential equation with regular singular points is Zariski-dense in its differential Galois group. This theorem was transposed to regular q -difference equations by Etingof (see [Eti95]) in 1995 and later to regular singular q -difference equations by Sauloy on the one hand (see [Sau99]) and by van der Put and Singer on the other hand (see [vdPS97]). The key to the extensions of Schlesinger's theorem to q -difference equations is a connection matrix introduced by Birkhoff playing the role of the monodromy in the differential case. Roughly speaking, this matrix connects 0 to ∞ : given a q -difference equation, we associate two bases of solutions, one at 0, the other one at ∞ and the connection matrix reflects the linear relations between these two bases of solutions. Likewise, for a Mahler equation which is regular singular at 0 and ∞ we can attach two bases of solutions, one at 0 and another one at ∞ . However, a problem arises for Mahler equations: these two bases of solutions can not be connected in general. Indeed, the basis of solutions at 0 consists of meromorphic functions on the open unit disk while the basis of solutions at ∞ consists of meromorphic functions on the complement of the closed unit disk and, by a theorem due to Randé (see [Ran92]), the unit circle is in general a natural boundary for these functions. In order to overcome this problem, our key idea is to use the point 1: we consider three bases of solutions, at 0, 1 and ∞ and we connect 0 to 1 on the one hand and 1 to ∞ on the other hand. In this way, we attach two connection matrices to any regular singular Mahler equation. This allows us to construct particular elements of

the Galois groupoid G of the category of regular singular Mahler equations. We will show that these elements generate a Zariski-dense subgroupoid of G .

We shall now describe more precisely the content of this paper. Part I contains prerequisites and first results about Mahler equations at the points 0, 1 and ∞ . Locally at 1, we can use known results about q -difference equations, see for instance [Sau00]. Indeed, locally at 1, a Mahler equation can be seen as a q -difference equation with the change of variables $z = \exp(u)$. In part II, we introduce the category of Fuchsian Mahler systems and regular singular Mahler systems at 0, 1 and ∞ . We prove that these categories are equivalent categories. In part III, we construct and describe the local Galois groups and groupoids at 0, 1 and ∞ of the category of regular singular Mahler systems, which are denoted by G_0 , G_1 and G_∞ respectively. Part IV deals with the global categories, we prove that these categories are equivalent Tannakian categories. In particular, we introduce the category of connections \mathcal{C} constructed from pairs of connection matrices attached to regular singular Mahler equations. In part V, we construct elements of the Galois groupoid G of the category of regular singular Mahler equations. A first family of elements come from the local Galois groupoids G_0 , G_1 and G_∞ . Thanks to the pair of connection matrices, we also build elements which make the link between G_0 and G_1 , denoted by Γ_{0, \tilde{z}_0} on the one hand, and between G_1 and G_∞ , denoted by $\Gamma_{\infty, \tilde{z}_\infty}$ on the other hand. We prove the main result of this paper:

Theorem 18.4 *The local Galois groupoids G_0, G_1, G_∞ and the Galoisian isomorphisms $\Gamma_{0, \tilde{z}_0}, \Gamma_{\infty, \tilde{z}_\infty}$ for all $\tilde{z}_0 \in \Sigma_0 \setminus \widetilde{E}_0, \tilde{z}_\infty \in \Sigma_\infty \setminus \widetilde{E}_\infty$ generate a Zariski-dense subgroupoid H of the Galois groupoid G of $\mathcal{C}_{E_0, E_\infty}$.*

Moreover, we know Zariski-dense subgroupoids of G_0, G_∞ (see 7.2, 8.2, 9.2) and of G_1 (see 10.2).

Acknowledgements. This work was performed within the framework of the LABEX MILYON (ANR-10-LABX-0070) of Université de Lyon, within the program “Investissements d’Avenir” (ANR-11-IDEX-0007) operated by the French National Research Agency (ANR).

Notations

We denote by $\mathbb{C}[[z]]$ the ring of formal power series with complex coefficients in the indeterminate z and by $\mathbb{C}\{z\}$ the ring of all power series in $\mathbb{C}[[z]]$ that are convergent in some neighborhood of the origin. We denote by $\mathbb{C}((z))$ the fraction field of $\mathbb{C}[[z]]$ (which is called the field of formal Laurent series) and by $\mathbb{C}(\{z\})$ the fraction field of $\mathbb{C}\{z\}$. In a similar way, $\mathbb{C}[[\frac{1}{z}]]$ is the ring of formal power series with complex coefficients in $1/z$ and $\mathbb{C}((\frac{1}{z}))$ is its fraction field. We denote by $\mathbb{C}\{\frac{1}{z}\}$ the convergent power series in $1/z$ and $\mathbb{C}(\{\frac{1}{z}\})$ its fraction field. Similarly, $\mathbb{C}\{z-1\}$ is the ring of all power series that are convergent in some neighborhood of 1 and $\mathbb{C}(\{z-1\})$ is its fraction field. If E is an open set of a Riemann surface, we denote by $\mathcal{M}(E)$ the field of meromorphic functions on E . We consider the universal cover of \mathbb{C}^* , denoted by $\widetilde{\mathbb{C}^*}$ and $\tilde{1}$ is the point $(1, 0)$ of this Riemann surface.

We denote by $\text{Vect}_{\mathbb{C}}^f$ the finite dimensional \mathbb{C} -vector spaces. If M is a matrix, $\text{rk}(M)$ is its rank, that is the dimension of the vector space generated by its columns.

We denote by $D(0, 1)$ the open unit disk and by $\overline{D}(0, 1)$ the closed unit disk.

If \mathcal{C} is a category, $\text{Obj}(\mathcal{C})$ and $\text{Mor}(\mathcal{C})$ denote respectively the objects and the morphisms of the category \mathcal{C} . If $A \in \text{Obj}(\mathcal{C})$ and $B \in \text{Obj}(\mathcal{C})$, we denote by $\text{Mor}_{\mathcal{C}}(A, B)$ the class of morphisms in the category \mathcal{C} from A to B . If $u \in \text{Mor}_{\mathcal{C}}(A, B)$, we write $u : A \rightarrow B$. If G is a groupoid and $\alpha, \beta \in \text{Obj}(\mathcal{C})$ then $G(\alpha, \beta)$ denote the morphisms of G from α to β .

Part I. Preliminary results

1. DIFFERENCE EQUATIONS, DIFFERENCE SYSTEMS AND DIFFERENCE MODULES

For more details, the reader is referred to [vdPS97].

1.1. Generalities

Let K be a field and $\phi : K \rightarrow K$ be an automorphism of K . We extend ϕ to the vector spaces of matrices with entries in K applying ϕ on each entry of the matrices.

1.1.a. The category of difference modules

A *difference module* over the difference field (K, ϕ) is a pair (M, Φ_M) consisting of a finite dimensional K -vector space M equipped with an automorphism

$$\Phi_M : M \rightarrow M$$

which is ϕ -linear, that is

$$\forall \lambda \in K, \forall x, y \in M, \Phi_M(x + \lambda y) = \Phi_M(x) + \phi(\lambda) \Phi_M(y).$$

The objects of the *category of difference modules* are the difference modules (M, Φ_M) and the morphisms f from the difference module (M, Φ_M) to the difference module (N, Φ_N) are the K -linear maps $f : M \rightarrow N$ such that

$$\Phi_N \circ f = f \circ \Phi_M.$$

Let

$$C_K = K^\phi = \{x \in K \mid \phi(x) = x\}$$

be the *constant field*. According to [vdPS97], the category of difference modules is a C_K -linear rigid abelian tensor category. The functor which forgets the difference structure i.e.

$$\left\{ \begin{array}{l} (M, \Phi_M) \rightsquigarrow M \\ f \rightsquigarrow f \end{array} \right.$$

from the category of difference modules to the category of finite dimensional K -vector spaces Vect_K^f is a fibre functor. Therefore, the category of difference modules is a Tannakian category over C_K (see [DM82]). For a better understanding on what follows, we give here more details on the tensor structure of this Tannakian category over C_K . The identity object of this category is the 1-dimensional K -vector space $\mathbf{1} = Ke$ equipped with the ϕ -linear automorphism $\Phi_{\mathbf{1}} : e \rightarrow e$. Let (M, Φ_M) and (N, Φ_N) be two objects of this category, the internal Hom is

$$\underline{\text{Hom}}((M, \Phi_M), (N, \Phi_N)) = (\text{Hom}_K(M, N), \Phi_{\text{Hom}_K(M, N)})$$

where $\text{Hom}_K(M, N)$ denotes the K -vector space of K -linear morphisms from M to N and $\Phi_{\text{Hom}_K(M, N)}$ is the ϕ -linear automorphism

$$\Phi_{\text{Hom}_K(M, N)} : \text{Hom}_K(M, N) \rightarrow \text{Hom}_K(M, N) \\ \sigma \mapsto \Phi_N \circ \sigma \circ \Phi_M^{-1}.$$

The dual of the object (M, Φ_M) is the object $\underline{\text{Hom}}((M, \Phi_M), (\mathbf{1}, \Phi_{\mathbf{1}}))$ that is

$$(\text{Hom}_K(M, K), \sigma \in \text{Hom}_K(M, K) \mapsto \sigma \circ \Phi_M^{-1} \in \text{Hom}_K(M, K)).$$

We notice that the category of difference modules is equivalent to its full subcategory whose objects are the K -vector spaces K^n , $n \in \mathbb{N}$. This category is therefore a Tannakian category over C_K and we denote it by $\text{Diff}(K, \phi)$.

1.1.b. *Difference systems and equations*

Let

$$L = \phi^n + a_1\phi^{n-1} + \dots + a_n$$

with $a_1, \dots, a_n \in K$, $a_n \neq 0$. Any *difference equation of order n*

$$Lf = \phi^n(f) + a_1\phi^{n-1}(f) + \dots + a_nf = 0$$

can be transformed into the difference system of rank n

$$\phi(Y) = A_L Y$$

with

$$A_L = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & 1 \\ -\frac{a_0}{a_n} & \dots & \dots & -\frac{a_{n-2}}{a_n} & -\frac{a_{n-1}}{a_n} \end{pmatrix} \in GL_n(K).$$

Conversely, by the cyclic vector lemma, any system

$$\phi(Y) = BY, \quad B \in GL_n(K)$$

is equivalent via the equivalence

$$A \sim (\phi(F))^{-1} AF, \quad F \in GL_n(K)$$

to a system of the form $\phi(Y) = A_L Y$. Thus, we can associate a difference equation with a difference system. In this article, we will focus on difference systems called Mahler systems but the results obtained can therefore be applied to Mahler equations.

The objects of the *category of difference systems* are the pairs (K^n, A) where $n \in \mathbb{N}$ and $A \in GL_n(K)$ and the morphisms from (K^{n_1}, A) to (K^{n_2}, B) are the $R \in \mathcal{M}_{n_2, n_1}(K)$ such that $\phi(R)A = BR$. To simplify notations, we will often denote by A the object (K^n, A) . This object is identified with the system $\phi(Y) = AY$.

1.1.c. *The category of difference systems and the category of difference modules are equivalent*

We can associate with the difference system

$$\phi(Y) = AY, \quad A \in GL_n(K)$$

the difference module $(K^n, \Phi_M) \in \text{Obj}(\text{Diff}(K, \phi))$ where

$$\Phi_M : Y \in K^n \mapsto A^{-1}\phi(Y) \in K^n$$

and with the morphism $R : A \rightarrow B$ in the category of difference systems, introduced in 1.1.b, the morphism

$$f : Y \in K^{n_1} \mapsto RY \in K^{n_2}$$

in the category $\text{Diff}(K, \phi)$. This provides an equivalence of abelian categories between the categories of difference systems and difference modules. We want to obtain an equivalence of tensor categories. In order to do that, we can consider the following tensor product on matrices called Kronecker product: if $A = (a_{i,j}) \in \mathcal{M}_{n_1, m_1}(K)$ and $B \in \mathcal{M}_{n_2, m_2}(K)$ then

$$A \otimes B = \begin{pmatrix} a_{1,1}B & \cdots & a_{1,m_1}B \\ \vdots & & \vdots \\ a_{n_1,1}B & \cdots & a_{n_1,m_1}B \end{pmatrix} \in \mathcal{M}_{n_1 n_2, m_1 m_2}(K).$$

Therefore, we define a tensor product on the category of difference systems as follows: if (K^{n_1}, A) and (K^{n_2}, B) are two objects then

$$(K^{n_1}, A) \otimes (K^{n_2}, B) = (K^{n_1 n_2}, A \otimes B)$$

and if R_1 et R_2 are two morphisms, their tensor product is $R_1 \otimes R_2$. The above construction provides an equivalence of abelian tensor categories between the category of difference systems and the category of difference modules. Therefore, the category of difference systems is a C_K -linear rigid abelian tensor category. It is a Tannakian category over C_K with the forgetful functor $(K^n, A) \rightsquigarrow K^n$, $R \rightsquigarrow R$.

As previously for the category of difference modules, let us explain the tensor structure inherited by the category of difference systems. The identity object is $1 \in K^*$. Let (K^{n_1}, A) and (K^{n_2}, B) be two objects of this category, the internal Hom is

$$\underline{\text{Hom}}((K^{n_1}, A), (K^{n_2}, B)) = (K^{n_1 n_2}, D_{A,B})$$

where $D_{A,B}$ is the inverse of the matrix representing the C_K -linear map

$$Y \in \mathcal{M}_{n_2, n_1}(K) \mapsto B^{-1} \phi(Y) A$$

in the canonical basis of $\mathcal{M}_{n_2, n_1}(K)$. The dual of the object (K^{n_1}, A) is

$$\underline{\text{Hom}}((K^{n_1}, A), (K, 1)) = (K^{n_1}, ({}^t A)^{-1}).$$

1.2. The category of Mahler systems with different base fields

1.2.a. The category $\mathcal{E}(K_{p^\infty})$

The category of Mahler systems with base field $K_{p^\infty} := \bigcup_{n \geq 0} K_{p^n}$ with $K_{p^n} = \mathbb{C}(z^{1/p^n})$ is the particular case of categories of difference systems introduced in 1.1.b where $K = K_{p^\infty}$ and the field automorphism of K is $\phi = \phi_p$ defined by

$$\phi_p : f(z) \in K \mapsto f(z^p) \in K.$$

We will denote it by $\mathcal{E}(K_{p^\infty})$. In this case, the constant field is $C_K = K^{\phi_p} = \mathbb{C}$. From the previous section, we know that it is a Tannakian category over \mathbb{C} .

1.2.b. The category \mathcal{E}

We will consider the category of Mahler systems whose base field is $K = \mathbb{C}(z)$ equipped with the injective endomorphism ϕ_p (which is not anymore surjective) i.e. the objects are the pairs $(\mathbb{C}(z)^n, A)$ where $n \in \mathbb{N}$ and $A \in GL_n(\mathbb{C}(z))$ and the morphisms from $(\mathbb{C}(z)^{n_1}, A) \in \text{Obj}(\mathcal{E})$ to $(\mathbb{C}(z)^{n_2}, B) \in \text{Obj}(\mathcal{E})$ are the $R \in \mathcal{M}_{n_2, n_1}(\mathbb{C}(z))$ such that $\phi_p(R)A = BR$. We denote by \mathcal{E} this category. It is a subcategory of $\mathcal{E}(K_{p^\infty})$.

Proposition 1.1. *The category \mathcal{E} is a Tannakian category over \mathbb{C} .*

Proof. The category \mathcal{E} is a rigid tensor subcategory of $\mathcal{E}(K_{p^\infty})$ because it is stable under tensor product, it contains the unit object, the internal Hom and every object has a dual (see 1.1.c for a description of these elements). To show that it is a Tannakian subcategory of $\mathcal{E}(K_{p^\infty})$, the only non trivial point which remains to be proved is the existence of kernels and cokernels in \mathcal{E} . Let us prove that every morphism in \mathcal{E} has a kernel in \mathcal{E} . The existence of cokernels will follow from the fact that duality exchanges kernels with cokernels. Let $A, B \in \text{Obj}(\mathcal{E})$ respectively of rank n_1 and n_2 and $R \in \text{Mor}_{\mathcal{E}}(A, B)$. The morphism R has a kernel $(C, K : C \rightarrow A)$ in the Tannakian category $\mathcal{E}(K_{p^\infty})$. There exists $n_0 \in \mathbb{N}$ such that $\phi_p^{n_0}(C), \phi_p^{n_0}(K)$ have entries in $\mathbb{C}(z)$. Let

$$F_A = \left(\prod_{i=1}^{n_0} \phi_p^{n_0-i}(A) \right)^{-1} \in GL_{n_1}(\mathbb{C}(z)).$$

The following diagram is commutative

$$\begin{array}{ccc}
 A & \xrightarrow{R} & B \\
 \uparrow F_A & & \uparrow F_B \\
 \phi_p^{n_0}(A) & \xrightarrow{\phi_p^{n_0}(R)} & \phi_p^{n_0}(B) \\
 \uparrow \phi_p^{n_0}(K) & \nearrow 0 & \\
 \phi_p^{n_0}(C) & &
 \end{array}$$

and one can check that $(\phi_p^{n_0}(C), F_A \phi_p^{n_0}(K) : \phi_p^{n_0}(C) \rightarrow A)$ is a kernel of R in the category \mathcal{E} . \square

We will also consider the category of Mahler systems whose base field is $K = \mathbb{C}(\{z\})$ (respectively $K = \mathbb{C}(\{z-1\})$ and $K = \mathbb{C}(\{\frac{1}{z}\})$) equipped with ϕ_p , we will denote it by $\mathcal{E}^{(0)}$ (respectively by $\mathcal{E}^{(1)}$, by $\mathcal{E}^{(\infty)}$). These are also Tannakian categories over \mathbb{C} . Indeed, for the categories $\mathcal{E}^{(0)}$ and $\mathcal{E}^{(\infty)}$ we can adapt the proof of Proposition 1.1 and for the category $\mathcal{E}^{(1)}$, $\phi_p : z \in \mathbb{C}(\{z-1\}) \mapsto z^p$ is an automorphism of $\mathbb{C}(\{z-1\})$ so it follows from 1.1.c.

2. REGULAR SINGULAR MAHLER SYSTEMS

Definition 2.1. A matrix A is *regular at* $x \in \mathbb{C}$ if its entries are analytic functions at x and $A(x) \in GL_n(\mathbb{C})$. The matrix A is *regular at* ∞ if $A(1/z)$ is regular at 0.

2.1. Neighbourhood of the point 0

We consider the Mahler system

$$(2) \quad \phi_p(Y) = AY$$

with $A \in GL_n(\mathbb{C}(\{z\}))$.

Definition 2.2. The system (2) is *Fuchsian at* 0 if A is regular at 0.

Definition 2.3. We say that the system (2) is *meromorphically equivalent at 0* to the system

$$\phi_p(Y) = BY$$

with $B \in GL_n(\mathbb{C}(\{z\}))$ if there exists $T \in GL_n(\mathbb{C}(\{z\}))$ such that

$$B = \phi_p(T)^{-1}AT.$$

The motivation behind this definition is the following: if Z is a solution of the Mahler system (2) and $T \in GL_n(\mathbb{C}(\{z\}))$ then $T^{-1}Z$ is a solution of the system $\phi_p(Y) = BY$.

Remark 2.4. Two systems of the form (2) are *meromorphically equivalent at 0* if and only if they are isomorphic in the category $\mathcal{E}^{(0)}$.

Definition 2.5. The system (2) is *regular singular at 0* if it is meromorphically equivalent at 0 to a Fuchsian one.

From now on, we fix a submultiplicative norm $\|\cdot\|$ on $\mathcal{M}_n(\mathbb{C})$.

Theorem 2.6. Assume that the system (2) is Fuchsian at 0. There exists a unique $F \in GL_n(\mathbb{C}[[z]])$ such that $F(0) = I_n$ and

$$(3) \quad \phi_p(F)^{-1}AF = A(0).$$

We have $F \in GL_n(\mathbb{C}\{z\})$.

Moreover, if we assume that $A \in GL_n(\mathbb{C}(z))$ then $F \in GL_n(\mathcal{M}(D(0,1)))$.

Proof. We are looking for a matrix $F \in GL_n(\mathbb{C}[[z]])$ such that

$$(4) \quad \begin{cases} AF = \phi_p(F)A(0) \\ F(0) = I_n \end{cases}$$

We write

$$F(z) = I_n + \sum_{k=1}^{+\infty} F_k z^k = \sum_{k=0}^{+\infty} F_k z^k$$

with $F_0 = I_n$ and

$$A(z) = \sum_{k=0}^{+\infty} A_k z^k.$$

From (4), we have

$$\begin{cases} \forall k \in \mathbb{N}, p \nmid k, \sum_{j=0}^k A_j F_{k-j} = 0 \\ \forall k \in \mathbb{N}, p \mid k, \sum_{j=0}^k A_j F_{k-j} = F_{k/p} A_0 \end{cases}.$$

Thus, this formal solution is defined by

$$\begin{cases} F_0 = I_n \\ \forall k \in \mathbb{N}, p \nmid k, F_k = -A_0^{-1} \left(\sum_{j=1}^k A_j F_{k-j} \right) \\ \forall i \in \mathbb{N}^*, F_{pi} = A_0^{-1} F_i A_0 - A_0^{-1} \left(\sum_{j=1}^{pi} A_j F_{pi-j} \right) \end{cases}.$$

Now, we have to prove the convergence of the formal solution. We write $f_k := \|F_k\|$, $a_k := \|A_k\|$ and $b := \|A_0^{-1}\|$. We introduce the sequence $(\bar{f}_n)_{n \geq 0}$ defined by

$$\begin{cases} \bar{f}_0 = f_0 \\ \forall n \in \mathbb{N}^*, \bar{f}_n = b \sum_{i=0}^{n-1} (a_{n-i} + a_0) \bar{f}_i \end{cases}.$$

By induction, we can show that for all $k \in \mathbb{N}$, $0 \leq f_k \leq \bar{f}_k$. We have

$$\bar{S}(z) := \sum_{j=0}^{+\infty} \bar{f}_j z^j = f_0 + b \sum_{j=1}^{+\infty} \sum_{i=0}^{j-1} (a_{j-i} + a_0) \bar{f}_i z^j = f_0 + b \sum_{i=0}^{+\infty} \sum_{j=i+1}^{+\infty} (a_{j-i} + a_0) \bar{f}_i z^j.$$

This implies

$$\bar{S}(z) = f_0 + b \bar{S}(z) \left(\sum_{j=1}^{+\infty} a_j z^j + a_0 \sum_{j=1}^{+\infty} z^j \right).$$

Therefore, $\bar{S}(z) = \frac{f_0}{1 - b \left(\sum_{j=1}^{+\infty} a_j z^j + a_0 \sum_{j=1}^{+\infty} z^j \right)}$ and \bar{S} is the expression as a power serie

around the center $z = 0$ of this holomorphic function. Thus, $\sum_{k=0}^{+\infty} f_k z^k$ has a non zero radius of convergence.

It remains to prove that if $A \in GL_n(\mathbb{C}(z))$ then $F \in GL_n(\mathcal{M}(D(0,1)))$. From the equation (3), for $z \in D(0,1)$, we have

$$F(z) = A^{-1}(z) \cdots A^{-1}(z^{p^k}) F(z^{p^{k+1}}) A_0^{k+1}.$$

This is a product of matrices with meromorphic function entries for a large enough k so the entries of F are meromorphic on $D(0,1)$. \square

Corollary 2.7. *If the system (2) is regular singular at 0 and $A \in GL_n(\mathbb{C}(z))$ then there exist $F_0 \in GL_n(\mathcal{M}(D(0,1)))$ and $A_0 \in GL_n(\mathbb{C})$ such that*

$$\phi_p(F_0)^{-1} A F_0 = A_0.$$

We highlight that we have $F_0 \in GL_n(\mathcal{M}(D(0,1)))$ and, generally, the unit circle is a natural boundary because, from a result of Bernard Randé in [Ran92], a solution of the Mahler equation (1) is rational or it has the unit circle as natural boundary.

Remark 2.8. *If the system (2) is regular singular at 0, we can reduce the study of its solutions to the study of the solutions of a Mahler system with constant entries. Indeed, Z is a solution of $\phi_p(Y) = A_0 Y$ if and only if $F_0 Z$ is a solution of $\phi_p(Y) = AY$.*

Remark 2.9. *In the category $\mathcal{E}^{(0)}$, the system (2) is regular singular at 0 if and only if it is isomorphic to a Mahler system with constant entries.*

2.2. Neighbourhood of the point ∞

We consider the Mahler system

$$(5) \quad \phi_p(Y) = AY$$

with $A \in GL_n(\mathbb{C}(\{\frac{1}{z}\}))$.

Definition 2.10. The system (5) is *Fuchsian at ∞* if A is regular at ∞ .

Definition 2.11. We say that the system (5) is *meromorphically equivalent at ∞* to the system

$$\phi_p(Y) = BY$$

with $B \in GL_n(\mathbb{C}(\{\frac{1}{z}\}))$ if there exists $T \in GL_n(\mathbb{C}(\{\frac{1}{z}\}))$ such that

$$B = \phi_p(T)^{-1} AT.$$

Remark 2.12. *Two systems of the form (5) are meromorphically equivalent at ∞ if and only if they are isomorphic in the category $\mathcal{E}^{(\infty)}$.*

Definition 2.13. The system (5) is *regular singular at ∞* if it is meromorphically equivalent at ∞ to a Fuchsian one.

Using the change of variables $z \rightarrow 1/z$, Theorem 2.6 gives the following result.

Theorem 2.14. *Assume that the system (5) is Fuchsian at ∞ . There exists a unique $F \in GL_n(\mathbb{C}[[\frac{1}{z}]])$ such that $F(\infty) = I_n$ and*

$$(6) \quad \phi_p(F)^{-1} AF = A(\infty).$$

We have $F \in GL_n(\mathbb{C}\{\frac{1}{z}\})$.

Moreover, if we assume that $A \in GL_n(\mathbb{C}(z))$ then $F \in GL_n(\mathcal{M}(\mathbb{P}^1(\mathbb{C}) \setminus \overline{D}(0,1)))$.

Corollary 2.15. *If the system (5) is regular singular at ∞ and $A \in GL_n(\mathbb{C}(z))$ then there exist $F_\infty \in GL_n(\mathcal{M}(\mathbb{P}^1(\mathbb{C}) \setminus \overline{D}(0,1)))$ and $A_\infty \in GL_n(\mathbb{C})$ such that*

$$\phi_p(F_\infty)^{-1} AF_\infty = A_\infty.$$

Remark 2.16. *In the category $\mathcal{E}^{(\infty)}$, the system (5) is regular singular at ∞ if and only if it is isomorphic to a Mahler system with constant entries.*

2.3. Neighbourhood of the point 1

We shall now study the regular singular Mahler equations at the point 1. We consider the Mahler system

$$(7) \quad \phi_p(Y) = AY$$

with $A \in GL_n(\mathbb{C}(\{z-1\}))$.

Definition 2.17. The system (7) is *Fuchsian at 1* if A is regular at 1.

Definition 2.18. We say that the system (7) is *meromorphically equivalent at 1* to the system

$$\phi_p(Y) = BY$$

with $B \in GL_n(\mathbb{C}(\{z-1\}))$ if there exists $T \in GL_n(\mathbb{C}(\{z-1\}))$ such that

$$B = \phi_p(T)^{-1} AT.$$

Remark 2.19. *Two systems of the form (7) are meromorphically equivalent at 1 if and only if they are isomorphic in the category $\mathcal{E}^{(1)}$.*

Definition 2.20. The system (7) is *regular singular at 1* if it is meromorphically equivalent at 1 to a Fuchsian one.

In order to study the system (7), we transform it into the q -difference system

$$(8) \quad X(pu) = B(u)X(u) \text{ with } B(u) = A(\exp(u))$$

with the change of variables $z = \exp(u)$. Locally, to transform the q -difference system (8) into the Mahler system (7), we can use the principal value Log because it is biholomorphic between a neighbourhood of 1 and a neighbourhood of 0. However, for global considerations, to go back to our initial system, we use the following logarithm which is holomorphic on the universal cover $\widetilde{\mathbb{C}^*} := \{(re^{ib}, b) \mid r > 0, b \in \mathbb{R}\}$ of \mathbb{C}^* :

$$\begin{aligned} \widetilde{\log} : \quad \widetilde{\mathbb{C}^*} &\rightarrow \mathbb{C} \\ (re^{ib}, b) &\mapsto \log(r) + ib \end{aligned}$$

It is moreover a biholomorphic function.

Notation 2.21. We define

$$\pi := \exp \circ \widetilde{\log}$$

and if W is a matrix with meromorphic entries on $V \subset \mathbb{C}$, for all \tilde{z} such that $\pi(\tilde{z}) \in V$,

$$\pi^*W(\tilde{z}) := W(\pi(\tilde{z})).$$

With an abuse of notation, we will also denote by ϕ_p the function

$$\begin{aligned} \phi_p : \quad \widetilde{\mathbb{C}^*} &\rightarrow \widetilde{\mathbb{C}^*} \\ (re^{ib}, b) &\mapsto (re^{ib}, b)^p \end{aligned}$$

where $(re^{ib}, b)^p := (r^p e^{ipb}, pb)$.

We notice that

$$p\widetilde{\log} = \widetilde{\log} \circ \phi_p.$$

We need the following lemma, proved in [Sau99, I.1.3.1].

Lemma 2.22. Let $B \in GL_n(\mathcal{M}(\mathbb{C}))$ be regular at 0. There exist $G \in GL_n(\mathcal{M}(\mathbb{C}))$ and $C_0 \in GL_n(\mathbb{C})$ such that

$$G(pu)^{-1}B(u)G(u) = C_0.$$

Theorem 2.23. Assume that the system (7) is Fuchsian at 1 and $A \in GL_n(\mathbb{C}(z))$. There exist $\widetilde{F}_1 \in GL_n(\mathcal{M}(\widetilde{\mathbb{C}^*}))$ and $C_0 \in GL_n(\mathbb{C})$ such that

$$(9) \quad \phi_p(\widetilde{F}_1)^{-1}(\pi^*A)\widetilde{F}_1 = C_0.$$

Proof. We transform the system (7) into the q -difference system (8). From the previous lemma, we can take $\widetilde{F}_1 := G \circ \widetilde{\log}$. \square

Corollary 2.24. If the system (7) is regular singular at 1 and $A \in GL_n(\mathbb{C}(z))$ then there exist $\widetilde{F}_1 \in GL_n(\mathcal{M}(\widetilde{\mathbb{C}^*}))$ and $A_1 \in GL_n(\mathbb{C})$ such that

$$\phi_p(\widetilde{F}_1)^{-1}(\pi^*A)\widetilde{F}_1 = A_1.$$

Remark 2.25. In the category $\mathcal{E}^{(1)}$, the system (7) is regular singular at 1 if and only if it is isomorphic to a Mahler system with constant entries.

3. MEROMORPHIC FUNCTIONS ON (A PART OF) THE UNIVERSAL COVER OF \mathbb{C}^*

One can identify meromorphic functions on \mathbb{C}^* with meromorphic functions on $\widetilde{\mathbb{C}^*}$ via

$$i: \mathcal{M}(\mathbb{C}^*) \hookrightarrow \mathcal{M}(\widetilde{\mathbb{C}^*}) .$$

$$f \mapsto f \circ \pi$$

We write

$$\pi^* \mathcal{M}(\mathbb{C}^*) := \{f \circ \pi \mid f \in \mathcal{M}(\mathbb{C}^*)\}$$

the image of i , it consists of the $g \in \mathcal{M}(\widetilde{\mathbb{C}^*})$ which are 2π -invariant in b . This identification between $\mathcal{M}(\mathbb{C}^*)$ and $\pi^* \mathcal{M}(\mathbb{C}^*)$ is compatible with $z \mapsto z^p$ i.e. there is the following commutative diagram

$$(10) \quad \begin{array}{ccc} \mathcal{M}(\mathbb{C}^*) & \xrightarrow{f(z) \mapsto f(z^p)} & \mathcal{M}(\mathbb{C}^*) . \\ \downarrow & & \downarrow \\ \mathcal{M}(\widetilde{\mathbb{C}^*}) & \xrightarrow{f(\tilde{z}) \mapsto f(\tilde{z}^p)} & \mathcal{M}(\widetilde{\mathbb{C}^*}) \end{array}$$

that is

$$\phi_p \circ \pi = \pi \circ \phi_p.$$

We also write

$$\pi^* \mathcal{M}(D(0, 1)) := \{f \circ \pi \mid f \in \mathcal{M}(D(0, 1))\}$$

and

$$\pi^* \mathcal{M}(\mathbb{P}^1(\mathbb{C}) \setminus \overline{D}(0, 1)) := \{f \circ \pi \mid f \in \mathcal{M}(\mathbb{P}^1(\mathbb{C}) \setminus \overline{D}(0, 1))\}.$$

We can check that

Lemma 3.1.

- (1) If $h \in \pi^* \mathcal{M}(D(0, 1))$ then $h \circ \phi_p \in \pi^* \mathcal{M}(D(0, 1))$.
- (2) If $h \in \pi^* \mathcal{M}(\mathbb{P}^1(\mathbb{C}) \setminus \overline{D}(0, 1))$ then $h \circ \phi_p \in \pi^* \mathcal{M}(\mathbb{P}^1(\mathbb{C}) \setminus \overline{D}(0, 1))$.

Lemma 3.2. *We have*

$$(11) \quad \mathcal{M}(\widetilde{\mathbb{C}^*}) \cap \pi^* \mathcal{M}(D(0, 1)) = \pi^* \mathcal{M}(\mathbb{C}),$$

$$(12) \quad \mathcal{M}(\widetilde{\mathbb{C}^*}) \cap \pi^* \mathcal{M}(\mathbb{P}^1(\mathbb{C}) \setminus \overline{D}(0, 1)) = \pi^* \mathcal{M}(\mathbb{P}^1(\mathbb{C}) \setminus \{0\})$$

and

$$(13) \quad \mathcal{M}(\widetilde{\mathbb{C}^*}) \cap \pi^* \mathcal{M}(D(0, 1)) \cap \pi^* \mathcal{M}(\mathbb{P}^1(\mathbb{C}) \setminus \overline{D}(0, 1)) = \pi^* \mathcal{M}(\mathbb{P}^1(\mathbb{C})) = \pi^* \mathbb{C}(z).$$

Proof. Let us prove (11). It is clear that

$$\pi^* \mathcal{M}(\mathbb{C}) \subset \mathcal{M}(\widetilde{\mathbb{C}^*}) \cap \pi^* \mathcal{M}(D(0, 1)).$$

It remains to show the reciprocal inclusion. Let $f \in \mathcal{M}(\widetilde{\mathbb{C}^*}) \cap \pi^* \mathcal{M}(D(0, 1))$. There exists $g \in \mathcal{M}(D(0, 1))$ such that $f = \pi^* g = g \circ \pi$. As $f = g \circ \pi = g \circ \exp \circ \widetilde{\log} \in \mathcal{M}(\widetilde{\mathbb{C}^*})$ and $\widetilde{\log}: \widetilde{\mathbb{C}^*} \rightarrow \mathbb{C}$ is a biholomorphic function then $g \circ \exp \in \mathcal{M}(\mathbb{C})$. Using that $\exp: \mathbb{C} \rightarrow \mathbb{C}^*$

- $C^{(0)} \in GL_n(\mathbb{C}(z))$ which can be written

$$C^{(0)} = u^{-k} T_{i_1, \underline{l}_1} D_{i_1, v_1} \cdots T_{i_r, \underline{l}_r} D_{i_r, v_r}$$

where $u, v_1, \dots, v_r \in \mathbb{C}(\{z\})$ have a valuation at 0 which is equal to 1, $k \in \mathbb{N}$ and $\underline{l}_1, \dots, \underline{l}_r \in \mathbb{C}^n \setminus \{0\}$.

- $R^{(0)} \in GL_n(\mathbb{C}(\{z\}))$ regular at 0.

Lemma 6.3. *If $M_1 \in GL_n(\mathbb{C}(\{z-1\}))$ then M_1 can be written*

$$M_1 = C^{(1)} R^{(1)}$$

with

- $C^{(1)} \in GL_n(\mathbb{C}(z))$ regular at 0 and ∞ which can be written

$$C^{(1)} = u^{-k} T_{i_1, \underline{l}_1} D_{i_1, u} \cdots T_{i_r, \underline{l}_r} D_{i_r, u}$$

where $k \in \mathbb{N}$, $\underline{l}_1, \dots, \underline{l}_r \in \mathbb{C}^n \setminus \{0\}$ and $u = \frac{z-1}{z+1}$.

- $R^{(1)} \in GL_n(\mathbb{C}(\{z-1\}))$ regular at 1.

Proof. To prove this, we adapt the proof of [Sau99, Corollaire 1, p.53]. Let $k \in \mathbb{N}$ be such that the entries of $u^k M_1 := M'_1$ are analytic at 1. Let $v_1(\det(M'_1))$ be the valuation at 1 of $\det(M'_1)$. If $v_1(\det(M'_1)) = 0$ then M'_1 is regular at 1 so we can take $R^{(1)} := M'_1$ and $C^{(1)} := u^{-k} I_n$. If $v_1(\det(M'_1)) > 0$, we do an induction on $v_1(\det(M'_1))$: we show that there exists a matrix N'_1 whose entries are analytic at 1 and such that $M'_1 = T_{i_1, \underline{l}_1} D_{i_1, u} N'_1$ with

$$v_1(\det(N'_1)) < v_1(\det(M'_1)).$$

If $v_1(\det(M'_1)) > 0$ then $M'_1(1)$ is not invertible so there exists

$$\underline{l} = (l_1, \dots, l_n) \in \mathbb{C}^n \setminus \{0\}$$

such that $\underline{l} M'_1(1) = 0 \in \mathbb{C}^n$. Let $i \in [1, n]$ be such that $l_i \neq 0$. Therefore, the entries of $N'_1 := D_{i, u}^{-1} T_{i, \underline{l}} M'_1$ are analytic at 1 and $v_1(\det(N'_1)) = v_1(\det(M'_1)) - 1$. \square

From [Sau99, Corollaire 3, p.54], we have the following result.

Lemma 6.4. *Let $C^{(0)}, C^{(\infty)} \in GL_n(\mathbb{C}(z))$. There exists $U \in GL_n(\mathbb{C}(z))$ such that $UC^{(0)}$ is regular at 0 and $UC^{(\infty)}$ is regular at ∞ .*

Proposition 6.5. *If the system $\phi_p(Y) = AY$ is regular singular at 0, 1 and ∞ then it is rationally equivalent to a Fuchsian system at 0, 1 and ∞ that is there exists $R \in GL_n(\mathbb{C}(z))$ such that the system*

$$\phi_p(Y) = BY \text{ with } B := \phi_p(R)^{-1} AR$$

is Fuchsian at 0, 1 and ∞ .

Proof. From Corollaries 2.7, 2.15 and Corollary 2.24 (using here the principal value Log instead of $\widetilde{\log}$), there exist $F_0 \in GL_n(\mathbb{C}(\{z\}))$ (respectively $F_1 \in GL_n(\mathbb{C}(\{z-1\}))$, $F_\infty \in GL_n(\mathbb{C}(\{\frac{1}{z}\}))$) and $A_0 \in GL_n(\mathbb{C})$ (respectively $A_1, A_\infty \in GL_n(\mathbb{C})$) such that for $i = 0, 1, \infty$,

$$\phi_p(F_i)^{-1} A F_i = A_i.$$

If we show that there exists $R \in GL_n(\mathbb{C}(z))$ such that for $i = 0, 1, \infty$, $K_i := R^{-1} F_i$ is regular at i then

$$B = \phi_p(R)^{-1} AR = \phi_p(K_i) A_i K_i^{-1}$$

will be regular at $0, 1$ and ∞ . From Lemma 6.2, F_0 can be written $C^{(0)}R^{(0)}$ with $C^{(0)} \in GL_n(\mathbb{C}(z))$ and $R^{(0)} \in GL_n(\mathbb{C}(\{z\}))$ regular at 0 . Similarly, with the change of variables $z \mapsto 1/z$, we can write $F_\infty = C^{(\infty)}R^{(\infty)}$ with $C^{(\infty)} \in GL_n(\mathbb{C}(z))$ and $R^{(\infty)} \in GL_n(\mathbb{C}(\{\frac{1}{z}\}))$ regular at ∞ . From Lemma 6.4, there exists $U \in GL_n(\mathbb{C}(z))$ such that $UC^{(0)}$ and $UC^{(\infty)}$ are respectively regular at 0 and ∞ . The entries of the matrix UF_1 are meromorphic at 1 . From Lemma 6.3, UF_1 can be written $C^{(1)}R^{(1)}$ with $C^{(1)} \in GL_n(\mathbb{C}(z))$ regular at $0, \infty$ and $R^{(1)}$ regular at 1 . The matrix $R := U^{-1}C^{(1)}$ satisfies the required properties. \square

Proposition 6.6. *The categories \mathcal{E}_{r_s} and \mathcal{E}_f are equivalent.*

Proof. The natural embedding $\mathcal{E}_f \rightarrow \mathcal{E}_{r_s}$ is an equivalence of categories. Indeed, it is clearly fully faithful and, by Proposition 6.5, it is also essentially surjective. \square

Part III. Local categories and local groupoids

7. PRELIMINARY RESULTS

7.1. On morphisms between constant systems

Lemma 7.1. *Let $p \in \mathbb{N}_{\geq 2}$, $A_0 \in GL_{n_1}(\mathbb{C})$ and $B_0 \in GL_{n_2}(\mathbb{C})$. If $T \in \mathcal{M}_{n_2, n_1}(\mathbb{C}(\{z\}))$ is such that $\phi_p(T)A_0 = B_0T$ then $T \in \mathcal{M}_{n_2, n_1}(\mathbb{C})$.*

Proof. We write $T(z) = \sum_{k \geq N} T_k z^k$ with $T_k \in \mathcal{M}_{n_2, n_1}(\mathbb{C})$ and $N = \min\{k \in \mathbb{Z} \mid T_k \neq 0\}$.

Thereby, we have

$$(14) \quad \sum_{k \geq N} T_k A_0 z^{pk} = \sum_{k \geq N} B_0 T_k z^k.$$

We obtain that $N = 0$ because, looking at the lower degree, $pN = N$ namely $N = 0$. Therefore, by the equality (14), all the T_k such that p does not divide k are equal to zero. Also, if p divides k then $B_0 T_k = T_{k/p} A_0$ and we iterate it until we obtain an integer $\frac{k}{p^j} \in \mathbb{N}$ which is not divisible by p so $T_k = B_0^{-j} T_{k/p^j} A_0^j = 0$. Hence $T(z) = T_0 \in \mathcal{M}_{n_2, n_1}(\mathbb{C})$. \square

We deduce the following result.

Lemma 7.2. *Let $p \in \mathbb{N}_{\geq 2}$, $A_\infty \in GL_{n_1}(\mathbb{C})$ and $B_\infty \in GL_{n_2}(\mathbb{C})$. If $T \in \mathcal{M}_{n_2, n_1}(\mathbb{C}(\{\frac{1}{z}\}))$ is such that $\phi_p(T)A_\infty = B_\infty T$ then $T \in \mathcal{M}_{n_2, n_1}(\mathbb{C})$.*

Lemma 7.3. *Let $p \in \mathbb{N}_{\geq 2}$, $A_1 \in GL_{n_1}(\mathbb{C})$, $B_1 \in GL_{n_2}(\mathbb{C})$ and \widetilde{S}_1 be a matrix of size $n_2 \times n_1$ with meromorphic entries at $\widetilde{1} := (1, 0) \in \widetilde{\mathbb{C}}^*$. If $\phi_p(\widetilde{S}_1)A_1 = B_1 \widetilde{S}_1$ then the entries of \widetilde{S}_1 are Laurent polynomials in $\widetilde{\log}$.*

Proof. Let $S_1 := \widetilde{S}_1 \circ \widetilde{\log}^{-1}$. Then,

$$\forall z \in \mathbb{C}, S_1(pz)A_1 = B_1 S_1(z).$$

Since the entries of S_1 are meromorphic functions at 0 , by [Sau03, 2.1.3.2], the entries of S_1 are Laurent polynomials. Therefore, the entries of $\widetilde{S}_1 = S_1 \circ \widetilde{\log}$ are Laurent polynomials in $\widetilde{\log}$. \square

Notation 7.4. *We denote by $\mathbb{C}[\widetilde{\log}, \widetilde{\log}^{-1}]$ the ring of Laurent polynomials in $\widetilde{\log}$.*

7.2. Finite dimensional complex representations of \mathbb{Z}

Details and proofs can be found in [Sau99, Annexe A]. Let \mathcal{R} be the category of finite dimensional complex representations of \mathbb{Z} . A representation ρ of \mathbb{Z} is entirely determined by $A = \rho(1) \in GL_n(\mathbb{C})$. Therefore, the objects of \mathcal{R} can be seen as pairs (\mathbb{C}^n, A) where $A \in GL_n(\mathbb{C})$ and the morphisms $F \in \text{Hom}_{\mathcal{R}}((\mathbb{C}^n, A), (\mathbb{C}^p, B))$ are the $F \in \mathcal{M}_{p,n}(\mathbb{C})$ such that $FA = BF$. It is a neutral Tannakian category over \mathbb{C} with the forgetful functor $\omega : (\mathbb{C}^n, A) \rightsquigarrow \mathbb{C}^n, F \rightsquigarrow F$ as fibre functor. Its Galois group is

$$\mathbb{Z}^{alg} := \text{Aut}^{\otimes}(\omega).$$

Notation 7.5. Let $A \in GL_n(\mathbb{C})$ and $A = A_s A_u$ its multiplicative Dunford decomposition. Let $(\gamma, \lambda) \in \text{Hom}_{gr}(\mathbb{C}^*, \mathbb{C}^*) \times \mathbb{C}$. We write $A^{(\gamma, \lambda)} := \gamma(A_s) A_u^\lambda = A_u^\lambda \gamma(A_s)$ where

- γ acts on the eigenvalues of A_s : if $A_s = Q \cdot \text{diag}(c_1, \dots, c_n) \cdot Q^{-1}$ then

$$\gamma(A_s) := Q \cdot \text{diag}(\gamma(c_1), \dots, \gamma(c_n)) \cdot Q^{-1}.$$

- $A_u^\lambda = \sum_{k \geq 0} \binom{\lambda}{k} (A_u - I_n)^k$.

The matrix $A^{(\gamma, \lambda)} \in GL_n(\mathbb{C})$ defines an automorphism of $\mathbb{C}^n = \omega(\mathbb{C}^n, A)$.

Proposition 7.6. There is a pro-algebraic group isomorphism

$$\begin{aligned} \text{Hom}_{gr}(\mathbb{C}^*, \mathbb{C}^*) \times \mathbb{C} &\rightarrow \mathbb{Z}^{alg} \\ (\gamma, \lambda) &\mapsto (\mathbb{C}^n, A) \mapsto A^{(\gamma, \lambda)} \end{aligned}$$

8. LOCALIZATION AT 0

8.1. The categories $\mathcal{E}_{rs}^{(0)}$ and $\mathcal{P}^{(0)}$

The category $\mathcal{E}_{rs}^{(0)}$ has the same objects as \mathcal{E}_{rs} but morphisms from A , of rank n_1 , to B , of rank n_2 , are all $R \in \mathcal{M}_{n_2, n_1}(\mathbb{C}(\{z\}))$ such that $\phi_p(R)A = BR$. From this equation satisfied by R , we deduce that

$$R \in \mathcal{M}_{n_2, n_1}(\mathcal{M}(D(0, 1))).$$

We denote by $\mathcal{P}^{(0)}$ the full subcategory of $\mathcal{E}_{rs}^{(0)}$ whose objects are the $A_0 \in GL_n(\mathbb{C})$.

By Lemma 7.1, we deduce that:

Proposition 8.1. If $R \in \text{Hom}_{\mathcal{P}^{(0)}}(A_0, B_0)$ with A_0 and B_0 respectively of rank n_1, n_2 then $R \in \mathcal{M}_{n_2, n_1}(\mathbb{C})$.

It is clear that:

Lemma 8.2. The categories \mathcal{R} and $\mathcal{P}^{(0)}$ are equivalent.

The following is an immediate consequence:

Proposition 8.3. The category $\mathcal{P}^{(0)}$ is a neutral Tannakian category over \mathbb{C} and the functor

$$\omega_{l,0} : \mathcal{P}^{(0)} \rightarrow \text{Vect}_{\mathbb{C}}^f \begin{cases} A_0 \rightsquigarrow \mathbb{C}^n \text{ with } n \text{ the rank of } A_0 \\ R \rightsquigarrow R \end{cases}$$

is a fibre functor for $\mathcal{P}^{(0)}$.

Proposition 8.4. The natural embedding $\mathcal{P}^{(0)} \rightarrow \mathcal{E}_{rs}^{(0)}$ is an equivalence of categories.

Proof. By definition, this functor is full and faithful. It is also essentially surjective according to Corollary 2.7. \square

Corollary 8.5. *The category $\mathcal{E}_{rs}^{(0)}$ is a Tannakian category over \mathbb{C} .*

8.2. Local Galois groupoid at 0

In the previous section, we have seen that the categories \mathcal{R} , $\mathcal{P}^{(0)}$ and $\mathcal{E}_{rs}^{(0)}$ are equivalent Tannakian categories over \mathbb{C} . Therefore, they have the same Galois group which is \mathbb{Z}^{alg} (see 7.2).

As a consequence, the *local Galois group at 0* is

$$G_0 = \mathbb{Z}^{\text{alg}}.$$

The *local Galois groupoid at 0* is a groupoid with one object $\omega_{l,0}$ and such that the morphisms from $\omega_{l,0}$ to $\omega_{l,0}$ are

$$G_0(\omega_{l,0}, \omega_{l,0}) := \mathbb{Z}^{\text{alg}}.$$

We will also denote it by G_0 .

9. LOCALIZATION AT ∞

9.1. The categories $\mathcal{E}_{rs}^{(\infty)}$ and $\mathcal{P}^{(\infty)}$

Similarly, we introduce the categories $\mathcal{E}_{rs}^{(\infty)}$ and $\mathcal{P}^{(\infty)}$. The category $\mathcal{E}_{rs}^{(\infty)}$ has the same objects as \mathcal{E}_{rs} but morphisms from A , of rank n_1 , to B , of rank n_2 , are the matrices $R \in \mathcal{M}_{n_2, n_1}(\mathbb{C}(\{\frac{1}{z}\}))$ such that $\phi_p(R)A = BR$. From this equation satisfied by R , we deduce that

$$R \in \mathcal{M}_{n_2, n_1}(\mathcal{M}(\mathbb{P}^1(\mathbb{C}) \setminus \overline{D}(0, 1))).$$

We denote by $\mathcal{P}^{(\infty)}$ the full subcategory of $\mathcal{E}_{rs}^{(\infty)}$ whose objects are the $A_\infty \in GL_n(\mathbb{C})$. By Lemma 7.2, we deduce that:

Proposition 9.1. *If $R \in \text{Hom}_{\mathcal{P}^{(\infty)}}(A_\infty, B_\infty)$ with A_∞ and B_∞ respectively of rank n_1 and n_2 then $R \in \mathcal{M}_{n_2, n_1}(\mathbb{C})$.*

As before with the localization at 0, the categories \mathcal{R} , $\mathcal{P}^{(\infty)}$ and $\mathcal{E}_{rs}^{(\infty)}$ are equivalent. These are Tannakian categories over \mathbb{C} . In particular, $\mathcal{P}^{(\infty)}$ is a neutral Tannakian category over \mathbb{C} with the fibre functor

$$\omega_{l,\infty} : \begin{cases} \mathcal{P}^{(\infty)} \rightarrow \text{Vect}_{\mathbb{C}}^f \\ \left\{ \begin{array}{l} A_\infty \rightsquigarrow \mathbb{C}^n \text{ with } n \text{ the rank of } A_\infty \\ R \rightsquigarrow R \end{array} \right. \end{cases}.$$

9.2. Local Galois groupoid at ∞

As previously, localizing at 0, the categories $\mathcal{E}^{(\infty)}$, $\mathcal{P}^{(\infty)}$ and \mathcal{R} have the same Galois group \mathbb{Z}^{alg} .

As a consequence, the *local Galois group at ∞* is

$$G_\infty = \mathbb{Z}^{\text{alg}}.$$

The *local Galois groupoid at ∞* is a groupoid with one object $\omega_{l,\infty}$ and such that the morphisms from $\omega_{l,\infty}$ to $\omega_{l,\infty}$ are

$$G_\infty(\omega_{l,\infty}, \omega_{l,\infty}) = \mathbb{Z}^{\text{alg}}.$$

We will also denote it by G_∞ .

10. LOCALIZATION AT 1

10.1. The categories $\mathcal{E}_{rs}^{(1)}$ and $\mathcal{P}^{(1)}$

Notation 10.1. We write

$$\Delta := \left\{ re^{it} \in \mathbb{C} \mid r > 0, t \in \left] -\frac{\pi}{p}, \frac{\pi}{p} \right[\right\}$$

and

$$\Lambda = \left\{ z \in \mathbb{C} \mid \Im(z) \in \left] -\frac{\pi}{p}, \frac{\pi}{p} \right[\right\}.$$

We introduce

$$\pi_1 : \begin{array}{ccc} \mathbb{C} \setminus \mathbb{R}^- \times \left] -\pi, \pi \right[& \rightarrow & \mathbb{C} \setminus \mathbb{R}^- \\ (re^{it}, t) & \mapsto & re^{it} \end{array}.$$

The category $\mathcal{E}_{rs}^{(1)}$ has the same objects as \mathcal{E}_{rs} but morphisms from A of rank n_1 to B of rank n_2 are all $R \in \mathcal{M}_{n_2, n_1}(\mathbb{C}(\{z-1\}))$ such that $\phi_p(R)A = BR$. From this equation satisfied by R , we deduce that $R \in \mathcal{M}_{n_2, n_1}(\mathcal{M}(\Delta))$.

We denote by $\mathcal{P}^{(1)}$ the category whose objects are all $A_1 \in GL_n(\mathbb{C})$ and morphisms from A_1 of rank n_1 to B_1 of rank n_2 are all matrices \tilde{R} of size $n_2 \times n_1$ whose entries are meromorphic functions at $\tilde{1} \in \widetilde{\mathbb{C}^\star}$ and such that

$$\phi_p(\tilde{R})A_1 = B_1\tilde{R}.$$

By Lemma 7.3, we deduce that:

Proposition 10.2. *If $R \in \text{Hom}_{\mathcal{P}^{(1)}}(A_1, B_1)$ with A_1 and B_1 respectively of rank n_1 and n_2 then the entries of R are Laurent polynomials in \log .*

Proposition 10.3. *The categories $\mathcal{E}^{(1)}$ and $\mathcal{P}^{(1)}$ are equivalent via the functor*

$$\mathcal{G} : \begin{array}{ccc} \mathcal{P}^{(1)} & \rightarrow & \mathcal{E}_{rs}^{(1)} \\ \left\{ \begin{array}{l} A_1 \rightsquigarrow A_1 \\ \tilde{R} \rightsquigarrow \tilde{R} \circ \pi_1^{-1} \end{array} \right. & & . \end{array}$$

Proof. This functor is full and faithful. It is also essentially surjective according to Corollary 2.24. \square

Lemma 10.4. *The category $\mathcal{P}^{(1)}$ and the category $\mathcal{P}^{(0)}$ introduced in [Sau03, 2.1.2] are isomorphic via the composition with \log :*

$$\begin{array}{ccc} \mathcal{P}^{(0)} & \rightarrow & \mathcal{P}^{(1)} \\ \left\{ \begin{array}{l} A_1 \rightsquigarrow A_1 \\ H \rightsquigarrow H \circ \log \end{array} \right. & & . \end{array}$$

According to [Sau03], the category $\mathcal{P}^{(0)}$ is a neutral Tannakian category over \mathbb{C} so:

Proposition 10.5. *The category $\mathcal{P}^{(1)}$ is a neutral Tannakian category over \mathbb{C} with the fiber functors,*

$$\omega_{i,1}^{(\tilde{c})} : \begin{array}{ccc} \mathcal{P}^{(1)} & \rightarrow & \text{Vect}_{\mathbb{C}}^f \\ \left\{ \begin{array}{l} A_1 \rightsquigarrow \mathbb{C}^n \text{ with } n \text{ the rank of } A_1 \\ \tilde{R} \rightsquigarrow \tilde{R}(\tilde{c}) \end{array} \right. & & , \end{array}$$

for all $\tilde{c} \in \widetilde{\mathbb{C}^\star} \setminus \{\tilde{1}\}$.

10.2. Local Galois groupoid at 1

From Lemma 10.4, we obtain that $\text{Aut}^\otimes(\omega_{l,1}^{(\tilde{c})})$ is a subgroup (and $\text{Iso}^\otimes(\omega_{l,1}^{(\tilde{c})}, \omega_{l,1}^{(\tilde{d})})$ a subset) of \mathbb{Z}^{alg} . We also have the following result.

Theorem 10.6. *With the identification of \mathbb{Z}^{alg} with $\text{Hom}_{\text{gr}}(\mathbb{C}^\star, \mathbb{C}^\star)$ given in Proposition 7.6, for all $\tilde{c}, \tilde{d} \in \widetilde{\mathbb{C}^\star} \setminus \{\tilde{1}\}$,*

$$\text{Iso}^\otimes(\omega_{l,1}^{(\tilde{c})}, \omega_{l,1}^{(\tilde{d})}) = \left\{ (\gamma, \lambda) \in \mathbb{Z}^{\text{alg}} \mid \gamma(p)\widetilde{\log}(\tilde{c}) = \widetilde{\log}(\tilde{d}) \right\}.$$

Notation 10.7. *We denote by G_1 the local Galois groupoid at 1. Its objects are the*

$$\omega_{l,1}^{(\tilde{c})}, \tilde{c} \in \widetilde{\mathbb{C}^\star} \setminus \{\tilde{1}\}$$

and its morphisms are the

$$G_1(\omega_{l,1}^{(\tilde{c})}, \omega_{l,1}^{(\tilde{d})}) = \text{Iso}^\otimes(\omega_{l,1}^{(\tilde{c})}, \omega_{l,1}^{(\tilde{d})}).$$

It is a transitive groupoid because there exists a morphism between each object. Therefore, we can take as local Galois group at 1 a group $G_1(\omega_{l,1}^{(\tilde{c})}, \omega_{l,1}^{(\tilde{c})})$ for a $\tilde{c} \in \widetilde{\mathbb{C}^\star} \setminus \{\tilde{1}\}$.

Corollary 10.8. *With the same identification, for $\tilde{c} \in \widetilde{\mathbb{C}^\star} \setminus \{\tilde{1}\}$, the local Galois group at 1 is*

$$G_1 := G_1(\omega_{l,1}^{(\tilde{c})}, \omega_{l,1}^{(\tilde{c})}) = \left\{ (\gamma, \lambda) \in \mathbb{Z}^{\text{alg}} \mid \gamma(p) = 1 \right\}.$$

Therefore,

$$G_1(\omega_{l,1}^{(\tilde{c})}, \omega_{l,1}^{(\tilde{c})}) \simeq \text{Hom}_{\text{gr}}(\mathbb{C}^\star/p^\mathbb{Z}, \mathbb{C}^\star) \times \mathbb{C} := G_{1,s} \times G_{1,u}.$$

The local Galois group at 1 obtained is the same group as the one obtained by Jacques Sauloy in [Sau03, 2.2.2.2]. Therefore, from [Sau03, 2.2.3], we know a Zariski-dense subgroup of this group:

Lemma 10.9 (elements of the unipotent component $G_{1,u}$ of G_1). *Let $A_1 \in \text{Obj}(\mathcal{P}^{(1)})$. Let $A_1 = A_{1,u}A_{1,s}$ be its Dunford decomposition. The element $1 \in \mathbb{C}$ which corresponds to $A_1 \rightsquigarrow A_{1,u} \in \text{Aut}^\otimes(\omega_{l,1}^{(\tilde{c})})$ generates a Zariski-dense subgroup of the unipotent component $G_{1,u}$.*

Notation 10.10. *Let $p \in \mathbb{N}_{\geq 2}$. We recall that all $z \in \mathbb{C}^\star$ can be uniquely written $z = up^x$ where u belongs to the unit circle and $x \in \mathbb{R}$. We define*

$$\begin{array}{ccc} \gamma_1 : \mathbb{C}^\star & \rightarrow & \mathbb{C}^\star \\ up^x & \mapsto & u \end{array} \quad \text{and} \quad \begin{array}{ccc} \gamma_2 : \mathbb{C}^\star & \rightarrow & \mathbb{C}^\star \\ up^x & \mapsto & e^{2i\pi x} \end{array}.$$

Lemma 10.11 (elements of the semi-simple component $G_{1,s}$ of G_1). *We denote by*

$$\overline{\gamma}_1, \overline{\gamma}_2 \in \text{Hom}_{\text{gr}}(\mathbb{C}^\star/p^\mathbb{Z}, \mathbb{C}^\star)$$

the morphisms induced by γ_1 and γ_2 respectively. They generate a Zariski-dense subgroup of $G_{1,s}$.

Theorem 10.12. *The subgroup of $\text{Hom}_{\text{gr}}(\mathbb{C}^\star/p^\mathbb{Z}, \mathbb{C}^\star) \times \mathbb{C}$ whose unipotent component is $\mathbb{Z} \subset \mathbb{C}$ and whose semi-simple component is generated by $\overline{\gamma}_1$ and $\overline{\gamma}_2$ is Zariski-dense in G_1 , the local Galois group at 1.*

Part IV. Global categories

11. THE CATEGORY \mathcal{E}_{rs} OF REGULAR SINGULAR SYSTEMS AT 0, 1 AND ∞

We will show that \mathcal{E}_{rs} is a Tannakian category over \mathbb{C} . From 6, it will also provide \mathcal{E}_f a structure of Tannakian category over \mathbb{C} .

Proposition 11.1. *The category \mathcal{E}_{rs} is a Tannakian category over \mathbb{C} .*

Proof. From 1.2.b, we know that \mathcal{E} is a Tannakian category over \mathbb{C} . The category \mathcal{E}_{rs} is a rigid tensor subcategory of \mathcal{E} because it is closed under tensor operations, it contains the unit object $\mathbf{1} = 1 \in \mathbb{C}^*$, the internal Hom and every object has a dual. To show that it is a Tannakian subcategory of \mathcal{E} , the only non trivial point which remains to be proved is the existence of kernels and cokernels in \mathcal{E}_{rs} . This follows from the lemma herebelow and the fact that duality in the Tannakian category \mathcal{E} exchanges kernels with cokernels. \square

Lemma 11.2. *All morphisms in the category \mathcal{E}_{rs} have kernels.*

Proof. We recall that the categories \mathcal{E} (introduced in 1.2.b), $\mathcal{E}^{(0)}$ (introduced in 1.2.b) and $\mathcal{E}_{rs}^{(0)}$ (introduced in 8.1) are Tannakian categories. We consider the following embeddings of categories

$$\begin{array}{ccc} \mathcal{E}_{rs} & \hookrightarrow & \mathcal{E} \\ \downarrow & & \downarrow \\ \mathcal{E}_{rs}^{(0)} & \hookrightarrow & \mathcal{E}^{(0)} \end{array} .$$

Let $R \in \mathcal{M}_{n_2, n_1}(\mathbb{C}(z))$ be a morphism in the category \mathcal{E}_{rs} from $A \in GL_{n_1}(\mathbb{C}(z))$ to $B \in GL_{n_2}(\mathbb{C}(z))$. We know that the morphism R has a kernel in the category \mathcal{E} , we denote it by

$$(15) \quad (C, K : C \rightarrow A).$$

To show that this kernel is also a kernel of R in the category \mathcal{E}_{rs} it remains to prove that the system $\phi_p(Y) = CY$ is regular singular at 0, 1 and ∞ . The morphism R has also a kernel in the category $\mathcal{E}_{rs}^{(0)}$, we denote it by

$$(16) \quad (C_0, K_0 : C_0 \rightarrow A)$$

so the system $\phi_p(Y) = C_0 Y$ is regular singular at 0. The kernels (15) and (16) are kernels of the morphism R in the category $\mathcal{E}^{(0)}$ so they are isomorphic that is there exists an invertible morphism $u_1 : C_0 \rightarrow C$ in the category $\mathcal{E}^{(0)}$ such that $Ku_1 = K$. Because of the fact that $u_1 \in \text{Mor}_{\mathcal{E}^{(0)}}(C_0, C)$, the entries of u_1 are meromorphic functions at 0 and u_1 satisfies

$$\phi_p(u_1)C_0 = Cu_1$$

thus the system $\phi_p(Y) = CY$ is also regular singular at 0. Similarly, we can show that the system $\phi_p(Y) = CY$ is regular singular at 1 and at ∞ so $C \in \text{Obj}(\mathcal{E}_{rs})$ and $(C, K : C \rightarrow A)$ is also a kernel of R in the category \mathcal{E}_{rs} . \square

12. THE CATEGORY \mathcal{C} OF CONNECTIONS

We define

$$\Sigma_0 := \left\{ (re^{ib}, b) \mid 0 < r < 1, b \in \mathbb{R} \right\} \subset \widetilde{\mathbb{C}}^*$$

and

$$\Sigma_\infty := \left\{ \left(r e^{ib}, b \right) \mid r > 1, b \in \mathbb{R} \right\} \subset \widetilde{\mathbb{C}}^\star.$$

We introduce the category \mathcal{C} of connections. Its objects are the

$$(A_0, A_1, A_\infty, \widetilde{M}_0, \widetilde{M}_\infty) \in GL_n(\mathbb{C})^3 \times GL_n(\mathcal{M}(\Sigma_0)) \times GL_n(\mathcal{M}(\Sigma_\infty))$$

such that

$$\begin{cases} \phi_p(\widetilde{M}_0) = A_1 \widetilde{M}_0 A_0^{-1} \\ \phi_p(\widetilde{M}_\infty) = A_1 \widetilde{M}_\infty A_\infty^{-1} \end{cases}$$

and such that there exist

$$\widetilde{W}_1 \in GL_n(\mathcal{M}(\widetilde{\mathbb{C}}^\star)), W_0 \in GL_n(\mathcal{M}(D(0, 1))), W_\infty \in GL_n(\mathcal{M}(\mathbb{P}^1(\mathbb{C}) \setminus \overline{D}(0, 1)))$$

such that

$$\begin{cases} \widetilde{W}_1 \widetilde{M}_0 = \pi^* W_0 \\ \widetilde{W}_1 \widetilde{M}_\infty = \pi^* W_\infty \end{cases}.$$

The integer n is called the *rank* of the object.

Morphisms from $(A_0, A_1, A_\infty, \widetilde{M}_0, \widetilde{M}_\infty) \in \text{Obj}(\mathcal{C})$ to $(B_0, B_1, B_\infty, \widetilde{N}_0, \widetilde{N}_\infty) \in \text{Obj}(\mathcal{C})$ respectively of rank n_1 and n_2 are the triples

$$(S_0, \widetilde{S}_1, S_\infty) \in \mathcal{M}_{n_2, n_1}(\mathbb{C}) \times \mathcal{M}_{n_2, n_1}(\mathbb{C}[\widetilde{\log}, \widetilde{\log}^{-1}]) \times \mathcal{M}_{n_2, n_1}(\mathbb{C})$$

such that

$$\begin{cases} S_0 A_0 = B_0 S_0 \\ \phi_p(\widetilde{S}_1) A_1 = B_1 \widetilde{S}_1 \\ S_\infty A_\infty = B_\infty S_\infty \\ \widetilde{S}_1 \widetilde{M}_0 = \widetilde{N}_0 S_0 \\ \widetilde{S}_1 \widetilde{M}_\infty = \widetilde{N}_\infty S_\infty \end{cases}.$$

Remark 12.1. From Lemmas 7.1, 7.2 and 7.3, this amounts to assuming that the entries of S_0 , \widetilde{S}_1 and S_∞ are meromorphic functions at 0, $\widetilde{1}$ and ∞ respectively.

13. THE CATEGORY \mathcal{S} OF SOLUTIONS

The objects of the category \mathcal{S} of solutions are the

$$(A_0, A_1, A_\infty, F_0, \widetilde{F}_1, F_\infty) \in GL_n(\mathbb{C})^3 \times GL_n(\mathcal{M}(D(0, 1))) \times GL_n(\mathcal{M}(\widetilde{\mathbb{C}}^\star)) \times GL_n(\mathcal{M}(\mathbb{P}^1(\mathbb{C}) \setminus \overline{D}(0, 1))).$$

such that

$$\begin{aligned} \pi^* A &:= \phi_p(\widetilde{F}_1) A_1 \widetilde{F}_1^{-1}, \\ A &= \phi_p(F_0) A_0 F_0^{-1}, \end{aligned}$$

and

$$A = \phi_p(F_\infty) A_\infty F_\infty^{-1}.$$

The integer n is called the *rank* of the object.

Morphisms from $(A_0, A_1, A_\infty, F_0, \widetilde{F}_1, F_\infty)$ to $(B_0, B_1, B_\infty, G_0, \widetilde{G}_1, G_\infty)$, two objects of \mathcal{S} respectively of rank n_1 and n_2 , are the

$$(R, S_0, \widetilde{S}_1, S_\infty) \in \mathcal{M}_{n_2, n_1}(\mathbb{C}(z)) \times \mathcal{M}_{n_2, n_1}(\mathbb{C}) \times \mathcal{M}_{n_2, n_1}(\mathbb{C}[\widetilde{\log}, \widetilde{\log}^{-1}]) \times \mathcal{M}_{n_2, n_1}(\mathbb{C})$$

such that

$$\begin{cases} S_0 A_0 = B_0 S_0 \\ \phi_p(\widetilde{S}_1) A_1 = B_1 \widetilde{S}_1 \\ S_\infty A_\infty = B_\infty S_\infty \\ RF_i = G_i S_i \text{ pour } i = 0, \infty \\ \widetilde{RF}_1 = \widetilde{G}_1 \widetilde{S}_1 \end{cases}$$

where we define $\widetilde{R} := \pi^* R$.

14. EQUIVALENCE BETWEEN THE CATEGORIES

Our goal is to obtain an equivalence between the categories \mathcal{C} and \mathcal{E}_{rs} .

Proposition 14.1. *The categories \mathcal{S} and \mathcal{E}_{rs} are equivalent by the functor*

$$\mathcal{F} : \mathcal{S} \rightarrow \mathcal{E}_{rs} \\ \left\{ \begin{array}{l} (A_0, A_1, A_\infty, F_0, \widetilde{F}_1, F_\infty) \rightsquigarrow A \text{ defined in the category } \mathcal{S} \text{ (in 13)} \\ (R, S_0, \widetilde{S}_1, S_\infty) \rightsquigarrow R \end{array} \right.$$

Proof. We check that \mathcal{F} is well-defined on objects. By the equalities $A = \phi_p(F_0) A_0 F_0^{-1}$, $A = \phi_p(F_\infty) A_\infty F_\infty^{-1}$ and $\pi^* A = \phi_p(\widetilde{F}_1) A_1 \widetilde{F}_1^{-1}$, we obtain respectively that the entries of $\pi^* A$ belong to $\pi^* \mathcal{M}(D(0, 1))$, $\pi^* \mathcal{M}(\mathbb{P}^1(\mathbb{C}) \setminus \overline{D}(0, 1))$ and $\mathcal{M}(\widetilde{\mathbb{C}}^*)$. By Lemma 3.2, $\pi^* A \in GL_n(\pi^* \mathbb{C}(z))$ so

$$A \in GL_n(\mathbb{C}(z)).$$

Moreover, the system $\phi_p(Y) = AY$ is regular singular at $0, 1, \infty$. The functor \mathcal{F} is also well-defined on morphisms because $R \in \mathcal{M}_{n_2, n_1}(\mathbb{C}(z))$ and it satisfies

$$\phi_p(R)A = \underbrace{\phi_p(R)\phi_p(F_0)}_{=\phi_p(G_0)S_0} A_0 F_0^{-1} = \phi_p(G_0) \underbrace{B_0 S_0 F_0^{-1}}_{=G_0^{-1}R} = BR.$$

The functor \mathcal{F} is essentially surjective because of Corollaries 2.7, 2.15, 2.24. It is also fully faithful. Indeed, keeping the notations introduced in 13, if $R \in \text{Hom}_{\mathcal{E}_{rs}}(A, B)$, $(R, G_0^{-1}RF_0, \widetilde{G}_1^{-1}\widetilde{RF}_1, G_\infty^{-1}RF_\infty)$ is the unique morphism of \mathcal{S} such that

$$\mathcal{F}(R, S_0, \widetilde{S}_1, S_\infty) = R$$

because it is uniquely determined and it satisfies

- for $i = 0, \infty$,

$$\phi_p(S_i) A_i = \phi_p(G_i)^{-1} \phi_p(R) \phi_p(F_i) A_i = B_i G_i^{-1} \underbrace{B^{-1} \phi_p(R) A F_i}_{=R} = B_i S_i$$

where we used the fact that $\phi_p(G_i) = B G_i B_i^{-1}$ and $\phi_p(F_i) = A F_i A_i^{-1}$ for the second equality. By Lemmas 7.1 and 7.2, we obtain that S_i has constant entries.

- Similarly, $\phi_p(\widetilde{S}_1) A_1 = B_1 \widetilde{S}_1$ and by Lemma 7.3, the entries of \widetilde{S}_1 are Laurent polynomials in \log .

□

Proposition 14.2. *The categories \mathcal{S} and \mathcal{C} are equivalent by the functor*

$$\mathcal{G} : \mathcal{S} \rightarrow \mathcal{C} \quad \left\{ \begin{array}{l} \left(A_0, A_1, A_\infty, F_0, \widetilde{F}_1, F_\infty \right) \rightsquigarrow \left(A_0, A_1, A_\infty, \widetilde{F}_1^{-1} \pi^* F_0, \widetilde{F}_1^{-1} \pi^* F_\infty \right) \\ \left(R, S_0, \widetilde{S}_1, S_\infty \right) \rightsquigarrow \left(S_0, \widetilde{S}_1, S_\infty \right) \end{array} \right. .$$

Proof. We write $\widetilde{F}_i := \pi^* F_i$ for $i = 0, \infty$ and $\widetilde{R} := \pi^* R$. The functor \mathcal{G} is clearly well-defined on objects. It is also well-defined on morphisms because if

$$\left(R, S_0, \widetilde{S}_1, S_\infty \right) \in \text{Mor}_{\mathcal{S}} \left(\left(A_0, A_1, A_\infty, F_0, \widetilde{F}_1, F_\infty \right), \left(B_0, B_1, B_\infty, G_0, \widetilde{G}_1, G_\infty \right) \right),$$

for $i = 0, \infty$, introducing $\widetilde{M}_i = \widetilde{F}_1^{-1} \widetilde{F}_i$ and $\widetilde{N}_i = \widetilde{G}_1^{-1} \widetilde{G}_i$,

$$\begin{aligned} \widetilde{S}_1 \widetilde{M}_i &= \underbrace{\widetilde{S}_1 \widetilde{F}_1^{-1}}_{=\widetilde{G}_1^{-1} \widetilde{R}} \widetilde{F}_i = \widetilde{N}_i S_i. \end{aligned}$$

The functor \mathcal{G} is an equivalence of categories because it is:

- essentially surjective. Here, each object $(A_0, A_1, A_\infty, \widetilde{M}_0, \widetilde{M}_\infty)$ of \mathcal{C} is of the form $G(A_0, A_1, A_\infty, F_0, \widetilde{F}_1, F_\infty)$. Indeed, we set $\widetilde{F}_1 := \widetilde{W}_1$ and $F_0 := W_0, F_\infty := W_\infty$ (notations introduced in the category of connections, 12).
- fully faithful. Let

$$\left(S_0, \widetilde{S}_1, S_\infty \right) \in \text{Hom}_{\mathcal{C}} \left(\left(A_0, A_1, A_\infty, \widetilde{M}_0, \widetilde{M}_\infty \right), \left(B_0, B_1, B_\infty, \widetilde{N}_0, \widetilde{N}_\infty \right) \right).$$

There exists exactly one morphism $(R, S_0, \widetilde{S}_1, S_\infty) \in \text{Mor}(\mathcal{S})$ such that

$$\mathcal{G}(R, S_0, \widetilde{S}_1, S_\infty) = (S_0, \widetilde{S}_1, S_\infty).$$

It is clear because $R = G_i S_i F_i^{-1}$ for $i = 0, \infty$ and

$$\pi^* R = \widetilde{G}_i S_i \widetilde{F}_i^{-1} = \widetilde{G}_1 \widetilde{S}_1 \widetilde{F}_1^{-1}.$$

Therefore, the entries of $\pi^* R$ belong to

$$\mathcal{M}(\widetilde{\mathbb{C}}^*) \cap \pi^* \mathcal{M}(D(0, 1)) \cap \pi^* \mathcal{M}(\mathbb{P}^1(\mathbb{C}) \setminus \overline{D}(0, 1)).$$

By Lemma 3.2, the entries of $\pi^* R$ belongs to $\pi^* \mathbb{C}(z)$ that is the entries of R are rational functions. □

From the two previous propositions, we have the following result.

Corollary 14.3. *The categories \mathcal{C} and \mathcal{E}_{rs} are equivalent.*

15. STRUCTURE OF NEUTRAL TANNAKIAN CATEGORY OVER \mathbb{C}

As is the case with \mathcal{E}_{rs} in 4 (and \mathcal{E}_f in 5), we equip the categories \mathcal{C} and \mathcal{S} with a tensor product on objects and morphisms. It is defined componentwise using the tensor product on matrices defined in 1.1.c. The equivalences defined in the previous section are compatible with these tensor structures. Since \mathcal{E}_{rs} is a rigid abelian tensor category (see 11), the categories \mathcal{C} and \mathcal{S} are also rigid abelian tensor categories.

Proposition 15.1. *For all $\tilde{a} \in \widetilde{\mathbb{C}^*} \setminus \{\tilde{1}\}$, we consider the functors*

$$\omega_0, \omega_\infty, \omega_1^{(\tilde{a})} : \mathcal{C} \rightarrow \text{Vect}_{\mathbb{C}}^f$$

defined by:

- if $\mathcal{X} \in \text{Obj}(\mathcal{C})$ is of rank n then for $i = 0, \infty$,

$$\omega_i(\mathcal{X}) = \mathbb{C}^n \text{ and } \omega_1^{(\tilde{a})}(\mathcal{X}) = \mathbb{C}^n,$$

- if $(S_0, \widetilde{S}_1, S_\infty) \in \text{Mor}(\mathcal{C})$ then for $i = 0, \infty$,

$$\omega_i(S_0, \widetilde{S}_1, S_\infty) = S_i \text{ and } \omega_1^{(\tilde{a})}(S_0, \widetilde{S}_1, S_\infty) = \widetilde{S}_1(\tilde{a}).$$

These are fibre functors and the category of connections \mathcal{C} is a neutral Tannakian category over \mathbb{C} .

Proof. The only non immediate point is the faithfulness of the functors $\omega_1^{(\tilde{a})}$, $\tilde{a} \in \widetilde{\mathbb{C}^*} \setminus \{\tilde{1}\}$. To prove it, we assume that $\widetilde{S}_1(\tilde{a}) = 0$ and we want to show that $\widetilde{S}_1 = 0$. From the equality $\phi_p(\widetilde{S}_1) = B_1 \widetilde{S}_1 A_1^{-1}$ it follows that $\widetilde{S}_1(\tilde{a}^{p^k}) = 0$ for all $k \in \mathbb{Z}$. Since $\tilde{a} \in \widetilde{\mathbb{C}^*} \setminus \{\tilde{1}\}$, the entries of \widetilde{S}_1 are Laurent polynomials in \log with an infinity of roots, thus they are equal to 0. \square

We can also show that \mathcal{S} is a Tannakian category over \mathbb{C} .

Part V. Mahler analogue of the density theorem of Schlesinger

16. THE GALOIS GROUPOID OF THE CATEGORY \mathcal{C}

Definition 16.1. The Galois groupoid of the category of connections \mathcal{C} , denoted by G , is such that:

- the objects are the fiber functors $\omega_0, \omega_\infty, \omega_1^{(\tilde{a})} : \mathcal{C} \rightarrow \text{Vect}_{\mathbb{C}}^f$ defined in Proposition 15.1,
- the morphisms between $\omega, \omega' \in \{\omega_0\} \cup \{\omega_\infty\} \cup \{\omega_1^{(\tilde{a})} \mid \tilde{a} \in \widetilde{\mathbb{C}^*} \setminus \{\tilde{1}\}\}$ are the isomorphisms of tensor functors from ω to ω' :

$$G(\omega, \omega') = \text{Iso}^\otimes(\omega, \omega').$$

Proposition 16.2. *The local Galois groupoids G_0 , G_1 and G_∞ can be identified with subgroupoids of the Galois groupoid G of the category \mathcal{C} .*

Proof. Let us prove that G_0 can be identified with a subgroupoid of G . We can see that $\omega_0 = \omega_{l,0} \mathcal{P}$ with

$$\begin{array}{ccc} \mathcal{P} : & \mathcal{C} & \rightarrow \mathcal{P}^{(0)} \\ & (A_0, A_1, A_\infty, \widetilde{M}_0, \widetilde{M}_\infty) \in \text{Obj}(\mathcal{C}) & \rightsquigarrow A_0 \\ & (S_0, \widetilde{S}_1, S_\infty) & \rightsquigarrow S_0 \end{array}$$

the functor “projection”. The restriction of the elements of $\text{Aut}^\otimes(\omega_{l,0})$ to \mathcal{C} gives a morphism of proalgebraic groups

$$\mathcal{P}^* : \text{Aut}^\otimes(\omega_{l,0}) \rightarrow \text{Aut}^\otimes(\omega_0)$$

and from [DM82, Proposition 2.21] it is a closed immersion of proalgebraic groups. \square

We also denote by G_0, G_1, G_∞ the subgroupoids of G which are identified with G_0, G_1, G_∞ by the previous proposition.

17. CONSTRUCTION OF ISOMORPHISMS OF TENSOR FUNCTORS

Let $(A_0, A_1, A_\infty, \widetilde{M}_0, \widetilde{M}_\infty) \in \text{Obj}(\mathcal{C})$. In order to construct elements of the Galois groupoid of \mathcal{C} , we need to evaluate the matrices \widetilde{M}_0 and \widetilde{M}_∞ . It is why we introduce the following definition.

Definition 17.1. The *singular locus* of a matrix M is

$$S(M) := \{\text{poles of } M\} \cup \{\text{poles of } M^{-1}\} = \{\text{poles of } M\} \cup \{\text{zeros of } \det M\}.$$

17.1. The category $\mathcal{C}_{E_0, E_\infty}$

Notation 17.2. Let E_0 (respectively E_∞) be a finite subset of $D(0, 1) \setminus \{0\}$ (respectively $\mathbb{C} \setminus \overline{D}(0, 1)$). We consider the full subcategory $\mathcal{C}_{E_0, E_\infty}$ of \mathcal{C} whose objects are the $(A_0, A_1, A_\infty, \widetilde{M}_0, \widetilde{M}_\infty) \in \text{Obj}(\mathcal{C})$ such that for $i = 0, \infty$,

$$S(\widetilde{M}_i) \subset \widetilde{E}_i := \left(E_i^{p\mathbb{Z}}, \arg(E_i^{p\mathbb{Z}}) + 2\pi\mathbb{Z} \right)$$

where for $z \in \mathbb{C}^*$, $\arg(z)$ denotes an argument of the complex number z and for $k \in \mathbb{Z}_{<0}$,

$$E_i^{p^k} = \{z \in \mathbb{C} \mid z^{p^{-k}} \in E_i\}.$$

Lemma 17.3. Every morphism in the category $\mathcal{C}_{E_0, E_\infty}$ has a kernel.

Proof. Let $(S_0, \widetilde{S}_1, S_\infty)$ be a morphism from $\mathcal{X} := (A_0, A_1, A_\infty, \widetilde{M}_0, \widetilde{M}_\infty)$ of rank n_1 to $\mathcal{Y} := (B_0, B_1, B_\infty, \widetilde{N}_0, \widetilde{N}_\infty)$ of rank n_2 in the category $\mathcal{C}_{E_0, E_\infty}$. By the equivalence between the categories \mathcal{E}_{r_s} and \mathcal{C} (see Corollary 14.3) and Lemma 11.2, there exist

$$\mathcal{X}' := (A'_0, A'_1, A'_\infty, \widetilde{M}'_0, \widetilde{M}'_\infty) \in \text{Obj}(\mathcal{C})$$

of rank $n_1 - r$ with $r = \text{rk}(S_0) = \text{rk}(\widetilde{S}_1) = \text{rk}(S_\infty)$ and

$$(K_0, \widetilde{K}_1, K_\infty) \in \text{Mor}(\mathcal{C})$$

such that

$$(17) \quad (\mathcal{X}', (K_0, \widetilde{K}_1, K_\infty)) : \mathcal{X}' \rightarrow \mathcal{X}$$

is a kernel of the morphism $(S_0, \widetilde{S}_1, S_\infty)$ in the category \mathcal{C} with

$$\text{rk}(K_0) = \text{rk}(\widetilde{K}_1) = \text{rk}(K_\infty) = n_1 - r.$$

It remains to show that \mathcal{X}' is an object of $\mathcal{C}_{E_0, E_\infty}$. Let us study the singular locus of \widetilde{M}'_0 . Given that $(K_0, \widetilde{K}_1, K_\infty) \in \text{Hom}_{\mathcal{C}}(\mathcal{X}', \mathcal{X})$, we have $\widetilde{K}_1 \widetilde{M}'_0 = \widetilde{M}_0 K_0$ that is

$$(18) \quad K_0 \widetilde{M}'_0{}^{-1} = \widetilde{M}_0^{-1} \widetilde{K}_1.$$

There exists a permutation matrix $T \in GL_{n_1}(\mathbb{C})$ such that

$$TK_0 = \begin{pmatrix} L_1 \\ L_2 \end{pmatrix} \text{ with } L_1 \in GL_{n_1-r}(\mathbb{C}) \text{ and } L_2 \in \mathcal{M}_{r,n_1-r}(\mathbb{C})$$

because $\text{rk}(K_0) = n_1 - r$. The equality (18) implies

$$\widetilde{M}_0'^{-1} = (L_1^{-1} \ 0) T \widetilde{M}_0^{-1} \widetilde{K}_1$$

so

$$(19) \quad \text{poles} \left(\widetilde{M}_0'^{-1} \right) \subset \widetilde{E}_0.$$

In order to finish the study of the singular locus of \widetilde{M}_0' , we show that

$$(20) \quad \text{zeros} \left(\det \left(\widetilde{M}_0'^{-1} \right) \right) \subset \widetilde{E}_0.$$

Let $U \in GL_{n_1}(\mathbb{C})$ be a permutation matrix such that

$$U \widetilde{K}_1 = \begin{pmatrix} \widetilde{L}_1 \\ \widetilde{L}_2 \end{pmatrix} \text{ with } \widetilde{L}_1 \in GL_{n_1-r} \left(\mathbb{C} \left[\log, \log^{-1} \right] \right) \text{ and } L_2 \in \mathcal{M}_{r,n_1-r} \left(\mathbb{C} \left[\log, \log^{-1} \right] \right).$$

By contradiction, assume that there exists

$$\tilde{a} \in \text{zero} \left(\det \left(\widetilde{M}_0'^{-1} \right) \right) \cap \left(\Sigma_0 \setminus \widetilde{E}_0 \right).$$

From (19), we know that \tilde{a} is not a pole of $\widetilde{M}_0'^{-1}$ so we can consider $\widetilde{M}_0'^{-1}(\tilde{a})$ and there exists $v \in \mathbb{C}^{n_1-r} \setminus \{0\}$ such that

$$\widetilde{M}_0'^{-1}(\tilde{a})v = 0 \in \mathbb{C}^{n_1-r}.$$

The equality (18) implies

$$0 = K_0 \widetilde{M}_0'^{-1}(\tilde{a})v = \widetilde{M}_0^{-1}(\tilde{a})U^{-1} \begin{pmatrix} \widetilde{L}_1(\tilde{a}) \\ \widetilde{L}_2(\tilde{a}) \end{pmatrix} v$$

so

$$\begin{pmatrix} \widetilde{L}_1(\tilde{a}) \\ \widetilde{L}_2(\tilde{a}) \end{pmatrix} v = 0$$

because $\widetilde{M}_0^{-1}(\tilde{a})U^{-1}$ is an invertible matrix. In particular, $\tilde{a} \in \text{zero} \left(\det \left(\widetilde{L}_1 \right) \right)$. From the equality

$$\phi_p \left(\widetilde{K}_1 \right) A'_1 = A_1 \widetilde{K}_1,$$

we have

$$\begin{pmatrix} \widetilde{L}_1(\tilde{a}^p) \\ \widetilde{L}_2(\tilde{a}^p) \end{pmatrix} A'_1 v = 0$$

so $\tilde{a}^p \in \text{zero} \left(\det \left(\widetilde{L}_1 \right) \right)$. By induction, it follows that for all $k \in \mathbb{N}$,

$$\tilde{a}^{p^k} \in \text{zero} \left(\det \left(\widetilde{L}_1 \right) \right).$$

As $\tilde{a} \neq \tilde{1}$, $\det \left(\widetilde{L}_1 \right)$ has an infinity of roots, it is a Laurent polynomial in \log so it is equal to 0. This contradicts the fact that \widetilde{L}_1 is an invertible matrix.

Therefore, from (19) and (20),

$$S\left(\widetilde{M}_0'\right) \subset \widetilde{E}_0$$

and, similarly, we can show that

$$S\left(\widetilde{M}_\infty'\right) \subset \widetilde{E}_\infty.$$

□

The category $\mathcal{C}_{E_0, E_\infty}$ is a subcategory of \mathcal{C} which is stable by tensor and abelian constructions, by the lemma hereabove it becomes clear that:

Theorem 17.4. *The category $\mathcal{C}_{E_0, E_\infty}$ is a neutral Tannakian category over \mathbb{C} .*

In what follows, we keep the same notations for the restrictions to $\mathcal{C}_{E_0, E_\infty}$ of the fibre functors $\omega_0, \omega_\infty, \omega_1^{(\tilde{a})}$, $\tilde{a} \in \widetilde{\mathbb{C}^*} \setminus \{\tilde{1}\}$ defined in Proposition 15.1.

Proposition 17.5. *If $\mathcal{X} = (A_0, A_1, A_\infty, \widetilde{M}_0, \widetilde{M}_\infty) \in \text{Obj}(\mathcal{C})$ then there exists a category $\mathcal{C}_{E_0, E_\infty}$ such that $\mathcal{X} \in \text{Obj}(\mathcal{C}_{E_0, E_\infty})$.*

Proof. Using the notations of the category \mathcal{C} , in 12, the matrix

$$(21) \quad A := \phi_p(W_0) A_0 W_0^{-1}$$

satisfies

$$(22) \quad A = \phi_p(W_\infty) A_\infty W_\infty^{-1}$$

and

$$(23) \quad \phi_p\left(\widetilde{W}_1\right) = \pi^* A \cdot \widetilde{W}_1 \cdot A_1^{-1}.$$

Therefore, from Lemma 3.2, $A \in GL_n(\mathbb{C}(z))$. From the equality $\widetilde{M}_0 = \widetilde{W}_1^{-1} \pi^* W_0$, we have

$$S\left(\widetilde{M}_0\right) \subset \left(S\left(\widetilde{W}_1\right) \cup S\left(\pi^* W_0\right)\right) \cap \Sigma_0.$$

From the equality (23), we have

$$\widetilde{W}_1 = \phi_p^{-1}(\pi^* A) \cdots \phi_p^{-n}(\pi^* A) \phi_p^{-n}\left(\widetilde{W}_1\right) A_1^{-n}.$$

For a $n \in \mathbb{N}$ big enough and using that $\widetilde{W}_1 \in GL_n\left(\mathcal{M}\left(\widetilde{\mathbb{C}^*}\right)\right)$ we obtain

$$S\left(\widetilde{W}_1\right) \cap \Sigma_0 \subset \left(\Delta^{p^{\mathbb{Z}}}, \arg\left(\Delta^{p^{\mathbb{Z}}}\right) + 2\pi\mathbb{Z}\right)$$

with $\Delta := S(A) \cap (D(0, 1) \setminus \{0\})$. Similarly, from the equality (21) we can show that

$$S\left(\pi^* W_0\right) \subset \left(\Delta^{p^{\mathbb{Z}}}, \arg\left(\Delta^{p^{\mathbb{Z}}}\right) + 2\pi\mathbb{Z}\right).$$

Therefore,

$$S\left(\widetilde{M}_0\right) \subset \left(\Delta^{p^{\mathbb{Z}}}, \arg\left(\Delta^{p^{\mathbb{Z}}}\right) + 2\pi\mathbb{Z}\right)$$

and Δ is by construction a finite subset of $D(0, 1) \setminus \{0\}$. The singular locus of \widetilde{M}_∞ satisfy the analogue property. □

17.2. Construction of isomorphisms of tensor functors

Proposition 17.6. *Let $\tilde{a} \in \Sigma_0 \setminus \widetilde{E}_0$. The natural transformation*

$$\Gamma_{0,\tilde{a}} : (A_0, A_1, A_\infty, \widetilde{M}_0, \widetilde{M}_\infty) \in \text{Obj}(\mathcal{C}_{E_0, E_\infty}) \rightsquigarrow \widetilde{M}_0(\tilde{a})$$

is an element of $\text{Iso}^\otimes(\omega_0, \omega_1^{(\tilde{a})})$.

Proof. To prove that $\Gamma_{0,\tilde{a}} \in \text{Iso}^\otimes(\omega_0, \omega_1^{(\tilde{a})})$, we have to verify the two following assumptions:

- Condition to be an isomorphism of functors. Let

$$\mathcal{X} := (A_0, A_1, A_\infty, \widetilde{M}_0, \widetilde{M}_\infty) \text{ and } \mathcal{Y} := (B_0, B_1, B_\infty, \widetilde{N}_0, \widetilde{N}_\infty)$$

be two objects of $\mathcal{C}_{E_0, E_\infty}$. The following diagram is commutative because in the category of connections $\widetilde{N}_0 S_0 = \widetilde{S}_1 \widetilde{M}_0$:

$$\begin{array}{ccc} \omega_0(A_0, A_1, A_\infty, \widetilde{M}_0, \widetilde{M}_\infty) & \xrightarrow{S_0} & \omega_0(B_0, B_1, B_\infty, \widetilde{N}_0, \widetilde{N}_\infty) \\ \widetilde{M}_0(\tilde{a}) \downarrow & & \downarrow \widetilde{N}_0(\tilde{a}) \\ \omega_1^{(\tilde{a})}(A_0, A_1, A_\infty, \widetilde{M}_0, \widetilde{M}_\infty) & \xrightarrow{\widetilde{S}_1(\tilde{a})} & \omega_1^{(\tilde{a})}(B_0, B_1, B_\infty, \widetilde{N}_0, \widetilde{N}_\infty) \end{array}$$

This shows that it is a morphism of functors. Moreover, it is an isomorphism because the $\widetilde{M}_0(\tilde{a})$ are isomorphisms.

- Condition of tensor compatibility. For all $\mathcal{X}, \mathcal{Y} \in \text{Obj}(\mathcal{C}_{E_0, E_\infty})$,

$$\Gamma_{0,\tilde{a}}(\mathcal{X}) \otimes \Gamma_{0,\tilde{a}}(\mathcal{Y}) = \widetilde{M}_0(\tilde{a}) \otimes \widetilde{N}_0(\tilde{a}) = (\widetilde{M}_0 \otimes \widetilde{N}_0)(\tilde{a}) = \Gamma_{0,\tilde{a}}(\mathcal{X} \otimes \mathcal{Y}).$$

□

Similarly, we have that:

Proposition 17.7. *Let $\tilde{b} \in \Sigma_\infty \setminus \widetilde{E}_\infty$. The natural transformation*

$$\Gamma_{0,\tilde{b}} : (A_0, A_1, A_\infty, \widetilde{M}_0, \widetilde{M}_\infty) \in \text{Obj}(\mathcal{C}_{E_0, E_\infty}) \rightsquigarrow \widetilde{M}_\infty(\tilde{b})$$

is an element of $\text{Iso}^\otimes(\omega_\infty, \omega_1^{(\tilde{b})})$.

We also have that:

Proposition 17.8. *The natural transformation*

$$(A_0, A_1, A_\infty, \widetilde{M}_0, \widetilde{M}_\infty) \in \text{Obj}(\mathcal{C}_{E_0, E_\infty}) \rightsquigarrow A_0$$

is an element of $\text{Aut}^\otimes(\omega_0)$.

The natural transformation

$$(A_0, A_1, A_\infty, \widetilde{M}_0, \widetilde{M}_\infty) \in \text{Obj}(\mathcal{C}_{E_0, E_\infty}) \rightsquigarrow A_\infty$$

is an element of $\text{Aut}^\otimes(\omega_\infty)$.

Let $\tilde{a} \in \widetilde{\mathbb{C}^*} \setminus (\widetilde{E}_0 \cup \widetilde{E}_\infty)$. The natural transformation

$$(A_0, A_1, A_\infty, \widetilde{M}_0, \widetilde{M}_\infty) \in \text{Obj}(\mathcal{C}_{E_0, E_\infty}) \rightsquigarrow A_1$$

is an element of $\text{Iso}^\otimes(\omega_1^{(\tilde{a})}, \omega_1^{(\tilde{a}^p)})$.

We will keep the same notations for the restriction of G to the Galois groupoid of $\mathcal{C}_{E_0, E_\infty}$.

Corollary 17.9. *The local Galois groupoids G_0, G_1, G_∞ , the $\Gamma_{0, \tilde{a}}, \tilde{a} \in \Sigma_0 \setminus \widetilde{E}_0$ (defined in Proposition 17.6) and the $\Gamma_{0, \tilde{b}}, \tilde{b} \in \Sigma_\infty \setminus \widetilde{E}_\infty$ (defined in Proposition 17.7) are elements of the Galois groupoid G of $\mathcal{C}_{E_0, E_\infty}$.*

In the next section, we will prove that they generate a Zariski-dense subgroupoid of the Galois groupoid G .

18. ANALOGUE OF THE DENSITY THEOREM OF SCHLESINGER FOR MAHLER EQUATIONS

18.1. Density criteria

We use the following results (see [Sau99, II.3.2.1]):

Theorem 18.1. *Let \mathcal{C} be a Tannakian category and ω a fibre functor for \mathcal{C} . Let $\langle \mathcal{X} \rangle$ be the Tannakian subcategory generated by the object \mathcal{X} of \mathcal{C} . Then the morphism*

$$\begin{aligned} \text{Aut}^\otimes(\omega|_{\langle \mathcal{X} \rangle}) &\rightarrow \text{Aut}(\omega(\mathcal{X})) \\ g &\mapsto g(\mathcal{X}) \end{aligned}$$

is injective. It identifies the Galois group $G(\mathcal{X}) = \text{Aut}^\otimes(\omega|_{\langle \mathcal{X} \rangle})$ of \mathcal{X} with an algebraic subgroup of $\text{Aut}(\omega(\mathcal{X})) \simeq GL_n(\mathbb{C})$ (n is the rank of \mathcal{X} namely the dimension of the \mathbb{C} -vector space $\omega(\mathcal{X})$).

The following result is a density criterion, we keep the notations introduced in the previous theorem:

Theorem 18.2. *Let H be a subgroup of $G = \text{Aut}^\otimes(\omega)$. If \mathcal{X} is an object of \mathcal{C} , we denote by $H(\mathcal{X})$ the image of H in $G(\mathcal{X})$.*

We assume that for all objects \mathcal{Y} of \mathcal{C} , for all $y \in \omega(\mathcal{Y})$, if the line $\mathbb{C}y$ is stable by $H(\mathcal{Y})$ then it is also stable by $G(\mathcal{Y})$.

Then, for all objects \mathcal{X} of \mathcal{C} , $H(\mathcal{X})$ is Zariski-dense in $G(\mathcal{X})$.

Herebelow, the analogue of the previous density criterion for groupoids:

Theorem 18.3. *We consider a transitive groupoid G whose objects are fiber functors ω_i for the Tannakian category \mathcal{C} and whose morphisms from the object ω_i to the object ω_j are the elements of $G(\omega_i, \omega_j) := \text{Iso}^\otimes(\omega_i, \omega_j)$. Let H be a subgroupoid of G which is transitive.*

We assume that for all objects \mathcal{Y} of \mathcal{C} , for all $y_i \in \omega_i(\mathcal{Y})$, if the lines $\mathbb{C}y_i$ are globally stable by $H(\mathcal{Y})$ then they are also globally stable by $G(\mathcal{Y})$.

Then, for all objects \mathcal{X} of \mathcal{C} , $H(\mathcal{X})$ is Zariski-dense in $G(\mathcal{X})$.

18.2. Density theorem in the regular singular case

We consider the following regular singular system at $0, 1$ and ∞

$$\phi_p(Y) = AY$$

with $A(z) \in GL_n(\mathbb{C}(z))$.

We recall that we restrict to the full subcategory $\mathcal{C}_{E_0, E_\infty}$ of \mathcal{C} .

Theorem 18.4. *The local Galois groupoids G_0, G_1, G_∞ and the Galoisian isomorphisms $\Gamma_{0, \tilde{z}_0}, \Gamma_{\infty, \tilde{z}_\infty}$ for all $\tilde{z}_0 \in \Sigma_0 \setminus \widetilde{E}_0, \tilde{z}_\infty \in \Sigma_\infty \setminus \widetilde{E}_\infty$ generate a Zariski-dense subgroupoid H of the Galois groupoid G of $\mathcal{C}_{E_0, E_\infty}$.*

Proof. To prove this theorem, we use the density criterion given in Theorem 18.3. Let

$$\mathcal{X} = (A_0, A_1, A_\infty, \widetilde{M}_0, \widetilde{M}_\infty)$$

be an object of $\mathcal{C}_{E_0, E_\infty}$. For each object $\omega_0, \omega_1^{(\tilde{c})}, \omega_\infty, \tilde{c} \in \widetilde{\mathbb{C}^\star} \setminus (\widetilde{E}_0 \cup \widetilde{E}_\infty \cup \{\tilde{1}\})$ of G , we choose a line $D_i \subset \omega_i(\mathcal{X}) = \mathbb{C}^n, i = 0, \infty$ and $\widetilde{D}_1^{(\tilde{c})} \subset \omega_1^{(\tilde{c})}(\mathcal{X}) = \mathbb{C}^n$. We assume that this family of lines is globally stable by H . We have to show that this family of lines is also globally stable by G .

We denote by $\rho : G \rightarrow GL(\omega(\mathcal{X}))$ the representation of G corresponding to \mathcal{X} by Tannaka duality.

The restriction $\rho|_{G_1} : G_1 \rightarrow GL(\omega(\mathcal{X}))$ is the representation corresponding to A_1 by

Tannaka duality for the category $\mathcal{P}^{(1)}$. The corestriction $\rho|_{G_1}^{GL(\widetilde{D}_1^{(\tilde{c})})}$ is a subrepresentation of rank 1 of $\rho|_{G_1}$. Therefore, by Tannaka duality, it comes from a subobject of rank 1 of A_1 in $\mathcal{P}^{(1)}$. This subobject is an object a_1 of $\mathcal{P}^{(1)}$ of rank 1 and a monomorphism $\tilde{d}_1 : a_1 \rightarrow A_1$ of $\mathcal{P}^{(1)}$. In short, $a_1 \in \mathbb{C}^\star$ and \tilde{d}_1 is a column matrix whose entries are Laurent polynomials in $\widetilde{\log}$ such that

$$(24) \quad \phi_p(\tilde{d}_1) a_1 = A_1 \tilde{d}_1$$

and

$$\omega_1^{(\tilde{c})}(\tilde{d}_1) \mathbb{C} = \tilde{d}_1(\tilde{c}) \mathbb{C} = \widetilde{D}_1^{(\tilde{c})}.$$

The restriction $\rho|_{G_0} : G_0 \rightarrow GL(\omega(\mathcal{X}))$ is a representation which corresponds to A_0 by Tannaka duality for the category $\mathcal{P}^{(0)}$. The corestriction $\rho|_{G_0}^{GL(\widetilde{D}_0)}$ is a subrepresentation of rank 1 of $\rho|_{G_0}$. Therefore, it comes from a subobject of rank 1 of A_0 in $\mathcal{P}^{(0)}$ namely an object a_0 of $\mathcal{P}^{(0)}$ of rank 1 and a monomorphism $d_0 : a_0 \rightarrow A_0$ of $\mathcal{P}^{(0)}$. In short, $a_0 \in \mathbb{C}^\star$ and $d_0 \in \mathbb{C}^n$ is such that

$$(25) \quad d_0 a_0 = A_0 d_0$$

and

$$\mathbb{C} d_0 = D_0.$$

Similarly, for the category $\mathcal{P}^{(\infty)}$ and the groupoid G_∞ , we have that

$$(26) \quad d_\infty a_\infty = A_\infty d_\infty$$

with $a_\infty \in \mathbb{C}^\star, d_\infty \in \mathbb{C}^n$ and

$$\mathbb{C} d_\infty = D_\infty.$$

For all $\tilde{a} \in \Sigma_0 \setminus \widetilde{E}_0$, the stability under $\Gamma_{0,\tilde{a}}$ of the family of lines means that

$$\widetilde{M}_0(\tilde{a}) D_0 = \widetilde{D}_1^{(\tilde{a})}.$$

Consequently, for all $\tilde{a} \in \Sigma_0 \setminus \widetilde{E}_0$, there exists $\widetilde{m}_0(\tilde{a}) \in \mathbb{C}$ such that $\widetilde{M}_0(\tilde{a}) d_0 = \widetilde{m}_0(\tilde{a}) \widetilde{d}_1(\tilde{a})$. Therefore, we obtain a function $\widetilde{m}_0 : \Sigma_0 \rightarrow \mathbb{C}$ such that

$$(27) \quad \widetilde{M}_0 d_0 = \widetilde{m}_0 \widetilde{d}_1.$$

Similarly, for all $\tilde{b} \in \Sigma_\infty \setminus \widetilde{E}_\infty$, $D_\infty = \widetilde{M}_\infty(\tilde{b})^{-1} D_1^{(\tilde{b})}$ and there exists $\widetilde{m}_\infty : \Sigma_\infty \rightarrow \mathbb{C}$ such that

$$(28) \quad \widetilde{M}_\infty d_\infty = \widetilde{m}_\infty \widetilde{d}_1.$$

Thus, we have:

- $\mathcal{X}' := (a_0, a_1, a_\infty, \widetilde{m}_0, \widetilde{m}_\infty)$ is an object of $\mathcal{C}_{E_0, E_\infty}$. Indeed, the equality (27) implies that \widetilde{m}_0 is a meromorphic function on Σ_0 . From the equality

$$\phi_p(\widetilde{M}_0) = A_1 \widetilde{M}_0 A_0^{-1},$$

given in 12, we have

$$\phi_p(\widetilde{M}_0) d_0 = A_1 \widetilde{M}_0 A_0^{-1} d_0$$

and the equalities (24), (25) imply

$$\phi_p(\widetilde{m}_0) = a_1 \widetilde{m}_0 a_0^{-1}.$$

The same argument shows that \widetilde{m}_∞ is a meromorphic function on Σ_∞ and satisfies

$$\phi_p(\widetilde{m}_\infty) = a_1 \widetilde{m}_\infty a_\infty^{-1}.$$

Moreover, the singular locus are such that $S(\widetilde{m}_i) \subset \widetilde{E}_i$, $i = 0, \infty$ (for more details see the following remark).

In order to prove that \mathcal{X}' is an object of $\mathcal{C}_{E_0, E_\infty}$, it remains to prove that there exists a non-zero $\widetilde{w}_1 \in \mathcal{M}(\mathbb{C}^*)$, $w_0 \in \mathcal{M}(D(0, 1))$ and $w_\infty \in \mathcal{M}(\mathbb{P}^1(\mathbb{C}) \setminus \overline{D}(0, 1))$ such that

$$\begin{cases} \widetilde{w}_1 \widetilde{m}_0 = \pi^* w_0 \\ \widetilde{w}_1 \widetilde{m}_\infty = \pi^* w_\infty \end{cases}$$

As $(A_0, A_1, A_\infty, \widetilde{M}_0, \widetilde{M}_\infty)$ is an object of $\mathcal{C}_{E_0, E_\infty}$, there exists

$$\begin{aligned} \widetilde{W}_1 &\in GL_n(\mathcal{M}(\mathbb{C}^*)), W_0 \in GL_n(\mathcal{M}(D(0, 1))), \\ W_\infty &\in GL_n(\mathcal{M}(\mathbb{P}^1(\mathbb{C}) \setminus \overline{D}(0, 1))) \end{aligned}$$

such that $\widetilde{W}_1 \widetilde{M}_0 = \pi^* W_0$ and $\widetilde{W}_1 \widetilde{M}_\infty = \pi^* W_\infty$. Therefore,

$$\begin{cases} \pi^* W_0 d_0 = \widetilde{W}_1 \widetilde{M}_0 d_0 = \widetilde{m}_0 \widetilde{W}_1 \widetilde{d}_1 \text{ on } \Sigma_0, \\ \pi^* W_\infty d_\infty = \widetilde{W}_1 \widetilde{M}_\infty d_\infty = \widetilde{m}_\infty \widetilde{W}_1 \widetilde{d}_1 \text{ on } \Sigma_\infty. \end{cases}$$

A non-zero entry of the column matrix $\widetilde{W}_1 \widetilde{d}_1$ satisfies the conditions to be \widetilde{w}_1 .

- $m = (d_0, \widetilde{d}_1, d_\infty) : \mathcal{X}' \rightsquigarrow \mathcal{X}$ is a morphism of $\mathcal{C}_{E_0, E_\infty}$. This is due to the fact that $d_0, d_\infty \in \mathbb{C}^n$, the entries of \widetilde{d}_1 are Laurent polynomials in $\widetilde{\log}$ and

$$\begin{cases} \widetilde{d}_1 \widetilde{m}_0 = \widetilde{M}_0 d_0 \\ \widetilde{d}_1 \widetilde{m}_\infty = \widetilde{M}_\infty d_\infty \end{cases}.$$

The family of lines, fixed by the subgroupoid H , are globally stable by the groupoid G because:

- for all $g_i \in \text{Aut}^\otimes(\omega_{i|\langle \mathcal{X} \rangle})$, $i = 0, \infty$, the following diagram is commutative

$$\begin{array}{ccc} \omega_i(\mathcal{X}') & \xrightarrow{g_i(\mathcal{X}')} & \omega_i(\mathcal{X}') \\ \omega_i(m)=d_i \downarrow & & \downarrow \omega_i(m)=d_i \\ \omega_i(\mathcal{X}) & \xrightarrow{g_i(\mathcal{X})} & \omega_i(\mathcal{X}) \end{array}$$

so, $g_i(\mathcal{X}) d_i = d_i \underbrace{g_i(\mathcal{X}')}_{\in \mathbb{C}^*} = g_i(\mathcal{X}') d_i$ and $g_i(\mathcal{X}) D_i = D_i$.

- for all $g_1 \in \text{Iso}^\otimes(\omega_1^{(\widetilde{c}_1)}|_{\langle \mathcal{X} \rangle}, \omega_1^{(\widetilde{c}_2)}|_{\langle \mathcal{X} \rangle})$, the following diagram is commutative

$$\begin{array}{ccc} \omega_1^{(\widetilde{c}_1)}(\mathcal{X}') & \xrightarrow{g_1(\mathcal{X}')} & \omega_1^{(\widetilde{c}_2)}(\mathcal{X}') \\ \omega_1^{(\widetilde{c}_1)}(m)=\widetilde{d}_1(\widetilde{c}_1) \downarrow & & \downarrow \omega_1^{(\widetilde{c}_2)}(m)=\widetilde{d}_1(\widetilde{c}_2) \\ \omega_1^{(\widetilde{c}_1)}(\mathcal{X}) & \xrightarrow{g_1(\mathcal{X})} & \omega_1^{(\widetilde{c}_2)}(\mathcal{X}) \end{array}$$

so, $g_1(\mathcal{X}) \widetilde{D}_1^{(\widetilde{c}_1)} = \widetilde{D}_1^{(\widetilde{c}_2)}$.

- similarly, for all $h_0 \in \text{Iso}^\otimes(\omega_{0|\langle \mathcal{X} \rangle}, \omega_1^{(\widetilde{a})}|_{\langle \mathcal{X} \rangle})$ and $h_\infty \in \text{Iso}^\otimes(\omega_{\infty|\langle \mathcal{X} \rangle}, \omega_1^{(\widetilde{b})}|_{\langle \mathcal{X} \rangle})$, we can prove that $h_0(\mathcal{X}) D_0 = \widetilde{D}_1^{(\widetilde{a})}$ and $h_\infty(\mathcal{X}) D_\infty = \widetilde{D}_1^{(\widetilde{b})}$.

□

Remark 18.5. Let us look at the singular locus of \widetilde{m}_0 . The equality (27),

$$\widetilde{m}_0^{-1} d_0 = \widetilde{M}_0^{-1} \widetilde{d}_1,$$

implies that

$$\text{poles}(\widetilde{m}_0^{-1}) \subset (\text{poles}(\widetilde{M}_0^{-1}) \cup \text{poles}(\widetilde{d}_1)) \cap \Sigma_0 \subset \widetilde{E}_0.$$

Also, $\text{zeros}(\widetilde{m}_0^{-1}) \subset \widetilde{E}_0$ because if $\widetilde{\alpha} \in \text{zeros}(\widetilde{m}_0^{-1}) \cap (\Sigma_0 \setminus \widetilde{E}_0)$ then $\widetilde{M}_0(\widetilde{\alpha})^{-1}$ is well-defined and invertible but $0 = \widetilde{M}_0(\widetilde{\alpha})^{-1} \widetilde{d}_1(\widetilde{\alpha})$ and $\widetilde{d}_1(\widetilde{\alpha}) \neq 0$. This implies that $\text{Vect}_{\mathbb{C}} \widetilde{d}_1(\widetilde{\alpha}) = D_1^{(\widetilde{\alpha})} = 0$, which is absurd. Therefore,

$$S(\widetilde{m}_0) \subset \widetilde{E}_0.$$

Similarly,

$$S(\widetilde{m}_\infty) \subset \widetilde{E}_\infty.$$

18.3. Density theorem in the regular case

Definition 18.6. We say that $A \in \text{Obj}(\mathcal{E}_{rs})$ is a *regular object of the category* \mathcal{E}_{rs} if the system $\phi_p(Y) = AY$ is meromorphically equivalent at 0, 1 and ∞ to a regular system respectively at 0, 1 and ∞ .

We say that $(A_0, A_1, A_\infty, \widetilde{M}_0, \widetilde{M}_\infty) \in \text{Obj}(\mathcal{C})$ of rank $n \in \mathbb{N}^*$ is a *regular object of the category* \mathcal{C} if $A_0 = A_1 = A_\infty = I_n$.

We denote by \mathcal{E}_r the full subcategory of \mathcal{E}_{rs} whose objects are the regular objects of \mathcal{E}_{rs} .

We denote by \mathcal{C}_r the full subcategory of \mathcal{C} whose objects are the regular objects of \mathcal{C} .

These categories are Tannakian subcategories of \mathcal{E}_{rs} and \mathcal{C} , respectively. We can check that the categories \mathcal{E}_r and \mathcal{C}_r are equivalent.

We denote by $\mathcal{C}_{r,E_0,E_\infty}$ the full subcategory of $\mathcal{C}_{E_0,E_\infty}$ whose objects are the regular objects. It is a neutral Tannakian category over \mathbb{C} . From Theorem 18.4, we obtain:

Corollary 18.7. *Let $\mathcal{X} = (I_n, I_n, I_n, \widetilde{M}_0, \widetilde{M}_\infty)$ be an object of the category $\mathcal{C}_{r,E_0,E_\infty}$. The subgroup H of $G = \text{Aut}^\otimes(\omega_1)$ containing all*

$$\begin{aligned} \mathcal{X} &\rightsquigarrow \widetilde{M}_0(\widetilde{a}_1)\widetilde{M}_0(\widetilde{a}_2)^{-1}; \\ \mathcal{X} &\rightsquigarrow \widetilde{M}_\infty(\widetilde{b}_1)\widetilde{M}_\infty(\widetilde{b}_2)^{-1} \end{aligned}$$

with $\widetilde{a}_1, \widetilde{a}_2 \in \Sigma_0 \setminus \widetilde{E}_0$ and $\widetilde{b}_1, \widetilde{b}_2 \in \Sigma_\infty \setminus \widetilde{E}_\infty$ is Zariski-dense in G .

Proof. It follows from Theorem 18.4 and the fact that in this case the local Galois groupoids G_0, G_1, G_∞ are trivial because $A_0 = A_1 = A_\infty = I_n$. \square

REFERENCES

- [AB17] Boris Adamczewski and Jason Bell. A problem about mahler functions. *Annali della Scuola Normale Superiore di Pisa - Classe di Scienze*, 17:1301–1355, 01 2017.
- [AF15] Boris Adamczewski and Colin Faverjon. Méthode de mahler : relations linéaires, transcendance et applications aux nombres automatiques. *Proceedings of the London Mathematical Society*, 115, 08 2015.
- [AF16] Boris Adamczewski and Colin Faverjon. Méthode de mahler, transcendance et relations linéaires : aspects effectifs. *Journal de Théorie des Nombres de Bordeaux*, 30, 10 2016.
- [AS92] Jean-Paul Allouche and Jeffrey Shallit. The ring of k -regular sequences. *Theoret. Comput. Sci.*, 98(2):163–197, 1992.
- [BCZ14] Richard Brent, Michael Coons, and Wadim Zudilin. Algebraic independence of mahler functions via radial asymptotics. *International Mathematics Research Notices*, 2016, 12 2014.
- [Bec94] Paul-Georg Becker. k -regular power series and Mahler-type functional equations. *J. Number Theory*, 49(3):269–286, 1994.
- [CDDM16] Frédéric Chyzak, Thomas Dreyfus, Philippe Dumas, and Marc Mezzarobba. Computing solutions of linear mahler equations. *Mathematics of Computation*, 87, 12 2016.
- [Cob68] Alan Cobham. On the hartmanis-stearns problem for a class of tag machines. *IEEE Conference Record of 1968 Ninth Annual Symposium on Switching and Automata Theory*, pages 51–60, 11 1968.
- [DHR15] Thomas Dreyfus, Charlotte Hardouin, and Julien Roques. Hypertranscendence of solutions of mahler equations. *Journal of the European Mathematical Society*, 20, 07 2015.
- [DM82] Pierre Deligne and James S. Milne. Tannakian categories, in hodge cycles and motives. *Lecture Notes in Mathematics*, 900, 01 1982.
- [Eti95] Pavel Etingof. Galois groups and connection matrices of q -difference equations. *Electronic Research Announcements of the American Mathematical Society [electronic only]*, 1, 01 1995.

- [Fer19] Gwladys Fernandes. Regular extensions and algebraic relations between values of mahler functions in positive characteristic. *Transactions of the American Mathematical Society*, 372:1, 01 2019.
- [Lox84] J. Loxton. A method of mahler in transcendence theory and some of its applications. *Bulletin of The Australian Mathematical Society - BULL AUST MATH SOC*, 29, 02 1984.
- [LP77] J.H. Loxton and A.J. Poorten. Transcendence and algebraic independence by a method of mahler. *Transcendence Theory: Advances and Applications*, 01 1977.
- [Mah29] Kurt Mahler. Arithmetische Eigenschaften der Lösungen einer Klasse von Funktionalgleichungen. *Math. Ann.*, 101(1):342–366, 1929.
- [Mah30a] Kurt Mahler. Arithmetische Eigenschaften einer Klasse transzendental-transzendenten Funktionen. *Math. Z.*, 32(1):545–585, 1930.
- [Mah30b] Kurt Mahler. Über das Verschwinden von Potenzreihen mehrerer Veränderlichen in speziellen Punktfolgen. *Math. Ann.*, 103(1):573–587, 1930.
- [MF80] Michel Mendès France. Nombres algébriques et théorie des automates. *L'Enseignement Mathématique. IIe Série*, 26, 01 1980.
- [Nis96] Kumiko Nishioka. *Mahler Functions and Transcendence*, volume 1631 of *Lecture Notes in Mathematics*. Springer-Verlag Berlin Heidelberg, 1996.
- [Pel11] Federico Pellarin. An introduction to mahler’s method for transcendence and algebraic independence, 2011.
- [Phi15] Patrice Philippon. Groupes de Galois et nombres automatiques. *J. Lond. Math. Soc. (2)*, 92(3):596–614, 2015.
- [Ran92] Bernard Randé. *Équations fonctionnelles de Mahler et applications aux suites p-régulières*. PhD thesis, Université Bordeaux 1, 1992.
- [Roq18] Julien Roques. On the algebraic relations between Mahler functions. *Trans. Amer. Math. Soc.*, 370(1):321–355, 2018.
- [Sau99] Jacques Sauloy. *Théorie de Galois des équations aux q-différences fuchsienues*. PhD thesis, Université Paul Sabatier, Toulouse, 1999.
- [Sau00] Jacques Sauloy. Systèmes aux q-différences singuliers réguliers: classification, matrice de connexion et monodromie. *Ann. Inst. Fourier (Grenoble)*, 50(4):1021–1071, 2000.
- [Sau03] Jacques Sauloy. Galois theory of Fuchsian q-difference equations. *Ann. Sci. École Norm. Sup. (4)*, 36(6):925–968 (2004), 2003.
- [Sch95] Ludwig Schlesinger. *Handbuch der Theorie der Linearen Differentialgleichungen*. Teubner, Leipzig, 1895.
- [vdPS97] Marius van der Put and Michael F. Singer. *Galois theory of difference equations*, volume 1666 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1997.

UNIV LYON, UNIVERSITÉ CLAUDE BERNARD LYON 1, CNRS UMR 5208, INSTITUT CAMILLE JORDAN,
 F-69622 VILLEURBANNE, FRANCE
E-mail address: poulet@math.univ-lyon1.fr