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# ON THE CAUCHY PROBLEM FOR A DERIVATIVE NONLINEAR SCHRÖDINGER EQUATION WITH NONVANISHING BOUNDARY CONDITIONS

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# ON THE CAUCHY PROBLEM FOR A DERIVATIVE NONLINEAR SCHRÖDINGER EQUATION WITH NONVANISHING BOUNDARY CONDITIONS

PHAN VAN TIN

ABSTRACT. In this paper we consider the Schrödinger equation with nonlinear derivative term. Our goal is to initiate the study of this equation with non vanishing boundary conditions. We obtain the local well posedness for the Cauchy problem on Zhidkov spaces  $X^k(\mathbb{R})$  and in  $\phi + H^k(\mathbb{R})$ . Moreover, we prove the existence of conservation laws by using localizing functions. Finally, we give explicit formulas for stationary solutions on Zhidkov spaces.

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## 1. INTRODUCTION

We are interested in the Cauchy problem for the following derivative nonlinear Schrödinger equation with nonvanishing boundary conditions:

$$\begin{cases} i\partial_t u + \partial^2 u = -iu^2\partial\bar{u}, \\ u(0) = u_0, \end{cases} \quad (1.1)$$

where  $u : \mathbb{R}_t \times \mathbb{R}_x \rightarrow \mathbb{C}$ ,  $\partial = \partial_x$  denotes derivative in space and  $\partial_t$  denotes derivative in time.

Our attention was drawn to this equation by the work of Hayashi and Ozawa [9] concerning the more general nonlinear Schrödinger equation

$$\begin{cases} i\partial_t u + \partial^2 u = i\lambda|u|^2\partial u + i\mu u^2\partial\bar{u} + f(u), \\ u(0) = u_0. \end{cases} \quad (1.2)$$

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When  $\lambda = 0$ ,  $\mu = -1$ ,  $f \equiv 0$ , then (1.2) reduces to (1.1). This type of equation is usually referred to as *derivative nonlinear Schrödinger equations*. It may appear in various areas of physics, e.g. in Plasma Physics for the propagation of Alfvén waves [12, 13].

Under Dirichlet boundary conditions in space, the Cauchy problem for (1.1) has been solved in [9]: local well-posedness holds in  $H^1(\mathbb{R})$ , i.e. for any  $u_0 \in H^1(\mathbb{R})$  there exists a unique solution  $u \in C(I, H^1(\mathbb{R}))$  of (1.1) on a maximal interval of time  $I$ . Moreover, we have continuous dependence with respect to the initial data, blow-up at the ends of the time interval of existence  $I$  if  $I$  is bounded and conservation of energy, mass and momentum.

The main difficulty is the appearance of the derivative term  $-iu^2\overline{u_x}$ . We cannot use the classical contraction method for this type of nonlinear Schrödinger equations. In [9] Hayashi and Ozawa use the Gauge transform to establish the equivalence of the local well-posedness between the equation (1.2) and a system of equations without derivative terms. By studying the Cauchy problem for this system, they obtain the associated results for (1.2). In [8], Hayashi and Ozawa construct a sequence of solutions of approximated equations and prove that this sequence is converging to a solution of (1.2), obtaining this way the local well-posedness of (1.2). The approximation method has also been used by Tsutsumi and Fukuda in [14, 15]. The difference between [8] and [14, 15] lies in the way of constructing the approximate equation. In [8], the authors use approximation on the non-linear term, whereas in [14, 15] the authors use approximation on the linear operator.

To our knowledge, the Cauchy problem for (1.1) has not been studied under non-zero boundary conditions, and our goal in this paper is to initiate this study. Note that non-zero boundary conditions on the whole space are much rarely considered in the literature around nonlinear dispersive equations than Dirichlet boundary conditions. In the case of the nonlinear Schrödinger equation with power-type nonlinearity, we refer to the works of Gérard [6, 7] for local well-posedness in the energy space and to the works of Gallo [4] and Zhidkov [16] for local well-posedness in Zhidkov spaces (see Section 2.1 for the definition of Zhidkov spaces) and Gallo [5] for local well-posedness in  $u_0 + H^1(\mathbb{R})$ . In this paper, using the method of Hayashi and Ozawa as in [9] on the Zhidkov-space  $X^k(\mathbb{R})$ , ( $k \geq 4$ ) and in the space  $\phi + H^k(\mathbb{R})$  ( $k = 1, 2$ ) for  $\phi$  in a Zhidkov space, we obtain the existence, uniqueness and continuous dependence on the initial data of solutions of (1.1) in these spaces. Using the transform

$$v = \partial u + \frac{i}{2}|u|^2u, \quad (1.3)$$

we see that if  $u$  is a solution of (1.1) then  $(u, v)$  is a solution of a system of two equations without derivative terms. It is easy to obtain the local wellposedness of this system on Zhidkov spaces. The main difficulty is how to obtain a solution of (1.1) from a solution of the system. Actually, we must prove that the relation (1.3) is conserved in time. The main difference in our setting with the setting in [10] is that we work on Zhidkov spaces instead of the space of localized functions  $H^1(\mathbb{R})$ . Our first main result is the following.

**Theorem 1.1.** *Let  $u_0 \in X^4(\mathbb{R})$ . Then there exists a unique maximal solution of (1.1)  $u \in C((-T_{min}, T_{max}), X^4(\mathbb{R})) \cap C^1((-T_{min}, T_{max}), X^2(\mathbb{R}))$ . Moreover,  $u$  satisfies the two following properties.*

- Blow-up alternative. *If  $T_{max}$  (resp.  $T_{min}$ )  $< +\infty$  then*

$$\lim_{t \rightarrow T_{max} \text{ (resp. } -T_{min})} \|u(t)\|_{X^4} = \infty.$$

- Continuity with respect to the initial data. *If  $u_0^n \in X^4(\mathbb{R})$  is such that  $u_0^n \rightarrow u_0$  in  $X^4(\mathbb{R})$  then for any subinterval  $[T_1, T_2] \subset (-T_{min}, T_{max})$  the associated solutions of equation (1.1) ( $u^n$ ) verify*

$$\lim_{n \rightarrow \infty} \|u^n - u\|_{L^\infty([T_1, T_2], X^4)} = 0.$$

To obtain the local wellposedness on  $\phi + H^k(\mathbb{R})$  for  $\phi$  in Zhidkov spaces  $X^l(\mathbb{R})$ , we need to use the following transform

$$v = \partial u + \frac{i}{2}u(|u|^2 - |\phi|^2) + \phi. \quad (1.4)$$

We see that if  $u$  is a solution of (1.1) then  $(u, v)$  is a solution of a system of two equations without the derivative terms. For technical reasons, we will need some regularity on  $\phi$  and we take  $l = 4$ . With a solution of the system in hand, we want to obtain a solution of (1.1). In practice, we need to prove that the relation (1.4) is conserved in time. Our main second result is the following.

**Theorem 1.2.** *Let  $\phi \in X^4(\mathbb{R})$  and  $u_0 \in \phi + H^2(\mathbb{R})$ . Then the problem (1.1) has a unique maximal solution  $u \in C((-T_{min}, T_{max}), \phi + H^2(\mathbb{R}))$  which is differentiable as a function of  $C((-T_{min}, T_{max}), \phi + L^2(\mathbb{R}))$  and such that  $u_t \in C((-T_{min}, T_{max}), L^2(\mathbb{R}))$ .*

*Moreover  $u$  satisfies the two following properties.*

(1) *If  $T_{max}$  (resp.  $T_{min}$ )  $< \infty$  then*

$$\lim_{t \rightarrow T_{max} \text{ (resp. } -T_{min})} (\|u(t) - \phi\|_{H^2(\mathbb{R})}) = \infty.$$

(2) *If  $(u_0^n) \subset \phi + H^2(\mathbb{R})$  is such that  $\|u_0^n - u_0\|_{H^2} \rightarrow 0$  as  $n \rightarrow \infty$  then for all  $[T_1, T_2] \subset (-T_{min}, T_{max})$  the associated solutions  $(u^n)$  of (1.1) satisfy*

$$\lim_{n \rightarrow \infty} \|u^n - u\|_{L^\infty([T_1, T_2], H^2)} = 0.$$

In the less regular space  $\phi + H^1(\mathbb{R})$ , we obtain the local well posedness under a smallness condition on the initial data. Our third main result is the following.

**Theorem 1.3.** *Let  $\phi \in X^4(\mathbb{R})$  such that  $\|\partial\phi\|_{H^2}$  is small enough,  $u_0 \in \phi + H^1(\mathbb{R})$  such that  $\|u_0 - \phi\|_{H^1(\mathbb{R})}$  is small enough. There exist  $T > 0$  and a unique solution  $u$  of (1.1) such that*

$$u - \phi \in C([-T, T], H^1(\mathbb{R})) \cap L^4([-T, T], W^{1, \infty}(\mathbb{R})).$$

In the proof of Theorem 1.3, the main difference with the case  $\phi + H^2(\mathbb{R})$  is that we use Strichartz estimates to prove the contractivity of a map on  $L^\infty([-T, T], L^2(\mathbb{R})) \cap L^4([-T, T], L^\infty(\mathbb{R}))$ . In the case of a general nonlinear term (as in (1.2)), our method is not working. The main reason is that we do not have a proper transform to give a system without derivative terms. Moreover, our method is not working if the initial data lies on  $X^1(\mathbb{R})$ . The main reason is that when we study the system of equations, we would have to study it on  $L^\infty(\mathbb{R})$ , but we know that the Schrödinger group is not bounded from  $L^\infty(\mathbb{R})$  to  $L^\infty(\mathbb{R})$ . Thus, the local wellposedness on less regular space is a difficult problem for nonlinear derivative Schrödinger equations.

To prove the conservation laws of (1.1), we need to use a localizing function, which is necessary for integrals to be well defined. Indeed, to obtain the conservation of the energy, using (1.1), at least formally, we have

$$\partial_t(|\partial u|^2) = \partial_x(F(u)) + \partial_t(G(u)),$$

for functions  $F$  and  $G$  which will be defined later. The important thing is that when  $u$  is not in  $H^1(\mathbb{R})$ , there are some terms in  $G(u)$  which do not belong to  $L^1(\mathbb{R})$ , hence, it is impossible to integrate the two sides as in the usual case. However, we can use a localizing function to deal with this problem. Similarly, we use the localizing function to prove the conservation of the mass and the momentum. The localizing function  $\chi$  is defined as follows

$$\chi \in C^1(\mathbb{R}), \quad \text{supp}\chi \subset [-2, 2], \quad \text{and } \chi = 1 \text{ on } [-1, 1]. \quad (1.5)$$

For all  $R > 0$ , we define

$$\chi_R(x) = \chi\left(\frac{x}{R}\right). \quad (1.6)$$

Our fourth main result is the following.

**Theorem 1.4.** *Let  $q_0 \in \mathbb{R}$  be a constant,  $u_0 \in q_0 + H^2(\mathbb{R})$  be such that  $|u_0|^2 - q_0^2 \in L^1(\mathbb{R})$  and  $u \in C((-T_{min}, T_{max}), q_0 + H^2(\mathbb{R}))$  be the associated solution of (1.1) given by Theorem 1.2. Then, we have*

$$M(u) := \lim_{R \rightarrow \infty} \int_{\mathbb{R}} (|u|^2 - q_0^2) \chi_R dx = M(u_0), \quad (1.7)$$

$$\begin{aligned} E(u) &:= \int_{\mathbb{R}} |\partial u|^2 dx + \frac{1}{2} \mathcal{I}m \int_{\mathbb{R}} (|u|^2 \bar{u} - q_0^3) \partial u dx \\ &\quad + \frac{1}{6} \int_{\mathbb{R}} (|u|^2 - |q_0|^2)^2 (|u|^2 + 2|q_0|^2) dx = E(u_0), \end{aligned} \quad (1.8)$$

$$P(u) := \frac{1}{2} \mathcal{I}m \int_{\mathbb{R}} (u - q_0) \partial \bar{u} dx - \int_{\mathbb{R}} \frac{1}{4} (|u|^2 - |q_0|^2)^2 dx = P(u_0) \quad . \quad (1.9)$$

for all  $t \in (-T_{min}, T_{max})$ .

*Remark 1.5.* (i) When  $q_0 = 0$ , we recover the classical conservation of mass, energy and momentum as usually defined.

(ii) Using the assumption  $|u_0|^2 - q_0^2 \in L^1(\mathbb{R})$ , we obtain

$$M(u_0) = \lim_{R \rightarrow \infty} \int_{\mathbb{R}} (|u_0|^2 - q_0^2) \chi_R dx = \int_{\mathbb{R}} (|u_0|^2 - q_0^2) dx.$$

Moreover, the existence of the limit  $\lim_{R \rightarrow \infty} \int_{\mathbb{R}} (|u|^2 - q_0^2) \chi_R dx$  does not imply that  $|u|^2 - q_0^2 \in L^1(\mathbb{R})$ . It means that the property  $|u|^2 - q_0^2 \in L^1(\mathbb{R})$  is not conserved in time.

In the classical Schrödinger equation, there are special solutions which are called *standing waves*. There are many works on standing waves (see e.g [11], [2] and the references therein). In [16], Zhidkov shows that there are two types of bounded solitary waves possessing limits as  $x \rightarrow \pm\infty$ . These are monotone solutions and solutions which have precisely one extreme point. They are called *kinks* and *soliton-like solutions*, respectively. In [16], Zhidkov studied the stability of kinks of classical Schrödinger equations. In [1], the authors have studied the stability of kinks in the energy space. To our knowledges, all these solitary waves are in Zhidkov spaces i.e the Zhidkov space is largest space we know to find special solutions. We want to investigate stationary solutions of (1.1) in Zhidkov spaces. Our fifth main result is the following.

**Theorem 1.6.** *Let  $\phi$  be a stationary solution of (1.1) (see Definition 5.1). Assume that  $\phi$  is not a constant function and satisfies*

$$\inf_{x \in \mathbb{R}} |\phi(x)| := m > 0$$

*Then  $\phi$  is of the form  $e^{i\theta} \sqrt{k}$  where*

$$k(x) = 2\sqrt{B} + \frac{-1}{\sqrt{\frac{5}{72B}} \cosh(2\sqrt{B}x) + \frac{5}{12\sqrt{B}}}, \quad \theta = \theta_0 - \int_x^\infty \left( \frac{B}{k(y)} - \frac{k(y)}{4} \right) dy,$$

*for some constants  $\theta_0 \in \mathbb{R}$ ,  $B > 0$ . Moreover, if  $\phi$  is a stationary solution of (1.1) such that  $\phi(\infty) = 0$  then  $\phi \equiv 0$  on  $\mathbb{R}$ .*

*Remark 1.7.* We have classified stationary solutions of (1.1) for the functions which are vanishing at infinity, and for the functions which are not vanishing on  $\overline{\mathbb{R}}$ . One question still unanswered is the class of stationary solutions of (1.1) vanishing at a point in  $\mathbb{R}$ .

This paper is organized as follows. In Section 2, we give the proof of local well posedness of solution of (1.1) on Zhidkov spaces. In Section 3, we prove the local well posedness on  $\phi + H^2(\mathbb{R})$  and  $\phi + H^1(\mathbb{R})$ , for  $\phi \in X^4(\mathbb{R})$  a given function. In Section 4, we give the proof of conservation laws when the initial data is in  $q_0 + H^2(\mathbb{R})$ , for a given constant  $q_0 \in \mathbb{R}$ . Finally, in Section 5, we have some results on stationary solutions of (1.1) on Zhidkov spaces.

**Notation.** *In this paper, we will use in the following the notation  $L$  for the linear part of the Schrödinger equation, that is*

$$L = i\partial_t + \partial^2.$$

*Moreover,  $C$  denotes various positive constants and  $C(R)$  denotes constants depending on  $R$ .*

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## 2. LOCAL EXISTENCE IN ZHIDKOV SPACES

In this section, we give the proof of Theorem 1.1.

**2.1. Preliminaries on Zhidkov spaces.** Before presenting our main results, we give some preliminaries. We start by recalling the definition of Zhidkov spaces, which were introduced by Peter Zhidkov in his pioneering works on Schrödinger equations with non-zero boundary conditions (see [16] and the references therein).

**Definition 2.1.** Let  $k \in \mathbb{N}$ ,  $k \geq 1$ . The *Zhidkov space*  $X^k(\mathbb{R})$  is defined by

$$X^k(\mathbb{R}) = \{u \in L^\infty(\mathbb{R}) : \partial u \in H^{k-1}(\mathbb{R})\}.$$

It is a Banach space when endowed with the norm

$$\|\cdot\|_{X^k} = \|\cdot\|_{L^\infty} + \sum_{\alpha=1}^k \|\partial^\alpha \cdot\|_{L^2}.$$

It was proved by Gallo [4, Theorem 3.1 and Theorem 3.2] that the Schrödinger operator defines a group on Zhidkov spaces. More precisely, we have the following result.

**Proposition 2.2.** *Let  $k \geq 1$  and  $u_0 \in X^k(\mathbb{R})$ . For  $t \in \mathbb{R}$  and  $x \in \mathbb{R}$ , the quantity*

$$S(t)u_0(x) := \begin{cases} e^{-i\pi/4}\pi^{-1/2} \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}} e^{(i-\varepsilon)z^2} u_0(x + 2\sqrt{t}z) dz & \text{if } t \geq 0, \\ e^{i\pi/4}\pi^{-1/2} \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}} e^{(-i-\varepsilon)z^2} u_0(x + 2\sqrt{-t}z) dz & \text{if } t \leq 0. \end{cases} \quad (2.1)$$

*is well-defined and  $S$  defines a strongly continuous group on  $X^k(\mathbb{R})$ . For all  $u_0 \in X^k(\mathbb{R})$  and  $t \in \mathbb{R}$  we have*

$$\|S(t)u_0\|_{X^k} \leq C(k)(1 + |t|^{1/4})\|u_0\|_{X^k}.$$

*The generator of the group  $(S(t))_{t \in \mathbb{R}}$  on  $X^k(\mathbb{R})$  is  $i\partial^2$  and its domain is  $X^{k+2}(\mathbb{R})$ .*

**Remark 2.3.** Since, for all  $\phi \in X^k(\mathbb{R})$ , we have  $\phi + H^k(\mathbb{R}) \subset X^k(\mathbb{R})$ , the uniqueness of solution in  $X^k(\mathbb{R})$  implies the uniqueness of solution in  $\phi + H^k(\mathbb{R})$ , and the existence of solution in  $\phi + H^k(\mathbb{R})$  implies the existence of solution in  $X^k(\mathbb{R})$ .

**2.2. From the equation to the system.** The equation (1.1) contains a spatial derivative of  $u$  in the nonlinear part, which makes it difficult to work with. In the following proposition, we indicate how to eliminate the derivative in the nonlinearity by introducing an auxiliary function and converting the equation into a system.

**Proposition 2.4.** *Let  $k \geq 2$ . Given  $u \in X^k(\mathbb{R})$ , we define  $v$  by*

$$v = \partial u + \frac{i}{2}|u|^2 u. \quad (2.2)$$

*Hence,  $v \in X^{k-1}(\mathbb{R})$ . Furthermore, if  $u$  satisfies the equation (1.1), then the couple  $(u, v)$  verifies the system*

$$\begin{cases} Lu = P_1(u, v), \\ Lv = P_2(u, v), \end{cases} \quad (2.3)$$

*where  $P_1$  and  $P_2$  are given by*

$$\begin{aligned} P_1(u, v) &= -iu^2 \bar{v} + \frac{1}{2}|u|^4 u, \\ P_2(u, v) &= i\bar{u}v^2 + \frac{3}{2}|u|^4 v + u^2|u|^2 \bar{v}. \end{aligned} \quad (2.4)$$

*Proof.* Let  $u$  be a solution of (1.1) and  $v$  be defined by (2.2). Then we have

$$Lu = -iu^2 \partial \bar{u} = -iu^2 \left( \bar{v} + \frac{i}{2}(|u|^2 \bar{u}) \right) = -iu^2 \bar{v} + \frac{1}{2}|u|^4 u,$$

which gives us the first equation in (2.3).

On the other hand, since  $L$  and  $\partial$  commute and  $u$  solves (1.1), we have

$$Lv = \partial(Lu) + \frac{i}{2}L(|u|^2 u) = \partial(-iu^2 \partial \bar{u}) + \frac{i}{2}L(|u|^2 u) = -i(u^2 \partial^2 \bar{u} + 2u|\partial u|^2) + \frac{i}{2}L(|u|^2 u). \quad (2.5)$$

Using

$$L(uv) = L(u)v + uL(v) + 2\partial u \partial v, \quad L(\bar{u}) = -\overline{Lu} + 2\partial^2 \bar{u}, \quad (2.6)$$

we have

$$\begin{aligned} L(|u|^2 u) &= L(u^2 \bar{u}) = L(u^2) \bar{u} + u^2 L(\bar{u}) + 2\partial(u^2) \partial \bar{u} \\ &= (2L(u)u + 2(\partial u)^2) \bar{u} + u^2 (-\overline{Lu} + 2\partial^2 \bar{u}) + 4u|\partial u|^2 \\ &= 2L(u)|u|^2 + 2\bar{u}(\partial u)^2 + 2u^2 \partial^2 \bar{u} - u^2 \overline{Lu} + 4u|\partial u|^2. \end{aligned} \quad (2.7)$$

We now recall that  $u$  verifies (1.1) to obtain

$$\frac{i}{2}L(|u|^2 u) = u^2 \partial \bar{u} |u|^2 + i\bar{u}(\partial u)^2 + iu^2 \partial^2 \bar{u} + \frac{1}{2}\partial u |u|^4 + 2iu|\partial u|^2. \quad (2.8)$$

Substituting in (2.5), we get

$$\begin{aligned} Lv &= -i(u^2\partial^2\bar{u} + 2u|\partial u|^2) + u^2\partial\bar{u}|u|^2 + i\bar{u}(\partial u)^2 + iu^2\partial^2\bar{u} + \frac{1}{2}\partial u|u|^4 + 2iu|\partial u|^2, \\ &= u^2\partial\bar{u}|u|^2 + i\bar{u}(\partial u)^2 + \frac{1}{2}\partial u|u|^4. \end{aligned}$$

Observe here that the second order derivatives of  $u$  have vanished and only first order derivatives remain. Therefore, using the expression of  $v$  given in (2.2) to substitute  $\partial u$ , we obtain by direct calculations

$$Lv = i\bar{u}v^2 + \frac{3}{2}|u|^4v + u^2|u|^2\bar{v},$$

which gives us the second equation in (2.3).  $\square$

**2.3. Resolution of the system.** We now establish the local well-posedness of the system (2.3) in Zhidkov spaces.

**Proposition 2.5.** *Let  $k \geq 3$ , and  $(u_0, v_0) \in X^k(\mathbb{R}) \times X^k(\mathbb{R})$ . There exist  $T_{min}, T^{max} > 0$  and a unique maximal solution  $(u, v)$  of system (2.3) such that  $(u, v) \in C((-T_{min}, T^{max}), X^k(\mathbb{R})) \cap C^1((-T_{min}, T^{max}), X^{k-2}(\mathbb{R}))$ . Furthermore the following properties are satisfied.*

- Blow-up alternative. *If  $T^{max}$  (resp.  $T_{min}$ )  $< \infty$  then*

$$\lim_{t \rightarrow T^{max} \text{ (resp. } T_{min})} (\|u(t)\|_{X^k} + \|v(t)\|_{X^k}) = \infty.$$

- Continuity with respect to the initial data. *If  $(u_0^n, v_0^n) \in X^k \times X^k$  is such that*

$$\|u_0^n - u_0\|_{X^k} + \|v_0^n - v_0\|_{X^k} \rightarrow 0$$

*then for any subinterval  $[T_1, T_2] \subset (-T_{min}, T^{max})$  the associated solution  $(u^n, v^n)$  of (2.3) satisfies*

$$\lim_{n \rightarrow \infty} (\|u^n - u\|_{L^\infty([T_1, T_2], X^k)} + \|v^n - v\|_{L^\infty([T_1, T_2], X^k)}) = 0.$$

*Proof.* Consider the operator  $A : D(A) \subset X^{k-2}(\mathbb{R}) \rightarrow X^{k-2}(\mathbb{R})$  defined by  $A = i\partial^2$  with domain  $D(A) = X^k(\mathbb{R})$ . From Proposition 2.2 we know that the operator  $A$  is the generator of the Schrödinger group  $S(t)$  on  $X^{k-2}(\mathbb{R})$ . From classical arguments (see [3, Lemma 4.1.1 and Corollary 4.1.8]) the couple  $(u, v) \in C((-T_{min}, T^{max}), X^k(\mathbb{R})) \cap C^1((-T_{min}, T^{max}), X^{k-2}(\mathbb{R}))$  solves (2.3) if and only if the couple  $(u, v) \in C((-T_{min}, T^{max}), X^k(\mathbb{R}))$  solves

$$\begin{cases} (u, v) = S(t)(u, v) - i \int_0^t S(t-s)P(u, v)(s)ds, \\ u(0) = u_0 \in X^k(\mathbb{R}), v(0) = v_0 \in X^k(\mathbb{R}), \end{cases} \quad (2.9)$$

where  $S(t)(u, v) := (S(t)u, S(t)v)$ ,  $P(u, v) = (P_1(u, v), P_2(u, v))$  and  $P_1$  and  $P_2$  are defined in (2.4). Consider  $P$  as a map from  $X^k(\mathbb{R}) \times X^k(\mathbb{R})$  into  $X^k(\mathbb{R}) \times X^k(\mathbb{R})$ . Since  $P_1$  and  $P_2$  are polynomial in  $u$  and  $v$ , the map  $P$  is Lipschitz continuous on bounded sets of  $X^k(\mathbb{R}) \times X^k(\mathbb{R})$ . The result then follows from standard arguments (see [3, Theorem 4.3.4 and Theorem 4.3.7]).  $\square$

**2.4. Preservation of the differential identity.** The following proposition establishes the link from (2.3) to (1.1) by showing preservation along the time evolution of the differential identity

$$v_0 = \partial u_0 + \frac{i}{2}|u_0|^2 u_0.$$

**Proposition 2.6.** *Let  $u_0, v_0 \in X^3(\mathbb{R})$  be such that*

$$v_0 = \partial u_0 + \frac{i}{2}u_0|u_0|^2.$$

*Then the associated solution  $(u, v) \in C((-T_{min}, T^{max}), X^3(\mathbb{R}) \times X^3(\mathbb{R}))$  obtained in Proposition 2.5 satisfies for all  $t \in (-T_{min}, T^{max})$  the differential identity*

$$v = \partial u + \frac{i}{2}|u|^2 u.$$

*Proof.* Given  $(u, v) \in C((-T_{min}, T^{max}), X^3(\mathbb{R}) \times X^3(\mathbb{R}))$  the solution of (2.3) obtained in Proposition 2.5, we define

$$w = \partial u + \frac{i}{2}|u|^2 u.$$

Our goal will be to show that  $w = v$ . We first have

$$\begin{aligned} Lu &= -iu^2 \bar{v} + \frac{1}{2}|u|^4 u \\ &= -iu^2(\bar{v} - \bar{w}) - iu^2 \bar{w} + \frac{1}{2}|u|^4 u \\ &= -iu^2(\bar{v} - \bar{w}) - iu^2 \partial \bar{u}. \end{aligned}$$

Applying  $L$  to  $w$  and using (2.7) and the expression previously obtained for  $Lu$ , we get

$$\begin{aligned} Lw &= \partial(Lu) + \frac{i}{2}L(|u|^2 u) \\ &= \partial(Lu) + \frac{i}{2}(2Lu|u|^2 + 2\bar{u}(\partial u)^2 + 2u^2 \partial^2 \bar{u} - u^2 \overline{Lu} + 4u|\partial u|^2) \\ &= \partial(-iu^2(\bar{v} - \bar{w}) - iu^2 \partial \bar{u}) + \frac{i}{2}\left(2(-iu^2 \partial \bar{u})|u|^2 + 2\bar{u}(\partial u)^2 - u^2 \overline{(-iu^2 \partial \bar{u})} + 2u^2 \partial^2 \bar{u} + 4u|\partial u|^2\right) \\ &\quad + \frac{i}{2}\left[2(-iu^2(\bar{v} - \bar{w}))|u|^2 - u^2 \overline{(-iu^2(\bar{v} - \bar{w}))}\right] \\ &= \left(-i\partial(u^2(\bar{v} - \bar{w})) + u^2|u|^2(\bar{v} - \bar{w}) + \frac{1}{2}|u|^4(v - w)\right) \\ &\quad + \left(-i\partial(u^2 \partial \bar{u}) + u^2 \partial \bar{u}|u|^2 + i\bar{u}(\partial u)^2 + \frac{1}{2}|u|^4 \partial u + iu^2 \partial^2 \bar{u} + 2iu|\partial u|^2\right) \\ &=: I_1 + I_2. \end{aligned}$$

As in the proof of Proposition 2.4, we obtain

$$I_2 = i\bar{u}w^2 + \frac{3}{2}|u|^4 w + |u|^2 u^2 \bar{w}.$$

Furthermore

$$\begin{aligned} I_1 &= \partial(-iu^2(\bar{v} - \bar{w})) + u^2|u|^2(\bar{v} - \bar{w}) + \frac{1}{2}|u|^4(v - w) \\ &= -iu^2 \partial(\bar{v} - \bar{w}) - 2iu \partial u(\bar{v} - \bar{w}) + u^2|u|^2(\bar{v} - \bar{w}) + \frac{1}{2}|u|^4(v - w). \end{aligned}$$

It follows that

$$Lw - Lv = I_1 + (I_2 - Lv) \tag{2.10}$$

$$= I_1 + i\bar{u}(w - v)(w + v) + \frac{3}{2}|u|^4(w - v) + |u|^2 u^2(\bar{w} - \bar{v}) \tag{2.11}$$

$$= (w - v)A_1 + (\bar{w} - \bar{v})A_2 - iu^2 \partial(\bar{v} - \bar{w}), \tag{2.12}$$

where  $A_1$  and  $A_2$  are polynomials of degree at most 4 in  $u, \partial u, v, \partial v$  and their complex conjugates. Hence,

$$(Lw - Lv)(\bar{w} - \bar{v}) = |w - v|^2 A_1 + (\bar{w} - \bar{v})^2 A_2 - iu^2 \frac{\partial(\bar{v} - \bar{w})^2}{2} := K, \tag{2.13}$$

where  $K$  is a polynomial of degree at most 6 in  $u, v, w, \partial u, \partial v, \partial w$  and their complex conjugates. Remembering that  $L = i\partial_t + \partial^2$ , and taking imaginary part in the two sides of (2.13) we obtain

$$\frac{1}{2}\partial_t |w - v|^2 + \mathcal{Im}(\partial((\partial w - \partial v)(\bar{w} - \bar{v}))) = \mathcal{Im}(K). \tag{2.14}$$

Let  $\chi : \mathbb{R} \rightarrow \mathbb{R}$  be a cut-off function such that

$$\chi \in C^1(\mathbb{R}), \quad \text{supp}(\chi) \subset [-2, 2], \quad \chi \equiv 1 \text{ on } (-1, 1), \quad 0 \leq \chi \leq 1, \quad |\chi'(x)|^2 \lesssim \chi(x) \text{ for all } x \in \mathbb{R}.$$

For each  $n \in \mathbb{N}$ , define

$$\chi_n(x) = \chi\left(\frac{x}{n}\right).$$



Multiplying the two sides of (2.14) with  $\chi_n$  and integrating in space we obtain

$$\frac{1}{2}\partial_t\|(w-v)\sqrt{\chi_n}\|_{L^2}^2 + \int_{\mathbb{R}} \mathcal{I}m(\partial((\partial w - \partial v)(\bar{w} - \bar{v})))\chi_n dx = \int_{\mathbb{R}} \mathcal{I}m(K)\chi_n dx. \quad (2.15)$$

For the right hand side, we have

$$\int_{\mathbb{R}} \mathcal{I}m(K)\chi_n dx = \mathcal{I}m \int_{\mathbb{R}} |w-v|^2 A_1 \chi_n dx + \mathcal{I}m \int_{\mathbb{R}} (\bar{w}-\bar{v})^2 A_2 \chi_n dx - \mathcal{I}m \int_{\mathbb{R}} iu^2 \frac{\partial((\bar{w}-\bar{w})^2)}{2} \chi_n dx,$$

and therefore

$$\left| \int_{\mathbb{R}} \mathcal{I}m(K)\chi_n dx \right| \leq \|(w-v)\sqrt{\chi_n}\|_{L^2}^2 (\|A_1\|_{L^\infty} + \|A_2\|_{L^\infty}) + \frac{1}{2} \left| \int_{\mathbb{R}} u^2 \partial((\bar{w}-\bar{w})^2) \chi_n dx \right|.$$

We now fix some arbitrary interval  $[-T_1, T_2]$  such that  $0 \in [-T_1, T_2] \subset (-T_{min}, T_{max})$  in which we will be working from now on, and we set

$$R = \|u\|_{L^\infty([T_1, T_2], X^3)} + \|v\|_{L^\infty([T_1, T_2], X^3)}.$$

From the fact that  $A_1$  and  $A_2$  are polynomials in  $u, \partial u, v, \partial v$  of degree at most 4, for all  $t \in [T_1, T_2]$  we have

$$\|A_1\|_{L^\infty} + \|A_2\|_{L^\infty} \leq C(R).$$

It follows that

$$\left| \int_{\mathbb{R}} \mathcal{I}m(K)\chi_n dx \right| \leq \|(w-v)\sqrt{\chi_n}\|_{L^2}^2 C(R) + \frac{1}{2} \left| \int_{\mathbb{R}} (\bar{w}-\bar{w})^2 (\partial(u^2)\chi_n + u^2 \partial\chi_n) dx \right|.$$

By definition of  $\chi$  we have

$$\begin{aligned} |\partial(u^2)\chi_n| &\leq C(R)\chi_n, \\ |u^2 \partial\chi_n| &\leq |u^2| \frac{1}{n} \left| \chi' \left( \frac{\cdot}{n} \right) \right| \leq \frac{1}{n} C(R) \sqrt{\chi \left( \frac{\cdot}{n} \right)} \leq C(R) \frac{1}{n} \sqrt{\chi_n(\cdot)}. \end{aligned}$$

Hence,

$$\begin{aligned} \left| \int_{\mathbb{R}} \mathcal{I}m(K)\chi_n dx \right| &\leq \|(w-v)\sqrt{\chi_n}\|_{L^2}^2 C(R) + \frac{C(R)}{n} \left| \int_{\mathbb{R}} (\bar{w}-\bar{w})^2 \sqrt{\chi_n} dx \right| \\ &\leq C(R) \|(w-v)\sqrt{\chi_n}\|_{L^2}^2 + \frac{C(R)^2}{n} \int_{\mathbb{R}} |v-w| \sqrt{\chi_n} dx \\ &\leq C(R) \|(w-v)\sqrt{\chi_n}\|_{L^2}^2 + \frac{C(R)^2}{n} \int_{-2n}^{2n} |v-w| \sqrt{\chi_n} dx \\ &\leq C(R) \|(w-v)\sqrt{\chi_n}\|_{L^2}^2 + \frac{C(R)^2}{n} \left( \int_{-2n}^{2n} (|v-w| \sqrt{\chi_n})^2 dx \right)^{\frac{1}{2}} \left( \int_{-2n}^{2n} dx \right)^{\frac{1}{2}} \\ &\leq C(R) \|(w-v)\sqrt{\chi_n}\|_{L^2}^2 + \frac{2C(R)^2}{\sqrt{n}} \|(w-v)\sqrt{\chi_n}\|_{L^2}. \end{aligned} \quad (2.16)$$

In addition, we have

$$\begin{aligned} \left| \int_{\mathbb{R}} \mathcal{I}m(\partial((\partial w - \partial v)(\bar{w} - \bar{v})))\chi_n dx \right| &= \left| \int_{\mathbb{R}} \mathcal{I}m(((\partial w - \partial v)(\bar{w} - \bar{v}))\chi_n') dx \right| \\ &= \left| \int_{\mathbb{R}} \mathcal{I}m \left( (\partial w - \partial v)(\bar{w} - \bar{v}) \frac{1}{n} \chi' \left( \frac{x}{n} \right) \right) dx \right| \\ &\leq \int_{\mathbb{R}} |\partial w - \partial v| |w-v| \frac{1}{n} \sqrt{\chi_n} dx \\ &\leq \frac{1}{n} \|\partial w - \partial v\|_{L^2} \|(w-v)\sqrt{\chi_n}\|_{L^2} \\ &\leq \frac{C(R)}{n} \|(w-v)\sqrt{\chi_n}\|_{L^2}. \end{aligned} \quad (2.17)$$

From (2.15), (2.16), (2.17) we obtain that

$$\partial_t \|(w-v)\sqrt{\chi_n}\|_{L^2}^2 \leq C(R)\|(w-v)\sqrt{\chi_n}\|_{L^2}^2 + \frac{C(R)}{\sqrt{n}}\|(w-v)\sqrt{\chi_n}\|_{L^2} \quad (2.18)$$

$$\leq C(R)\|(w-v)\sqrt{\chi_n}\|_{L^2}^2 + \frac{C(R)}{\sqrt{n}} \quad (2.19)$$

where we have used the Cauchy inequality  $|x| \leq \frac{|x|^2+1}{2}$ . Define the function  $g : [-T_1, T_2]$  by

$$g = \|(w-v)\sqrt{\chi_n}\|_{L^2}^2.$$

Then by definition of  $w$  we have  $g(t=0) = 0$ . Furthermore, from (2.19) we have

$$\partial_t g \leq C(R)g + \frac{C(R)}{\sqrt{n}}.$$

By Gronwall inequality for all  $t \in [-T_1, T_2]$  we have

$$g \leq \frac{C(R)}{\sqrt{n}} \exp(C(R)(T_2 + T_1)) \leq \frac{C(R)}{\sqrt{n}}. \quad (2.20)$$

Assume by contradiction that there exist  $t$  and  $x$  such that

$$w(t, x) \neq v(t, x).$$

By continuity of  $v$  and  $w$ , there exists  $\varepsilon > 0$  such that (for  $n > |x|$ ) we have

$$g(t) = \|(w-v)\sqrt{\chi_n}\|_{L^2}^2 > \varepsilon.$$

Since  $\varepsilon > 0$  is independent of  $n$ , we obtain a contradiction with (2.20) when  $n$  is large enough. Therefore for all  $t$  and  $x$ , we have

$$v(t, x) = w(t, x),$$

which concludes the proof.  $\square$

**2.5. From the system to the equation.** With Proposition 2.6 in hand, we give the proof of Theorem 1.1.

*Proof of Theorem 1.1.* We start by defining  $v_0$  by

$$v_0 = \partial u_0 + \frac{i}{2}|u_0|^2 u_0 \in X^3(\mathbb{R}).$$

From Proposition 2.5 there exists a unique maximal solution  $(u, v) \in C((-T_{min}, T^{max}), X^3(\mathbb{R}) \times X^3(\mathbb{R})) \cap C^1((-T_{min}, T^{max}), X^1(\mathbb{R}) \times X^1(\mathbb{R}))$  of the system (2.3) associated with  $(u_0, v_0)$ . From Proposition 2.6, for all  $t \in (-T_{min}, T^{max})$  we have

$$v = \partial u + \frac{i}{2}|u|^2 u. \quad (2.21)$$

It follows that

$$Lu = -iu^2 \bar{v} + \frac{1}{2}|u|^4 u = -iu^2 \partial \bar{u},$$

and therefore  $u$  is a solution of (1.1) on  $(-T_{min}, T^{max})$ . Furthermore

$$u \in C((-T_{min}, T^{max}), X^3(\mathbb{R})) \cap C^1((-T_{min}, T^{max}), X^1(\mathbb{R})).$$

To obtain the desired regularity on  $u$ , we observe that, since  $v$  has the same regularity as  $u$ , and verifies (2.21), we have

$$\partial u = v - \frac{i}{2}|u|^2 u \in C((-T_{min}, T^{max}), X^3(\mathbb{R})) \cap C^1((-T_{min}, T^{max}), X^1(\mathbb{R}))$$

This implies that

$$u \in C((-T_{min}, T^{max}), X^4(\mathbb{R})) \cap C^1((-T_{min}, T^{max}), X^2(\mathbb{R})).$$

This proves the existence part of the result. Uniqueness is a direct consequence from Proposition 2.4 and Proposition 2.5.

To prove the blow-up alternative, assume that  $T^{max} < \infty$ . Then from Proposition 2.5 we have

$$\lim_{t \rightarrow T^{max}} (\|u(t)\|_{X^3(\mathbb{R})} + \|v(t)\|_{X^3(\mathbb{R})}) = \infty$$

On the other hand, from the differential identity (2.21) we obtain

$$\lim_{t \rightarrow T^{max}} (\|u(t)\|_{X^3(\mathbb{R})} + \|\partial u(t)\|_{X^3(\mathbb{R})}) = \infty.$$

It follows that

$$\lim_{t \rightarrow T^{max}} \|u(t)\|_{X^4(\mathbb{R})} = \infty.$$

Finally, we establish the continuity with respect to the initial data. Take a subinterval  $[T_1, T_2] \subset (-T_{min}, T^{max})$ , and a sequence  $(u_0^n) \in X^4(\mathbb{R})$  such that  $u_0^n \rightarrow u_0$  in  $X^4$ . Let  $u_n$  be the solution of (1.1) associated with  $u_0^n$  and define  $v_n$  by

$$v_n = \partial u_n + \frac{i}{2}|u_n|^2 u_n. \quad (2.22)$$

By Proposition 2.5 the couple  $(u_n, v_n)$  is the unique maximal solution of system (2.3) in

$$C((-T_{min}, T^{max}), X^3(\mathbb{R}) \times X^3(\mathbb{R})) \cap C^1((-T_{min}, T^{max}), X^1(\mathbb{R}) \times X^1(\mathbb{R})).$$

Moreover, we have

$$\lim_{n \rightarrow +\infty} (\|u_n - u\|_{L^\infty([T_1, T_2], X^3)} + \|v_n - v\|_{L^\infty([T_1, T_2], X^3)}) = 0 \quad (2.23)$$

Since  $v$  and  $v_n$  verify the differential identity (2.22), we have

$$\partial(u_n - u) = (v_n - v) - \frac{i}{2}(|u_n|^2 u_n - |u|^2 u).$$

Therefore we have

$$\lim_{n \rightarrow +\infty} \|u_n - u\|_{L^\infty([T_1, T_2], X^4)} = 0,$$

which completes the proof.  $\square$

### 3. RESULTS ON THE SPACE $\phi + H^k(\mathbb{R})$ FOR $\phi \in X^k(\mathbb{R})$

In this section, we give the proof of Theorem 1.2 and Theorem 1.3. For  $k \geq 1$ , let  $\phi \in X^k(\mathbb{R})$ .

#### 3.1. The local well posedness on $\phi + H^2(\mathbb{R})$ .

##### 3.1.1. From the equation to the system.

**Proposition 3.1.** *If  $u$  is a solution of (1.1) then for  $v$  defined by*

$$v = \partial u + \frac{i}{2}u(|u|^2 - |\phi|^2) + \phi. \quad (3.1)$$

*the couple  $(u, v)$  verifies the system*

$$\begin{cases} Lu = Q_1(u, v, \phi), \\ Lv = Q_2(u, v, \phi), \end{cases} \quad (3.2)$$

*where  $Q_1$  and  $Q_2$  are given by*

$$\begin{aligned} Q_1(u, v, \phi) &= -iu^2(\bar{v} - \bar{\phi}) + \frac{1}{2}u|u|^2(|u|^2 - |\phi|^2), \\ Q_2(u, v, \phi) &= \partial^2 \phi + u^2|u|^2(\bar{v} - \bar{\phi}) + i\bar{u}((v - \phi)^2 - i(v - \phi)u(|u|^2 - |\phi|^2)) \\ &\quad + \frac{1}{2}|u|^4(v - \phi) - \frac{1}{2}u^2|\phi|^2(\bar{v} - \bar{\phi}) - \frac{i}{2}\partial^2(|\phi|^2)u - i\partial(|\phi|^2)(v - \phi) - \frac{i}{2}u(|u|^2 - |\phi|^2). \end{aligned} \quad (3.3)$$

*The functions  $Q_1(u, v, \phi)$  and  $Q_2(u, v, \phi)$  are polynomials of degree at most 5 in  $u, v, \phi, \partial\phi, \partial^2\phi$ . Set  $\tilde{u} = u - \phi$ ,  $\tilde{v} = v - \phi$ . The functions  $u, v$  are solutions of the system (3.2) if and only if the functions  $\tilde{u}, \tilde{v}$  are solutions of the following system*

$$\begin{cases} L\tilde{u} = \tilde{Q}_1(\tilde{u}, \tilde{v}, \phi), \\ L\tilde{v} = \tilde{Q}_2(\tilde{u}, \tilde{v}, \phi), \end{cases} \quad (3.4)$$

*where*

$$\begin{aligned} \tilde{Q}_1(\tilde{u}, \tilde{v}, \phi) &:= Lu - L\phi = Q_1(u, v, \phi) - \partial^2 \phi = Q_1(\tilde{u} + \phi, \tilde{v} + \phi, \phi) - \partial^2 \phi, \\ \tilde{Q}_2(\tilde{u}, \tilde{v}, \phi) &:= Lv - L\phi = Q_2(u, v, \phi) - \partial^2 \phi. \end{aligned}$$

*Proof.* Since  $u$  is a solution of (1.1) we have

$$Lu = -iu^2\partial\bar{u} = -iu^2\left(\bar{v} + \frac{i}{2}\bar{u}(|u|^2 - |\phi|^2) - \bar{\phi}\right) = -iu^2(\bar{v} - \bar{\phi}) + \frac{1}{2}u|u|^2(|u|^2 - |\phi|^2),$$

which gives us the first equation in (3.2). On the other hand, applying  $L$  to  $v$ , we obtain

$$Lv = \partial(Lu) + \frac{i}{2}L(|u|^2u) - \frac{i}{2}L(|\phi|^2u) + L(\phi) \quad (3.5)$$

$$= \partial(-iu^2\partial\bar{u}) + \frac{i}{2}L(u^2\bar{u}) - \frac{i}{2}L(|\phi|^2u) + \partial^2\phi \quad (3.6)$$

$$= -2iu|\partial u|^2 - iu^2\partial^2\bar{u} + \partial^2\phi + \frac{i}{2}L(u^2\bar{u}) - \frac{i}{2}L(|\phi|^2u). \quad (3.7)$$

As in the proof of Proposition 2.4, we use (2.6) to get

$$L(|\phi|^2u) = L(u)|\phi|^2 + L(|\phi|^2)u + 2\partial(|\phi|^2)\partial u \quad (3.8)$$

$$= |\phi|^2(-iu^2\partial\bar{u}) + \partial^2(|\phi|^2)u + 2\partial u\partial(|\phi|^2). \quad (3.9)$$

Recall from (2.8) in the proof of Proposition (1.1) that

$$\frac{i}{2}L(|u|^2u) = u^2\partial\bar{u}|u|^2 + i\bar{u}(\partial u)^2 + iu^2\partial^2\bar{u} + \frac{1}{2}\partial u|u|^4 + 2iu|\partial u|^2.$$

Combining the previous identities, we obtain

$$\begin{aligned} Lv &= -2iu|\partial u|^2 - iu^2\partial^2\bar{u} + \partial^2\phi + u^2\partial\bar{u}|u|^2 + i\bar{u}(\partial u)^2 \\ &\quad + iu^2\partial^2\bar{u} + \frac{1}{2}\partial u|u|^4 + 2iu|\partial u|^2 - \frac{i}{2}(-iu^2\partial\bar{u}|\phi|^2 + \partial^2(|\phi|^2)u + 2\partial u\partial(|\phi|^2)) \\ &= \partial^2\phi + u^2\partial\bar{u}|u|^2 + i\bar{u}(\partial u)^2 + \frac{1}{2}|u|^4\partial u - \frac{1}{2}u^2\partial\bar{u}|\phi|^2 - \frac{i}{2}\partial^2(|\phi|^2)u - i\partial(|\phi|^2)\partial u. \end{aligned}$$

Using the differential identity (3.1), we get

$$\begin{aligned} Lv &= \partial^2\phi + u^2|u|^2\left(\bar{v} - \bar{\phi} + \frac{i}{2}\bar{u}(|u|^2 - |\phi|^2)\right) + i\bar{u}\left(v - \phi - \frac{i}{2}u(|u|^2 - |\phi|^2)\right)^2 \\ &\quad + \frac{1}{2}|u|^4\left(v - \phi - \frac{i}{2}u(|u|^2 - |\phi|^2)\right) - \frac{1}{2}u^2|\phi|^2\left(\bar{v} - \bar{\phi} + \frac{i}{2}\bar{u}(|u|^2 - |\phi|^2)\right) - \frac{i}{2}\partial^2(|\phi|^2)u \\ &\quad - i\partial(|\phi|^2)\left(v - \phi - \frac{i}{2}u(|u|^2 - |\phi|^2)\right) \\ &= \left(\partial^2\phi + u^2|u|^2(\bar{v} - \bar{\phi}) + i\bar{u}\left((v - \phi)^2 - i(v - \phi)u(|u|^2 - |\phi|^2)\right)\right) \\ &\quad + \frac{1}{2}|u|^4(v - \phi) - \frac{1}{2}u^2|\phi|^2(\bar{v} - \bar{\phi}) - \frac{i}{2}\partial^2(|\phi|^2)u - i\partial(|\phi|^2)(v - \phi) - \frac{i}{2}u(|u|^2 - |\phi|^2) \\ &\quad + \left(\frac{i}{2}u|u|^4(|u|^2 - |\phi|^2) - \frac{i}{4}u|u|^2(|u|^2 - |\phi|^2)^2 - \frac{i}{4}u|u|^4(|u|^2 - |\phi|^2) \right. \\ &\quad \left. - \frac{i}{4}u|u|^2|\phi|^2(|u|^2 - |\phi|^2)\right). \end{aligned}$$

Observing that

$$\frac{i}{2}u|u|^4(|u|^2 - |\phi|^2) - \frac{i}{4}u|u|^2(|u|^2 - |\phi|^2)^2 - \frac{i}{4}u|u|^4(|u|^2 - |\phi|^2) - \frac{i}{4}u|u|^2|\phi|^2(|u|^2 - |\phi|^2) = 0,$$

we obtain the second equation in (3.2).  $\square$

**3.1.2. Resolution of the system.** From similar arguments to the one used for the proof of Proposition 2.5, we may obtain the following local well-posedness result.

**Proposition 3.2.** *Let  $k \geq 1$ ,  $\phi \in X^{k+2}$ ,  $\tilde{u}_0, \tilde{v}_0 \in H^k(\mathbb{R})$ . There exist  $T_{min}, T_{max} > 0$  and a unique maximal solution  $(\tilde{u}, \tilde{v})$  of the system (3.4) such that  $\tilde{u}, \tilde{v} \in C((-T_{min}, T_{max}), H^k(\mathbb{R})) \cap C^1((-T_{min}, T_{max}), H^{k-2}(\mathbb{R}))$ . Furthermore the following properties are satisfied.*

- Blow-up alternative. If  $T^{max}$  (resp.  $T_{min}$ )  $< +\infty$  then

$$\lim_{t \rightarrow T^{max} \text{ (resp. } T_{min})} (\|\tilde{u}\|_{H^k} + \|\tilde{v}\|_{H^k}) = \infty.$$

- Continuity with respect to the initial data. If  $\tilde{u}_0^n, \tilde{v}_0^n \in H^k(\mathbb{R})$  are such that

$$\|\tilde{u}_0^n - \tilde{u}_0\|_{H^k} + \|\tilde{v}_0^n - \tilde{v}_0\|_{H^k} \rightarrow 0$$

then for any subinterval  $[T_1, T_2] \subset (-T_{min}, T^{max})$  the associated solution  $(\tilde{u}^n, \tilde{v}^n)$  of (3.4) satisfies

$$\lim_{n \rightarrow +\infty} (\|\tilde{u}^n - \tilde{u}\|_{L^\infty([T_1, T_2], H^k)} + \|\tilde{v}^n - \tilde{v}\|_{L^\infty([T_1, T_2], H^k)}) = 0.$$

**3.1.3. Preservation of a differential identity.** Given well-posedness of the system (3.2), we need to show preservation of the differential identity to go back to (1.1). This is the object of the following proposition.

**Proposition 3.3.** Let  $\phi \in X^4(\mathbb{R})$  and  $\tilde{u}_0, \tilde{v}_0 \in H^2(\mathbb{R})$  such that the condition

$$\tilde{v}_0 = \partial \tilde{u}_0 + \frac{i}{2}(\tilde{u}_0 + \phi)(|\tilde{u}_0 + \phi|^2 - |\phi|^2) + \partial \phi \quad (3.10)$$

is verified. Then the associated solutions  $\tilde{u}, \tilde{v}$  obtained in Proposition 3.2 also satisfy (3.10) for all  $t \in (-T_{min}, T^{max})$ .

*Proof.* We define

$$\tilde{w} = \partial \tilde{u} + \frac{i}{2}(\tilde{u} + \phi)(|\tilde{u} + \phi|^2 - |\phi|^2) + \partial \phi. \quad (3.11)$$

Set  $u = \tilde{u} + \phi$ ,  $v = \tilde{v} + \phi$ ,  $w = \tilde{w} + \phi$ . We have

$$w = \partial u + \frac{i}{2}u(|u|^2 - |\phi|^2) + \phi.$$

Since  $\tilde{u}, \tilde{v}$  is a solution of (3.4), we have  $u, v$  is a solution of (3.2).

$$Lu = -iu^2(\bar{v} - \bar{w}) - iu^2(\bar{w} - \bar{\phi}) + \frac{1}{2}u|u|^2(|u|^2 - |\phi|^2) = -iu^2(\bar{v} - \bar{w}) + H,$$

where we have defined

$$H = -iu^2(\bar{w} - \bar{\phi}) + \frac{1}{2}u|u|^2(|u|^2 - |\phi|^2).$$

Applying  $L$  to  $w$  and using (2.7) and the previously expression obtained for  $Lu$ , we get

$$\begin{aligned} Lw &= \partial(Lu) + \frac{i}{2}L(|u|^2u) - \frac{i}{2}L(|\phi|^2u) + L(\phi) \\ &= \partial(Lu) + \frac{i}{2} \left( 2L(u)|u|^2 + 2\bar{u}(\partial u)^2 + 2u^2\partial^2\bar{u} - u^2\overline{L(u)} + 4u|\partial u|^2 \right) \\ &\quad - \frac{i}{2} \left( |\phi|^2L(u) + u\partial^2(|\phi|^2) + 2\partial u\partial(|\phi|^2) \right) \\ &= \partial(-iu^2(\bar{v} - \bar{w})) + \partial H \\ &\quad + \frac{i}{2} \left( 2H|u|^2 - 2iu^2|u|^2(\bar{v} - \bar{w}) + 2\bar{u}(\partial u)^2 + 2u^2\partial^2\bar{u} - u^2(i\bar{u}^2(v - w) + \overline{H}) + 4u|\partial u|^2 \right) \\ &\quad - \frac{i}{2} \left( -iu^2(\bar{v} - \bar{w})|\phi|^2 + |\phi|^2H + u\partial^2(|\phi|^2) + 2\partial u\partial(|\phi|^2) \right) \\ &= -i\partial(u^2(\bar{v} - \bar{w})) + u^2|u|^2(\bar{v} - \bar{w}) + \frac{1}{2}|u|^4(v - w) - \frac{1}{2}u^2(\bar{v} - \bar{w}) + K, \end{aligned}$$

where  $K$  depends on  $u$ ,  $w$  and  $\phi$  but not on  $v$  and is given by

$$K = \partial^2\phi + u^2\partial\bar{u}|u|^2 + i\bar{u}(\partial u)^2 + \frac{1}{2}|u|^4\partial u - \frac{1}{2}u^2\partial\bar{u}|\phi|^2 - i\partial(|\phi|^2)\partial u.$$

Using the differential identity (3.11) to replace  $\partial u$ , we obtain for  $K$  the following

$$\begin{aligned} K &= \partial^2 \phi + u^2 |u|^2 \left( \bar{w} - \bar{\phi} + \frac{i}{2} \bar{u} (|u|^2 - |\phi|^2) \right) + i \bar{u} \left( w - \phi - \frac{i}{2} u (|u|^2 - |\phi|^2) \right)^2 \\ &\quad + \frac{1}{2} |u|^4 \left( w - \phi - \frac{i}{2} u (|u|^2 - |\phi|^2) \right) - \frac{1}{2} u^2 |\phi|^2 \left( \bar{w} - \bar{\phi} + \frac{i}{2} \bar{u} (|u|^2 - |\phi|^2) \right) - \frac{i}{2} \partial^2 (|\phi|^2) u \\ &\quad - i \partial (|\phi|^2) \left( w - \phi - \frac{i}{2} u (|u|^2 - |\phi|^2) \right) \\ &= \partial^2 \phi + u^2 |u|^2 (\bar{w} - \bar{\phi}) + i \bar{u} ((w - \phi)^2 - i(w - \phi)u(|u|^2 - |\phi|^2)) \\ &\quad + \frac{1}{2} |u|^4 (w - \phi) - \frac{1}{2} |\phi|^2 |u|^2 (\bar{w} - \bar{\phi}) - \frac{i}{2} \partial^2 (|\phi|^2) u - i \partial (|\phi|^2) (w - \phi) - \frac{i}{2} u (|u|^2 - |\phi|^2). \end{aligned}$$

As a consequence, we arrive for  $L(w) - L(v)$  at the following expression:

$$\begin{aligned} Lw - Lv &= -i \partial (u^2 (\bar{v} - \bar{w})) + u^2 |u|^2 (\bar{v} - \bar{w}) + \frac{1}{2} |u|^4 (v - w) - \frac{1}{2} u^2 (\bar{v} - \bar{w}) + (K - L(v)) \\ &= -i u^2 \partial (\bar{v} - \bar{w}) + A(v - w) + B(\bar{v} - \bar{w}), \end{aligned}$$

where  $A, B$  are polynomials in  $u, v, \phi, \partial \phi, \partial^2 \phi$ . It implies that

$$L(\tilde{w} - \tilde{v}) = -i(\tilde{u} + \phi)^2 \partial (\bar{\tilde{v}} - \bar{\tilde{w}}) + A(\tilde{v} - \tilde{w}) + B(\bar{\tilde{v}} - \bar{\tilde{w}}). \quad (3.12)$$

Multiplying two sides of (3.12) by  $\bar{\tilde{w}} - \bar{\tilde{v}}$ , taking the imaginary part, and integrating over space with integration by part for the first term of right hand side of (3.12), we obtain

$$\frac{d}{dt} \|\tilde{w} - \tilde{v}\|_{L^2}^2 \lesssim (\|\tilde{u} + \phi\|_{L^\infty} + \|\partial \tilde{u} + \partial \phi\|_{L^\infty} + \|A\|_{L^\infty} + \|B\|_{L^\infty}) \|\tilde{w} - \tilde{v}\|_{L^2}^2.$$

By Grönwall's inequality we obtain

$$\|\tilde{w} - \tilde{v}\|_{L^2}^2 \leq \|\tilde{w}(0) - \tilde{v}(0)\|_{L^2}^2 \times \exp(C \int_0^t (\|\tilde{u} + \phi\|_{L^\infty} + \|\partial \tilde{u} + \partial \phi\|_{L^\infty} + \|A\|_{L^\infty} + \|B\|_{L^\infty}) ds).$$

Using the fact that  $\tilde{w}(0) = \tilde{v}(0)$ , we obtain  $\tilde{w} = \tilde{v}$ , for all  $t$ . It implies that

$$\tilde{v} = \partial \tilde{u} + \frac{i}{2} (\tilde{u} + \phi) (|\tilde{u} + \phi|^2 - |\phi|^2) + \partial \phi.$$

This complete the proof of Proposition 3.3.  $\square$

**3.1.4. From the system to the equation.** With local well-posedness of the system and preservation of the differential identity in hand, we may now go back to the original equation and finish the proof of Theorem 1.2.

*Proof of Theorem 1.2.* Let  $\phi \in X^4(\mathbb{R})$ . We define  $v_0 \in X^1(\mathbb{R})$ ,  $\tilde{u}_0 \in H^2(\mathbb{R})$  and  $\tilde{v}_0 \in H^1(\mathbb{R})$  in the following way:

$$v_0 = \partial u_0 + \frac{i}{2} u_0 (|u_0|^2 - |\phi|^2) + \phi, \quad \tilde{u}_0 = u_0 - \phi, \quad \text{and } \tilde{v}_0 = v_0 - \phi.$$

We have

$$\tilde{v}_0 = \partial \tilde{u}_0 + \frac{i}{2} (\tilde{u}_0 + \phi) (|\tilde{u}_0 + \phi|^2 - |\phi|^2) + \partial \phi.$$

From Proposition 3.2 there exists a unique maximal solution  $\tilde{u}, \tilde{v} \in C((-T_{min}, T_{max}), H^1(\mathbb{R})) \cap C^1((-T_{min}, T_{max}), H^{-1}(\mathbb{R}))$  of (3.4). Let  $\tilde{u}_0^n \in H^3(\mathbb{R})$  be such that

$$\|\tilde{u}_0^n - \tilde{u}_0\|_{H^2(\mathbb{R})} \rightarrow 0$$

as  $n \rightarrow \infty$ . Define  $\tilde{v}_0^n \in H^2(\mathbb{R})$  by

$$\tilde{v}_0^n = \partial \tilde{u}_0^n + \frac{i}{2} (\tilde{u}_0^n + \phi) (|\tilde{u}_0^n + \phi|^2 - |\phi|^2) + \partial \phi.$$

From Proposition 3.2, there exists a unique solution maximal solution.

$$\tilde{u}^n, \tilde{v}^n \in C((-T_{min}^n, T_{max}^n), H^2(\mathbb{R})) \cap C^1((-T_{min}^n, T_{max}^n), L^2(\mathbb{R}))$$

of the system (3.4). Let  $[-T_1, T_2] \subset (-T_{min}, T^{max})$  be any closed interval. From [3, proposition 4.3.7], for  $n \geq N_0$  large enough, we have  $[-T_1, T_2] \subset (-T_{min}^n, T_{max}^n)$ . By Proposition 3.3, for  $n \geq N_0$ ,  $t \in [-T_1, T_2]$ , we have

$$\tilde{v}^n = \partial \tilde{u}^n + \frac{i}{2}(\tilde{u}^n + \phi)(|\tilde{u}^n + \phi|^2 - |\phi|^2) + \partial \phi.$$

By Proposition 3.2, we have

$$\lim_{n \rightarrow +\infty} \sup_{t \in [T_1, T_2]} (\|\tilde{u}^n(t) - \tilde{u}(t)\|_{H^1(\mathbb{R})} + \|\tilde{v}^n - \tilde{v}(t)\|_{H^1(\mathbb{R})}) \rightarrow 0.$$

Letting  $n \rightarrow +\infty$ , we obtain that for all  $t \in [-T_1, T_2]$ , and then for all  $t \in (-T_{min}, T^{max})$ :

$$\tilde{v} = \partial \tilde{u} + \frac{i}{2}(\tilde{u} + \phi)(|\tilde{u} + \phi|^2 - |\phi|^2) + \partial \phi.$$

It follows that

$$\partial \tilde{u} \in C((-T_{min}, T^{max}), H^1(\mathbb{R})) \cap C^1((-T_{min}, T^{max}), H^{-1}(\mathbb{R})).$$

Hence we have

$$\tilde{u} \in C((-T_{min}, T^{max}), H^2(\mathbb{R})) \cap C^1((-T_{min}, T^{max}), L^2(\mathbb{R}))$$

Define  $u \in \phi + C((-T_{min}, T^{max}), \phi + H^2(\mathbb{R})) \cap C^1((-T_{min}, T^{max}), L^2(\mathbb{R}))$  by

$$u = \phi + \tilde{u}.$$

and define  $v \in \phi + C((-T_{min}, T^{max}), \phi + H^1(\mathbb{R})) \cap C^1((-T_{min}, T^{max}), H^{-1}(\mathbb{R}))$  by

$$v = \tilde{v} + \phi = \partial u + \frac{i}{2}u(|u|^2 - |\phi|^2) + \phi.$$

Since  $\tilde{u}, \tilde{v}$  are solution of system (3.4),  $u, v$  are solutions of the system (3.2). Therefore,

$$Lu = Q_1(u, v) = Q_1\left(u, \partial u + \frac{i}{2}u(|u|^2 - |\phi|^2) + \phi\right) = -iu^2 \partial \bar{u}.$$

This establishes the existence of a solution to (1.1). To prove uniqueness, assume that  $U \in \phi + C((-T_{min}, T^{max}), H^2(\mathbb{R})) \cap C^1((-T_{min}, T^{max}), L^2(\mathbb{R}))$  is another solution of (1.1). Set  $V = \partial U + \frac{i}{2}U(|U|^2 - |\phi|^2) + \phi$ , and  $\tilde{U} = U - \phi$ ,  $\tilde{V} = V - \phi$ . We see that  $\tilde{U}, \tilde{V} \in C((-T_{min}, T^{max}), H^1(\mathbb{R})) \cap C^1((-T_{min}, T^{max}), H^{-1}(\mathbb{R}))$  are solutions of the system (3.4). From the uniqueness statement in Proposition 3.2 we obtain  $\tilde{U} = \tilde{u}$ . Hence,  $u = U$ , which proves uniqueness. The blow-up alternative and continuity with respect to the initial data are proved using similar arguments as in the proof of Theorem 1.1. This completes the proof of Theorem 1.2.  $\square$

**3.2. The local well posedness on  $\phi + H^1(\mathbb{R})$ .** In this section, we give the proof of Theorem 1.3, using the method of Hayashi and Ozawa [10]. As in Section 3.1.1, we work with the system (3.4).

**3.2.1. Resolution of the system.** Since we are working in the less regular space  $\phi + H^1(\mathbb{R})$ , we cannot use Proposition 3.2. Instead, we establish the following result using Strichartz estimate.

**Proposition 3.4.** *Consider the system (3.4). Let  $\phi \in X^2(\mathbb{R})$ ,  $\tilde{u}_0, \tilde{v}_0 \in L^2(\mathbb{R})$ . There exists  $R > 0$  such that if  $\|\tilde{u}_0\|_{L^2} + \|\tilde{v}_0\|_{L^2} < R$  then there exists  $T > 0$  and a unique solution  $\tilde{u}, \tilde{v}$  of the system (3.4) verifying*

$$\tilde{u}, \tilde{v} \in C([-T, T], L^2) \cap L^4([-T, T], L^\infty).$$

*Moreover, we have the following continuous dependence on initial data property: If  $(\tilde{u}_0^n, \tilde{v}_0^n) \in L^2(\mathbb{R}) \times L^2(\mathbb{R})$  is a sequence such that  $\|\tilde{u}_0^n\|_2 + \|\tilde{v}_0^n\|_2 < R$  and  $\|\tilde{u}_0^n - u_0\|_2 + \|\tilde{v}_0^n - v_0\|_2 \rightarrow 0$  then the associated solutions  $(\tilde{u}^n, \tilde{v}^n)$  such that*

$$\|\tilde{u}^n - \tilde{u}\|_{L^\infty L^2 \cap L^4 L^\infty} + \|\tilde{v}^n - \tilde{v}\|_{L^\infty L^2 \cap L^4 L^\infty} \rightarrow 0,$$

*where  $T$  is the time of existence of  $\tilde{u}, \tilde{v}$  and we have used the following notation:*

$$L^\infty L^2 = L^\infty([-T, T], L^2(\mathbb{R})), \quad L^4 L^\infty = L^4([-T, T], L^\infty(\mathbb{R}))$$

*and the norm on  $L^\infty L^2 \cap L^4 L^\infty$  is defined, as usual for the intersection of two Banach spaces, as the sum of the norms on each space.*

*Proof.* Let  $\tilde{Q}_1, \tilde{Q}_2$  as in system (3.4). By direct calculations, we have

$$\tilde{Q}_1(\tilde{u}, \tilde{v}, \phi) = -i(\tilde{u} + \phi)^2 \tilde{v} + \frac{1}{2}(\tilde{u} + \phi)|(\tilde{u} + \phi)|^2(|\tilde{u} + \phi|^2 - |\phi|^2) - \partial^2 \phi, \quad (3.13)$$

$$\begin{aligned} \tilde{Q}_2(\tilde{u}, \tilde{v}, \phi) &= (\tilde{u} + \phi)^2 |\tilde{u} + \phi|^2 \tilde{v} + i(\tilde{u} + \phi)((\tilde{v})^2 - i\tilde{v}(\tilde{u} + \phi)(|\tilde{u} + \phi|^2 - |\phi|^2)) \\ &\quad + \frac{1}{2}|\tilde{u} + \phi|^4 \tilde{v} - \frac{1}{2}(\tilde{u} + \phi)|\phi|^2 \tilde{v} - \frac{i}{2}\partial^2(|\phi|^2)(\tilde{u} + \phi) - i\partial(|\phi|^2)\tilde{v} \\ &\quad - \frac{i}{2}(\tilde{u} + \phi)(|\tilde{u} + \phi|^2 - |\phi|^2). \end{aligned} \quad (3.14)$$

Consider the following problem

$$(\tilde{u}, \tilde{v}) = S(t)(\tilde{u}_0, \tilde{v}_0) - i \int_0^t S(t-s)\tilde{Q}(\tilde{u}, \tilde{v}, \phi) ds \quad (3.15)$$

where  $\tilde{Q} = (\tilde{Q}_1, \tilde{Q}_2)$ . Let

$$\Phi(\tilde{u}, \tilde{v}) = S(t)(\tilde{u}_0, \tilde{v}_0) - i \int_0^t S(t-s)\tilde{Q} ds.$$

Assume that  $\|\tilde{u}_0\|_{L^2(\mathbb{R})} + \|\tilde{v}_0\|_{L^2(\mathbb{R})} \leq \frac{R}{4}$  for  $R > 0$  small enough. For  $T > 0$  we define the space  $X_{T,R}$  by

$$X_{T,R} = \{(\tilde{u}, \tilde{v}) \in C([-T, T], L^2(\mathbb{R})) \cap L^4([-T, T], L^\infty(\mathbb{R})) : \|\tilde{u}\|_{L^\infty L^2 \cap L^4 L^\infty} + \|\tilde{v}\|_{L^\infty L^2 \cap L^4 L^\infty} \leq R\}.$$

We are going to prove that for  $R, T$  small enough the map  $\Phi$  is a contraction from  $X_{T,R}$  to  $X_{T,R}$ .

We first prove that for  $R, T$  small enough,  $\Phi$  maps  $X_{T,R}$  into  $X_{T,R}$ . Let  $(\tilde{u}, \tilde{v}) \in X_{T,R}$ . By Strichartz estimates we have

$$\begin{aligned} \|\Phi(\tilde{u}, \tilde{v})\|_{L^\infty L^2 \cap L^4 L^\infty} &\lesssim \|(\tilde{u}_0, \tilde{v}_0)\|_{L^2 \times L^2} + \|\tilde{Q}\|_{L^1 L^2 \times L^1 L^2}, \\ &\lesssim \frac{R}{4} + (\|\tilde{Q}_1\|_{L^1 L^2} + \|\tilde{Q}_2\|_{L^1 L^2}). \end{aligned}$$

We have

$$\begin{aligned} \|\tilde{Q}_1\|_{L^1 L^2} &\lesssim \| |\tilde{u} + \phi|^2 \tilde{v} \|_{L^1 L^2} + \| |\tilde{u} + \phi|^3 (|\tilde{u}|^2 + |\tilde{u}||\phi|) \|_{L^1 L^2} + \|\partial^2 \phi\|_{L^1 L^2} \\ &\lesssim \|\tilde{v}\|_{L^2 L^2} \|\tilde{u} + \phi\|_{L^4 L^\infty}^2 + \|\tilde{u} + \phi\|_{L^4 L^\infty}^3 (\|\tilde{u}\|_{L^4 L^\infty} \|\tilde{u}\|_{L^\infty L^2} + \|\tilde{u}\|_{L^\infty L^2} \|\phi\|_{L^4 L^\infty}) + \|\partial^2 \phi\|_{L^1 L^2} \\ &\lesssim \|\tilde{v}\|_{L^\infty L^2} (2T)^{\frac{1}{2}} (\|\tilde{u}\|_{L^4 L^\infty} + \|\phi\|_{L^4 L^\infty})^2 \\ &\quad + (\|\tilde{u}\|_{L^4 L^\infty} + \|\phi\|_{L^4 L^\infty})^3 \left( \|\tilde{u}\|_{L^4 L^\infty} \|\tilde{u}\|_{L^\infty L^2} + \|\tilde{u}\|_{L^\infty L^2} \|\phi\|_{L^\infty} (2T)^{\frac{1}{4}} \right) + \|\partial^2 \phi\|_{L^1 L^2} \\ &\lesssim (2T)^{\frac{1}{2}} R (R + \|\phi\|_{L^\infty} (2T)^{\frac{1}{4}})^2 + (R + \|\phi\|_{L^\infty} (2T)^{\frac{1}{4}})^3 \left( R^2 + R \|\phi\|_{L^\infty} (2T)^{\frac{1}{4}} \right) + (2T) \|\partial^2 \phi\|_{L^2(\mathbb{R})} \\ &\lesssim \frac{R}{4}, \end{aligned}$$

for  $T, R$  small enough. Since  $\tilde{Q}_2$  contains polynomial of order at most 5, we also have

$$\|\tilde{Q}_2\|_{L^1 L^2} \lesssim \frac{R}{4}$$

for  $T, R$  small enough. Therefore, for  $T, R$  small enough, we have

$$\|\Phi(\tilde{u}, \tilde{v})\|_{(L^\infty L^2 \cap L^4 L^\infty)^2} \leq \frac{3R}{4} < R.$$

Hence,  $\Phi$  maps from  $X_{T,R}$  into itself.

We now show that for  $T, R$  small enough, the map  $\Phi$  is a contraction from  $X_{T,R}$  to itself. Indeed, let  $(u_1, v_1), (u_2, v_2) \in X_{T,R}$ . By Strichartz estimates we have

$$\begin{aligned} \|\Phi(u_1, v_1) - \Phi(u_2, v_2)\|_{L^\infty L^2 \cap L^4 L^\infty} &= \left\| \int_0^t S(t-s) \left( \tilde{Q}(u_1, v_1) - \tilde{Q}(u_2, v_2) \right) ds \right\|_{(L^\infty L^2 \cap L^4 L^\infty)^2}, \\ &\lesssim \|\tilde{Q}_1(u_1, v_1) - \tilde{Q}_1(u_2, v_2)\|_{L^1 L^2} + \|\tilde{Q}_2(u_1, v_1) - \tilde{Q}_2(u_2, v_2)\|_{L^1 L^2}. \end{aligned}$$

Using the same kind of arguments as before we obtain that  $\Phi$  is a contraction on  $X_{T,R}$ . Therefore, using the Banach fixed-point theorem, there exist  $T > 0$  and a unique solution  $\tilde{u}, \tilde{v} \in C([-T, T], L^2(\mathbb{R})) \cap L^4([-T, T], L^\infty(\mathbb{R}))$  of the problem (3.15). This implies by classical arguments



that  $\tilde{u}, \tilde{v}$  is solution of the system (3.4) such that  $\tilde{u}, \tilde{v} \in C([-T, T], L^2\mathbb{R}) \cap C^1([-T, T], H^{-2}(\mathbb{R}))$ . We now prove the uniqueness property of the solution. Assume that  $(\tilde{u}_1, \tilde{v}_1)$  and  $(\tilde{u}_2, \tilde{v}_2)$  are two solutions of the system (3.4) in  $C([-T, T], L^2(\mathbb{R})) \cap L^4([-T, T], L^\infty(\mathbb{R}))$ . We also have  $(\tilde{u}_1, \tilde{v}_1), (\tilde{u}_2, \tilde{v}_2) \in C([-T, T], L^2(\mathbb{R})) \cap C^1([-T, T], H^{-2}(\mathbb{R}))$ . Therefore,  $(\tilde{u}_1, \tilde{v}_1)$  and  $(\tilde{u}_2, \tilde{v}_2)$  also satisfy the equation (3.15). By uniqueness property of solutions of (3.15) we obtain that  $\tilde{u}_1 = \tilde{u}_2$  and  $\tilde{v}_1 = \tilde{v}_2$ .

It remains to prove the continuous dependence on initial data. Assume that  $(u_0^n, v_0^n) \in L^2(\mathbb{R}) \times L^2(\mathbb{R})$  is such that

$$\|u_0^n - \tilde{u}_0\|_{L^2(\mathbb{R})} + \|v_0^n - \tilde{v}_0\|_{L^2(\mathbb{R})} \rightarrow 0,$$

as  $n \rightarrow \infty$ . In particular, for  $n$  large enough, we have

$$\|u_0^n\|_{L^2(\mathbb{R})} + \|v_0^n\|_{L^2(\mathbb{R})} < R.$$

There exists a unique maximal solution  $(u^n, v^n)$  of system (3.4), and we may assume that for  $n$  large enough,  $(u^n, v^n)$  is defined on  $[-T, T]$ . Assume that  $T$  small enough such that

$$\|\tilde{u}\|_{L^\infty L^2 \cap L^4 L^\infty} + \|\tilde{v}\|_{L^\infty L^2 \cap L^4 L^\infty} + \sup_n (\|u^n\|_{L^\infty L^2 \cap L^4 L^\infty} + \|v^n\|_{L^\infty L^2 \cap L^4 L^\infty}) \leq 2R. \quad (3.16)$$

The functions  $(\tilde{u}, \tilde{v})$  are solutions of the following system

$$(\tilde{u}, \tilde{v}) = S(t)(\tilde{u}_0, \tilde{v}_0) - i \int_0^t S(t-s)(\tilde{Q}_1(\tilde{u}, \tilde{v}, \phi), \tilde{Q}_2(\tilde{u}, \tilde{v}, \phi)).$$

Similarly,  $(u^n, v^n)$  are solutions of the following system

$$(u^n, v^n) = S(t)(u_0^n, v_0^n) - i \int_0^t S(t-s)(\tilde{Q}_1(u^n, v^n, \phi), \tilde{Q}_2(u^n, v^n, \phi)).$$

Hence,

$$(u^n - \tilde{u}, v^n - \tilde{v}) = S(t)(u_0^n - \tilde{u}_0, v_0^n - \tilde{v}_0) - i \int_0^t S(t-s)(\tilde{Q}_1(u^n, v^n, \phi) - \tilde{Q}_1(\tilde{u}, \tilde{v}, \phi), \tilde{Q}_2(u^n, v^n, \phi) - \tilde{Q}_2(\tilde{u}, \tilde{v}, \phi)).$$

Using Strichartz estimates and (3.16), for all  $t \in [-T, T]$  and  $R, T$  small enough, we have

$$\begin{aligned} & \|u^n - \tilde{u}\|_{L^\infty L^2 \cap L^4 L^\infty} + \|v^n - \tilde{v}\|_{L^\infty L^2 \cap L^4 L^\infty} \\ & \lesssim \|u_0^n - \tilde{u}_0\|_{L^2} + \|v_0^n - \tilde{v}_0\|_{L^2} \\ & \quad + \|\tilde{Q}_1(u^n, v^n, \phi) - \tilde{Q}_1(\tilde{u}, \tilde{v}, \phi)\|_{L^1 L^2} + \|\tilde{Q}_2(u^n, v^n, \phi) - \tilde{Q}_2(\tilde{u}, \tilde{v}, \phi)\|_{L^1 L^2} \\ & \lesssim \|u_0^n - \tilde{u}_0\|_{L^2} + \|v_0^n - \tilde{v}_0\|_{L^2} \\ & \quad + R(\|u^n - \tilde{u}\|_{L^\infty L^2 \cap L^4 L^\infty} + \|v^n - \tilde{v}\|_{L^\infty L^2 \cap L^4 L^\infty}). \end{aligned}$$

For  $R < \frac{1}{2}$  small enough, we have

$$\frac{1}{2}(\|u^n - \tilde{u}\|_{L^\infty L^2 \cap L^4 L^\infty} + \|v^n - \tilde{v}\|_{L^\infty L^2 \cap L^4 L^\infty}) \leq \|u_0^n - \tilde{u}_0\|_{L^2(\mathbb{R})} + \|v_0^n - \tilde{v}_0\|_{L^2(\mathbb{R})}.$$

Letting  $n \rightarrow +\infty$  we obtain the desired result.  $\square$

By similar arguments we obtain the following result in higher regularity.

**Proposition 3.5.** *Consider the system (3.4). Let  $\phi \in X^4(\mathbb{R})$  and  $\tilde{u}_0, \tilde{v}_0 \in H^2(\mathbb{R})$  such that  $\|\tilde{u}_0\|_{H^2(\mathbb{R})} + \|\tilde{v}_0\|_{H^2(\mathbb{R})} < R$  small enough. Then, there exist  $T = T(R)$  and a unique solution  $\tilde{u}, \tilde{v} \in C([-T, T], H^2(\mathbb{R})) \cap L^4([-T, T], W^{2,\infty}(\mathbb{R}))$ .*

**3.2.2. Preservation of a differential identity.** By Proposition 3.3 the solutions obtained by Proposition 3.5 satisfy the following property.

**Proposition 3.6.** *Let  $\phi \in X^4(\mathbb{R})$ ,  $\tilde{u}_0, \tilde{v}_0 \in H^2(\mathbb{R})$  such that*

$$\tilde{v}_0 = \partial \tilde{u}_0 + \frac{i}{2}(\tilde{u}_0 + \phi)(|\tilde{u}_0 + \phi|^2 - |\phi|^2) + \partial \phi.$$

*Then the associated solutions  $\tilde{u}, \tilde{v}$  of (3.4) satisfy the following condition for all  $t \in [-T, T]$*

$$\tilde{v} = \partial \tilde{u} + \frac{i}{2}(\tilde{u} + \phi)(|\tilde{u} + \phi|^2 - |\phi|^2) + \partial \phi.$$

**3.2.3. From the system to the equation.** In this section, we prove Theorem 1.3 by getting back to the equation.

*Proof of Theorem 1.3.* Let  $\phi \in X^4(\mathbb{R})$  such that  $\|\partial\phi\|_{L^2}$  is small enough. Let  $u_0 \in \phi + H^1(\mathbb{R})$  be such that  $\|u_0 - \phi\|_{H^1}$  small enough. Set  $v_0 = \partial u_0 + \frac{i}{2}u_0(|u_0|^2 - |\phi|^2) + \phi$ ,  $\tilde{u}_0 = u_0 - \phi$  and  $\tilde{v}_0 = v_0 - \phi$ . We have

$$\tilde{v}_0 = \partial\tilde{u}_0 + \frac{i}{2}(\tilde{u}_0 + \phi)(|\tilde{u}_0 + \phi|^2 - |\phi|^2) + \partial\phi.$$

Furthermore,  $\tilde{u}_0 \in H^1(\mathbb{R})$ ,  $\tilde{v}_0 \in L^2(\mathbb{R})$ . We have

$$\|\tilde{u}_0\|_{L^2(\mathbb{R})} + \|\tilde{v}_0\|_{L^2(\mathbb{R})} \lesssim \|\tilde{u}_0\|_{H^1(\mathbb{R})} + \|\partial\phi\|_{L^2},$$

which is small enough by the assumption. By Proposition 3.4, there exists  $T > 0$  and a unique solution  $\tilde{u}, \tilde{v} \in C([-T, T], L^2(\mathbb{R})) \cap L^4([-T, T], L^\infty)$  of the system (3.4). Let  $u_0^n \in H^3(\mathbb{R})$  be such that  $\|u_0^n\|_{H^3(\mathbb{R})}$  small enough and  $\|u_0^n - \tilde{u}_0\|_{H^1(\mathbb{R})} \rightarrow 0$  as  $n \rightarrow +\infty$ . Set

$$v_0^n = \partial u_0^n + \frac{i}{2}(u_0^n + \phi)(|u_0^n + \phi|^2 - |\phi|^2) + \partial\phi.$$

We have

$$\|\tilde{u}_0^n\|_{H^2(\mathbb{R})} + \|\tilde{v}_0^n\|_{H^2(\mathbb{R})} \lesssim \|\tilde{u}_0\|_{H^3(\mathbb{R})} + \|\partial\phi\|_{H^2},$$

which is small enough by the assumption. Let  $(u^n, v^n)$  be the  $H^2(\mathbb{R})$  solution of the system (3.4) obtained by Proposition 3.5 with data  $(u_0^n, v_0^n)$ . By Proposition 3.6 we have

$$v^n = \partial u^n + \frac{i}{2}(u^n + \phi)(|u^n + \phi|^2 - |\phi|^2) + \partial\phi. \quad (3.17)$$

Furthermore, we have

$$\|u_0^n - \tilde{u}_0\|_{L^2(\mathbb{R})} + \|v_0^n - \tilde{v}_0\|_{L^2(\mathbb{R})} \rightarrow 0.$$

From the continuous dependence on the initial data obtained in Proposition 3.4,  $(u^n, v^n), (\tilde{u}, \tilde{v})$  are solutions of the system (3.4) on  $[-T, T]$  for  $n$  large enough, and

$$\|u^n - \tilde{u}\|_{L^\infty L^2 \cap L^4 L^\infty} + \|v^n - \tilde{v}\|_{L^\infty L^2 \cap L^4 L^\infty} \rightarrow 0$$

as  $n \rightarrow \infty$ . Let  $n \rightarrow \infty$  on the two sides of (3.17), we obtain for all  $t \in [-T, T]$

$$\tilde{v} = \partial\tilde{u} + \frac{i}{2}(\tilde{u} + \phi)(|\tilde{u} + \phi|^2 - |\phi|^2) + \partial\phi, \quad (3.18)$$

which make sense in  $H^{-1}(\mathbb{R})$ . From (3.18) we see that  $\partial\tilde{u} \in C([-T, T], L^2(\mathbb{R}))$  and (3.18) makes sense in  $L^2(\mathbb{R})$ . Then  $\tilde{u} \in C([-T, T], H^1(\mathbb{R})) \cap L^4([-T, T], L^\infty)$ . By the Sobolev embedding of  $H^1(\mathbb{R})$  in  $L^\infty(\mathbb{R})$  we obtain that

$$\begin{aligned} \|(\tilde{u} + \phi)(|\tilde{u} + \phi|^2 - |\phi|^2)\|_{L^4 L^\infty} &\lesssim \| |\tilde{u} + \phi| |\tilde{u}| |\tilde{u}| + |\phi| \| \lesssim \|\tilde{u}\|_{L^4 L^\infty} (\|\tilde{u}\|_{L^\infty L^\infty} + \|\phi\|_{L^\infty L^\infty})^2 \\ &\lesssim \|\tilde{u}\|_{L^4 L^\infty} (\|\tilde{u}\|_{L^\infty H^1} + \|\phi\|_{L^\infty L^\infty})^2 < \infty. \end{aligned}$$

Hence,  $(\tilde{u} + \phi)(|\tilde{u} + \phi|^2 - |\phi|^2) \in L^4 L^\infty$ . From (3.18) we obtain that  $\partial\tilde{u} \in L^4 L^\infty$  which implies  $\tilde{u} \in L^4([-T, T], W^{1, \infty}(\mathbb{R}))$ . Set  $u = \tilde{u} + \phi$ ,  $v = \tilde{v} + \phi$ , then  $u - \phi \in C([-T, T], H^1(\mathbb{R})) \cap L^4([-T, T], W^{1, \infty}(\mathbb{R}))$  and  $v - \phi \in C([-T, T], L^2(\mathbb{R})) \cap L^4([-T, T], L^\infty(\mathbb{R}))$ . Moreover,

$$v = \partial u + \frac{i}{2}u(|u|^2 - |\phi|^2) + \phi.$$

Since  $u, v$  solve (3.2), we have

$$Lu = Q_1(u, v, \phi) = Q_1\left(u, \partial u + \frac{i}{2}u(|u|^2 - |\phi|^2) + \phi, \phi\right) = -iu^2\partial\bar{u}.$$

The existence of a solution of the equation (1.1) follows. To prove the uniqueness property, assume that  $U \in C([-T, T], \phi + H^1(\mathbb{R})) \cap L^4([-T, T], \phi + W^{1, \infty}(\mathbb{R}))$  is an other solution of the equation (1.1). Set  $V = \partial U + \frac{i}{2}U(|U|^2 - |\phi|^2) + \phi$  and  $\tilde{U} = U - \phi$ ,  $\tilde{V} = V - \phi$ . Hence  $\tilde{U} \in C([-T, T], H^1(\mathbb{R})) \cap L^4([-T, T], W^{1, \infty}(\mathbb{R}))$  and  $\tilde{V} \in C([-T, T], L^2(\mathbb{R})) \cap L^4([-T, T], L^\infty(\mathbb{R}))$ . Moreover,  $\tilde{U}, \tilde{V}$  is solution of the system (3.4). By the uniqueness of solutions of (3.4), we obtain that  $\tilde{U} = \tilde{u}$ . Hence,  $u = U$ , which complete the proof.  $\square$

*Remark 3.7.* In [6], Gérard gives the proof of local well posedness of solutions of the Gross-Pitaevskii in energy space, using some properties of the energy space. More precisely, he proved that there exists a unique maximal solution  $u \in C((-T_{min}, T_{max}), \mathcal{E})$  of the problem

$$u(t) = S(t)u_0 - i \int_0^t S(t-s)(u(s)(|u(s)|^2 - 1)) ds, \quad (3.19)$$

where  $S(t)$  is the Schrödinger group,  $u_0 \in \mathcal{E}$  is given and  $\mathcal{E}$  is the energy space which is defined by

$$\mathcal{E} := \{u \in H_{loc}^1(\mathbb{R}) : \partial u \in L^2(\mathbb{R}), |u|^2 - 1 \in L^2(\mathbb{R})\}.$$

The proof of Gérard is in dimension 2 and 3. We can give a simple proof of this result in one dimension. Indeed, we see that  $u_0 \in \mathcal{E} \subset X^1(\mathbb{R})$ , then, it is easy to prove that there exists a unique maximal solution  $u \in C((-T_{min}, T_{max}), X^1(\mathbb{R}))$ . Set

$$w(t, x) = u(t, x) - u_0(x).$$

Consider the following problem

$$w(t) = -i \int_0^t S(t-s)((u_0 + w(s))(|u_0 + w(s)|^2 - 1)) ds. \quad (3.20)$$

We can check that the function  $P : H^1(\mathbb{R}) \rightarrow H^1(\mathbb{R})$  defined by

$$P(w) = (u_0 + w)(|u_0 + w|^2 - 1),$$

is Lipschitz continuous on bounded set of  $H^1(\mathbb{R})$ . Thus, by elementary arguments, there exists unique maximal solution  $w \in C((-T_{min}, T_{max}), H^1(\mathbb{R}))$  of the Cauchy problem (3.20). It implies that there exists a unique maximal solution  $u \in C((-T_{min}, T_{max}), u_0 + H^1(\mathbb{R}))$  of the Cauchy problem (3.19). Using the fact that

$$u_0 + H^1(\mathbb{R}) \subset \mathcal{E} \subset X^1(\mathbb{R}),$$

we obtain that there exists a unique maximal solution  $u \in C((-T_{min}, T_{max}), \mathcal{E})$  of the Cauchy problem (3.19).

#### 4. CONSERVATION OF THE MASS, THE ENERGY AND THE MOMENTUM

In this section, we prove Theorem 1.4. Let  $q_0 \in \mathbb{R}$  and  $u \in q_0 + H^2(\mathbb{R})$  be a solution of (1.1). Let  $\chi$  and  $\chi_R$  be the functions defined as in (1.5) and (1.6). We have

$$\|\partial \chi_R\|_{L^2(\mathbb{R})} = \left( \int_{\mathbb{R}} \frac{1}{R} \chi' \left( \frac{x}{R} \right) \right)^{\frac{1}{2}} = \frac{1}{R^{\frac{1}{2}}} \|\chi\|_{L^2(\mathbb{R})} \rightarrow 0 \text{ as } R \rightarrow +\infty. \quad (4.1)$$

By the continuous depend on initial data property of solution, we can assume that

$$u \in C((-T_{min}, T_{max}), q_0 + H^3(\mathbb{R})).$$

It is enough to prove (1.7), (1.8) and (1.9) for any closed interval  $[-T_0, T_1] \in (-T_{min}, T_{max})$ . Let  $T_0 > 0, T_1 > 0$  be such that  $[-T_0, T_1] \subset (-T_{min}, T_{max})$ . Let  $M > 0$  be defined by

$$M = \sup_{t \in [-T_0, T_1]} \|u - q_0\|_{H^2(\mathbb{R})}.$$

**4.1. Conservation of mass.** Multiply the two sides of (1.1) with  $\bar{u}$  and take imaginary part to obtain

$$\operatorname{Re}(u_t \bar{u}) + \operatorname{Im}(\partial^2 u \bar{u}) + \operatorname{Re}(|u|^2 u \partial \bar{u}) = 0.$$

This implies that

$$\begin{aligned} 0 &= \frac{1}{2} \partial_t (|u|^2) + \partial(\operatorname{Im}(\partial u \bar{u})) + \frac{1}{4} \partial(|u|^4) \\ &= \frac{1}{2} \partial_t (|u|^2 - q_0^2) + \partial(\operatorname{Im}(\partial u \bar{u})) + \frac{1}{4} \partial(|u|^4 - q_0^4). \end{aligned}$$

By multiplying the two sides with  $\chi_R$ , integrating on space, and integrating by part we have

$$0 = \partial_t \int_{\mathbb{R}} \frac{1}{2}(|u|^2 - q_0^2) \chi_R dx - \int_{\mathbb{R}} \mathcal{I}m(\partial u \bar{u}) \partial \chi_R - \int_{\mathbb{R}} \frac{(|u|^4 - q_0^4)}{4} \partial \chi_R dx \quad (4.2)$$

$$= \partial_t \int_{\mathbb{R}} \frac{1}{2}(|u|^2 - q_0^2) \chi_R dx - \int_{\mathbb{R}} \left( \mathcal{I}m(\partial u \bar{u}) + \frac{1}{4}(|u|^4 - q_0^4) \right) \partial \chi_R dx. \quad (4.3)$$

Denote the second term of (4.3) by  $K$ , using (4.1), we have

$$|K| \leq \|\mathcal{I}m(\partial u \bar{u}) + \frac{1}{4}(|u|^4 - q_0^4)\|_{L^2} \|\partial \chi_R\|_{L^2} \lesssim C(M) \frac{1}{R^{\frac{1}{2}}} \rightarrow 0 \text{ as } R \rightarrow \infty.$$

Thus, by integrating from 0 to  $t$  and taking  $R$  to infinity, using the assumption  $|u_0|^2 - q_0^2 \in L^1(\mathbb{R})$  we obtain

$$\lim_{R \rightarrow \infty} \int_{\mathbb{R}} \frac{1}{2}(|u|^2 - q_0^2) \chi_R dx = \lim_{R \rightarrow \infty} \int_{\mathbb{R}} \frac{1}{2}(|u_0|^2 - q_0^2) \chi_R dx = \frac{1}{2} \int_{\mathbb{R}} (|u_0|^2 - q_0^2) dx. \quad (4.4)$$

Thus, we obtain the conservation of the mass (1.7).

**4.2. Conservation of energy.** Now, we prove the conservation of the energy. Since  $u$  solves (1.1), after elementary calculations, we have

$$\partial_t(|\partial u|^2) = \partial \left( 2\mathcal{R}e(\partial u \partial_t \bar{u}) + \mathcal{R}e(u^2(\partial \bar{u})^2) - |\partial u|^2|u|^2 - |u|^4 \mathcal{I}m(\bar{u} \partial u) \right) + |u|^4 \partial \mathcal{I}m(\bar{u} \partial u) + 2\mathcal{I}m(|u|^2 \partial \bar{u} u_t). \quad (4.5)$$

Recall that we have

$$\partial \mathcal{I}m(\partial u \bar{u}) = -\frac{1}{2} \partial_t(|u|^2) - \frac{1}{4} \partial(|u|^4). \quad (4.6)$$

Moreover, we have

$$\partial_t \mathcal{I}m(|u|^2 u \partial \bar{u}) = 4\mathcal{I}m(u_t |u|^2 \partial \bar{u}) + \partial \mathcal{I}m(|u|^2 u \partial_t \bar{u}).$$

It follows that

$$2\mathcal{I}m(|u|^2 u_t \partial \bar{u}) = \frac{1}{2} \left( \partial_t \mathcal{I}m(|u|^2 u \partial \bar{u}) - \partial \mathcal{I}m(|u|^2 u \partial_t \bar{u}) \right). \quad (4.7)$$

From (4.5), (4.6) and (4.7) we have

$$\begin{aligned} \partial_t(|\partial u|^2) &= \partial \left( 2\mathcal{R}e(\partial u \partial_t \bar{u}) + \mathcal{R}e(u^2(\partial \bar{u})^2) - |\partial u|^2|u|^2 - |u|^4 \mathcal{I}m(\partial u \bar{u}) - \frac{1}{2} \mathcal{I}m(|u|^2 u \partial_t \bar{u}) \right) \\ &\quad + \frac{1}{2} \partial_t \mathcal{I}m(|u|^2 u \partial \bar{u}) - \frac{1}{8} \partial(|u|^8) - \frac{1}{6} \partial_t(|u|^6). \end{aligned}$$

Hence,

$$\begin{aligned} &\partial_t \left( |\partial u|^2 - \frac{1}{2} \mathcal{I}m((|u|^2 u - q_0^3) \partial \bar{u}) + \frac{1}{6}(|u|^6 - q_0^6) \right) \\ &= \partial \left( 2\mathcal{R}e(\partial u \partial_t \bar{u}) + \mathcal{R}e(u^2(\partial \bar{u})^2) - |\partial u|^2|u|^2 - |u|^4 \mathcal{I}m(\partial u \bar{u}) - \frac{1}{2} \mathcal{I}m(|u|^2 u \partial_t \bar{u}) - \frac{1}{8}(|u|^8 - q_0^8) \right) + \frac{1}{2} q_0^3 \mathcal{I}m \partial_t \overline{\partial(u - q_0)}. \end{aligned}$$

By multiplying the two sides with  $\chi_R$  then integrating in space and integrating by part we obtain

$$\begin{aligned} &\partial_t \int_{\mathbb{R}} \left( |\partial u|^2 - \frac{1}{2} \mathcal{I}m((|u|^2 u - q_0^3) \partial \bar{u}) + \frac{1}{6}(|u|^6 - q_0^6) \right) \chi_R dx \\ &= - \int_{\mathbb{R}} \partial \chi_R \left( 2\mathcal{R}e(\partial u \partial_t \bar{u}) + \mathcal{R}e(u^2(\partial \bar{u})^2) - |\partial u|^2|u|^2 - |u|^4 \mathcal{I}m(\partial u \bar{u}) - \frac{1}{2} \mathcal{I}m(|u|^2 u \partial_t \bar{u}) - \frac{1}{8}(|u|^8 - q_0^8) \right) dx \\ &\quad - \frac{q_0^3}{2} \mathcal{I}m \partial_t \int_{\mathbb{R}} \overline{(u - q_0)} \partial \chi_R dx. \end{aligned}$$

Integrating from 0 to  $t$  we obtain

$$\int_{\mathbb{R}} \left( |\partial u|^2 - \frac{1}{2} \mathcal{I}m((|u|^2 u - q_0^3) \partial \bar{u}) + \frac{1}{6} (|u|^6 - q_0^6) \right) \chi_R dx \quad (4.8)$$

$$- \int_{\mathbb{R}} \left( |\partial u_0|^2 - \frac{1}{2} \mathcal{I}m((|u_0|^2 u_0 - q_0^3) \partial \bar{u}_0) + \frac{1}{6} (|u_0|^6 - q_0^6) \right) \chi_R dx \quad (4.9)$$

$$\begin{aligned} &= \int_0^t \int_{\mathbb{R}} \partial \chi_R (2 \mathcal{R}e(\partial u \partial_t \bar{u}) + \mathcal{R}e(u^2 (\partial \bar{u})^2) - |u|^2 |\partial u|^2 - |u|^4 \mathcal{I}m(\partial u \bar{u})) \\ &\quad - \frac{1}{2} \mathcal{I}m(|u|^2 u \partial_t \bar{u}) - \frac{1}{8} (|u|^8 - q_0^8) \Big) dx ds \end{aligned} \quad (4.10)$$

$$- \frac{q_0^3}{2} \left( \mathcal{I}m \int_{\mathbb{R}} \overline{(u - q_0)} \partial \chi_R dx - \mathcal{I}m \int_{\mathbb{R}} \overline{(u_0 - q_0)} \partial \chi_R dx \right) \quad (4.11)$$

Denoting the term (4.10) by  $A$ , using (4.1), we have

$$|A| \leq \|\partial \chi_R\|_{L^2} \|2 \mathcal{R}e(\partial u \partial_t \bar{u}) + \mathcal{R}e(u^2 (\partial \bar{u})^2) - |u|^2 |\partial u|^2 - |u|^4 \mathcal{I}m(\partial u \bar{u})\|_{L^2} \quad (4.12)$$

$$- \frac{1}{2} \mathcal{I}m(|u|^2 u \partial_t \bar{u}) - \frac{1}{8} (|u|^8 - q_0^8) \|_{L^2} \quad (4.13)$$

$$\lesssim C(M) \|\partial \chi_R\|_{L^2} \rightarrow 0 \text{ as } R \rightarrow \infty. \quad (4.14)$$

Moreover, using (4.1) again, we have

$$\left| \mathcal{I}m \int_{\mathbb{R}} \overline{(u - q_0)} \partial \chi_R dx \right| \leq \|u - q_0\|_{L^2} \|\partial \chi_R\|_{L^2} \lesssim C(M) \|\partial \chi_R\|_{L^2} \rightarrow 0 \text{ as } R \rightarrow \infty. \quad (4.15)$$

$$\left| \mathcal{I}m \int_{\mathbb{R}} \overline{(u_0 - q_0)} \partial \chi_R dx \right| \leq \|u_0 - q_0\|_{L^2} \|\partial \chi_R\|_{L^2} \lesssim C(M) \|\partial \chi_R\|_{L^2} \rightarrow 0 \text{ as } R \rightarrow \infty. \quad (4.16)$$

To deal with the term (4.8), we need to divide into two terms. First, using  $u \in q_0 + H^3(\mathbb{R})$ , as  $R \rightarrow \infty$ , we have

$$\int_{\mathbb{R}} \left( |\partial u|^2 - \frac{1}{2} \mathcal{I}m((|u|^2 u - q_0^3) \partial \bar{u}) \right) \chi_R dx \rightarrow \int_{\mathbb{R}} \left( |\partial u|^2 - \frac{1}{2} \mathcal{I}m((|u|^2 u - q_0^3) \partial \bar{u}) \right) dx. \quad (4.17)$$

Second, by easy calculations, we have

$$\frac{1}{6} \int_{\mathbb{R}} (|u|^6 - q_0^6) \chi_R dx \quad (4.18)$$

$$= \frac{1}{6} \int_{\mathbb{R}} [(|u|^2 - q_0^2)(|u|^4 + q_0^2 |u_0|^2 - 2q_0^4) + 3q_0^4 (|u|^2 - q_0^2)] \chi_R dx$$

$$= \frac{1}{6} \int_{\mathbb{R}} (|u|^2 - q_0^2)^2 (|u|^2 + 2q_0^2) \chi_R dx \quad (4.19)$$

$$+ \frac{1}{2} \int_{\mathbb{R}} (|u|^2 - q_0^2) \chi_R dx. \quad (4.20)$$

Denote the term (4.19) by  $B$ , we have

$$B \rightarrow \frac{1}{6} \int_{\mathbb{R}} (|u|^2 - q_0^2)(|u|^4 + q_0^2 |u_0|^2 - 2q_0^4) dx \text{ as } R \rightarrow +\infty. \quad (4.21)$$

The term (4.20) is treated using conservation of mass (4.4). Finally, we have

$$\lim_{R \rightarrow \infty} \frac{1}{6} \int_{\mathbb{R}} (|u|^6 - q_0^6) \chi_R dx = \frac{1}{6} \int_{\mathbb{R}} (|u|^2 - q_0^2)^2 (|u|^2 + 2q_0^2) dx + \frac{q_0^4}{2} \lim_{R \rightarrow \infty} \int_{\mathbb{R}} (|u_0|^2 - q_0^2) \chi_R dx. \quad (4.22)$$

Combining (4.22), (4.17) we have

$$\begin{aligned} \lim_{R \rightarrow \infty} (\text{the term (4.8)}) &= \int_{\mathbb{R}} |\partial u|^2 - \frac{1}{2} \mathcal{I}m(|u|^2 u - q_0^3) \partial \bar{u}) dx + \frac{1}{6} \int_{\mathbb{R}} (|u|^2 - q_0^2)^2 (|u|^2 + 2q_0^2) dx \\ &\quad + \frac{1}{2} \lim_{R \rightarrow \infty} \int_{\mathbb{R}} (|u_0|^2 - q_0^2) \chi_R dx. \end{aligned} \quad (4.23)$$

Similarly,

$$\begin{aligned} \lim_{R \rightarrow \infty} (\text{the term (4.9)}) &= \int_{\mathbb{R}} |\partial u_0|^2 - \frac{1}{2} \mathcal{I}m(|u_0|^2 u_0 - q_0^3) \partial \bar{u}_0) dx + \frac{1}{6} \int_{\mathbb{R}} (|u_0|^2 - q_0^2)^2 (|u_0|^2 + 2q_0^2) dx \\ &\quad + \frac{1}{2} \lim_{R \rightarrow \infty} \int_{\mathbb{R}} (|u_0|^2 - q_0^2) \chi_R dx. \end{aligned} \quad (4.24)$$

Combined (4.8)-(4.16), (4.23) and (4.24), we have

$$\begin{aligned} &\int_{\mathbb{R}} |\partial u|^2 - \frac{1}{2} \mathcal{I}m(|u|^2 u - q_0^3) \partial \bar{u}) dx + \frac{1}{6} \int_{\mathbb{R}} (|u|^2 - q_0^2)^2 (|u|^2 + 2q_0^2) dx \\ &\quad - \int_{\mathbb{R}} |\partial u_0|^2 - \frac{1}{2} \mathcal{I}m(|u_0|^2 u_0 - q_0^3) \partial \bar{u}_0) dx + \frac{1}{6} \int_{\mathbb{R}} (|u_0|^2 - q_0^2)^2 (|u_0|^2 + 2q_0^2) dx \\ &= 0 \end{aligned}$$

This implies (1.8).

**4.3. Conservation of momentum.** Now, we prove (1.9). Multiplying the two sides of (1.1) with  $-\partial \bar{u}$  and taking real part we obtain

$$\begin{aligned} 0 &= -\mathcal{R}e(iu_t \partial \bar{u} + \partial^2 u \partial \bar{u} + iu^2 (\partial \bar{u})^2) \\ &= \mathcal{I}m(u_t \partial \bar{u}) + \mathcal{I}m(u^2 (\partial \bar{u})^2) - \frac{1}{2} \partial (|\partial u|^2). \end{aligned} \quad (4.25)$$

Moreover, by elementary calculation, we have

$$\partial_t \mathcal{I}m(u \partial \bar{u}) = 2\mathcal{I}m(u_t \partial \bar{u}) + \partial \mathcal{I}m(u \partial_t \bar{u}).$$

Replacing  $\mathcal{I}m(u_t \partial \bar{u}) = \frac{1}{2} (\partial_t \mathcal{I}m(u \partial \bar{u}) - \partial \mathcal{I}m(u \partial_t \bar{u}))$  in (4.25), we obtain that

$$\begin{aligned} 0 &= \left( \frac{1}{2} \partial_t \mathcal{I}m(u \partial \bar{u}) - \frac{1}{2} \partial \mathcal{I}m(u \partial_t \bar{u}) \right) + 2\mathcal{R}e(u \partial \bar{u}) \mathcal{I}m(u \partial \bar{u}) - \frac{1}{2} \partial (|\partial u|^2) \\ &= \partial_t \left[ \frac{1}{2} \mathcal{I}m(u \partial \bar{u}) - \frac{1}{4} (|u|^4 - q_0^4) \right] + \partial \left[ \mathcal{I}m(|u|^2 u \partial \bar{u}) - \frac{1}{2} |\partial u|^2 - \frac{1}{6} (|u|^6 - q_0^6) \right]. \end{aligned}$$

Multiply two sides by  $\chi_R$  and integral on space, using integral by part, we have

$$\begin{aligned} 0 &= \partial_t \int_{\mathbb{R}} \left[ \frac{1}{2} \mathcal{I}m(u \partial \bar{u}) - \frac{1}{4} (|u|^4 - q_0^4) \right] \chi_R dx - \int_{\mathbb{R}} \left[ \mathcal{I}m(|u|^2 u \partial \bar{u}) - \frac{1}{2} |\partial u|^2 - \frac{1}{6} (|u|^6 - q_0^6) \right] \partial \chi_R dx \\ &= \partial_t \int_{\mathbb{R}} \left[ \frac{1}{2} \mathcal{I}m(u \partial \bar{u}) - \frac{1}{4} (|u|^2 - q_0^2)^2 - \frac{1}{2} q_0^2 (|u|^2 - q_0^2) \right] \chi_R dx - \int_{\mathbb{R}} \left[ \mathcal{I}m(|u|^2 u \partial \bar{u}) - \frac{1}{2} |\partial u|^2 - \frac{1}{6} (|u|^6 - q_0^6) \right] \partial \chi_R dx. \end{aligned} \quad (4.26)$$

Denoting the second term of (4.26) by  $D$ , we have

$$|D| \leq \left\| \mathcal{I}m(|u|^2 u \partial \bar{u}) - \frac{1}{2} |\partial u|^2 - \frac{1}{6} (|u|^6 - q_0^6) \right\|_{L^2} \|\partial \chi_R\|_{L^2} \lesssim C(M) \|\partial \chi_R\|_{L^2} \rightarrow 0 \text{ as } R \rightarrow \infty. \quad (4.27)$$

Integrating from 0 to  $t$  the two sides of (4.26) and taking  $R$  to infinity, using (4.27), we have

$$\int_{\mathbb{R}} \left[ \frac{1}{2} \mathcal{I}m(u \partial \bar{u}) - \frac{1}{4} (|u|^2 - q_0^2)^2 \right] dx - \frac{q_0^2}{2} \lim_{R \rightarrow \infty} \int_{\mathbb{R}} (|u|^2 - q_0^2) \chi_R dx \quad (4.28)$$

$$= \int_{\mathbb{R}} \left[ \frac{1}{2} \mathcal{I}m(u_0 \partial \bar{u}_0) - \frac{1}{4} (|u_0|^2 - q_0^2)^2 \right] dx - \frac{q_0^2}{2} \lim_{R \rightarrow \infty} \int_{\mathbb{R}} (|u_0|^2 - q_0^2) \chi_R dx. \quad (4.29)$$

Combined (4.29) and (4.4), we obtain the conservation of momentum (1.9), which completes the proof of Theorem 1.4.

## 5. STATIONARY SOLUTIONS

In this section, we give the proof of Theorem 1.6. We start by the following definition of stationary solutions of (1.1).

**Definition 5.1.** The stationary solutions of (1.1) are functions  $\phi \in X^2(\mathbb{R})$  satisfying

$$\phi_{xx} + i\phi^2 \bar{\phi}_x = 0. \quad (5.1)$$

*Proof of Theorem 1.6.* Let  $\phi$  be a non constant solution of (5.1) such that  $m = \inf_{x \in \mathbb{R}} |\phi(x)| > 0$ . From (5.1), we have  $\phi \in X^3(\mathbb{R})$ . Using the assumptions on  $\phi$  we can write  $\phi$  as

$$\phi(x) = R(x) e^{i\theta(x)}$$

where  $R > 0$  and  $R, \theta \in C^2(\mathbb{R})$  are real-valued functions. We have

$$\begin{aligned} \phi_x &= e^{i\theta} (R_x + i\theta_x R), \\ \phi_{xx} &= e^{i\theta} (R_{xx} + 2iR_x \theta_x + iR\theta_{xx} - R\theta_x^2). \end{aligned}$$

Hence, since  $\phi$  satisfies (5.1) we obtain

$$0 = (R_{xx} - R\theta_x^2 + R^3\theta_x) + i(2R_x\theta_x + R\theta_{xx} + R^2R_x).$$

It is equivalent to

$$0 = R_{xx} - R\theta_x^2 + R^3\theta_x, \quad (5.2)$$

$$0 = 2R_x\theta_x + R\theta_{xx} + R^2R_x. \quad (5.3)$$

The equation (5.3) is equivalent to

$$0 = \partial_x \left( R^2\theta_x + \frac{1}{4}R^4 \right).$$

Hence there exists  $B \in \mathbb{R}$  such that

$$B = R^2\theta_x + \frac{1}{4}R^4. \quad (5.4)$$

This implies

$$\theta_x = \frac{B}{R^2} - \frac{R^2}{4}. \quad (5.5)$$

Substituting the above equality in (5.2) we obtain

$$\begin{aligned} 0 &= R_{xx} - R \left( \frac{B}{R^2} - \frac{R^2}{4} \right)^2 + R^3 \left( \frac{B}{R^2} - \frac{R^2}{4} \right) \\ &= R_{xx} - \frac{B^2}{R^3} - \frac{5R^5}{16} + \frac{3BR}{2}. \end{aligned} \quad (5.6)$$

We prove that the set  $V = \{x \in \mathbb{R} : R_x(x) \neq 0\}$  is dense in  $\mathbb{R}$ . Indeed, assume there exists  $x \in \mathbb{R} \setminus \bar{V}$ . Thus, there exists  $\varepsilon$  such that  $B(x, \varepsilon) \in \mathbb{R} \setminus \bar{V}$ . It implies that for all  $y \in B(x, \varepsilon)$ , we have  $R_x(y) = 0$ . Hence,  $R$  is a constant function on  $B(x, \varepsilon)$ . By uniqueness of  $C^2$  solution of (5.6), we have  $R$  is constant function on  $\mathbb{R}$ . By (5.5),  $\theta_x$  is constant. Thus,  $\phi(x)$  is of form  $Ce^{ikx}$ , for some constants

$C, K \in \mathbb{R}$ . If  $k = 0$  it is a constant and if  $k \neq 0$  it is not in  $X^1(\mathbb{R})$ , which contradicts the assumption of  $\phi$ . From (5.6), we have

$$\begin{aligned} 0 &= R_x \left( R_{xx} - \frac{B^2}{R^3} - \frac{5R^5}{16} + \frac{3BR}{2} \right) \\ &= \frac{d}{dx} \left[ \frac{1}{2}R_x^2 + \frac{B^2}{2R^2} - \frac{5}{96}R^6 + \frac{3B}{4}R^2 \right]. \end{aligned}$$

Hence there exists  $a \in \mathbb{R}$  such that

$$a = \frac{1}{2}R_x^2 + \frac{B^2}{2R^2} - \frac{5}{96}R^6 + \frac{3B}{4}R^2.$$

It is equivalent to

$$\begin{aligned} 0 &= R_x^2 R^2 + B^2 - \frac{5}{48}R^8 + \frac{3B}{2}R^4 - 2aR^2, \\ &= \frac{1}{4}[(R^2)_x]^2 + B^2 - \frac{5}{48}R^8 + \frac{3B}{2}R^4 - 2aR^2. \end{aligned}$$

Set  $k = R^2$ . We have

$$0 = \frac{1}{4}k_x^2 + B^2 - \frac{5}{48}k^4 + \frac{3B}{2}k^2 - 2ak. \quad (5.7)$$

Differentiating the two sides of (5.7) we have

$$0 = k_x \left( \frac{k_{xx}}{2} - \frac{5}{12}k^3 + 3Bk - 2a \right)$$

On the other hand, since  $k_x = 2R_x R \neq 0$  for a.e  $x$  in  $\mathbb{R}$ , we obtain the following equation for a.e  $x$  in  $\mathbb{R}$ , hence, by continuity of  $k$ , it is true for all  $x$  in  $\mathbb{R}$ :

$$0 = \frac{k_{xx}}{2} - \frac{5}{12}k^3 + 3Bk - 2a. \quad (5.8)$$

Now, using Lemma 5.2 we have  $k - 2\sqrt{B} \in H^3(\mathbb{R})$ . Combining with (5.8) we obtain  $a = \frac{4B\sqrt{B}}{3}$ . Set  $h = k - 2\sqrt{B}$ . Then from (5.8)  $h \in H^3(\mathbb{R})$  solves

$$\begin{cases} 0 = h_{xx} - \frac{5}{6}h^3 - 5\sqrt{B}h^2 - 4Bh, \\ h > m^2 - 2\sqrt{B}. \end{cases} \quad (5.9)$$

The equation (5.9) can be explicitly solved, and we find

$$h = \frac{-1}{\sqrt{\frac{5}{72B}} \cosh(2\sqrt{B}x) + \frac{5}{12\sqrt{B}}}.$$

This implies

$$k = 2\sqrt{B} + h = 2\sqrt{B} + \frac{-1}{\sqrt{\frac{5}{72B}} \cosh(2\sqrt{B}x) + \frac{5}{12\sqrt{B}}}.$$

Furthermore, using  $\theta_x = \frac{B}{k} - \frac{k}{4}$ , there exists  $\theta_0 \in \mathbb{R}$  such that

$$\theta(x) = \theta_0 - \int_x^\infty \left( \frac{B}{k} - \frac{k}{4} \right) dy.$$

Now, assume that  $\phi$  is a solution of (5.1) such that  $\phi(\infty) = 0$ . We prove  $\phi \equiv 0$  on  $\mathbb{R}$ . Multiplying the two side of (5.1) with  $\bar{\phi}$  then taking the imaginary part we obtain

$$\partial_x \mathcal{I}m(\phi_x \bar{\phi}) + \frac{1}{4} \partial_x (|\phi|^4) = 0$$

On the other hand,  $\phi(\infty) = \phi_x(\infty) = 0$  then on  $\mathbb{R}$  we have

$$\mathcal{I}m(\phi_x \bar{\phi}) + \frac{1}{4} |\phi|^4 = 0. \quad (5.10)$$

If there exists  $y_0$  such that  $\phi_x(y_0) = 0$  then from (5.10) we have  $\phi(y_0) = 0$ . By the uniqueness of Cauchy problem we obtain  $\phi \equiv 0$  on  $\mathbb{R}$ . Otherwise,  $\phi_x$  is non vanishing on  $\mathbb{R}$ . From now on, we



will consider this case. Multiplying the two sides of (5.1) with  $\overline{\phi_x}$  then taking the real part, we have

$$\begin{aligned} 0 &= \operatorname{Re}(\phi_{xx}\overline{\phi_x}) - \operatorname{Im}(\phi^2\overline{\phi_x^2}) \\ &= \frac{1}{2} \frac{d}{dx} |\phi_x|^2 - 2\operatorname{Re}(\phi\overline{\phi_x})\operatorname{Im}(\phi\overline{\phi_x}) \\ &= \frac{1}{2} \frac{d}{dx} |\phi_x|^2 - \partial_x(|\phi|^2) \frac{1}{4} |\phi|^4 \\ &= \frac{d}{dx} \left( \frac{1}{2} |\phi_x|^2 - \frac{1}{12} |\phi|^6 \right). \end{aligned}$$

It implies that

$$|\phi_x|^2 - \frac{1}{6} |\phi|^6 = 0.$$

Hence, since  $\phi_x$  is non vanishing,  $\phi$  is also non vanishing on  $\mathbb{R}$ . We can write  $\phi = \rho e^{i\theta}$  for  $\rho > 0$ ,  $\rho, \theta \in C^2(\mathbb{R})$ . Replacing  $\phi = \rho e^{i\theta}$  in (5.1) we have

$$0 = (-\rho\theta_x^2 + \rho_{xx} + \rho^3\theta_x) + i(2\rho_x\theta_x + \rho\theta_{xx} + \rho^2\rho_x).$$

It implies that

$$0 = -\rho\theta_x^2 + \rho_{xx} + \rho^3\theta_x. \quad (5.11)$$

Replacing  $\phi = \rho e^{i\theta}$  in (5.10) we have

$$0 = \rho^2\theta_x + \frac{1}{4}\rho^4.$$

Then  $\theta_x = -\frac{1}{4}\rho^2$ , replacing this equality in (5.11) we obtain

$$0 = \rho_{xx} - \frac{5}{16}\rho^5.$$

Multiplying the two sides of the above equality with  $\rho_x$  we obtain

$$0 = \rho_{xx}\rho_x - \frac{5}{16}\rho^5\rho_x = \frac{d}{dx} \left( \frac{1}{2}\rho_x^2 - \frac{5}{96}\rho^6 \right).$$

Hence,

$$0 = \rho_x^2 - \frac{5}{48}\rho^6.$$

Moreover,  $\phi$  is non vanishing on  $\mathbb{R}$  then  $\rho > 0$  and then  $\rho_x$  is not change sign on  $\mathbb{R}$ . If  $\rho_x > 0$  then since  $\rho(\infty) = 0$  we have  $\rho < 0$  on  $\mathbb{R}$ , a contradiction. Hence,  $\rho_x < 0$  and  $\rho_x = -\sqrt{\frac{5}{48}}\rho^3$ . From this we easily check that

$$\rho^2(x) = \frac{1}{\rho(0)^2 + \sqrt{\frac{5}{12}}x},$$

which implies the contradiction, for the right hand side is not a continuous function on  $\mathbb{R}$ . This complete the proof.  $\square$

**Lemma 5.2.** *The following is true:*

$$k - 2\sqrt{B} \in L^2(\mathbb{R}), \quad k \in X^3(\mathbb{R}).$$

*Proof.* Using  $\phi \in L^\infty(\mathbb{R})$  we obtain  $k \in L^\infty(\mathbb{R})$ . On the other hand, since  $\phi \in X^3(\mathbb{R})$ , we have  $\phi_x \in L^2(\mathbb{R})$ ,  $\phi_{xx} \in L^2(\mathbb{R})$  and it easy to see that

$$\begin{aligned} |\phi_x|^2 &= \frac{k_x^2}{4k} + k\theta_x^2 \in L^1(\mathbb{R}), \\ |\phi_{xx}|^2 &= \left| \frac{k_x\theta_x}{\sqrt{k}} + \theta_{xx}\sqrt{k} \right|^2 + \left| \frac{k_{xx}}{2\sqrt{k}} - \sqrt{k}\theta_x^2 - \frac{k_x^2}{4k\sqrt{k}} \right|^2 \in L^1(\mathbb{R}). \end{aligned}$$

This implies

$$\begin{aligned} \frac{k_x}{2\sqrt{k}} &\in L^2(\mathbb{R}) \text{ and } \sqrt{k}\theta_x \in L^2(\mathbb{R}) \\ \frac{k_x\theta_x}{\sqrt{k}} + \theta_{xx}\sqrt{k} &\in L^2(\mathbb{R}) \text{ and } \frac{k_{xx}}{2\sqrt{k}} - \sqrt{k}\theta_x^2 - \frac{k_x^2}{4k\sqrt{k}} \in L^2(\mathbb{R}). \end{aligned}$$

Using  $\sqrt{m} < k < \|k\|_{L^\infty}$ ,  $\theta_x = \frac{4B-k^2}{4k} \in L^\infty(\mathbb{R})$ ,  $k_x = 2RR_x \in L^\infty$  (indeed  $|\phi_x|^2 = |R_x|^2 + |R\theta_x|^2 \in L^\infty(\mathbb{R})$ ) we have

$$\begin{aligned} k_x &\in L^2 \text{ and } \theta_x \in L^2, \\ \theta_{xx} &\in L^2 \text{ and } k_{xx} \in L^2. \end{aligned}$$

By using  $\theta_x = \frac{4B-k^2}{4k} \in L^2(\mathbb{R})$ , we have  $4B - k^2 \in L^2(\mathbb{R})$ . Thus,  $B \geq 0$  and  $2\sqrt{B} - k \in L^2(\mathbb{R})$ . If  $B = 0$  then  $k \in L^2(\mathbb{R})$ , hence,  $R \in L^2(\mathbb{R})$ . It implies that  $R \in H^1(\mathbb{R})$ , which contradicts the assumption of  $\phi$ . Thus,  $B > 0$ . It remains to prove that  $k_{xxx} \in L^2(\mathbb{R})$ . Indeed, from  $\phi_{xxx} \in L^2(\mathbb{R})$  we have

$$|\phi_{xxx}|^2 = |\theta_{xxx}\sqrt{k} + \mathcal{M}|^2 + \left| \frac{k_{xxx}}{2\sqrt{k}} + \mathcal{N} \right|^2 \in L^1(\mathbb{R}) \quad (5.12)$$

where  $\mathcal{M}, \mathcal{N}$  are functions of  $\theta, \theta_x, \theta_{xx}, k, k_x, k_{xx}$ . We can easily check that  $\mathcal{M}, \mathcal{N} \in L^2(\mathbb{R})$ . Hence, from (5.12) and the facts that  $\theta_x \in H^1(\mathbb{R})$ ,  $k \in X^2(\mathbb{R})$ ,  $k$  bounded from below we obtain  $\theta_{xxx}, k_{xxx} \in L^2(\mathbb{R})$ . This implies the desired results.  $\square$

From now on, we will denote  $\phi_B$  is the stationary solution of (5.1) given by Theorem 1.6 with  $\theta_0 = 0$ . We have the following asymptotic properties for  $\phi_B$  at  $\infty$ .

**Proposition 5.3.** *Let  $B > 0$  and  $\phi_B$  be kink solution of (1.1). Then for  $x > 0$ , we have*

$$|\phi_B - \sqrt{2\sqrt{B}}| \lesssim e^{-\sqrt{B}x}.$$

As consequence  $\phi_B$  converges to  $\sqrt{2\sqrt{B}}$  as  $x$  tends to  $\infty$  and there exists limit of  $\phi_B$  as  $x$  tends to  $-\infty$ .

*Proof.* We recall that

$$\phi_B = e^{i\theta}\sqrt{k}, \quad (5.13)$$

$$k(x) = 2\sqrt{B} + \frac{-1}{\sqrt{\frac{5}{72B} \cosh(2\sqrt{B}x) + \frac{5}{12\sqrt{B}}}}, \quad (5.14)$$

$$\theta(x) = - \int_x^\infty \frac{B}{k(y)} - \frac{k(y)}{4} dy. \quad (5.15)$$

From (5.14) we have

$$|k - 2\sqrt{B}| \lesssim e^{-2\sqrt{B}x}.$$

Hence, for all  $x \in \mathbb{R}$  we have

$$|\phi_B(x) - \sqrt{2\sqrt{B}}| \lesssim |e^{i\theta(x)}\sqrt{k(x)} - \sqrt{k(x)}| + |\sqrt{k(x)} - \sqrt{2\sqrt{B}}| \quad (5.16)$$

$$\lesssim \|k\|_{L^\infty}^{\frac{1}{2}} |e^{i\theta(x)} - 1| + e^{-\sqrt{B}x} \quad (5.17)$$

Moreover, for  $x > 0$ , we have

$$\begin{aligned} |e^{i\theta(x)} - 1| &\leq |\theta(x)| \leq \int_x^\infty \left| \frac{B}{k} - \frac{k}{4} \right| dx \\ &\leq \int_x^\infty \left| \frac{B}{k} - \frac{\sqrt{B}}{2} \right| + \left| \frac{\sqrt{B}}{2} - \frac{k}{4} \right| dx \\ &\lesssim \int_x^\infty |k - 2\sqrt{B}| dx \lesssim \int_x^\infty e^{-2\sqrt{B}x} dx \lesssim e^{-2\sqrt{B}x}. \end{aligned}$$

Combining with (5.17) we obtain

$$|\phi_B(x) - \sqrt{2\sqrt{B}}| \lesssim e^{-\sqrt{B}x}.$$

As consequence  $\phi_B$  converges to  $\sqrt{2\sqrt{B}}$  as  $x$  tends to  $\infty$ . It is easy to check that  $\theta$  converges to some constant when  $x$  tends to  $-\infty$ , hence, there exists limit of  $\phi_B$  when  $x$  tends to  $-\infty$ . It is the desired result.  $\square$

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