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# Exact Recovery Analysis of Non-Negative Orthogonal Greedy Algorithms

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**Abstract**— It is well-known that Orthogonal Matching Pursuit (OMP) recovers the exact support of  $K$ -sparse signals under the condition  $\mu < 1/(2K - 1)$  where  $\mu$  denotes the mutual coherence of the dictionary. In this communication, we show that under the same condition and if the unknown  $K$ -sparse signal is non-negative, the weights of the atoms selected by OMP are non-negative at any of the first  $K$  iterations. Therefore, the generalized version of OMP to the non-negative setting (NNOMP) identifies with OMP, which allows us to establish an exact recovery analysis of NNOMP under the mutual coherence condition. We further establish a similar analysis of the non-negative extension of Orthogonal Least Squares (OLS), and discuss open issues related to the elaboration of weakened guarantees as compared to mutual coherence guarantees.

## 1 Introduction

Greedy algorithms are popular iterative schemes for sparse signal reconstruction. Their principle is to sequentially select atoms in a given dictionary and to update the sparse approximation coefficients by solving a least-square problem whenever a new atom is selected. Orthogonal Matching Pursuit (OMP) [1] is a well-known greedy algorithm in which at each iteration, the atom having the largest inner product with the current residual is selected. Then, an Unconstrained Least Squares (ULS) problem is solved to update the sparse approximation coefficients.

In the theoretical viewpoint, OMP is guaranteed to exactly recover the support of  $K$ -sparse representations in  $K$  steps (irrespective of the magnitude of the non-zero coefficients in the  $K$ -sparse representation) when the mutual coherence of the dictionary is lower than  $1/(2K - 1)$  [2]. Moreover, this condition was proved to be sharp [3].

## 2 Sign preservation with OMP

Our first contribution states that under the same condition and for  $K$ -sparse representations with non-negative weights, the weights of the atoms selected by OMP are non-negative at any iteration.

**Theorem 1.** [4, Corollary 3.1] *Consider a dictionary  $\mathbf{H}$  whose mutual coherence satisfies  $\mu < 1/(2K - 1)$ . Let  $\mathbf{y} = \mathbf{H}\mathbf{x}$  be a  $K$ -sparse representation with non-negative weights  $\mathbf{x}$ . Then, OMP recovers the support of  $\mathbf{x}$  in  $K$  iterations, and at each iteration  $k = 1, \dots, K$ , the weights of selected atoms are non-negative.*

## 3 $K$ -step analysis of NNOMP

Non-Negative OMP (NNOMP) was first introduced by Bruckstein *et al.* as a generalization of OMP to address sparse reconstruction under non-negativity constraints [5]. Similar to OMP, the empty support is used for initialization of NNOMP. At each iteration, NNOMP selects the atom having the most positive inner product with the current residual, the atoms yielding negative inner products being ignored. Then, the non-negative weights are updated by solving a Non-Negative Least Squares (NNLS) problem. It is noticeable that unlike ULS problems, NNLS problems do not have a closed-form solution. Therefore, a subroutine must be called at each iteration of NNOMP. Note also that since NNLS are constrained least-square problems, some non-negativity constraints may be activated, corresponding to the cancellation of some weights in the sparse approximation. Hence, more than  $K$  iterations might be necessary to reach a solution that is exactly  $K$ -sparse. The reader is referred to [6] for further developments about fully recursive implementations of NNOMP.

On the theoretical side, Bruckstein *et al.* conjectured that NNOMP is guaranteed to exactly recover the support of non-negative  $K$ -sparse representations in  $K$  steps when  $\mu < 1/(2K - 1)$  [5, Theorem 3], but the rigorous proof was not provided. In [4], we have proved this result as a corollary of Theorem 1. In a nutshell, Theorem 1 states that the ULS solution related to the support found at each iteration of OMP is non-negative, which implies that the NNLS solution related to the same support coincides with this ULS solution. Since the NNOMP selection rule is close to that of OMP, we could thus deduce that the NNOMP and OMP iterates coincide, which yields the following  $K$ -step recovery result.

**Theorem 2.** [4, Corollary 3.2] *Consider a dictionary  $\mathbf{H}$  whose mutual coherence satisfies  $\mu < 1/(2K - 1)$ . Let  $\mathbf{y} = \mathbf{H}\mathbf{x}$  be a  $K$ -sparse representation with non-negative weights  $\mathbf{x}$ . Then, the NNOMP iterates identify with those of OMP. Thus, NNOMP recovers the support of  $\mathbf{x}$  in  $K$  iterations.*

Details of these results can be found in our technical report [4]. Note that in [4], we also carry out unified analysis of the non-negative extensions of Orthogonal Least Squares (OLS) [7] proposed in [8]. Open issues related to the elaboration of weakened theoretical guarantees as compared to mutual coherence guarantees will be addressed at the iTWIST workshop.

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