On primary pseudo-polynomials (Around Ruzsa's Conjecture)

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Abstract

Every polynomial $P(X) \in \mathbb{Z}[X]$ satisfies the congruences $P(n+m) \equiv P(n) \mod m$ for all integers $n, m \geq 0$. An integer valued sequence $(a_n)_{n\geq 0}$ is called a pseudo-polynomial when it satisfies these congruences. Hall characterized pseudo-polynomials and proved that they are not necessarily polynomials. A long standing conjecture of Ruzsa says that a pseudo-polynomial a_n is a polynomial as soon as $\limsup_n |a_n|^{1/n} < e$. Under this growth assumption, Perelli and Zannier proved that the generating series $\sum_{n=0}^{\infty} a_n z^n$ is a G-function. A primary pseudo-polynomial is an integer valued sequence $(a_n)_{n\geq 0}$ such that $a_{n+p} \equiv a_n \mod p$ for all integers $n \geq 0$ and all prime numbers p. The same conjecture has been formulated for them, which implies Ruzsa's, and this paper revolves around this conjecture. We obtain a Hall type characterization of primary pseudo-polynomials and draw various consequences from it. We give a new proof and generalize a result due to Zannier that any primary pseudo-polynomial with an algebraic generating series is a polynomial. This leads us to formulate a conjecture on diagonals of rational fractions and primary pseudo-polynomials, which is related to classic conjectures of Christol and van der Poorten. We make the Perelli-Zannier Theorem effective. We prove a Pólya type result: if there exists a function F analytic in a right-half plane with not too large exponential growth (in a precise sense) and such that for all large n the primary pseudo-polynomial $a_n = F(n)$, then a_n is a polynomial. Finally, we show how to construct a non-polynomial primary pseudo-polynomial starting from any primary pseudo-polynomial generated by a G-function different of $1/(1-x)$.

1 Introduction

Let $P(X)$ be a polynomial in $\mathbb{Z}[X]$. For all distinct integers m and n, the integer $m-n$ divides $P(m)-P(n)$. Equivalently, for all integers n and k with $k \neq 0$, we have the congruence $P(n+k) \equiv P(n) \mod k$.

In the sense of Hall [6], a sequence $(a_n)_{n\geq 0} \in \mathbb{Z}^{\mathbb{N}}$ is said to be a *pseudo-polynomial* when the following property holds: for any integers $n \geq 0$ and $k \geq 1$, we have $a_{n+k} \equiv a_n \mod k$.

Note that an integer-valued polynomial is not necessarily a pseudo-polynomial as $X(X+1)/2$ shows, see below. Pseudo-polynomials have long been studied for themselves, but they have also found recent applications in analytic number theory [9, 10].

For every non-negative integer k, we consider the polynomial

$$P_k(X) = \frac{X(X-1)\cdots(X-k+1)}{k!},$$

and $P_0(X) := 1$, whose integer values are binomial coefficients. It is well known [15, Problem 85] that integer valued polynomials are $\mathbb{Z}$-linear combinations of the P_k's. It turns out that those polynomials also lead to a characterization of pseudo-polynomials. Hall proved in [6] that a sequence $(a_n)_{n\geq 0}$ is a pseudo-polynomial if and only if there exists a sequence of integers $(b_n)_{n\geq 0}$ such that, for every positive integer n, b_n is a multiple of $d_n := \mathrm{lcm}\{1, 2, \ldots, n\}$ (with $d_0 := 1$) and we have

$$a_n = \sum_{k=0}^{\infty} b_k P_k(n). \tag{1.1}$$

(For each given $n \geq 0$, the sum is finite, more precisely it runs from $k = 0$ to $k = n$.) Given a sequence $(a_n)_{n\geq 0} \in \mathbb{C}^{\mathbb{N}}$, we define its *binomial transform* $(b_n)_{n\geq 0} \in \mathbb{C}^{\mathbb{N}}$ as

$$\forall n \geq 0, \quad b_n := \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} a_k. \tag{1.2}$$

It is well-known that $(a_n)_{n\geq 0}$ can be recovered from $(b_n)_{n\geq 0}$ by

$$\forall n \geq 0, \quad a_n = \sum_{k=0}^{n} \binom{n}{k} b_k, \tag{1.3}$$

i.e. the binomial transform is "almost" involutive. Hence Hall proved that a sequence $(a_n)_{n\geq 0}$ is a pseudo-polynomial if and only if its binomial transform $(b_n)_{n\geq 0}$ satisfies

$$\forall n \geq 0,\ d_n \,|\, b_n. \tag{1.4}$$

Because of (1.2) and (1.3), observe that in (1.1) the sequence $(b_n)_{n\geq 0}$ is uniquely determined by the sequence $(a_n)_{n\geq 0}$, so that as claimed above $\frac{1}{2}X(X+1) = P_1(X) + P_2(X)$ is not a pseudo-polynomial because $d_2 = 2$.

An important property of the binomial transform is that they also lead to a characterization of polynomials. Let $(a_n)_{n\geq 0}$ be a complex sequence and $(b_n)_{n\geq 0}$ its binomial transform. Then the following assertions are equivalent:

- there exists $P(X) \in \mathbb{C}[X]$ such that $a_n = P(n)$ for all integers $n \geq 0$;
- $b_n = 0$ for all n large enough.

(We can replace $\mathbb{C}[X]$ by $\mathbb{Q}[X]$ when the sequences take values in $\mathbb{Q}$.)

In the sequel, we shall often say that an integer valued sequence $(a_n)_{n\geq 0}$ is "a polynomial" when there exists a polynomial $P(X) \in \mathbb{Q}[X]$ such that $a_n = P(n)$ for all non-negative integers n.

We recall that, for all non-negative integers n, we have

$$d_n = \prod_{p\leq n} p^{\lfloor \log_p(n)\rfloor} \leq 3^n \quad \text{and} \quad d_n^{1/n} \underset{n\to+\infty}{\longrightarrow} e,$$

by the Prime Number Theorem. Taking $b_n := n!$ in (1.4), we see from (1.3) that the resulting pseudo-polynomial a_n is simply equal to $\lfloor n!e\rfloor$ for $n \geq 1$, which is obviously not a polynomial. With $b_n := d_n$ in (1.4), we obtain from (1.3) another pseudo-polynomial of slower growth $a_n := \sum_{k=0}^{n} \binom{n}{k} d_k \leq 4^n$; since $a_n \geq 2^n$ for all $n \geq 0$, this is not a polynomial either.

Following Hall, we say that a non-polynomial pseudo-polynomial is a *genuine* pseudo-polynomial. The search of minimal growth conditions that can be attained by genuine pseudo-polynomials has been the subject of many papers. Hall [6] and Ruzsa [17] independently proved that if

$$\limsup_{n\to+\infty} |a_n|^{1/n} < e - 1, \tag{1.5}$$

then $(a_n)_{n\geq 0}$ is a polynomial. Using his characterization (1.4), Hall [6, p. 76] sketched an inductive construction of a genuine pseudo-polynomial $(a_n)_{n\geq 0}$ such that

$$\limsup_{n\to+\infty} |a_n|^{1/n} \leq e.$$

Ruzsa proposed the following conjecture.

Conjecture 1 (Ruzsa)**.** *Let $(a_n)_{n\geq 0} \in \mathbb{Z}^{\mathbb{N}}$ be a pseudo-polynomial such that*

$$\limsup_{n\to+\infty} |a_n|^{1/n} < e. \tag{1.6}$$

Then $(a_n)_{n\geq 0}$ is a polynomial.

In fact, many results towards Ruzsa's conjecture have been proven for sequences we shall call *primary pseudo-polynomial* (for lack of better terminology).

Definition 1. *A sequence $(a_n)_{n\geq 0} \in \mathbb{Z}^{\mathbb{N}}$ is said to be a primary pseudo-polynomial when the following property holds: for any integer $n \geq 0$ and any prime number p, $a_{n+p} \equiv a_n \mod p$.*

The set of primary pseudo-polynomials is a ring for the term-wise sum and product of sequences in $\mathbb{Z}^{\mathbb{N}}$, with the null and unit sequences defined with all terms equal to 0 and all terms equal to 1 respectively. A pseudo-polynomial is a primary pseudo-polynomial but the converse is false (see the comments following Theorem 1 below). Many authors delt with the following conjecture, the truth of which would imply that of Ruzsa.

Conjecture 2. *Let $(a_n)_{n\geq 0} \in \mathbb{Z}^{\mathbb{N}}$ be a primary pseudo-polynomial such that*

$$\limsup_{n\to+\infty} |a_n|^{1/n} < e. \tag{1.7}$$

Then $(a_n)_{n\geq 0}$ is a polynomial.

As in the case of pseudo-polynomials, $< e$ cannot be replaced by $\leq e$, and we refer again to the comments following Theorem 1 for a proof of this.

Perelli and Zannier [14] proved a highly non-trivial property: under the growth condition (1.6) in Conjecture 2, the primary pseudo-polynomial sequence $(a_n)_{n\geq 0}$ satisfies a linear recurrence with polynomials coefficients; see (1.10) below. In other words, the generating function $f_a(x) := \sum_{n=0}^{\infty} a_n x^n \in \mathbb{Z}[[x]]$ satisfies a linear differential equation with coefficients in $\mathbb{Z}[x]$. Hence, f_a is a G-function ([1]). Perelli and Zannier also proved a form of Conjecture 2 under a stronger assumption than (1.6), *i.e.* with e replaced by $e^{0.66}$. Zannier [21] was even able to replace $e^{0.66}$ by $e^{0.75}$.

Zannier [21, p. 398] also proved that we can omit the growth condition if we further assume that $f_a(x)$ is algebraic. That is, if $(a_n)_{n\geq 0}$ is a primary pseudo-polynomial such that $f_a(x)$ is algebraic then $(a_n)_{n\geq 0}$ is a polynomial. Diagonals of rational fractions form an intermediate class between algebraic series and G-functions. By definition, the diagonal of a multivariate power series $\sum_{n_1,\ldots,n_k\geq 0} u_{n_1,\ldots,n_k} z_1^{n_1}\cdots z_k^{n_k}$ is defined by $\sum_{n=0}^{\infty} u_{n,n,\ldots,n} z^n$. A classical result of Furstenberg [5] says that algebraic series over a field coincide with diagonals of rational functions in two variables. It would be very interesting to know whether the following conjecture holds true.

Conjecture 3. *Let $(a_n)_{n\geq 0} \in \mathbb{Z}^{\mathbb{N}}$ be a primary pseudo-polynomial such that its generating series is the diagonal of a rational fraction. Then $(a_n)_{n\geq 0}$ is a polynomial.*

Moreover, diagonals of rational functions are globally bounded G-functions in the sense of Christol, who conjectured that the converse holds (see [2]). In particular, G-functions with integer coefficients are globally bounded. Therefore given the theorem of Perelli and Zannier (recalled below), Christol's Conjecture and Conjecture 3 would together imply Conjecture 2 and Ruzsa's Conjecture 1. See also related comments in [21, pp. 392–393].

Furthermore, Conjecture 3 is implied by the following special case of van der Poorten's Conjecture [19, p. 13] : Given $f(x)$ the diagonal of a rational fraction, if for almost all primes p the reduction of $f(x)$ modulo p is a rational fraction, then $f(x)$ is a rational fraction.

Hence Conjecture 3 could be seen as an intermediate step towards the proof of Rusza's conjecture. Note that Christol's Conjecture together with van der Poorten's Conjecture imply Rusza's Conjecture.

[1] A power series $\sum_{n=0}^{\infty} a_n x^n \in \overline{\mathbb{Q}}[[x]]$ is said to be a G-function when it is solution of a non-zero linear differential equation over $\overline{\mathbb{Q}}(x)$ (D-finiteness), and the maximum of the modulus of all the Galoisian conjuguates of $a_0,\ldots,a_n$ as well as the positive denominator of $a_0,\ldots,a_n$ are both bounded for all $n\geq 0$ by C^{n+1}, for some $C\geq 1$. For instance, any D-finite series in $\mathbb{Z}[[x]]$ with positive radius of convergence is a G-function. A power series $\sum_{n=0}^{\infty} \frac{a_n}{n!} z^x \in \overline{\mathbb{Q}}[[x]]$ is said to be an E-function when $\sum_{n=0}^{\infty} a_n x^n$ is a G-function. See [1, 18] for the properties satisfied by these functions.

In this paper, we are interested in the properties of primary pseudo-polynomials and of their generating functions. We now present our four main results, make comments about their significance and mention further open problems.

• ***A Hall type characterization of primary pseudo-polynomials.*** We shall first prove an analogue (*i.e.* Eq. (1.8) below) of Hall's characterization for pseudo-polynomials and deduce some consequences of it. We set $P_0 = P_1 := 1$ and, for $n \geq 2$,

$$P_n := \prod_{p \leq n} p,$$

where the product is over prime numbers. By the Prime Number Theorem, we have $P_n^{1/n} \to e$ as n tends to $+\infty$. We also say that a primary pseudo-polynomial which is not a polynomial is a *genuine* primary pseudo-polynomial.

Theorem 1. *We have the following.*

(*i*) *A sequence $(a_n)_{n\geq 0} \in \mathbb{Z}^{\mathbb{N}}$ is a primary pseudo-polynomial if and only if its binomial transform $(b_n)_{n\geq 0} \in \mathbb{Z}^{\mathbb{N}}$ satisfies*

$$\forall n \geq 0,\ P_n \,|\, b_n. \tag{1.8}$$

(*ii*) *Given a genuine primary pseudo-polynomial $(a_n)_{n\geq 0}$, then*

$$\liminf_{n\to+\infty} |b_n|^{1/n} \geq e.$$

(*iii*) *If a primary pseudo-polynomial $(a_n)_{n\geq 0}$ satisfies $\limsup_{n\to+\infty} |a_n|^{1/n} < e - 1$, then $(a_n)_{n\geq 0}$ is a polynomial.*

(*iv*) *Given any function $\varphi : \mathbb{N} \to \mathbb{R}$ with $\varphi(0) = 1$, there exists a genuine primary pseudo-polynomial $(A_n)_{n\geq 0}$ such that $\varphi(n) \leq A_n \leq \varphi(n) + 2P_n$ for all $n \in \mathbb{N}$.*

We recall that

$$d_n = \prod_{p \leq n} p^{\lfloor \log_p(n) \rfloor}.$$

Hence, for all $n \geq 0$, P_n divides d_n, but obviously d_n divides P_n for no $n \geq 4$. Choosing $b_n := P_n$ in (1.8), the resulting sequence in (1.3) $a_n := \sum_{k=0}^{n} \binom{n}{k} P_k$ is a primary pseudo-polynomial, but not a pseudo-polynomial because it does not satisfy Hall's criterion (1.4). Under the assumption in (*ii*), if we also assume that b_n is eventually of the same sign, then $\liminf_{n\to+\infty} |a_n|^{1/n} \geq e + 1$ because $a_n = \sum_{k=0}^{n} \binom{n}{k} b_k$. Consequently, any putative counter-example $(a_n)_{n\geq 0}$ to Conjecture 2 must be such that its binomial transform $(b_n)_{n\geq 0}$ changes sign infinitely often. A similar remark applies to Ruzsa's Conjecture 1.

Assertion (*iii*) is the analogue of the Hall-Ruzsa result recalled at Eq. (1.5).

In (*iv*), given φ, the existence of sequence $(A_n)_{n\geq 0}$ is proved constructively by an inductive process. An important consequence of (*iv*) is the existence of a genuine primary

pseudo-polynomial of any growth $\varphi(n) > 0$ provided $\liminf_n \varphi(n)^{1/n} \geq e$. In particular, with $\varphi(n) = P_n$, we deduce that $< e$ cannot be replaced by $\leq e$ on the right-hand side of (1.7) in Conjecture 2.

• ***Primary pseudo-polynomials with an algebraic generating series.*** The generating functions f_a and f_b of the sequences $(a_n)_{n\geq 0}$ and its binomial transform $(b_n)_{n\geq 0}$ satisfy the relations

$$f_b(x) = \frac{1}{1+x} f_a\Big(\frac{x}{1+x}\Big) \quad \text{and} \quad f_a(x) = \frac{1}{1-x} f_b\Big(\frac{x}{1-x}\Big). \tag{1.9}$$

In particular $f_a(x)$ is algebraic over $\mathbb{Q}(x)$ if and only if $f_b(x)$ is algebraic over $\mathbb{Q}(x)$.

Theorem 2. *Let $(a_n)_{n\geq 0} \in \mathbb{Z}^{\mathbb{N}}$ be a primary pseudo-polynomial, and let $(b_n)_{n\geq 0} \in \mathbb{Z}^{\mathbb{N}}$ be its binomial transform.*

(i) Assume there exists $m \geq 0$ such that $f_a^{(m)}(x)$ is algebraic over $\mathbb{Q}(x)$. Then $(a_n)_{n\geq 0}$ is a polynomial, and thus $f_a(x) \in \mathbb{Q}(x)$.

(ii) If $f_b(x)$ is algebraic over $\mathbb{Q}(x)$, then $f_b(x)$ is in $\mathbb{Z}[x]$.

As said above, the case $m = 0$ in (i) was proved by Zannier in [21, p. 398]. We shall present a different proof of this case. It implies that if there exists a counter example to Conjecture 2 or to Ruzsa's Conjecture 1, then its generating function $f_a(x)$ is transcendental over $\mathbb{C}(x)$.

• ***An effective version of a result of Perelli and Zannier.*** In [14], Perelli and Zannier sketched the proof of the following result.

Theorem (Perelli-Zannier)**.** *Let $(a_n)_{n\geq 0}$ be a primary pseudo-polynomial such that there exist $c > 0$ and $1 < \delta < e$ such that $|a_n| \leq c\delta^n$ for all $n \geq 0$. Then there exist an integer $S \geq 0$ and $S+1$ polynomials $p_0(X), \ldots, p_S(X) \in \mathbb{Z}[X]$ not all zero such that, for all $n \geq 0$, we have*

$$\sum_{j=0}^{S} p_j(n) a_{n+j} = 0. \tag{1.10}$$

In other words, $f_a(x)$ is D-finite, and even a G-function.

Perelli and Zannier mentioned that it would be possible to provide upper bounds for S and the degree/height of the $p_j(X)$ in terms of δ, but they did not write them down. We make more precise their theorem as follows, where given $Q(X) = \sum_j q_j X^j \in \mathbb{C}[X]$, we set $H(Q) := \max_j |q_j|$.

Theorem 3. *Let $(a_n)_{n\geq 0}$ be a primary pseudo-polynomial such that there exist $c > 0$ and $1 < \delta < e$ such that $|a_n| \leq c\delta^n$ for all $n \geq 0$. Then, there exists an effectively computable*

constant $H(c,\delta) \geq 1$ *such that a non-trivial linear recurrence for* $(a_n)_{n\geq 0}$ *as in* (1.10) *holds with*

$$\begin{cases} \max_j \deg(p_j) \leq \max\left(0, \left\lceil \frac{5\log(\delta)-1}{1-\log(\delta)} \right\rceil\right), \\ \max_j H(p_j) \leq H(c,\delta), \\ S \leq \log\big(H(c,\delta)\big)/\log(\delta). \end{cases} \tag{1.11}$$

Moreover, the Perelli-Zannier Theorem is best possible in the sense that its conclusion does not necessarily hold if e *is replaced by any larger number in the assumption* $1 < \delta < e$.

To prove the final statement in Theorem 3, we take $\varphi(n) := P_n$ in Theorem 1(iv): we obtain a genuine primary pseudo-polynomial A_n such that $|A_n|^{1/n} \to e$. Hence $(A_n)_{n\geq 0}$ does not satisfy a non-zero linear recurrence with coefficients polynomials over $\mathbb{Q}$. ([2])

Lower and upper bounds for the function $H(c,\delta)$ are given in (4.8) and (4.10) respectively in §4.3. Our bound for $\max \deg(p_j)$ in (1.11) is obviously not optimal; in fact, at the cost of more complicated computations, Perelli-Zannier [14] and then Zannier [21] obtained better bounds when $\delta \leq e^{0.66}$ and $\delta \leq e^{0.75}$.

The classification of primary pseudo-polynomials with a D-finite generating series is an open problem. As shown by the above example $(A_n)_{n\geq 0}$, Theorem 1 rules out the possibility that *every* primary pseudo-polynomial satisfies a linear recurrence with coefficients polynomials over $\mathbb{Q}$. Another example is the primary pseudo-polynomial $D_n := \sum_{k=0}^n \binom{n}{k} P_k$: it cannot satisfy such a linear recurrence because otherwise

$$P_n = \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} D_k$$

would satisfy one as well, which is not possible because $P_n^{1/n} \to e$. Since $P_n \geq 0$ and $P_n = e^{n+o(n)}$, a simple analytic argument shows that $D_n = (e+1)^{n+o(n)}$. More specifically, it would be interesting to know if there is any genuine primary pseudo-polynomial $(a_n)_{n\geq 0}$ such that $f_a(x)$ is a G-function. There exist primary pseudo-polynomials $(a_n)_{n\geq 0}$ such that f_a is D-finite but is not a G-function. For instance,

$$e_n := \lfloor (n+1)!e \rfloor = \sum_{k=0}^{n+1} \binom{n+1}{k} k! \quad (n \geq 0)$$

is a primary pseudo-polynomial by Theorem 1, and for all $n \geq 0$,

$$e_{n+2} = (n+4)e_{n+1} - (n+2)e_n \quad (e_0 = 2, e_1 = 5),$$

[2]Indeed, if a solution $(a_n)_{n\geq 0}$ of a linear recurrence with coefficients polynomials over $\mathbb{Q}$ is such that $|a_n|^{1/n} \to \alpha$ finite, then α is an algebraic number.

Moreover, if a sequence of *rational numbers* satisfies a non-zero linear recurrence of minimal order with coefficients polynomials over $\mathbb{C}$, then these coefficients are necessarily polynomials over $\mathbb{Q}$, up to a common non-zero multiplicative constant. Hence $(d_n)_{n\geq 0}$ and $(P_n)_{n\geq 0}$ do not satisfy any non-zero linear recurrence with coefficients polynomials over $\mathbb{C}$.

so that $\sum_{n=0}^{\infty} e_n x^n$ is D-finite. A method to obtain further examples is presented at the end of §6.

• ***A Pólya type result for primary pseudo-polynomials.*** Perelli and Zannier also proved in [13] that if a (primary) pseudo-polynomial $a_n = F(n)$ for some entire function F such that

$$\limsup_{R\to\infty} \frac{1}{R} \log \max_{|x|=R} |F(x)| < \log(e+1),$$

then $(a_n)_{n\geq 0}$ is a polynomial. We prove here a result of a similar flavor with a different analyticity condition.

Theorem 4. *Let $(a_n)_{n\geq 0} \in \mathbb{Z}^{\mathbb{N}}$ be a primary pseudo-polynomial. Let us assume that there exists $F(x)$ analytic in a right-half plane $\Re(x) > u$ such that $a_n = F(n)$ for all $n > u$, and $c > 0, 0 < \rho < \log(2\sqrt{e})$ such that*

$$\big|F(x)\big| \leq c \cdot e^{\rho \Re(x)} \tag{1.12}$$

for $\Re(x) > u$. Then $F(x)$ is in $\mathbb{Q}[x]$ and $(a_n)_{n\geq 0}$ is a polynomial.

We have $\log(2\sqrt{e}) \approx 1.193$ while $\log(e+1) \approx 1.313$. In the proof, we shall obtain that $(a_n)_{n\geq 0}$ is a polynomial before proving that F is a polynomial. Because of the Perelli-Zannier Theorem recalled before Theorem 3, the assumptions of Theorem 4 are in fact natural in the context of Ruzsa's Conjecture 1. Indeed, for any given G-function $f(x) := \sum_{n=0}^{\infty} v_n x^n \in \overline{\mathbb{Q}}[[x]]$, there exists a function

$$\lambda(x) := \sum_{j=1}^{p} c_j(x) \cdot e^{\rho_j x},$$

for some functions $c_j(x)$ analytic in $\Re(x) > u$ and of polynomial growth (at most), and such that $v_n = \lambda(n)$ for all $n > u$; the numbers $e^{-\rho_j}$ are the finite singularities of $f(x)$ (see [3, §7.1] for details). Notice that a bound involving $e^{\rho_j \Re(x)}$ is *a priori* different of a bound involving $|e^{\rho_j x}| = e^{\Re(\rho_j x)}$, but they are the same when $\rho_j \in \mathbb{R}$. In particular, if all the singularities of $f(x)$ are positive real numbers, then a bound as in (1.12) holds for $\lambda(x)$ for some $\rho \in \mathbb{R}$.

In Theorem 1(iv), take $\varphi(n) = \delta^n$ with $e < \delta < 2\sqrt{e}$. This yields a genuine primary pseudo-polynomial $(a_n)_{n\geq 0}$ such that $a_n = \delta^{n+o(n)}$ as $n \to +\infty$. Theorem 4 thus implies that there is no function $F(x)$ analytic in a right-half plane on which (1.12) holds, and such that $a_n = F(n)$ for all large n.

Perelli-Zannier's result and Theorem 4 are similar to Pólya's celebrated theorem: if an entire function $F(x)$ is such that

$$\limsup_{R\to+\infty} \frac{1}{R} \log \max_{|x|=R} |F(x)| < \log(2) \quad \text{and} \quad F(\mathbb{N}) \subset \mathbb{Z},$$

then $F(x)$ is a polynomial. See [20] for a recent survey on Pólya type results, where a connection with Rusza's Conjecture 1 is also mentioned.

Theorems 1, 2, 3 and 4 are proved in §2, §3, §4 and §5 respectively. In §6, we present a method to construct a genuine primary pseudo-polynomial starting from every primary pseudo-polynomial generated by a G-function (Theorem 5).

2 Proof of Theorem 1

(i) Let $(a_n)_{n\geq 0}$ be a sequence of integers and consider its binomial transform $(b_n)_{n\geq 0}$.

Assume that for every non-negative integer n, P_n divides b_n. Let p be a fixed prime number. Hence, for every integer $n \geq p$, p divides b_n. For every non-negative integer n, it yields

$$\begin{aligned} a_{n+p} &= \sum_{k=0}^{n+p} \binom{n+p}{k} b_k \\ &\equiv \sum_{k=0}^{p-1} \binom{n+p}{k} b_k \mod p \\ &\equiv \sum_{k=0}^{p-1} \binom{n}{k} b_k \mod p \\ &\equiv a_n \mod p, \end{aligned}$$

where we used Lucas' congruence for binomial coefficients: for every u, v in $\{0, \ldots, p-1\}$ and every non-negative integers m and ℓ, we have

$$\binom{u+mp}{v+\ell p} \equiv \binom{u}{v}\binom{m}{\ell} \mod p.$$

It follows that $(a_n)_{n\geq 0}$ is a primary pseudo-polynomial.

Conversely, assume that $(a_n)_{n\geq 0}$ is a primary pseudo-polynomial. Let p be a prime number and $n \geq p$ be an integer. It suffices to show that p divides b_n.

Write $n = v + mp$ with v in $\{0, \dots, p-1\}$ and $m \geq 1$. We obtain that

$$
\begin{aligned}
& b_{v+mp} \\
&= \sum_{k=0}^{v+mp} (-1)^{v+mp-k} \binom{v+mp}{k} a_k \\
&= \sum_{u=0}^{p-1} \sum_{\ell=0}^{m-1} (-1)^{v-u} (-1)^{(m-\ell)p} \binom{v+mp}{u+\ell p} a_{u+\ell p} + \sum_{u=0}^{v} (-1)^{v-u} \binom{v+mp}{u+mp} a_{u+mp} \\
&\equiv \sum_{u=0}^{p-1} \sum_{\ell=0}^{m-1} (-1)^{v-u} (-1)^{(m-\ell)p} \binom{v}{u} \binom{m}{\ell} a_u + \sum_{u=0}^{v} (-1)^{v-u} \binom{v}{u} a_u \mod p \\
&\equiv \sum_{u=0}^{v} (-1)^{v-u} \binom{v}{u} a_u \sum_{\ell=0}^{m} (-1)^{(m-\ell)p} \binom{m}{\ell} \mod p \quad \left(\binom{v}{u} = 0 \text{ for } u = v+1, \dots, p-1 \right) \\
&\equiv \sum_{u=0}^{v} (-1)^{v-u} \binom{v}{u} a_u (1 + (-1)^p)^m \mod p \\
&\equiv 0 \mod p \qquad (m \geq 1).
\end{aligned}
$$

Hence p divides b_n as expected. It follows that, for every non-negative integer n, P_n divides b_n. The first equivalence in Theorem 1 is proved.

(*ii*) The binomial transform $(b_n)_{n\geq 0}$ is eventually 0 if, and only if there exists a polynomial $Q(X)$ in $\mathbb{Q}[X]$ such that $a_n = Q(n)$ for every non-negative integer n. Hence, if the latter is false, then $(b_n)_{n\geq 0}$ is not eventually 0 and, since P_n divides b_n, it follows that

$$\liminf_{n\to+\infty} |b_n|^{1/n} \geq e$$

because $P_n^{1/n} \to e$ as $n \to +\infty$.

(*iii*) If the primary pseudo-polynomial $(a_n)_{n\geq 0}$ is not a polynomial, then by (*ii*) above, $\liminf_{n\to+\infty} |b_n|^{1/n} \geq e$. But since

$$b_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} a_k,$$

the assumption $\limsup_{n\to+\infty} |a_n|^{1/n} < e - 1$ implies that

$$\limsup_{n\to+\infty} |b_n|^{1/n} \leq \limsup_{n\to+\infty} \left(\sum_{k=0}^{n} \binom{n}{k} |a_k| \right)^{1/n} < e.$$

This proves that $(a_n)_{n\geq 0}$ is a polynomial.

(*iv*) The following argument generalizes Hall's (sketchy) construction of a genuine pseudo-polynomial with growth $\leq e^{n+o(n)}$ in [6, p. 76]. By (*i*), we know that any sequence of integers $(B_n)_{n\geq 0}$ such that P_n divides B_n defines a primary pseudo-polynomial

$A_n := \sum_{k=0}^{n} \binom{n}{k} B_k$, which is not a polynomial if (and only if) $B_n \neq 0$ for infinitely many n. Since A_{n-1} depends only on $B_0, B_1, \ldots, B_{n-1}$, we will recursively construct $B_n \neq 0$ and thus A_n. We have $A_0 = B_0$: choosing $B_0 = 1$, we have $\varphi(0) = A_0 = \varphi(0) + 2P_0$. Let $n \geq 0$ and let us assume that we have constructed $B_0, B_1, \ldots, B_{n-1}$ all non-zero and such that P_k divides B_k and $\varphi(k) \leq A_k \leq \varphi(k) + 2P_k$ for all $k \in \{0, \ldots, n-1\}$.

We want to construct an integer $B_n \neq 0$ such that $P_n \mid B_n$ and $\varphi(n) \leq A_n \leq \varphi(n) + 2P_n$. To do this, we first set $C_n := \sum_{k=0}^{n-1} \binom{n}{k} B_k$, so that we will have $A_n = B_n + C_n$. We now perform the euclidean division of C_n by P_n: we have $C_n = u_n P_n + v_n$ with $u_n \in \mathbb{Z}, v_n \in \mathbb{N}$ and $0 \leq v_n < P_n$. We set $B_n := w_n P_n \neq 0$ where the non-zero integer w_n is defined as follows: if $\lceil \frac{\varphi(n) - v_n}{P_n} \rceil \neq u_n$, we take

$$w_n = \left\lceil \frac{\varphi(n) - v_n}{P_n} \right\rceil - u_n,$$

while if $\lceil \frac{\varphi(n) - v_n}{P_n} \rceil = u_n$, we take $w_n = 1$. Since $A_n = (u_n + w_n)P_n + v_n$, we see that $\varphi(n) \leq A_n \leq \varphi(n) + P_n$ in the former case, while $\varphi(n) + P_n \leq A_n \leq \varphi(n) + 2P_n$ in the latter case. This finishes the recursive construction of a genuine primary pseudo-polynomial $(A_n)_{n \geq 0}$ such that $\varphi(n) \leq A_n \leq \varphi(n) + 2P_n$ for all integers $n \geq 0$.

3 Proof of Theorem 2

From the proof of Theorem 1(i), we see that for any given prime number p, the assertions "for all $n \geq 0$, $a_{n+p} \equiv a_n \mod p$" and "for all $n \geq p$, p divides b_n" are equivalent. It follows that the assertions "for all $p \in \mathcal{P}$ and all $n \geq 0$, $a_{n+p} \equiv a_n \mod p$" and "for all $p \in \mathcal{P}$ and all $n \geq p$, p divides b_n" are equivalent, where $\mathcal{P}$ is a same set of prime numbers, and this generalizes Theorem 1(i). We shall in fact prove Theorem 2 under the weaker assumption that there exists an infinite set $\mathcal{P}$ of prime numbers such that for all $p \in \mathcal{P}$ and all $n \geq 0$, $a_{n+p} \equiv a_n \mod p$.

Given $u \in \mathbb{Z}$ and a prime number p, we set $u_{|p} := u \mod p$. Given a power series $F(x) := \sum_{n=0}^{\infty} u_n x^n$ with integer coefficients, we set

$$F_{|p}(x) := \sum_{n=0}^{\infty} u_{n|p} x^n \in \mathbb{F}_p[[x]].$$

We shall first prove (ii) for the series $f_b(x) := \sum_{n=0}^{\infty} b_n x^n$. Let $\mathcal{P}$ denote an infinite set of prime numbers such that for all $n \geq 0$ and all $p \in \mathcal{P}$, we have $a_{n+p} \equiv a_n \mod p$. As already said, this is equivalent to the fact that for all $p \in \mathcal{P}$ and all $n \geq p$, p divides b_n. It follows in particular that for any $p \in \mathcal{P}$, $f_{b|p}(x)$ is a polynomial in $\mathbb{F}_p[x]$ of degree at most $p-1$. For simplicity, we denote by $Q_p(x)$ this polynomial, and by q_p its degree.

Let us now assume that $f_b(x)$ is algebraic over $\mathbb{Q}(x)$. If $f_b(x)$ is a constant, there is nothing else to prove. We now assume that $f_b(x)$ is not a constant so that it has degree $d \geq 1$. There exist an integer $\delta \in \{1, \ldots, d-1\}$, some integers $0 \leq j_1 < j_2 < \ldots < j_\delta \leq$

$d-1$, and some polynomials $A_d(x)$, $A_{j_1}(x), \dots, A_{j_\delta}(x) \in \mathbb{Z}[x]$ all not identically zero such that

$$A_d f_b^d = \sum_{\ell=1}^{\delta} A_{j_\ell} f_b^{j_\ell} \tag{3.1}$$

in $\mathbb{Z}[[x]]$.

We fix $p \in \mathcal{P}$ such that $p > H$ where H is the maximum of the modulus of the coefficients of $A_d(x)$, $A_{j_1}(x), \dots, A_{j_\delta}(x)$. It follows that

$$\deg(A_{d|p}) = \deg(A_d),\ \deg(A_{j_1|p}) = \deg(A_{j_1}),\ \dots,\ \deg(A_{j_\delta|p}) = \deg(A_{j_\delta}). \tag{3.2}$$

We deduce from the reduction of (3.1) mod p that

$$A_{d|p} Q_p^d = \sum_{\ell=1}^{\delta} A_{j_\ell|p} Q_p^{j_\ell} \tag{3.3}$$

in $\mathbb{F}_p[[x]]$, and in fact in $\mathbb{F}_p[x]$ because $Q_p(x) \in \mathbb{F}_p[x]$.

Case 1). If Q_p is identically zero, this means that p divides the coefficients b_n for all $n \geq 0$.

Case 2). If Q_p is not identically zero, we deduce from (3.2) and (3.3) that

$$\begin{aligned} \deg(A_d) + dq_p &\leq \max\big(\deg(A_{j_1}) + j_1 q_p, \deg(A_{j_2}) + j_2 q_p, \dots, \deg(A_{j_\delta}) + j_\delta q_p\big) \\ &\leq \max\big(\deg(A_{j_1}), \deg(A_{j_2}), \dots, \deg(A_{j_\delta})\big) + (d-1)q_p. \end{aligned}$$

Hence

$$q_p \leq \max\big(\deg(A_{j_1}), \deg(A_{j_2}), \dots, \deg(A_{j_\delta})\big) - \deg(A_d) =: N.$$

It follows that for any $n > N$, p divides b_n, where N is *independent* of p.
Since $p \in \mathcal{P}$ was simply assumed larger than a quantity H depending only on f_b, the conclusion of Case 1 and Case 2 is that for any $p \in \mathcal{P}$ such that $p > H$ and any $n > N$, p divides b_n. Since $\mathcal{P}$ is infinite, b_n is divisible by infinitely many primes when $n > N$. Hence $b_n = 0$ for all $n > N$ and $f_b(x) = \sum_{n=0}^{N} b_n x^n \in \mathbb{Z}[x]$, as expected.

Let us now prove (i) in the case $m = 0$. If $f_a(x)$ is algebraic over $\mathbb{Q}(x)$, then $f_b(x)$ as well by (1.9). Hence $f_b(x) \in \mathbb{Z}[x]$ by (ii) just proven, *i.e.* there exists an integer M such that $b_n = 0$ if $n > M$. Since, for all $n \geq 0$, we have

$$a_n = \sum_{k=0}^{n} \binom{n}{k} b_k = \sum_{k=0}^{\min(n,M)} \binom{n}{k} b_k,$$

it follows that for all $n \geq M$, we have $a_n = Q(n)$ with $Q(X) = \sum_{k=0}^{M} \binom{X}{k} b_k \in \mathbb{Q}[X]$.

We now prove (i) for any integer $m \geq 0$. We need the following simple lemma.

Lemma 1. *Let $R(X) \in \mathbb{Q}(X)$ be such that $R(n) \in \mathbb{Z}$ for infinitely many integers. Then, $R(X) \in \mathbb{Q}[X]$.*

Proof. We write $R = A/B$ with $A, B \in \mathbb{Q}[X]$. We assume that $\deg(B) \geq 1$ otherwise there is nothing to prove. There exist $U, V \in \mathbb{Q}[X]$ such that $A = UB + V$ and $\deg(V) < \deg(B)$. Let $w \in \mathbb{Z} \setminus \{0\}$ be such that $wU, wV \in \mathbb{Z}[X]$. Let $\mathcal{N}$ be the infinite set of integers n such that $R(n) \in \mathbb{Z}$; without loss of generality, we can assume that $\mathcal{N}$ contains infinitely many positive integers. For every $n \in \mathcal{N}$, we have $wV(n)/B(n) = wR(n) - wU(n) \in \mathbb{Z}$. But $\lim_{x\to+\infty} wV(x)/B(x) = 0$. Hence there exists M such that $n \in \mathcal{N}$ and $n \geq M$ imply that $wV(n)/B(n) = 0$. Therefore, wV has infinitely many roots: it must be the null polynomial, so that $R = U \in \mathbb{Q}[X]$. □

Now, we have

$$f_a^{(m)}(x) = \sum_{n=0}^{\infty} (n+m)(n+m-1)\cdots(n+1)a_{n+m}x^n.$$

Since $(a_n)_{n\geq 0}$ is a primary pseudo-polynomial, this is also the case of

$$((n+m)(n+m-1)\cdots(n+1)a_{n+m})_{n\geq 0}$$

because it is a product of two primary pseudo-polynomials. Since $f_a^{(m)}(x)$ is algebraic over $\mathbb{Q}(x)$, $((n+m)(n+m-1)\cdots(n+1)a_{n+m})_{n\geq 0}$ is a polynomial by the already proven case $m = 0$ of Theorem 2(i). Hence $(a_{n+m})_{n\geq 0}$ is a rational fraction, so that by Lemma 1, $(a_{n+m})_{n\geq 0}$ is a polynomial. This completes the proof of Case (i).

4 Proof of Theorem 3

The proof of the Perelli-Zannier Theorem is based on the following lemma proved in [14]. We shall also use it.

Lemma 2. *For $\underline{k} = (k_j)_{j\geq 0} \in (\mathbb{R}^+)^{\mathbb{N}}$, an integer $N \geq 1$, set*

$$A(N, \underline{k}) := \{(x_1, \ldots, x_N) \in \mathbb{Z}^N : |x_j| \leq k_j \quad \text{and} \quad \forall p, \forall n \leq N - p,\ x_{n+p} \equiv x_n \mod p\}.$$

Then

$$\#A(N, \underline{k}) \leq \prod_{j=1}^{N} \left(1 + \frac{2k_j}{P_{j-1}}\right).$$

Let $R \geq 1, H \geq 0, D \geq 1$ be integers. Let $q_0, \ldots, q_R \in \mathbb{Z}[X]$ with $\max_j H(q_j) \leq H$ and $\max_j \deg(q_j) \leq D - 1$. Considering the coefficients of the q_j's as indeterminates, there are $(2H+1)^{RD}$ functions F of the form

$$F(n) := \sum_{j=0}^{R-1} q_j(n) a_{n+j}. \tag{4.1}$$

Such a function satisfies $|F(n)| \le cRDH(n+1)^D\delta^{n+R}$ for all $n \ge 0$ and $F(n+p) \equiv F(n)$ mod p for all prime number p and all $n \ge 0$. Hence for all $N \ge 1$, $(F(0), \dots, F(N-1)) \in A(N, \underline{K})$ where $K_j := cRDHj^D\delta^{j+R-1}$. Therefore, given N, if

$$\prod_{j=1}^{N}\left(1+\frac{2K_j}{P_{j-1}}\right) < (2H+1)^{RD}, \tag{4.2}$$

there exists *two* different functions F_1 and F_2 of the form (4.1) such that $F_1(n) = F_2(n)$ for all $n \in \{0, 1, \dots, N-1\}$. Hence the function $G_N := F_1 - F_2$ is of the form (4.1)

$$G_N(n) = \sum_{j=0}^{R-1} q_j(n)a_{n+j},$$

with $q_0(X), \dots, q_{R-1}(X) \in \mathbb{Z}[X]$ not all identically zero, with $G_N(n) = 0$ for every integer n in $\{0, 1, \dots, N-1\}$ and $|G_N(n)| \le 2cRDH(n+1)^D\delta^{n+R}$ for all $n \ge 0$. Note that G_N depends on N which is fixed but can be as large as desired in this construction.

Eq. (4.2) holds if we assume the stronger condition

$$\prod_{j=1}^{\infty}\left(1+\frac{2K_j}{P_{j-1}}\right) \le H^{RD}, \tag{4.3}$$

because the assumption $\delta < e$ implies the convergence of the product

$$\Phi(D, x) := \prod_{j=1}^{\infty}\left(1+x\frac{j^D\delta^j}{P_{j-1}}\right)$$

for all $x \ge 0$ and all $D \ge 0$, and obviously $1+x\frac{j^D\delta^j}{P_{j-1}} \ge 1$. We shall provide an upper bound for $\Phi(D, x)$ in §4.2, from which we shall deduce values of H, R and D such that (4.3) holds. It is important to observe here that (4.3) does not depend on N.

4.1 Proof that $G_N(n) = 0$ for all $n \ge 0$

Following the Perelli-Zannier method, we now want to prove that, provided N is large enough, $G_N(n)$ vanishes for all $n \ge 0$. Assume this is not the case. Then for every N, let $M_N \ge N$ denote the largest integer such that $G_N(0) = G_N(1) = \ldots = G_N(M_N) = 0$ but $G_N(M_N+1) \ne 0$. We fix α in $\left(0, \frac{2}{\log(\delta)} - 2\right)$.

We shall first prove that $G_N(m) = 0$ for m in $I := [2M_N, (2+\alpha)M_N]$. Let $m \in I$. First assume that p is a prime and $p < M_N$: since $G_N(0) = \cdots = G_N(p) = 0$ and $G_N(n+p) \equiv G_N(n) \mod p$, p divides $G_N(m)$. Assume now that $M_N \le p \le m/2$, so that

$$0 \le m - 2p \le m - 2M_N \le \alpha M_N \le M_N,$$

hence $G_N(m) \equiv G_N(m-2p) = 0 \mod p$. Assume to finish that $m - M_N < p \leq m$ (such primes have not yet been considered) so that $0 \leq m-p < M_N$ and $G_N(m) \equiv G_N(m-p) \equiv 0 \mod p$. It follows that $G_N(m)$ si divisible by $P_{m/2}P_m/P_{m-M_N}$.

Therefore, if $G_N(m) \neq 0$ for some $m \in I$, then

$$|G_N(m)| \geq e^{m/2+M_N+o(M_N)} \geq e^{2M_N+o(M_N)},$$

where $o(M_N)$ denotes a term such that $o(M_N)/M_N$ becomes arbitrarily small when N is taken arbitrarily large. But on the other hand, we know that, for any $m \in I$,

$$|G_N(m)| \leq 2cRDH(m+1)^D\delta^{m+R} \leq 2cRDH(2M_N + \alpha M_N + 1)^D\delta^R\delta^{(2+\alpha)M_N}.$$

We recall that we assume that H, R and D are such that (4.3) holds, which is independent of N. Hence we can let $N \to +\infty$, hence *a fortiori* $M_N \to +\infty$ so that the above lower and upper bounds for $G_N(m) \neq 0$ imply that $e^2 \leq \delta^{2+\alpha}$, *i.e.* that

$$\alpha \geq \frac{2}{\log(\delta)} - 2,$$

which is contrary to the assumption on α. Hence, provided N is large enough, we have $G_N(m) = 0$ for all m in I.

We thus have $G_N(m) = 0$ for all integers m in $[0, \ldots, M_N]$ or $[2M_N, (2+\alpha)M_N]$. It follows that for any $p \leq (1+\alpha)M_N - 1$, p divides $G_N(M_N+1)$. Indeed, if $p \leq M_N$, we write $M_N + 1 = n + p$ for some $n \leq M_N - 2$ so that $G_N(M_N+1) \equiv G(n) = 0 \mod p$, while if $M_N < p \leq (1+\alpha)M_N - 1$, we have $0 = G_N(M_N + 1 + p) \equiv G(M_N+1) \mod p$ because $M_N + 1 + p \in [2M_N, (2+\alpha)M_N]$. Hence, because $G_N(M_N+1) \neq 0$, we have

$$|G_N(M_N+1)| \geq P_{(1+\alpha)M_N-1} \geq e^{(1+\alpha)M_N+o(M_N)}.$$

On the other hand,

$$|G_N(M_N+1)| \leq c2RDH(M_N+2)^D\delta^{M_N+1+R}.$$

As above, we take N large enough so that these two bounds imply that $\alpha \leq \log(\delta) - 1$, which is impossible because $\log(\delta) - 1 < 0$ while α was chosen positive.

Therefore, there is no such M_N such that $G(M_N+1) \neq 0$, so that $G(n) = 0$ for all integer $n \geq 0$.

4.2 Upper bound for $\Phi(D, x)$

In this section, $x > 0$ is a fixed real parameter and $D \geq 1$ is a fixed integer. We fix $\varepsilon > 0$ such that $\delta < e - \varepsilon$, and we let $\omega = \delta/(e-\varepsilon) < 1$. By the Prime Number Theorem, we have $P_{j-1} \geq (e-\varepsilon)^j$ for all $j > J = J(\varepsilon)$ so that

$$\Phi(D,x) := \prod_{j=1}^{\infty}\left(1 + xj^D\frac{\delta^j}{P_{j-1}}\right) \leq \prod_{j=1}^{J}\left(1 + xj^D\frac{\delta^j}{P_{j-1}}\right)\prod_{j=J+1}^{\infty}\left(1 + xj^D\omega^j\right).$$

We have

$$\prod_{j=1}^{J}\left(1+xj^D\frac{\delta^j}{P_{j-1}}\right)\leq\prod_{j=1}^{J}\left(1+xj^D\delta^j\right)\leq\prod_{j=1}^{J}\left(1+(1+x)j^D\delta^j\right)\leq\prod_{j=1}^{J}\left(2(1+x)j^D\delta^j\right),$$

since $(1+x)j^D\delta^j\geq 1$ for every j in $\{1,\ldots,J\}$. Hence,

$$\prod_{j=1}^{J}\left(2(1+x)j^D\delta^j\right)=2^J(1+x)^JJ!^D\delta^{J(J+1)/2}\leq 2^J(1+x)^JJ!^D\delta^{J^2}.$$

In addition, we have

$$\prod_{j=J+1}^{\infty}\left(1+xj^D\omega^j\right)\leq\prod_{j=1}^{\infty}\left(1+xj^D\omega^j\right),$$

which yields

$$\Phi(D,x)\leq 2^J(1+x)^JJ!^D\delta^{J^2}\prod_{j=1}^{\infty}\left(1+xj^D\omega^j\right).$$

We now bound the infinite product $\Psi(D,x):=\prod_{j=1}^{\infty}\left(1+xj^D\omega^j\right)$. The maximum of the function $t\mapsto t^D\omega^{t/2}$ is $m(D):=(2D/(e\log(1/\omega)))^D$, attained at $j_0:=2D/\log(1/\omega)$. Hence, for all $j\geq j_0$, we have $j^D\leq m(D)\omega^{-j/2}$. Moreover, $t\mapsto t^D\omega^{t/2}$ is increasing on $[0,j_0]$ and $m(D)\leq j_0^D$. Hence,

$$\begin{aligned}\Psi(D,x)&\leq\prod_{1\leq j<j_0}\left(1+xj^D\omega^j\right)\prod_{j\geq j_0}^{\infty}\left(1+xm(D)\omega^{j/2}\right)\\&\leq\left(1+(1+x)j_0^D\right)^{\lfloor j_0\rfloor}\prod_{j=1}^{\infty}\left(1+xm(D)\omega^{j/2}\right)\\&\leq\left(2(1+x)j_0^D\right)^{\lfloor j_0\rfloor}\prod_{j=1}^{\infty}\left(1+xj_0^D\omega^{j/2}\right).\end{aligned}$$

We now bound $\prod_{j=1}^{\infty}\left(1+xj_0^D\omega^{j/2}\right)$. We set $y:=xj_0^D$. Since $t\mapsto\omega^t$ is decreasing on $[0,\infty)$, we have

$$\begin{aligned}\log\left(\prod_{j=1}^{\infty}\left(1+y\omega^{j/2}\right)\right)&\leq\int_0^{+\infty}\log(1+y\omega^{t/2})dt\\&\leq\frac{2}{\log(1/\omega)}\int_0^y\frac{\log(1+u)}{u}du,\quad(u:=y\omega^{t/2})\\&\leq-\frac{2\mathrm{Li}_2(-y)}{\log(1/\omega)}.\end{aligned}$$

Here, we use the dilogarithm $\mathrm{Li}_2(z) := -\int_0^z \log(1-x)/x dx$ defined for $z \in \mathbb{C} \setminus (1, +\infty)$ using the principal branch of log in the integral; see [11, p. 1, (1.4)]. Here, we want to use it for large negative values $-y$. For this, we use the identity (see [11, p. 4, (1.7)])

$$\mathrm{Li}_2(-y) = -\frac{1}{2}\log(y)^2 - \mathrm{Li}_2(-1/y) - \zeta(2), \quad y > 0$$

which yields

$$-2\mathrm{Li}_2(-y) \leq \log(y)^2 + 4\zeta(2), \quad y \geq 1,$$

because $\mathrm{Li}_2(-1/y) + \zeta(2) \leq 2\zeta(2)$ when $y \geq 1$. We obtain that for $y \geq 1$,

$$\prod_{j=1}^{\infty} \left(1 + y\omega^{j/2}\right) \leq c_0 e^{\log(y)^2/\log(1/\omega)},$$

with $c_0 := \exp(4\zeta(2)/\log(1/\omega)) \geq 1$.

Putting all the pieces together, we finally obtain the bound

$$1 \leq \Phi(D, x) \leq 2^J (1+x)^J J!^D \delta^{J^2} (2(1+x) j_0^D)^{\lfloor j_0 \rfloor} c_0 e^{\log(x j_0^D)^2/\log(1/\omega)}, \tag{4.4}$$

where we recall that $\omega = \delta/(e-\varepsilon)$ where $\varepsilon > 0$ is such that $\delta < e - \varepsilon$.

4.3 Conclusion of the proof

For ease of reading, we set d, r, h for D, R, H. We want to find conditions on d, r and h such that $\Phi(d, x) \leq h^{rd}$ when $x = 2crdh\delta^{r-1}$ (which corresponds to (4.3)). It will be enough to find conditions on d, r, h and ε such that the right-hand side of (4.4) is $\leq h^{rd}$.

From now on, we set $\ell := \log(\delta) < 1$. We assume that d and ρ depend on ℓ but are independent of h, and we let $r := \lfloor \rho \log(h) \rfloor + 1$. Since J and j_0 are also fixed, when $h \to +\infty$, we have

$$\log\left(c_0 2^J (1+x)^J J!^d \delta^{J^2} (2(1+x) j_0^d)^{\lfloor j_0 \rfloor} e^{\log(x j_0^d)^2/\log(1/\omega)}\right) \sim \frac{(1+\rho\ell)^2}{\log(1/\omega)} \log(h)^2,$$

while

$$\log(h^{rd}) \sim d\rho \log(h)^2.$$

Hence for our goal, it suffices to choose d, ρ and ε such that

$$\frac{(1+\rho\ell)^2}{\log(1/\omega)} < d\rho. \tag{4.5}$$

Recall that $\omega = \delta/(e-\varepsilon)$ so that $\log(1/\omega) \to 1 - \ell$ as $\varepsilon \to 0$. So, by choosing d and ρ such that

$$\frac{(1+\rho\ell)^2}{1-\ell} \leq d\rho, \tag{4.6}$$

we can choose $\varepsilon > 0$ such that Eq. (4.5) holds true. Eq. (4.6) is equivalent to

$$\ell^2\rho^2 + (2\ell - d(1-\ell))\rho + 1 \leq 0,$$

which defines a polynomial in ρ whose discriminant is $\Delta := d(1-\ell)(d(1-\ell) - 4\ell)$. Taking $d := \max(1, \lceil \frac{4\ell}{1-\ell} \rceil)$ ensures that (4.6) holds true for any choice of $\rho > 0$ in

$$\left[\frac{d(1-\ell) - 2\ell - \sqrt{\Delta}}{2\ell^2}, \frac{d(1-\ell) - 2\ell + \sqrt{\Delta}}{2\ell^2} \right].$$

For simplicity, we also restrict ρ to be $\leq 1/\ell$ which is possible because the product of those roots is $1/\ell^2$ and, since $d(1-\ell) \geq 4\ell$, we have

$$\frac{d(1-\ell) - 2\ell + \sqrt{\Delta}}{2\ell^2} \geq \frac{1}{\ell} \quad \text{and} \quad \frac{d(1-\ell) - 2\ell - \sqrt{\Delta}}{2\ell^2} \leq \frac{1}{\ell}.$$

With such choices of ε, d and ρ, we now define $H(c, \delta)$ as the smallest integer $h \geq 1$ such that

$$c_0 2^J (1+x)^J J!^d \delta^{J^2} (2(1+x) j_0^d)^{\lfloor j_0 \rfloor} e^{\log(x j_0^d)^2 / \log(1/\omega)} \leq h^{rd}, \tag{4.7}$$

where $x = 2crdh\delta^{r-1}$. We then obtain (1.11) with $\max \deg(p_j) \leq d-1 = \max(0, \lceil \frac{5\log(\delta)-1}{1-\log(\delta)} \rceil)$ and $S \leq r-1 \leq \lfloor \rho \log(h) \rfloor \leq \log(H(c,\delta))/\ell$. Notice that $H(c, \delta)$ also depends on the choice of ρ and we now explain how to bound it.

The left-hand side of (4.7) is an increasing function of $h \geq 1$, which appears in the expressions of $r := \lfloor \rho \log(h) \rfloor + 1$ and $x := 2crdh\delta^{r-1}$. Hence,

$$H(c,\delta)^{d(1+\lfloor \rho \log(H(c,\delta)) \rfloor)}$$

is larger than the value A (which is ≥ 1) of the left-hand side of (4.7) at $h = 1$, in which case $r = 1$ and $x = 2cd$. It follows that $\log(A) \leq d \log(H(c,\delta)) + d\rho \log(H(c,\delta))^2$, so that

$$H(c,\delta) \geq \exp\Big(\frac{\sqrt{d^2 + 4d\rho \log(A)} - d}{d\rho} \Big). \tag{4.8}$$

Since $A \to +\infty$ when $\delta \to e$ and $\varepsilon \to 0$ (because of the term $1/\log(\omega)$), $H(c,\delta)$ can be very large.

We now explain how to bound $Y := H(c,\delta)$ from above. We assume that $H(c,\delta) \geq 2$ otherwise there is nothing else to do. The left-hand side of (4.7) is greater than or equal to 1, so that we can take the logarithms of both sides. After some transformations, we obtain a function $S(h) \geq 0$ for all $h \geq 1$ (which could be explicited) such that Y is the smallest integer $h \geq 1$ such that

$$S(h) \leq \left(\rho d - \frac{(1+\rho\ell)^2}{\log(1/\omega)} \right) \log(h)^2. \tag{4.9}$$

Recall that

$$\gamma := \rho d - \frac{(1+\rho\ell)^2}{\log(1/\omega)} > 0.$$

Moreover, there exist α and β that depend on ρ, d, δ and ε (and could be explicited as well) such that $S(h) \le \alpha \log(h) + \beta$ for all $h \ge 1$. Since $Y \ge 2$ is the smallest integer such that (4.9) holds, we have

$$\gamma \log(Y-1)^2 < S(Y-1) \le \alpha \log(Y-1) + \beta.$$

Hence, $\log(Y-1)$ is smaller than the largest solution of the quadratic equation

$$\gamma X^2 - \alpha X - \beta = 0,$$

so that finally

$$H(c,\delta) \le 1 + \exp\left(\frac{\alpha + \sqrt{\alpha^2 + 4\beta\gamma}}{2\gamma}\right). \tag{4.10}$$

5 Proof of Theorem 4

Let $F(z)$ be as in the theorem such that $F(n) = a_n$ for all $n > u$. Notice that $\widetilde{a}_n := a_{n+\lceil u \rceil + 2}$, $n \ge 0$, is a primary peudo-polynomial; it is a polynomial if and only if $(a_n)_{n\ge 0}$ is a polynomial. The function $\widetilde{F}(z) := F(z + \lceil u \rceil + 2)$ is analytic in $\Re(z) > -2$, satisfies $|\widetilde{F}(z)| \le \tilde{c} \cdot \exp(\rho \Re(z))$ in $\Re(z) > -2$ for some constant $\tilde{c} > 0$, and $\widetilde{a}_n = \widetilde{F}(n)$ for all $n \ge 0$. Moreover, $F(z)$ is in $\mathbb{Q}[z]$ if and only if $\widetilde{F}(z)$ is in $\mathbb{Q}[z]$. Therefore, without loss of generality, we can and will assume that $F(z)$ is analytic in $\Re(z) > -2$ and that $F(n) = a_n$ for all integers $n \ge 0$.

Let $\mathcal{C}_n$ denote the circle of center n and radius n oriented in the direct sense. The function $F(z-1)$ being analytic in $\Re(z) > -1$, the residue theorem yields

$$\frac{(n-1)!}{2i\pi} \int_{\mathcal{C}_n} \frac{F(z-1)}{(z-1)\cdots(z-n)} dz = \sum_{k=0}^{n-1} (-1)^{n-k-1} \binom{n-1}{k} a_k := b_{n-1} \in \mathbb{Z}.$$

We parametrize the circle $\mathcal{C}_n$ as $n + ne^{2ix} = 2n\cos(x)e^{ix}$ for $x \in [-\pi/2, \pi/2]$. Nörlund [12, p. 387] proved that

$$\left|\frac{(n-1)!}{(z-1)\cdots(z-n)}\right| \le c_1(n) e^{-2n\cos(x)\psi(x)}, \quad z = 2n\cos(x)e^{ix} \in \mathcal{C}_n,$$

where $c_1(n) > 0$ is bounded above by some polynomial in n and

$$\psi(x) := \cos(x)\log(2\cos(x)) + x\sin(x).$$

(See also the proof given in [16].) The minimum on $[-\pi/2, \pi/2]$ of $\psi(x)$ is $\log(2)$ at $x = 0$.

We have

$$|F(z-1)| \leq c \cdot e^{\rho \Re(z-1)} = ce^{-\rho} \cdot e^{2n\rho \cos(x)^2}, \quad z = 2n\cos(x)e^{ix} \in \mathcal{C}_n.$$

Hence

$$|b_{n-1}| \leq c_2(n) \max_{x \in [-\pi/2, \pi/2]} e^{2n\cos(x)(\rho\cos(x) - \psi(x))}$$

where $c_2(n) > 0$ is bounded above by some polynomial in n. Notice that $0 \leq \cos(x) \leq 1$ on $[-\pi/2, \pi/2]$. Recall also that $\rho > 0$. Hence if x is such that $\rho\cos(x) - \psi(x) \geq 0$, then

$$\cos(x)(\rho\cos(x) - \psi(x)) \leq \rho - \psi(x) \leq \rho - \log(2),$$

whereas if x is such that $\rho\cos(x) - \psi(x) \leq 0$, then $\cos(x)(\rho\cos(x) - \psi(x)) \leq 0$. Therefore, for all $x \in [-\pi/2, \pi/2]$,

$$e^{2\cos(x)(\rho\cos(x) - \psi(x))} \leq \max(1, e^{2(\rho - \log(2))})$$

and $|b_{n-1}| \leq c_2(n)\max(1, e^{2(\rho-\log(2))})^n$. Since $2(\rho - \log(2)) < 1$, it follows that

$$\limsup_{n \to +\infty} |b_n|^{1/n} < e.$$

But because $(a_n)_{n \geq 0}$ is a primary pseudo-polynomial, we know by Theorem 1(ii) that if b_n is not eventually equal to 0 then

$$\liminf_{n \to +\infty} |b_n|^{1/n} \geq e.$$

This implies that b_n is indeed eventually equal to 0, thus that there exist $P(X) \in \mathbb{Q}[X]$ and an integer $N \geq 0$ such that $a_n = P(n)$ for all $n \geq 0$.

Consider now the function $g(z) := F(z) - P(z)$ which is analytic in $\Re(z) > -2$ (at least), and such that $g(n) = 0$ for every integer $n \geq 0$. Moreover, since $\rho > 0$, there exists a constant $d > 0$ such that $|g(z)| \leq d \cdot \exp(\rho|z|)$ for any z such that $\Re(z) > 0$. Since $\rho < \frac{1}{2} + \log(2) < \pi$, we can then apply a classical result of Carlson (see Hardy [7, p. 328]) and deduce that $g(z) = 0$ identically. Hence $F(z)$ reduces to a polynomial function in $\mathbb{Q}[z]$. This completes the proof of Theorem 4.

6 Construction of genuine primary pseudo-polynomials

We conclude this paper by presenting a method to construct a non-polynomial primary pseudo-polynomial starting from a given primary pseudo-polynomial $(a_n)_{n \geq 0}$ such that $a_0 = 1$. The justification of the method uses a non-trivial property satisfied by E-functions.

Let as usual $(b_n)_{n \geq 0}$ be the binomial transform (1.2) of $(a_n)_{n \geq 0}$. Let

$$F_b(x) := \sum_{n=0}^{\infty} \frac{b_n}{n!} x^n.$$

We assume that $a_0 = 1$, so that $b_0 = 1$ as well. We define the sequence $(c_n)_{n\geq 0}$ formally by

$$F_c(x) := \sum_{n=0}^{\infty} \frac{c_n}{n!} x^n = \frac{1}{F_b(x)}.$$

Let us now define $(u_n)_{n\geq 0}$ as the inverse binomial transform (1.3) of $(c_n)_{n\geq 0}$, *i.e.*

$$u_n := \sum_{k=0}^{n} \binom{n}{k} c_k.$$

Then we have the following.

Theorem 5. *Let $(a_n)_{n\geq 0} \in \mathbb{Z}^{\mathbb{N}}$ be a primary pseudo-polynomial such that $a_0 = 1$. Then, $(u_n)_{n\geq 0} \in \mathbb{Z}^{\mathbb{N}}$ is a primary pseudo-polynomial. Moreover, assuming also that $f_a(x)$ is a G-function, if $\limsup_n |u_n|^{1/n} < e$, then $a_n = u_n = 1$ for all $n \geq 0$.*

Consequently, if $f_a(x)$ is a G-function not equal to $(1-x)^{-1}$, then $(u_n)_{n\geq 0} \in \mathbb{Z}^{\mathbb{N}}$ is a primary pseudo-polynomial such that $\limsup_n |u_n|^{1/n} \geq e$ (hence not a polynomial). We explain after the proof of the theorem why u_n grows like $n!$ in this case.

So far, the assumption that $f_a(x)$ is a G-function is known to be satisfied only when $\limsup_n |a_n|^{1/n} < e$ (when the Perelli-Zannier Theorem can be applied), which in turn implies that $(a_n)_{n\geq 0}$ should be a polynomial by Conjecture 2. Hence, in practice the second assertion of Theorem 5 is useful only when $(a_n)_{n\geq 0}$ is already known to be a polynomial in which case $F_b(x) \in \mathbb{Q}[x]$. For instance, if $a_n = n+1$, then

$$\sum_{n=0}^{\infty} \frac{c_n}{n!} x^n = \frac{1}{1+x} \quad \text{and} \quad u_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} k!. \tag{6.1}$$

Proof of Theorem 5. We first prove that for all $n \geq 0$, $c_n \in \mathbb{Z}$ and for every prime $p \leq n$, p divides c_n. By definition of the c_n's, we have

$$1 = \left(\sum_{n=0}^{\infty} \frac{b_n}{n!} x^n \right) \left(\sum_{n=0}^{\infty} \frac{c_n}{n!} x^n \right) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\sum_{k=0}^{n} \binom{n}{k} b_k c_{n-k} \right) x^n,$$

so that $c_0 = 1$ and, for every integer $n \geq 1$, we have

$$\sum_{k=0}^{n} \binom{n}{k} b_k c_{n-k} = 0.$$

This yields the recursive relation (because $b_0 = 1$)

$$c_n = -\sum_{k=1}^{n} \binom{n}{k} b_k c_{n-k}, \quad (n \geq 1).$$

It follows that, for all $n \geq 0$, c_n is an integer.

Let n be a positive integer such that, for every positive integer $m < n$ and every prime number $p \leq m$, p divides c_m. Consider a prime number $p \leq n$ and an integer k in $\{1, \ldots, n\}$. If $p \leq k$ then p divides b_k. If $p \leq n-k$, then p divides c_{n-k}. If $p > \max(k, n-k)$, then p divides $\binom{n}{k}$ because p divides $n!$ but neither $k!$ nor $(n-k)!$. In all cases, p divides $\binom{n}{k} b_k c_{n-k}$, so that p divides c_n. By strong induction on n, it follows that, for every integer $n \geq 0$ and every prime $p \leq n$, p divides c_n (this property holds trivially if $n = 0$ or 1). By Theorem 1(i), the sequence $(u_n)_{n\geq 0} \in \mathbb{Z}^{\mathbb{N}}$ is a primary pseudo-polynomial. Notice that so far we had no need to assume that f_a is a G-function.

We now complete the proof of Theorem 5 when f_a is also a G-function. We first observe that $F_b(x)$ is an E-function: indeed, $f_b(x)$ is a G-function because

$$f_b(x) := \sum_{n=0}^{\infty} b_n x^n = \frac{1}{1+x} f_a\left(\frac{x}{1+x}\right)$$

and $f_a(x)$ is a G-function. ([3]) Let us assume that $\limsup_n |u_n|^{1/n} < e$. By the Perelli-Zannier Theorem quoted in the Introduction, $f_u(x) := \sum_{n=0}^{\infty} u_n x^n$ is a G-function. From the equation $c_n := \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} u_k$ $(\forall n \geq 0)$, we deduce that

$$f_c(x) := \sum_{n=0}^{\infty} c_n x^n = \frac{1}{1+x} f_u\left(\frac{x}{1+x}\right)$$

is also a G-function. Hence $F_c(x) := 1/F_b(z)$ is also an E-function. Therefore, $F_b(x)$ is a unit of the ring of E-functions, *i.e.* it is of the form $\beta e^{\alpha z}$ with $\alpha \in \overline{\mathbb{Q}}$ and $\beta \in \overline{\mathbb{Q}}^*$ by [1, p. 717] (see also Footnote 4). Therefore, $b_n = \beta \alpha^n$ for all $n \geq 0$, with the usual convention that $\alpha^0 = 1$ if $\alpha = 0$. Since all primes $p \leq n$ divide b_n, we deduce that $\alpha = 0$ is the only possibility. Hence $1 = b_0 = \beta$ and $b_n = 0$ for all $n \geq 0$, so that $a_n = 1$ for all $n \geq 0$. It now remains to observe the following facts: "$a_n = 1$ for all $n \geq 0$" is equivalent to "$b_0 = 1$ and $b_n = 0$ for all $n \geq 1$" which is equivalent to "$c_0 = 1$ and $c_n = 0$ for all $n \geq 1$", which in turn is equivalent to "$u_n = 1$ for all $n \geq 0$". □

To conclude, let us explain how to obtain the asymptotic behavior of u_n as $n \to +\infty$. Since

$$F_u(x) := \sum_{n=0}^{\infty} \frac{u_n}{n!} x^n = e^x F_c(x) = e^x / F_b(x),$$

the asymptotic behavior of $u_n/n!$ is determined by the zeroes of smallest modulus of $F_b(x)$, when it has at least one. Notice that an E-function $F(x)$ with no zero in $\mathbb{C}$ must be of the form $\beta e^{\alpha x}$ with $\alpha \in \overline{\mathbb{Q}}$ and $\beta \in \overline{\mathbb{Q}}^*$. Indeed, an E-function is an entire function satisfying $|F(x)| \ll e^{\rho|x|}$ for some $\rho > 0$, so that Hadamard's factorization theorem yields $F(x) = \beta x^m e^{\alpha x} \prod_{j\geq 1} \left((1 - \frac{x}{x_j}) e^{x/x_j}\right)$ where the x_j's are the zeroes of $F(x)$, $m \in \mathbb{N}$ and

[3] Given a G-function $f(x)$ and $\alpha(x)$ an algebraic function over $\overline{\mathbb{Q}}(x)$ regular at $x = 0$, $f(x\alpha(x))$ is a G-function.

$\alpha, \beta \in \mathbb{C}$. ([4]) Therefore, a "no zero" assumption implies that $F(x) = \beta e^{\alpha x}$ with $\beta \neq 0$, and since $F(x) \in \overline{\mathbb{Q}}[[x]]$, α, β must be in $\overline{\mathbb{Q}}$. Coming back to $F_b(x)$, we have seen during the proof of Theorem 5 that this case implies that $\alpha = 0, \beta = 1$, hence that $f_a(x) = (1-x)^{-1}$.

Therefore, assuming that $f_a(x)$ is a G-function different of $(1-x)^{-1}$, the E-function $F_b(x)$ has at least one zero. Let $x_1, \ldots, x_m$ denote the zeroes of $F_b(x)$ of the same modulus which is the smallest amongst all modulus of the zeroes. Classical transfer theorems in [4, Chapter VI] enable to deduce the asymptotic behavior of u_n. For instance, if the x_j's are simple zeroes of $F_b(x)$, then

$$u_n = n! \sum_{j=1}^{m} \frac{e^{x_j}}{F_b'(x_j)} \frac{1+o(1)}{x_j^n}.$$

This is coherent with (6.1) above (where $F_b(x) = 1 + x$) because we can rewrite it as

$$u_n = (-1)^n n! \sum_{k=0}^{n} \frac{(-1)^{n-k}}{(n-k)!} \sim \frac{(-1)^n}{e} n!, \quad n \to +\infty.$$

Because of the different arguments of the x_j's, oscillations can occur. In presence of zeroes of $F_b(x)$ of higher multiplicities, similar but more complicated expressions can be given. Finally, even though $F_b(x)/e^x$ is an E-function, we don't expect $e^x/F_b(x)$ to be D-finite in general, ([5]) but this is obviously the case if $F_b(x)$ is a polynomial.

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[4] This argument also explains the characterization of the units of the ring of E-functions: we simply have to observe that if an E-function $F(x)$ is a unit, *i.e.* that $1/F(x)$ is an E-function, then it does not vanish anywhere on $\mathbb{C}$ because an E-function is an entire function.

[5] A classical result of Harris-Sibuya [8] ensures that if y and $1/y$ are both holonomic, then y'/y is an algebraic function.

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