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# On primary pseudo-polynomials (Around Ruzsa's Conjecture)

É. Delaygue and T. Rivoal

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## Abstract

Every polynomial  $P(X) \in \mathbb{Z}[X]$  satisfies the congruences  $P(n+m) \equiv P(n) \pmod{m}$  for all integers  $n, m \geq 0$ . An integer valued sequence  $(a_n)_{n \geq 0}$  is called a pseudo-polynomial when it satisfies these congruences. Hall characterized pseudo-polynomials and proved that they are not necessarily polynomials. A long standing conjecture of Ruzsa says that a pseudo-polynomial  $a_n$  is a polynomial as soon as  $\limsup_n |a_n|^{1/n} < e$ . Under this growth assumption, Perelli and Zannier proved that the generating series  $\sum_{n=0}^{\infty} a_n z^n$  is a  $G$ -function. A primary pseudo-polynomial is an integer valued sequence  $(a_n)_{n \geq 0}$  such that  $a_{n+p} \equiv a_n \pmod{p}$  for all integers  $n \geq 0$  and all prime numbers  $p$ . The same conjecture has been formulated for them, which implies Ruzsa's, and this paper revolves around this conjecture. We obtain a Hall type characterization of primary pseudo-polynomials and draw various consequences from it. We prove that any primary pseudo-polynomial with an algebraic generating series is a polynomial. We make the Perelli-Zannier Theorem effective and show that its conclusion does not necessarily hold if  $\limsup_n |a_n|^{1/n} \leq e$ . We prove a Pólya type result: if there exists a function  $F$  analytic in a right-half plane with not too large exponential growth (in a precise sense) and such that for all large  $n$  the primary pseudo-polynomial  $a_n = F(n)$ , then  $a_n$  is a polynomial. Finally, we show how to construct a non-polynomial primary pseudo-polynomial starting from any primary pseudo-polynomial generated by a  $G$ -function different of  $1/(1-x)$ .

## 1 Introduction

Given a sequence  $(a_n)_{n \geq 0} \in \mathbb{C}^{\mathbb{N}}$ , we define its *binomial transform*  $(b_n)_{n \geq 0} \in \mathbb{C}^{\mathbb{N}}$  as

$$\forall n \geq 0, \quad b_n := \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} a_k. \quad (1.1)$$

It is well-known that  $(a_n)_{n \geq 0}$  can be recovered from  $(b_n)_{n \geq 0}$  by

$$\forall n \geq 0, \quad a_n = \sum_{k=0}^n \binom{n}{k} b_k, \quad (1.2)$$

ie, the binomial transform is “almost” involutive. An important property of the binomial transform is that “there exists  $P(X) \in \mathbb{C}[X]$  such that  $a_n = P(n)$  for all  $n$  large enough” if and only if “ $b_n = 0$  for all  $n$  large enough”. In the sequel, we will say that a sequence  $a_n$  is “eventually in  $\mathbb{A}[n]$ ” (where  $\mathbb{A}$  is a given ring) when there exist a polynomial  $P(X) \in \mathbb{A}[X]$  and an integer  $N$  such that  $a_n = P(n)$  for all  $n \geq N$ .

A sequence  $(a_n)_{n \geq 0} \in \mathbb{Z}^{\mathbb{N}}$  is said to be a *pseudo-polynomial* when the following property holds: for any integers  $n, k \geq 0$ ,  $a_{n+k} \equiv a_n \pmod{k}$ . The terminology comes from the fact that for any  $P(X) \in \mathbb{Z}[X]$ , the sequence  $(P(n))_{n \geq 0}$  is a pseudo-polynomial. Note that if  $P(X) \in \mathbb{C}[X]$  is such that the sequence  $(P(n))_{n \geq 0}$  is a pseudo-polynomial, then  $P(X) \in \mathbb{Q}[X]$ , but  $P(X)$  does not necessarily belong to  $\mathbb{Z}[X]$  as  $P(X) = \frac{1}{2}X(X+1)$  shows. Pseudo-polynomials have long been studied for themselves, but they have also found recent applications in analytic number theory [8, 9].

We now set  $d_n := \text{lcm}\{1, 2, \dots, n\}$  for  $n \geq 1$  and  $d_0 := 1$ . We recall that  $d_n = \prod_{p \leq n} p^{\lfloor \log_p(n) \rfloor} \leq 3^n$  for all  $n$  and that  $d_n^{1/n} \rightarrow e$  as  $n \rightarrow +\infty$  (by the Prime Number Theorem). Hall [5] proved the following fundamental property: a sequence  $(a_n)_{n \geq 0} \in \mathbb{Z}^{\mathbb{N}}$  is a pseudo-polynomial if and only if its binomial transform  $(b_n)_{n \geq 0} \in \mathbb{Z}^{\mathbb{N}}$  satisfies

$$\forall n \geq 0, d_n \mid b_n. \tag{1.3}$$

Using this characterization with  $b_n := n!$  in (1.3), we see from (1.2) that the resulting pseudo-polynomial  $a_n$  is simply equal to  $[n!e]$  for  $n \geq 1$  (but not for  $n = 0$ ), which is obviously not a polynomial. With  $b_n := d_n$  in (1.3), we obtain from (1.2) another pseudo-polynomial of slower growth  $a_n := \sum_{k=0}^n \binom{n}{k} d_k \leq 4^n$ ; since  $a_n \geq 2^n$  for all  $n \geq 0$ , this is not a polynomial either.

The search of minimal growth conditions that can be attained by non-polynomial pseudo-polynomials has been the subject of many papers. Hall [5] and Ruzsa [16] independently proved that if

$$\limsup_{n \rightarrow +\infty} |a_n|^{1/n} < e - 1 \tag{1.4}$$

then  $a_n$  is eventually in  $\mathbb{Q}[n]$ . Ruzsa proposed the following

**Conjecture 1** (Ruzsa). *Let  $(a_n)_{n \geq 0} \in \mathbb{Z}^{\mathbb{N}}$  be a pseudo-polynomial such that*

$$\limsup_{n \rightarrow +\infty} |a_n|^{1/n} < e. \tag{1.5}$$

*Then  $a_n$  is eventually in  $\mathbb{Q}[n]$ .*

Using his characterization (1.3), Hall [5, p. 76] sketched an inductive construction of a non-polynomial pseudo-polynomial  $(a_n)_{n \geq 0}$  such that  $\limsup_{n \rightarrow +\infty} |a_n|^{1/n} \leq e$ , showing that the upper bound  $< e$  is best possible in Ruzsa’s conjecture. Perelli and Zanier [13] then proved a highly non-trivial property: under the growth condition (1.5) in Ruzsa’s Conjecture, the pseudo-polynomial sequence  $(a_n)_{n \geq 0}$  satisfies a linear recurrence with polynomial coefficients; see (1.9) below. In other words, the generating function  $f_a(x) := \sum_{n=0}^{\infty} a_n x^n \in \mathbb{Z}[[x]]$  satisfies a linear differential equation with coefficients in  $\mathbb{Z}[x]$ .

Hence,  $f_a$  is a  $G$ -function <sup>(1)</sup>. Perelli and Zannier also proved a form of Ruzsa's Conjecture 1 under a stronger assumption than (1.5), ie with  $e$  replaced by  $e^{0.66}$ .

In fact, many results towards Ruzsa's conjecture have been proven for sequences we shall call *primary pseudo-polynomial* (for lack of better terminology).

**Definition 1.** A sequence  $(a_n)_{n \geq 0} \in \mathbb{Z}^{\mathbb{N}}$  is said to be a *primary pseudo-polynomial* when the following property holds: for any integer  $n \geq 0$  and any prime number  $p$ ,  $a_{n+p} \equiv a_n \pmod{p}$ .

The set of primary pseudo-polynomials is a ring for the term by term sum and product of sequences in  $\mathbb{Z}^{\mathbb{N}}$ , with the null and unit sequences defined with all terms equal to 0 and all terms equal to 1 respectively. A pseudo-polynomial is a primary pseudo-polynomial but the converse is false (see the comments following Theorem 1 below). Many authors dealt with the following conjecture, the truth of which would imply that of Ruzsa.

**Conjecture 2.** Let  $(a_n)_{n \geq 0} \in \mathbb{Z}^{\mathbb{N}}$  be a primary pseudo-polynomial such that

$$\limsup_{n \rightarrow +\infty} |a_n|^{1/n} < e. \quad (1.6)$$

Then  $a_n$  is eventually in  $\mathbb{Q}[n]$ .

As in the case of pseudo-polynomials,  $< e$  cannot be replaced by  $\leq e$ , and we refer again to the comments following Theorem 1 for a proof of this. In fact, the above quoted results of Perelli and Zannier hold more generally for primary pseudo-polynomials, and Zannier [19] was even able to replace  $e^{0.66}$  by  $e^{0.75}$ .

In this paper, we are interested in the properties of primary pseudo-polynomials and of their generating functions. We now present our four main results, make comments about their significance and mention further open problems.

• **A Hall type characterization of primary pseudo-polynomials.** We shall first prove an analogue (ie Eq. (1.7) below) of Hall's characterization for pseudo-polynomials and deduce some consequences of it. We set  $P_n := \prod_{p \leq n} p$  for  $n \geq 2$  and  $P_0 = P_1 := 1$ , where the product is over prime numbers. By the Prime Number Theorem,  $P_n^{1/n} \rightarrow e$  as  $n \rightarrow +\infty$ .

**Theorem 1.** (i) A sequence  $(a_n)_{n \geq 0} \in \mathbb{Z}^{\mathbb{N}}$  is primary pseudo-polynomial if and only if its binomial transform  $(b_n)_{n \geq 0} \in \mathbb{Z}^{\mathbb{N}}$  is such that

$$\forall n \geq 0, P_n \mid b_n. \quad (1.7)$$

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<sup>1</sup>A power series  $\sum_{n=0}^{\infty} a_n x^n \in \overline{\mathbb{Q}}[[x]]$  is said to be a  $G$ -function when it is solution of a non-zero linear differential equation over  $\overline{\mathbb{Q}}(x)$  ( $D$ -finiteness), and the maximum of the modulus of all the Galoisian conjugates of  $a_0, \dots, a_n$  as well as the positive denominator of  $a_0, \dots, a_n$  are both bounded for all  $n \geq 0$  by  $C^{n+1}$ , for some  $C \geq 1$ . For instance, any  $D$ -finite series in  $\mathbb{Z}[[x]]$  with positive radius of convergence is a  $G$ -function. A power series  $\sum_{n=0}^{\infty} \frac{a_n}{n!} z^n \in \overline{\mathbb{Q}}[[z]]$  is said to be an  $E$ -function when  $\sum_{n=0}^{\infty} a_n x^n$  is a  $G$ -function. See [1, 17] for the properties satisfied by these functions.

(ii) Given a primary pseudo-polynomial  $(a_n)_{n \geq 0}$ , if there is no  $Q(X) \in \mathbb{Q}[X]$  such that  $a_n = Q(n)$  for all  $n$  large enough, then

$$\liminf_{n \rightarrow +\infty} |b_n|^{1/n} \geq e.$$

(iii) If a primary pseudo-polynomial  $(a_n)_{n \geq 0}$  is such that  $\limsup_{n \rightarrow +\infty} |a_n|^{1/n} < e - 1$ , then  $a_n$  is eventually a polynomial.

(iv) Given any function  $\varphi : \mathbb{N} \rightarrow \mathbb{R}$  with  $\varphi(0) = 1$ , there exists a non-polynomial primary pseudo-polynomial  $(A_n)_{n \geq 0}$  such that  $\varphi(n) \leq A_n \leq \varphi(n) + 2P_n$  for all  $n \in \mathbb{N}$ .

We recall that  $d_n = \prod_{p \leq n} p^{\lfloor \log_p(n) \rfloor}$ . Hence, for all  $n \geq 0$ ,  $P_n$  divides  $d_n$ , but obviously  $d_n$  divides  $P_n$  for no  $n \geq 4$ . Choosing  $b_n := P_n$  in (1.7), the resulting sequence in (1.2)  $a_n := \sum_{k=0}^n \binom{n}{k} P_k$  is a primary pseudo-polynomial, but not a pseudo-polynomial because it does not satisfy Hall's criterion (1.3). Under the assumption in (ii), if we also assume that  $b_n$  is eventually of the same sign, then  $\liminf_{n \rightarrow +\infty} |a_n|^{1/n} \geq e + 1$  because  $a_n = \sum_{k=0}^n \binom{n}{k} b_k$ . Consequently, any putative counter-example  $(a_n)_{n \geq 0}$  to Conjecture 2 must be such that its binomial transform  $(b_n)_{n \geq 0}$  changes sign infinitely often. A similar remark applies to Ruzsa's Conjecture 1.

Assertion (iii) is the analogue of the Hall-Ruzsa result recalled at Eq. (1.4).

In (iv), given  $\varphi$ , the existence of sequence  $(A_n)_{n \geq 0}$  is proved constructively by an inductive process. An important consequence of (iv) is the existence of a non-polynomial primary pseudo-polynomial of any growth  $\varphi(n) > 0$  provided  $\liminf_n \varphi(n)^{1/n} \geq e$ . In particular, with  $\varphi(n) = P_n$ , we deduce that  $< e$  cannot be replaced by  $\leq e$  on the right-hand side of (1.6) in Conjecture 2.

• **Primary pseudo-polynomials with an algebraic generating series.** The generating functions  $f_a$  and  $f_b$  of the sequences  $(a_n)_{n \geq 0}$  and its binomial transform  $(b_n)_{n \geq 0}$  satisfy the relations

$$f_b(x) = \frac{1}{1+x} f_a\left(\frac{x}{1+x}\right), \quad f_a(x) = \frac{1}{1-x} f_b\left(\frac{x}{1-x}\right). \quad (1.8)$$

In particular  $f_a(x)$  is algebraic over  $\mathbb{Q}(x)$  if and only if  $f_b(x)$  is algebraic over  $\mathbb{Q}(x)$ .

**Theorem 2.** Let  $(a_n)_{n \geq 0} \in \mathbb{Z}^{\mathbb{N}}$  be a primary pseudo-polynomial, and let  $(b_n)_{n \geq 0} \in \mathbb{Z}^{\mathbb{N}}$  be its binomial transform.

- (i) If  $f_a(x)$  is algebraic over  $\mathbb{Q}(x)$ , then  $a_n$  is eventually in  $\mathbb{Q}[n]$  (and thus  $f_a(x) \in \mathbb{Q}(x)$ ).
- (ii) If  $f_b(x)$  is algebraic over  $\mathbb{Q}(x)$ , then  $f_b(x)$  is in  $\mathbb{Z}[x]$ .

Theorem 2 implies in particular that the Taylor coefficients of an irrational algebraic function over  $\mathbb{Q}(x)$  do not define a primary pseudo-polynomial, and *a fortiori* not a pseudo-polynomial either. It also implies that if there exists a counter example to Conjecture 2 or to Ruzsa's Conjecture 1, then its generating function  $f_a(x)$  is transcendental over  $\mathbb{C}(x)$ . A slight generalization of Theorem 2(i) is presented in §3.

By definition, the diagonal of a multivariate power series  $\sum_{n_1, \dots, n_k \geq 0} u_{n_1, \dots, n_k} z_1^{n_1} \cdots z_k^{n_k}$  is the univariate power series  $\sum_{n=0}^{\infty} u_{n, n, \dots, n} z^n$ . A classical result of Furstenberg [4] says that algebraic series coincide with diagonals of rational functions in two variables. It would be very interesting to know if the conclusions of Theorem 2 also hold when  $f_a$  and  $f_b$  are assumed to be diagonals of rational functions of an arbitrary (but fixed) number of variables (which they are, or not, simultaneously). Moreover, diagonal of rational functions are globally bounded  $G$ -functions in the sense of Christol, who conjectured that the converse holds (see [2]). In particular,  $G$ -functions with integer coefficients are globally bounded. Therefore given the Perelli-Zannier Theorem quoted just below, Christol's Conjecture and the above putative generalization of Theorem 2 would together imply Conjecture 2 and Ruzsa's Conjecture 1. See also related comments in [19, pp. 392–393].

• **An effective version of a result of Perelli and Zannier.** In [13], Perelli and Zannier sketched the proof of the following result.

**Theorem** (Perelli-Zannier). *Let  $(a_n)_{n \geq 0}$  be a primary pseudo-polynomial such that there exist  $c > 0$  and  $1 < \delta < e$  such that  $|a_n| \leq c\delta^n$  for all  $n \geq 0$ . Then there exist an integer  $S \geq 0$  and  $S + 1$  polynomials  $p_0(X), \dots, p_S(X) \in \mathbb{Z}[X]$  not all zero such that*

$$\sum_{j=0}^S p_j(n) a_{n+j} = 0. \quad (1.9)$$

*In other words,  $f_a(x)$  is  $D$ -finite, and even a  $G$ -function.*

Perelli and Zannier mentioned that it would be possible to provide upper bounds for  $S$  and the degree/height of the  $p_j(X)$  in terms of  $\delta$ , but they did not write them down. We make more precise their theorem as follows, where given  $Q(X) = \sum_j q_j X^j \in \mathbb{C}[X]$ , we set  $H(Q) := \max_j |q_j|$ .

**Theorem 3.** *In the conditions and notations of the Perelli-Zannier Theorem, for any  $\delta \in (1, e)$ , there exists an effectively computable constant  $H(\delta) \geq 1$  such that a non-trivial linear recurrence for  $(a_n)_{n \geq 0}$  as in (1.9) holds with*

$$\begin{cases} \max_j \deg(p_j) \leq \max \left( 0, \left\lceil \frac{5 \log(\delta) - 1}{1 - \log(\delta)} \right\rceil \right), \\ \max_j H(p_j) \leq H(\delta), \\ S \leq \log(H(\delta)) / \log(\delta). \end{cases} \quad (1.10)$$

*Moreover, the Perelli-Zannier Theorem is best possible in the sense that its conclusion does not necessarily hold if  $\delta = e$ .*

To prove the final statement in Theorem 3, we take  $\varphi(n) := P_n$  in Theorem 1(iv): we obtain a non-polynomial primary pseudo-polynomial  $A_n$  such that  $|A_n|^{1/n} \rightarrow e$ . Hence

$(A_n)_{n \geq 0}$  does not satisfy a non-zero linear recurrence with coefficients in  $\mathbb{Q}[n]$ . <sup>(2)</sup>

Lower and upper bounds for the function  $H(\delta)$  are given in (4.8) and (4.10) respectively in §4.3. Our bound for  $\max \deg(p_j)$  in (1.10) is obviously not optimal; in fact, at the cost of more complicated computations, Perelli-Zannier [13] and then Zannier [19] obtained better bounds when  $\delta \leq e^{0.66}$  and  $\delta \leq e^{0.75}$ .

The classification of primary pseudo-polynomials with a  $D$ -finite generating series is an open problem. As shown by the above example  $(A_n)_{n \geq 0}$ , Theorem 1 rules out the possibility that *every* primary pseudo-polynomial satisfies a linear recurrence with coefficients in  $\mathbb{Q}[n]$ . Another example is the primary pseudo-polynomial  $D_n := \sum_{k=0}^n \binom{n}{k} P_k$ : it cannot satisfy such a linear recurrence because otherwise  $P_n = \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} D_k$  would satisfy one as well, which is not possible because  $P_n^{1/n} \rightarrow e$ . Since  $P_n \geq 0$  and  $P_n = e^{n+o(n)}$ , a simple analytic argument shows that  $D_n = (e+1)^{n+o(n)}$ . More specifically, it would be interesting to know if there is any non-polynomial primary pseudo-polynomial  $(a_n)_{n \geq 0}$  such that  $f_a(x)$  is a  $G$ -function. There exist primary pseudo-polynomials  $(a_n)_{n \geq 0}$  such that  $f_a$  is  $D$ -finite but is not a  $G$ -function. For instance,  $e_n := \lfloor (n+1)!e \rfloor = \sum_{k=0}^{n+1} \binom{n+1}{k} k!$  ( $n \geq 0$ ) is a primary pseudo-polynomial by Theorem 1, and for all  $n \geq 0$ ,  $e_{n+2} = (n+4)e_{n+1} - (n+2)e_n$  ( $e_0 = 2, e_1 = 5$ ), so that  $\sum_{n=0}^{\infty} e_n x^n$  is  $D$ -finite. A method to obtain further examples is presented at the end of §6.

• **A Pólya type result for primary pseudo-polynomials.** Perelli and Zannier also proved in [12] that if a (primary) pseudo-polynomial  $a_n = F(n)$  for some entire function  $F$  such that  $\limsup_{R \rightarrow \infty} \frac{1}{R} \log \max_{|x|=R} |F(x)| < \log(e+1)$ , then  $a_n$  is in  $\mathbb{Q}[n]$ . We prove here a result of a similar flavor with a different analyticity condition.

**Theorem 4.** *Let  $(a_n)_{n \geq 0} \in \mathbb{Z}^{\mathbb{N}}$  be a primary pseudo-polynomial. Let us assume that there exists  $F(x)$  analytic in a right-half plane  $\Re(x) > u$  such that  $a_n = F(n)$  for all  $n > u$ , and  $c > 0, 0 < \rho < \log(2\sqrt{e})$  such that*

$$|F(x)| \leq c \cdot e^{\rho \Re(x)} \tag{1.11}$$

for  $\Re(x) > u$ . Then  $F(x)$  is in  $\mathbb{Q}[x]$  and  $a_n$  is eventually in  $\mathbb{Q}[n]$ .

We have  $\log(2\sqrt{e}) \approx 1.193$  while  $\log(e+1) \approx 1.313$ . In the proof, we shall obtain that  $a_n$  is eventually a polynomial before proving that  $F$  is a polynomial. Because of the Perelli-Zannier Theorem recalled before Theorem 3, the assumptions of Theorem 4 are in fact natural in the context of Ruzsa's Conjecture 1. Indeed, for any given  $G$ -function  $f(x) := \sum_{n=0}^{\infty} v_n x^n \in \overline{\mathbb{Q}}[[x]]$ , there exists a function  $\lambda(x) := \sum_{j=1}^p c_j(x) \cdot e^{\rho_j x}$  for some functions  $c_j(x)$  analytic in  $\Re(x) > u$  and of polynomial growth (at most), and such that  $v_n = \lambda(n)$  for all  $n > u$ ; the numbers  $e^{-\rho_j}$  are the finite singularities of  $f(x)$  (see [14, §7.1] for details). Notice that a bound involving  $e^{\rho_j \Re(x)}$  is *a priori* different of a bound involving

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<sup>2</sup>Indeed, if a solution  $(a_n)_{n \geq 0}$  of a linear recurrence with coefficients in  $\mathbb{Q}[n]$  is such that  $|a_n|^{1/n} \rightarrow \alpha$  finite, then  $\alpha$  is an algebraic number. Moreover, if a sequence of *rational numbers* satisfies a non-zero linear recurrence of minimal order with coefficients in  $\mathbb{C}[n]$ , then these coefficients are necessarily in  $\mathbb{Q}[n]$ , up to a common non-zero multiplicative constant. Hence  $(A_n)_{n \geq 0}$ ,  $(d_n)_{n \geq 0}$  and  $(P_n)_{n \geq 0}$  do not satisfy any non-zero linear recurrence with coefficients in  $\mathbb{C}[n]$ .

$|e^{\rho_j x}| = e^{\Re(\rho_j x)}$ , but they are the same when  $\rho_j \in \mathbb{R}$ . In particular, if all the singularities of  $f(x)$  are positive real numbers, then a bound as in (1.11) holds for  $\lambda(x)$  for some  $\rho \in \mathbb{R}$ .

In Theorem 1(iv), take  $\varphi(n) = \delta^n$  with  $e < \delta < 2\sqrt{e}$ . This yields of non-polynomial primary pseudo-polynomial  $(a_n)_{n \geq 0}$  such that  $a_n = \delta^{n+o(n)}$  as  $n \rightarrow +\infty$ . Theorem 4 thus implies that there is no function  $F(x)$  analytic in a right-half plane on which (1.11) holds, and such that  $a_n = F(n)$  for all large  $n$ .

Perelli-Zannier's result and Theorem 4 are similar to Pólya's celebrated theorem: if an entire function  $F(x)$  is such that  $\limsup_{R \rightarrow +\infty} \frac{1}{R} \max_{|x|=R} |F(x)| < \log(2)$  and  $F(\mathbb{N}) \subset \mathbb{Z}$ , then  $F(x)$  is a polynomial. See [18] for a recent survey on Pólya type results, where a connection with Rusza's Conjecture 1 is also mentioned.

Theorems 1, 2, 3 and 4 are proved in §2, §3, §4 and §5 respectively. In §6, we present a method to construct a non-polynomial primary pseudo-polynomial starting from every primary pseudo-polynomial generated by a  $G$ -function (Theorem 6).

## 2 Proof of Theorem 1

(i) Let  $(a_n)_{n \geq 0}$  be a sequence of integers and consider its binomial transform  $(b_n)_{n \geq 0}$ .

Assume that for every non-negative integer  $n$ ,  $P_n$  divides  $b_n$ . Let  $p$  be a fixed prime number. Hence, for every integer  $n \geq p$ ,  $p$  divides  $b_n$ . For every non-negative integer  $n$ , it yields

$$\begin{aligned} a_{n+p} &= \sum_{k=0}^{n+p} \binom{n+p}{k} b_k \\ &\equiv \sum_{k=0}^{p-1} \binom{n+p}{k} b_k \pmod{p} \\ &\equiv \sum_{k=0}^{p-1} \binom{n}{k} b_k \pmod{p} \\ &\equiv a_n \pmod{p}, \end{aligned}$$

where we used Lucas' congruence for binomial coefficients: for every  $u, v$  in  $\{0, \dots, p-1\}$  and every non-negative integers  $m$  and  $\ell$ , we have

$$\binom{u+mp}{v+\ell p} \equiv \binom{u}{v} \binom{m}{\ell} \pmod{p}.$$

It follows that  $(a_n)_{n \geq 0}$  is a primary pseudo-polynomial.

Conversely, assume that  $(a_n)_{n \geq 0}$  is a primary pseudo-polynomial. Let  $p$  be a prime number and  $n \geq p$  be an integer. It suffices to show that  $p$  divides  $b_n$ .



Write  $n = v + mp$  with  $v$  in  $\{0, \dots, p-1\}$  and  $m \geq 1$ . We obtain that

$$\begin{aligned}
& b_{v+mp} \\
&= \sum_{k=0}^{v+mp} (-1)^{v+mp-k} \binom{v+mp}{k} a_k \\
&= \sum_{u=0}^{p-1} \sum_{\ell=0}^{m-1} (-1)^{v-u} (-1)^{(m-\ell)p} \binom{v+mp}{u+\ell p} a_{u+\ell p} + \sum_{u=0}^v (-1)^{v-u} \binom{v+mp}{u+mp} a_{u+mp} \\
&\equiv \sum_{u=0}^{p-1} \sum_{\ell=0}^{m-1} (-1)^{v-u} (-1)^{(m-\ell)p} \binom{v}{u} \binom{m}{\ell} a_u + \sum_{u=0}^v (-1)^{v-u} \binom{v}{u} a_u \pmod{p} \\
&\equiv \sum_{u=0}^v (-1)^{v-u} \binom{v}{u} a_u \sum_{\ell=0}^m (-1)^{(m-\ell)p} \binom{m}{\ell} \pmod{p} \quad \left( \binom{v}{u} = 0 \text{ for } u = v+1, \dots, p-1 \right) \\
&\equiv \sum_{u=0}^v (-1)^{v-u} \binom{v}{u} a_u (1 + (-1)^p)^m \pmod{p} \\
&\equiv 0 \pmod{p} \quad (m \geq 1).
\end{aligned}$$

Hence  $p$  divides  $b_n$  as expected. It follows that, for every non-negative integer  $n$ ,  $P_n$  divides  $b_n$ . The first equivalence in Theorem 1 is proved.

(ii) The binomial transform  $(b_n)_{n \geq 0}$  is eventually 0 if, and only if there exists a polynomial  $Q(X)$  in  $\mathbb{Q}[X]$  such that  $a_n = Q(n)$  for every large enough non-negative integer  $n$ . Hence, if the latter is false, then  $(b_n)_{n \geq 0}$  is not eventually 0 and, since  $P_n$  divides  $b_n$ , it follows that

$$\liminf_{n \rightarrow +\infty} |b_n|^{1/n} \geq e$$

because  $P_n^{1/n} \rightarrow e$  as  $n \rightarrow +\infty$ .

(iii) If the primary pseudo-polynomial  $a_n$  is not eventually a polynomial, then by (ii) above,  $\liminf_{n \rightarrow +\infty} |b_n|^{1/n} \geq e$ . But since  $b_n = \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} a_k$ , the assumption  $\limsup_{n \rightarrow +\infty} |a_n|^{1/n} < e - 1$  implies that

$$\limsup_{n \rightarrow +\infty} |b_n|^{1/n} \leq \limsup_{n \rightarrow +\infty} \left( \sum_{k=0}^n \binom{n}{k} |a_k| \right)^{1/n} < e.$$

This proves that  $a_n$  is eventually a polynomial.

(iv) The following argument generalizes Hall's (sketchy) construction of a non-polynomial pseudo-polynomial with growth  $\leq e^{n+o(n)}$  in [5, p. 76]. By (i), we know that any sequence of integers  $(B_n)_{n \geq 0}$  such that  $P_n | B_n$  defines a primary pseudo-polynomial  $A_n := \sum_{k=0}^n \binom{n}{k} B_k$ , which is not eventually a polynomial if (and only if)  $B_n \neq 0$  for infinitely many  $n$ . Since  $A_{n-1}$  depends only on  $B_0, B_1, \dots, B_{n-1}$ , we will recursively construct  $B_n \neq 0$  and thus  $A_n$ . We have  $A_0 = B_0$ : choosing  $B_0 = 1$ , we have  $\varphi(0) = A_0 = \varphi(0) + 2P_0$ . Let  $n \geq 0$  and

let us assume that we have constructed  $B_0, B_1, \dots, B_{n-1}$  all non-zero and such that  $P_k | B_k$  and  $\varphi(k) \leq A_k \leq \varphi(k) + 2P_k$  for all  $k \in \{0, \dots, n-1\}$ .

We want to construct an integer  $B_n \neq 0$  such that  $P_n | B_n$  and  $\varphi(n) \leq A_n \leq \varphi(n) + 2P_n$ . To do this, we first set  $C_n := \sum_{k=0}^{n-1} \binom{n}{k} B_k$ , so that we will have  $A_n = B_n + C_n$ . We now perform the euclidean division of  $C_n$  by  $P_n$ : we have  $C_n = u_n P_n + v_n$  with  $u_n \in \mathbb{Z}, v_n \in \mathbb{N}$  and  $0 \leq v_n < P_n$ . We set  $B_n := w_n P_n \neq 0$  where the non-zero integer  $w_n$  is defined as follows: if  $\lceil \frac{\varphi(n) - v_n}{P_n} \rceil \neq u_n$ , we take

$$w_n = \left\lceil \frac{\varphi(n) - v_n}{P_n} \right\rceil - u_n,$$

while if  $\lceil \frac{\varphi(n) - v_n}{P_n} \rceil = u_n$ , we take  $w_n = 1$ . Since  $A_n = (u_n + w_n)P_n + v_n$ , we see that  $\varphi(n) \leq A_n \leq \varphi(n) + P_n$  in the former case, while  $\varphi(n) + P_n \leq A_n \leq \varphi(n) + 2P_n$  in the latter case. This finishes the recursive construction of a non-polynomial primary pseudo-polynomial  $(A_n)_{n \geq 0}$  such that  $\varphi(n) \leq A_n \leq \varphi(n) + 2P_n$  for all integer  $n \geq 0$ .

### 3 Proof of Theorem 2

From the proof of Theorem 1(i), we see that for any given prime number  $p$ , the assertions “for all  $n \geq 0, a_{n+p} \equiv a_n \pmod{p}$ ” and “for all  $n \geq p, p | b_n$ ” are equivalent. It follows that the assertions “for all  $p \in \mathcal{P}$  and all  $n \geq 0, a_{n+p} \equiv a_n \pmod{p}$ ” and “for all  $p \in \mathcal{P}$  and all  $n \geq p, p$  divides  $b_n$ ” are equivalent, where  $\mathcal{P}$  is a same set of prime numbers, and this generalizes Theorem 1(i). We shall in fact prove Theorem 2 under the weaker assumption that there exists an infinite set  $\mathcal{P}$  of prime numbers such that for all  $p \in \mathcal{P}$  and all  $n \geq 0, a_{n+p} \equiv a_n \pmod{p}$ .

Given  $u \in \mathbb{Z}$  and a prime number  $p$ , we set  $u|_p := u \pmod{p}$ . Given a power series  $F(x) := \sum_{n=0}^{\infty} u_n x^n \in \mathbb{Z}[[x]]$ , we set  $F|_p(x) := \sum_{n=0}^{\infty} u_n|_p x^n \in \mathbb{F}_p[[x]]$ .

We shall first prove (ii) for the series  $f_b(x) := \sum_{n=0}^{\infty} b_n x^n$ . Let  $\mathcal{P}$  denote an infinite set of prime numbers such that for all  $n \geq 0$  and all  $p \in \mathcal{P}$ , we have  $a_{n+p} \equiv a_n \pmod{p}$ . As already said, this is equivalent to the fact that for all  $p \in \mathcal{P}$  and all  $n \geq p$ ,  $p$  divides  $b_n$ . It follows in particular that for any  $p \in \mathcal{P}$ ,  $f_b|_p(x)$  is a polynomial in  $\mathbb{F}_p[x]$  of degree at most  $p-1$ . For simplicity, we denote by  $Q_p(x)$  this polynomial, and by  $q_p$  its degree.

Let us now assume that  $f_b(x)$  is algebraic over  $\mathbb{Q}(x)$ . If  $f_b(x)$  is a constant, there is nothing else to prove. We now assume that  $f_b(x)$  is not a constant so that it has degree  $d \geq 1$ . There exist an integer  $\delta \in \{1, \dots, d-1\}$ , some integers  $0 \leq j_1 < j_2 < \dots < j_\delta \leq d-1$ , and some polynomials  $A_d(x), A_{j_1}(x), \dots, A_{j_\delta}(x) \in \mathbb{Z}[x]$  all not identically zero such that

$$A_d f_b^d = \sum_{\ell=1}^{\delta} A_{j_\ell} f_b^{j_\ell} \tag{3.1}$$

in  $\mathbb{Z}[[x]]$ .

We fix  $p \in \mathcal{P}$  such that  $p > H$  where  $H$  is the maximum of the modulus of the coefficients of  $A_d(x), A_{j_1}(x), \dots, A_{j_\delta}(x)$ . It follows that

$$\deg(A_{d|p}) = \deg(A_d), \deg(A_{j_1|p}) = \deg(A_{j_1}), \dots, \deg(A_{j_\delta|p}) = \deg(A_{j_\delta}). \quad (3.2)$$

We deduce from the reduction of (3.1) mod  $p$  that

$$A_{d|p}Q_p^d = \sum_{\ell=1}^{\delta} A_{j_\ell|p}Q_p^{j_\ell} \quad (3.3)$$

in  $\mathbb{F}_p[[x]]$ , and in fact in  $\mathbb{F}_p[x]$  because  $Q_p(x) \in \mathbb{F}_p[x]$ .

Case 1). If  $Q_p$  is identically zero, this means that  $p$  divides the coefficients  $b_n$  for all  $n \geq 0$ .

Case 2). If  $Q_p$  is not identically zero, we deduce from (3.2) and (3.3) that

$$\begin{aligned} \deg(A_d) + dq_p &\leq \max(\deg(A_{j_1}) + j_1q_p, \deg(A_{j_2}) + j_2q_p, \dots, \deg(A_{j_\delta}) + j_\delta q_p) \\ &\leq \max(\deg(A_{j_1}), \deg(A_{j_2}), \dots, \deg(A_{j_\delta})) + (d-1)q_p. \end{aligned}$$

Hence

$$q_p \leq \max(\deg(A_{j_1}), \deg(A_{j_2}), \dots, \deg(A_{j_\delta})) - \deg(A_d) =: N.$$

It follows that for any  $n > N$ ,  $p$  divides  $b_n$ , where  $N$  is *independent* of  $p$ .

Since  $p \in \mathcal{P}$  was simply assumed larger than a quantity  $H$  depending only on  $f_b$ , the conclusion of Case 1 and Case 2 is that for any  $p \in \mathcal{P}$  such that  $p > H$  and any  $n > N$ ,  $p$  divides  $b_n$ . Since  $\mathcal{P}$  is infinite,  $b_n$  is divisible by infinitely many primes when  $n > N$ . Hence  $b_n = 0$  for all  $n > N$  and  $f_b(x) = \sum_{n=0}^N b_n x^n \in \mathbb{Z}[x]$ , as expected.

Let us now prove (i). If  $f_a(x)$  is algebraic over  $\mathbb{Q}(x)$ , then  $f_b(x)$  as well by (1.8). Hence  $f_b(x) \in \mathbb{Z}[x]$  by (ii) just proven, *i.e.* there exists an integer  $M$  such that  $b_n = 0$  if  $n > M$ . Since, for all  $n \geq 0$ ,

$$a_n = \sum_{k=0}^n \binom{n}{k} b_k = \sum_{k=0}^{\min(n, M)} \binom{n}{k} b_k,$$

it follows that for all  $n \geq M$ , we have  $a_n = Q(n)$  with  $Q(X) = \sum_{k=0}^M \binom{X}{k} b_k \in \mathbb{Q}[X]$ .

The first part of Theorem 2 can be generalized as follows.

**Theorem 5.** *Let  $(a_n)_{n \geq 0} \in \mathbb{Z}^{\mathbb{N}}$  be a primary pseudo-polynomial. Assume there exists  $m \geq 0$  such that  $f_a^{(m)}(x)$  is algebraic over  $\mathbb{Q}(x)$ . Then  $a_n$  is eventually in  $\mathbb{Q}[n]$ , and thus  $f_a(x) \in \mathbb{Q}(x)$ .*

The proof uses the following simple lemma.

**Lemma 1.** *Let  $R(X) \in \mathbb{Q}(X)$  be such that  $R(n) \in \mathbb{Z}$  for infinitely many integers. Then,  $R(X) \in \mathbb{Q}[X]$ .*

*Proof.* We write  $R = \frac{A}{B}$  with  $A, B \in \mathbb{Q}[X]$ . We assume that  $\deg(B) \geq 1$  otherwise there is nothing to prove. There exist  $U, V \in \mathbb{Q}[X]$  such that  $A = UB + V$  and  $\deg(V) < \deg(B)$ . Let  $w \in \mathbb{Z} \setminus \{0\}$  be such that  $wU, wV \in \mathbb{Z}[X]$ . Let  $\mathcal{N}$  be the infinite set of integers  $n$  such that  $R(n) \in \mathbb{Z}$ ; without loss of generality, we can assume that  $\mathcal{N}$  contains infinitely many positive integers. For every  $n \in \mathcal{N}$ , we have  $\frac{wV(n)}{B(n)} = wR(n) - wU(n) \in \mathbb{Z}$ . But  $\lim_{x \rightarrow +\infty} \frac{wV(x)}{B(x)} = 0$ . Hence there exists  $M$  such that  $n \in \mathcal{N}$  and  $n \geq M$  imply that  $\frac{wV(n)}{B(n)} = 0$ . Therefore,  $wV$  has infinitely many roots: it must be the null polynomial, so that  $R = U \in \mathbb{Q}[X]$ .  $\square$

*Proof of Theorem 5.* We have

$$f_a^{(m)}(x) = \sum_{n=0}^{\infty} (n+m)(n+m-1) \cdots (n+1) a_{n+m} x^n.$$

Since  $(a_n)_{n \geq 0}$  is a primary pseudo-polynomial, this is also the case of  $((n+m)(n+m-1) \cdots (n+1) a_{n+m})_{n \geq 0}$  because it is a product of two primary pseudo-polynomials. Since  $f_a^{(m)}(x)$  is algebraic over  $\mathbb{Q}(x)$ ,  $(n+m)(n+m-1) \cdots (n+1) a_{n+m}$  is eventually in  $\mathbb{Q}[n]$  by Theorem 2. Hence  $a_{n+m}$  is eventually in  $\mathbb{Q}(n)$ , so that by Lemma 1,  $a_{n+m}$  is eventually in  $\mathbb{Q}[n]$ . This completes the proof.  $\square$

## 4 Proof of Theorem 3

The proof of the Perelli-Zannier Theorem is based on the following lemma proved in [13]. We shall also use it.

**Lemma 2.** For  $\underline{k} = (k_j)_{j \geq 0} \in (\mathbb{R}^+)^{\mathbb{N}}$ , an integer  $N \geq 1$ , set

$$A(N, \underline{k}) := \{(x_1, \dots, x_N) \in \mathbb{Z}^N : |x_j| \leq k_j \quad \text{and} \quad \forall p, \forall n \leq N-p, x_{n+p} \equiv x_n \pmod{p}\}.$$

Then

$$\#A(N, \underline{k}) \leq \prod_{j=1}^N \left(1 + \frac{2k_j}{P_{j-1}}\right).$$

Let  $R \geq 1, H \geq 0, D \geq 1$  be integers. Let  $q_0, \dots, q_R \in \mathbb{Z}[X]$  with  $\max_j H(q_j) \leq H$  and  $\max_j \deg(q_j) \leq D-1$ . Considering the coefficients of the  $q_j$ 's as indeterminates, there are  $(2H+1)^{RD}$  functions  $F$  of the form

$$F(n) := \sum_{j=0}^{R-1} q_j(n) a_{n+j}. \tag{4.1}$$

Such a function satisfies  $|F(n)| \leq cRDH(n+1)^D \delta^{n+R}$  for all  $n \geq 0$  and  $F(n+p) \equiv F(n) \pmod{p}$  for all prime number  $p$  and all  $n \geq 0$ . Hence for all  $N \geq 1$ ,  $(F(0), \dots, F(N-1)) \in$

$A(N, \underline{K})$  where  $K_j := cRDHj^D\delta^{j+R-1}$ . Therefore, given  $N$ , if

$$\prod_{j=1}^N \left(1 + \frac{2K_j}{P_{j-1}}\right) < (2H+1)^{RD} \quad (4.2)$$

there exists *two* different functions  $F_1$  and  $F_2$  of the form (4.1) such that  $F_1(n) = F_2(n)$  for all  $n \in \{0, 1, \dots, N-1\}$ . Hence the function  $G_N := F_1 - F_2$  is of the form (4.1)  $G_N(n) = \sum_{j=0}^{R-1} q_j(n)a_{n+j}$  with  $q_0(X), \dots, q_{R-1}(X) \in \mathbb{Z}[X]$  not all identically zero, with  $|G_N(n)| \leq 2cRDH(n+1)^D\delta^{n+R}$  for all  $n \geq 0$  and  $G_N(n) = 0$  for every integer  $n$  in  $\{0, 1, \dots, N-1\}$ . Note that  $G_N$  depends on  $N$  which is fixed but can be as large as desired in this construction.

Eq. (4.2) holds if we assume the stronger condition

$$\prod_{j=1}^{\infty} \left(1 + \frac{2K_j}{P_{j-1}}\right) \leq H^{RD}, \quad (4.3)$$

because the assumption  $\delta < e$  implies the convergence of the product

$$\Phi(D, x) := \prod_{j=1}^{\infty} \left(1 + x \frac{j^D \delta^j}{P_{j-1}}\right)$$

for all  $x \geq 0$  and all  $D \geq 0$ , and obviously  $1 + x \frac{j^D \delta^j}{P_{j-1}} \geq 1$ . We shall provide an upper bound for  $\Phi(D, x)$  in §4.2, from which we shall deduce values of  $H, R, D$  such that (4.3) holds. It is important to observe here that (4.3) does not depend on  $N$ .

#### 4.1 Proof that $G_N(n) = 0$ for all $n \geq 0$

Following the Perelli-Zannier method, we now want to prove that, provided  $N$  is large enough,  $G_N(n)$  vanishes for all  $n \geq 0$ . Assume this is not the case. Then for every  $N$ , let  $M_N \geq N$  denote the largest integer such that  $G_N(0) = G_N(1) = \dots = G_N(M_N) = 0$  but  $G_N(M_N + 1) \neq 0$ . We fix  $\alpha \in (0, \frac{2}{\log(\delta)} - 2)$ .

We shall first prove that  $G_N(m) = 0$  for  $m$  in  $I := [2M_N, (2 + \alpha)M_N]$ . Let  $m \in I$ . First assume that  $p$  is a prime and  $p < M_N$ : since  $G_N(0) = \dots = G_N(p) = 0$  and  $G_N(n+p) \equiv G_N(n) \pmod{p}$ ,  $p$  divides  $G_N(m)$ . Assume now that  $M_N \leq p \leq m/2$ , so that

$$0 \leq m - 2p \leq m - 2M_N \leq \alpha M_N \leq M_N,$$

hence  $G_N(m) \equiv G_N(m - 2p) = 0 \pmod{p}$ . Assume to finish that  $m - M_N < p \leq m$  (such primes have not yet been considered) so that  $0 \leq m - p < M_N$  and  $G_N(m) \equiv G_N(m - p) \equiv 0 \pmod{p}$ . It follows that  $G_N(m)$  is divisible by  $P_{m/2}P_m/P_{m-M_N}$ .

Therefore, if  $G_N(m) \neq 0$  for some  $m \in I$ , then

$$|G_N(m)| \geq e^{m/2 + M_N + o(M_N)} \geq e^{2M_N + o(M_N)},$$

where  $o(M_N)$  denotes a term such that  $o(M_N)/M_N$  becomes arbitrarily small when  $N$  is taken arbitrarily large. But on the other hand, we know that, for any  $m \in I$ ,

$$|G_N(m)| \leq 2cRDH(m+1)^D \delta^{m+R} \leq 2cRDH(2M_N + \alpha M_N + 1)^D \delta^R \delta^{(2+\alpha)M_N}.$$

We recall that we assume that  $H, R, D$  are such that (4.3) holds, which is independent of  $N$ . Hence we can let  $N \rightarrow +\infty$ , hence *a fortiori*  $M_N \rightarrow +\infty$  so that the above lower and upper bounds for  $G_N(m) \neq 0$  imply that  $e^2 \leq \delta^{2+\alpha}$ , *i.e.* that  $\alpha \geq \frac{2}{\log(\delta)} - 2$ , which is contrary to the assumption on  $\alpha$ . Hence, provided  $N$  is large enough, we have  $G_N(m) = 0$  for all  $m \in I$ .

We thus have  $G_N(m) = 0$  for all integers  $m$  in  $[0, \dots, M_N]$  or  $[2M_N, (2 + \alpha)M_N]$ . It follows that for any  $p \leq (1 + \alpha)M_N - 1$ ,  $p$  divides  $G_N(M_N + 1)$ . Indeed, if  $p \leq M_N$ , we write  $M_N + 1 = n + p$  for some  $n \leq M_N - 2$  so that  $G_N(M_N + 1) \equiv G(n) = 0 \pmod{p}$ , while if  $M_N < p \leq (1 + \alpha)M_N - 1$ , we have  $0 = G_N(M_N + 1 + p) \equiv G(M_N + 1) \pmod{p}$  because  $M_N + 1 + p \in [2M_N, (2 + \alpha)M_N]$ . Hence, because  $G_N(M_N + 1) \neq 0$ , we have

$$|G_N(M_N + 1)| \geq P_{(1+\alpha)M_N-1} \geq e^{(1+\alpha)M_N+o(M_N)}.$$

On the other hand,

$$|G_N(M_N + 1)| \leq c2RDH(M_N + 2)^D \delta^{M_N+1+R}.$$

As above, we take  $N$  large enough so that these two bounds imply that  $\alpha \leq \log(\delta) - 1$ , which is impossible because  $\log(\delta) - 1 < 0$  while  $\alpha$  was chosen positive.

Therefore, there is no such  $M_N$  such that  $G(M_N + 1) \neq 0$ , so that  $G(n) = 0$  for all integer  $n \geq 0$ .

## 4.2 Upper bound for $\Phi(D, x)$

In this section,  $x > 0$  is a fixed real parameter and  $D \geq 1$  is a fixed integer. We fix  $\varepsilon > 0$  such that  $\delta < e - \varepsilon$ , and we let  $\omega = \delta/(e - \varepsilon) < 1$ . By the Prime Number Theorem,  $P_{j-1} \geq (e - \varepsilon)^j$  for all  $j > J = J(\varepsilon)$  so that

$$\Phi(D, x) := \prod_{j=1}^{\infty} \left(1 + xj^D \frac{\delta^j}{P_{j-1}}\right) \leq \prod_{j=1}^J \left(1 + xj^D \frac{\delta^j}{P_{j-1}}\right) \prod_{j=J+1}^{\infty} (1 + xj^D \omega^j).$$

We have

$$\prod_{j=1}^J \left(1 + xj^D \frac{\delta^j}{P_{j-1}}\right) \leq \prod_{j=1}^J (1 + xj^D \delta^j) \leq \prod_{j=1}^J (1 + (1+x)j^D \delta^j) \leq \prod_{j=1}^J (2(1+x)j^D \delta^j),$$

since  $(1+x)j^D \delta^j \geq 1$  for every  $j$  in  $\{1, \dots, J\}$ . Hence,

$$\prod_{j=1}^J (2(1+x)j^D \delta^j) = 2^J (1+x)^J J!^D \delta^{J(J+1)/2} \leq 2^J (1+x)^J J!^D \delta^{J^2}.$$

In addition, we have

$$\prod_{j=J+1}^{\infty} (1 + xj^D \omega^j) \leq \prod_{j=1}^{\infty} (1 + xj^D \omega^j),$$

which yields

$$\Phi(D, x) \leq 2^J (1+x)^J J!^D \delta^{J^2} \prod_{j=1}^{\infty} (1 + xj^D \omega^j).$$

We now bound the infinite product  $\Psi(D, x) := \prod_{j=1}^{\infty} (1 + xj^D \omega^j)$ . The maximum of the function  $t \mapsto t^D \omega^{t/2}$  is  $m(D) := (2D/(e \log(1/\omega)))^D$ , attained at  $j_0 := 2D/\log(1/\omega)$ . Hence, for all  $j \geq j_0$ , we have  $j^D \leq m(D) \omega^{-j/2}$ . Moreover,  $t \mapsto t^D \omega^{t/2}$  is increasing on  $[0, j_0]$  and  $m(D) \leq j_0^D$ . Hence,

$$\begin{aligned} \Psi(D, x) &\leq \prod_{1 \leq j < j_0} (1 + xj^D \omega^j) \prod_{j \geq j_0} (1 + xm(D) \omega^{j/2}) \\ &\leq (1 + (1+x)j_0^D)^{\lfloor j_0 \rfloor} \prod_{j=1}^{\infty} (1 + xm(D) \omega^{j/2}) \\ &\leq (2(1+x)j_0^D)^{\lfloor j_0 \rfloor} \prod_{j=1}^{\infty} (1 + xj_0^D \omega^{j/2}). \end{aligned}$$

We now bound  $\prod_{j=1}^{\infty} (1 + xj_0^D \omega^{j/2})$ . We set  $y := xj_0^D$ . Since  $t \mapsto \omega^t$  is decreasing on  $[0, \infty)$ , we have

$$\begin{aligned} \log \left( \prod_{j=1}^{\infty} (1 + y \omega^{j/2}) \right) &\leq \int_0^{+\infty} \log(1 + y \omega^{t/2}) dt \\ &\leq \frac{2}{\log(1/\omega)} \int_0^y \frac{\log(1+u)}{u} du \quad (u := y \omega^{t/2}) \\ &\leq -\frac{2\text{Li}_2(-y)}{\log(1/\omega)}. \end{aligned}$$

Here, we use the dilogarithm  $\text{Li}_2(z) := -\int_0^z \log(1-x)/x dx$  defined for  $z \in \mathbb{C} \setminus (1, +\infty)$  using the principal branch of  $\log$  in the integral; see [10, p. 1, (1.4)]. Here, we want to use it for large negative values  $-y$ . For this, we use the identity (see [10, p. 4, (1.7)])

$$\text{Li}_2(-y) = -\frac{1}{2} \log(y)^2 - \text{Li}_2(-1/y) - \zeta(2), \quad y > 0$$

which yields

$$-2\text{Li}_2(-y) \leq \log(y)^2 + 4\zeta(2), \quad y \geq 1,$$

because  $\text{Li}_2(-1/y) + \zeta(2) \leq 2\zeta(2)$  when  $y \geq 1$ . We obtain that for  $y \geq 1$ ,

$$\prod_{j=1}^{\infty} (1 + y \omega^{j/2}) \leq c_0 e^{\log(y)^2 / \log(1/\omega)},$$

with  $c_0 := \exp(4\zeta(2)/\log(1/\omega)) \geq 1$ .

Putting all the pieces together, we finally obtain the bound

$$1 \leq \Phi(D, x) \leq 2^J (1+x)^J J!^D \delta^{J^2} (2(1+x)j_0^D)^{\lfloor j_0 \rfloor} c_0 e^{\log(xj_0^D)^2/\log(1/\omega)}, \quad (4.4)$$

where we recall that  $\omega = \delta/(e - \varepsilon)$  where  $\varepsilon > 0$  is such that  $\delta < e - \varepsilon$ .

### 4.3 Conclusion of the proof

For ease of reading, we set  $d, r, h$  for  $D, R, H$ . We want to find conditions on  $d, r$  and  $h$  such that  $\Phi(d, x) \leq h^{rd}$  when  $x = 2crdh\delta^{r-1}$  (which corresponds to (4.3)). It will be enough to find conditions on  $d, r, h$  and  $\varepsilon$  such that the right-hand side of (4.4) is  $\leq h^{rd}$ .

From now on, we set  $\ell := \log(\delta) < 1$ . We assume that  $d$  and  $\rho$  depend on  $\ell$  but are independent of  $h$ , and we let  $r := \lfloor \rho \log(h) \rfloor + 1$ . Since  $J$  and  $j_0$  are also fixed, when  $h \rightarrow +\infty$ , we have

$$\log \left( c_0 2^J (1+x)^J J!^d \delta^{J^2} (2(1+x)j_0^d)^{\lfloor j_0 \rfloor} e^{\log(xj_0^d)^2/\log(1/\omega)} \right) \sim \frac{(1+\rho\ell)^2}{\log(1/\omega)} \log(h)^2,$$

while

$$\log(h^{rd}) \sim d\rho \log(h)^2.$$

Hence for our goal, it suffices to choose  $d, \rho$  and  $\varepsilon$  such that

$$\frac{(1+\rho\ell)^2}{\log(1/\omega)} < d\rho. \quad (4.5)$$

Recall that  $\omega = \delta/(e - \varepsilon)$  so that  $\log(1/\omega) \rightarrow 1 - \ell$  as  $\varepsilon \rightarrow 0$ . So, by choosing  $d$  and  $\rho$  such that

$$\frac{(1+\rho\ell)^2}{1-\ell} \leq d\rho, \quad (4.6)$$

we can choose  $\varepsilon > 0$  such that Eq. (4.5) holds true. Eq. (4.6) is equivalent to

$$\ell^2 \rho^2 + (2\ell - d(1-\ell))\rho + 1 \leq 0,$$

which defines a polynomial in  $\rho$  whose discriminant is  $\Delta := d(1-\ell)(d(1-\ell) - 4\ell)$ . Taking  $d := \max(1, \lceil \frac{4\ell}{1-\ell} \rceil)$  ensures that (4.6) holds true for any choice of  $\rho > 0$  in

$$\left[ \frac{d(1-\ell) - 2\ell - \sqrt{\Delta}}{2\ell^2}, \frac{d(1-\ell) - 2\ell + \sqrt{\Delta}}{2\ell^2} \right].$$

For simplicity, we also restrict  $\rho$  to be  $\leq 1/\ell$  which is possible because the product of those roots is  $1/\ell^2$  and, since  $d(1-\ell) \geq 4\ell$ , we have

$$\frac{d(1-\ell) - 2\ell + \sqrt{\Delta}}{2\ell^2} \geq \frac{1}{\ell} \quad \text{and} \quad \frac{d(1-\ell) - 2\ell - \sqrt{\Delta}}{2\ell^2} \leq \frac{1}{\ell}.$$



With such choices of  $\varepsilon$ ,  $d$  and  $\rho$ , we now define  $H(\delta)$  as the smallest integer  $h \geq 1$  such that

$$c_0 2^J (1+x)^J J!^d \delta^{J^2} (2(1+x)j_0^d)^{\lfloor j_0 \rfloor} e^{\log(xj_0^d)^2 / \log(1/\omega)} \leq h^{rd}, \quad (4.7)$$

where  $x = 2crdh\delta^{r-1}$ . We then obtain (1.10) with  $\max \deg(p_j) \leq d-1 = \max(0, \lceil \frac{5 \log(\delta)-1}{1-\log(\delta)} \rceil)$  and  $S \leq r-1 \leq \lfloor \rho \log(h) \rfloor \leq \log(H(\delta))/\ell$ . Notice that  $H(\delta)$  also depends on the choice of  $\rho$  and we now explain how to bound it.

The left-hand side of (4.7) is an increasing function of  $h \geq 1$ , which appears in the expressions of  $r := \lfloor \rho \log(h) \rfloor + 1$  and  $x := 2crdh\delta^{r-1}$ . Hence,  $H(\delta)^{d(1+\lfloor \rho \log(H(\delta)) \rfloor)}$  is larger than the value  $A$  (which is  $\geq 1$ ) of the left-hand side of (4.7) at  $h = 1$ , in which case  $r = 1$  and  $x = 2cd$ . It follows that  $\log(A) \leq d \log(H(\delta)) + d\rho \log(H(\delta))^2$ , so that

$$H(\delta) \geq \exp\left(\frac{\sqrt{d^2 + 4d\rho \log(A)} - d}{d\rho}\right). \quad (4.8)$$

Since  $A \rightarrow +\infty$  when  $\delta \rightarrow e$  and  $\varepsilon \rightarrow 0$  (because of the term  $1/\log(\omega)$ ),  $H(\delta)$  can be very large.

We now explain how to bound  $Y := H(\delta)$  from above. We assume that  $H(\delta) \geq 2$  otherwise there is nothing else to do. The left-hand side of (4.7) is  $\geq 1$ , so that we can take the logarithms of both sides. After some transformations, we obtain a function  $S(h) \geq 0$  for all  $h \geq 1$  (which could be explicitated) such that  $Y$  is the smallest integer  $h \geq 1$  such that

$$S(h) \leq \left(\rho d - \frac{(1+\rho\ell)^2}{\log(1/\omega)}\right) \log(h)^2. \quad (4.9)$$

Recall that  $\gamma := \rho d - \frac{(1+\rho\ell)^2}{\log(1/\omega)} > 0$ . Moreover, there exist  $\alpha, \beta$  that depend on  $\rho, d, \delta, \varepsilon$  (and could be explicitated as well) such that  $S(h) \leq \alpha \log(h) + \beta$  for all  $h \geq 1$ . Since  $Y \geq 2$  is the smallest integer such that (4.9) holds, we have

$$\gamma \log(Y-1)^2 < s(Y-1) \leq \alpha \log(Y-1) + \beta.$$

Hence,  $\log(Y-1)$  is smaller than the largest solution of the quadratic equation  $\gamma X^2 - \alpha X - \beta = 0$ , so that finally

$$H(\delta) \leq 1 + \exp\left(\frac{\alpha + \sqrt{\alpha^2 + 4\beta\gamma}}{2\gamma}\right). \quad (4.10)$$

## 5 Proof of Theorem 4

Let  $F(z)$  be as in the theorem such that  $F(n) = a_n$  for all  $n > u$ . Notice that  $\tilde{a}_n := a_{n+\lfloor u \rfloor+2}$ ,  $n \geq 0$ , is a primary pseudo-polynomial; it is eventually in  $\mathbb{Q}[n]$  if and only if  $a_n$  is eventually in  $\mathbb{Q}[n]$ . The function  $\tilde{F}(z) := F(z + \lfloor u \rfloor + 2)$  is analytic in  $\Re(z) > -2$ , satisfies  $|\tilde{F}(z)| \leq \tilde{c} \cdot \exp(\rho \Re(z))$  in  $\Re(z) > -2$  for some constant  $\tilde{c} > 0$ , and  $\tilde{a}_n = \tilde{F}(n)$  for all  $n \geq 0$ . Moreover,  $F(z)$  is in  $\mathbb{Q}[z]$  if and only if  $\tilde{F}(z)$  is in  $\mathbb{Q}[z]$ . Therefore, without loss of

generality, we can and will assume that  $F(z)$  is analytic in  $\Re(z) > -2$  and that  $F(n) = a_n$  for all integers  $n \geq 0$ .

Let  $\mathcal{C}_n$  denote the circle of center  $n$  and radius  $n$  oriented in the direct sense. The function  $F(z-1)$  being analytic in  $\Re(z) > -1$ , the residue theorem yields

$$\frac{(n-1)!}{2i\pi} \int_{\mathcal{C}_n} \frac{F(z-1)}{(z-1)\cdots(z-n)} dz = \sum_{k=0}^{n-1} (-1)^{n-k-1} \binom{n-1}{k} a_k := b_{n-1} \in \mathbb{Z}.$$

We parametrize the circle  $\mathcal{C}_n$  as  $n + ne^{2ix} = 2n \cos(x)e^{ix}$  for  $x \in [-\pi/2, \pi/2]$ . Nörlund [11, p. 387] proved that

$$\left| \frac{(n-1)!}{(z-1)\cdots(z-n)} \right| \leq c_1(n) e^{-2n \cos(x)\psi(x)}, \quad z = 2n \cos(x)e^{ix} \in \mathcal{C}_n,$$

where  $c_1(n) > 0$  is bounded above by some polynomial in  $n$  and

$$\psi(x) := \cos(x) \log(2 \cos(x)) + x \sin(x).$$

(See also the proof given in [15].) The minimum on  $[-\pi/2, \pi/2]$  of  $\psi(x)$  is  $\log(2)$  at  $x = 0$ .

We have

$$|F(z-1)| \leq c \cdot e^{\rho \Re(z-1)} = ce^{-\rho} \cdot e^{2n\rho \cos(x)^2}, \quad z = 2n \cos(x)e^{ix} \in \mathcal{C}_n.$$

Hence

$$|b_{n-1}| \leq c_2(n) \max_{x \in [-\pi/2, \pi/2]} e^{2n \cos(x)(\rho \cos(x) - \psi(x))}$$

where  $c_2(n) > 0$  is bounded above by some polynomial in  $n$ . Notice that  $0 \leq \cos(x) \leq 1$  on  $[-\pi/2, \pi/2]$ . Recall also that  $\rho > 0$ . Hence if  $x$  is such that  $\rho \cos(x) - \psi(x) \geq 0$ , then  $\cos(x)(\rho \cos(x) - \psi(x)) \leq \rho - \psi(x) \leq \rho - \log(2)$ , whereas if  $x$  is such that  $\rho \cos(x) - \psi(x) \leq 0$ , then  $\cos(x)(\rho \cos(x) - \psi(x)) \leq 0$ . Therefore, for all  $x \in [-\pi/2, \pi/2]$ ,

$$e^{2 \cos(x)(\rho \cos(x) - \psi(x))} \leq \max(1, e^{2(\rho - \log(2))})$$

and  $|b_{n-1}| \leq c_2(n) \max(1, e^{2(\rho - \log(2))})^n$ . Since  $2(\rho - \log(2)) < 1$ , it follows that

$$\limsup_{n \rightarrow +\infty} |b_n|^{1/n} < e.$$

But because  $(a_n)_{n \geq 0}$  is a primary pseudo-polynomial, we know by Theorem 1(ii) that if  $b_n$  is not eventually equal to 0 then

$$\liminf_{n \rightarrow +\infty} |b_n|^{1/n} \geq e.$$

This implies that  $b_n$  is indeed eventually equal to 0, thus that there exist  $P(X) \in \mathbb{Q}[X]$  and an integer  $N \geq 0$  such that  $a_n = P(n)$  for all  $n \geq N$ .

Consider now the function  $g(z) := F(z+N) - P(z+N)$  which is analytic in  $\Re(z) > -2$  (at least), and such that  $g(n) = 0$  for every integer  $n \geq 1$ . Moreover, since  $\rho > 0$ , there exists a constant  $d > 0$  such that  $|g(z)| \leq d \cdot \exp(\rho|z|)$  for any  $z$  such that  $\Re(z) > 0$ . Since  $\rho < \frac{1}{2} + \log(2) < \pi$ , we can then apply a classical result of Carlson (see Hardy [6, p. 328]) and deduce that  $g(z) = 0$  identically. Hence  $F(z)$  reduces to a polynomial function in  $\mathbb{Q}[z]$ . This completes the proof of Theorem 4.

## 6 Construction of non-polynomial primary pseudo-polynomials

We conclude this paper by presenting a method to construct a non-polynomial primary pseudo-polynomial starting from a given primary pseudo-polynomial  $(a_n)_{n \geq 0}$  such that  $a_0 = 1$ . The justification of the method uses a non-trivial property satisfied by  $E$ -functions.

Let as usual  $(b_n)_{n \geq 0}$  be the binomial transform (1.1) of  $(a_n)_{n \geq 0}$ . Let  $F_b(x) := \sum_{n=0}^{\infty} \frac{b_n}{n!} x^n$ . We assume that  $a_0 = 1$ , so that  $b_0 = 1$  as well. We define the sequence  $(c_n)_{n \geq 0}$  formally by

$$F_c(x) := \sum_{n=0}^{\infty} \frac{c_n}{n!} x^n = \frac{1}{F_b(x)}.$$

Let us now define  $(u_n)_{n \geq 0}$  as the inverse binomial transform (1.2) of  $(c_n)_{n \geq 0}$ , *i.e.*

$$u_n := \sum_{k=0}^n \binom{n}{k} c_k.$$

Then we have the following.

**Theorem 6.** *Let  $(a_n)_{n \geq 0} \in \mathbb{Z}^{\mathbb{N}}$  be a primary pseudo polynomial such that  $a_0 = 1$ . Then,  $(u_n)_{n \geq 0} \in \mathbb{Z}^{\mathbb{N}}$  is a primary pseudo-polynomial. Moreover, assuming also that  $f_a(x)$  is a  $G$ -function, if  $\limsup_n |u_n|^{1/n} < e$ , then  $a_n = u_n = 1$  for all  $n \geq 0$ .*

Consequently, if  $f_a(x)$  is a  $G$ -function not equal to  $\frac{1}{1-x}$ , then  $(u_n)_{n \geq 0} \in \mathbb{Z}^{\mathbb{N}}$  is a primary pseudo-polynomial such that  $\limsup_n |u_n|^{1/n} \geq e$  (hence not a polynomial). We explain after the proof of the theorem why  $u_n$  grows like  $n!$  in this case.

So far, the assumption that  $f_a(x)$  is a  $G$ -function is known to be satisfied only when  $\limsup_n |a_n|^{1/n} < e$  (when the Perelli-Zannier Theorem can be applied), which in turn implies that  $a_n$  should eventually be in  $\mathbb{Q}[n]$  by Conjecture 2. Hence, in practice the second assertion of Theorem 6 is useful only when  $a_n$  is already known to be eventually in  $\mathbb{Q}[n]$  in which case  $F_b(x) \in \mathbb{Q}[x]$ : For instance, if  $a_n = n + 1$ , then

$$\sum_{n=0}^{\infty} \frac{c_n}{n!} x^n = \frac{1}{1+x} \quad \text{and} \quad u_n = \sum_{k=0}^n (-1)^k \binom{n}{k} k!. \quad (6.1)$$

*Proof of Theorem 6.* We first prove that for all  $n \geq 0$ ,  $c_n \in \mathbb{Z}$  and for every prime  $p \leq n$ ,  $p$  divides  $c_n$ . By definition of the  $c_n$ 's, we have

$$1 = \left( \sum_{n=0}^{\infty} \frac{b_n}{n!} x^n \right) \left( \sum_{n=0}^{\infty} \frac{c_n}{n!} x^n \right) = \sum_{n=0}^{\infty} \frac{1}{n!} \left( \sum_{k=0}^n \binom{n}{k} b_k c_{n-k} \right) x^n,$$

so that  $c_0 = 1$  and, for every integer  $n \geq 1$ , we have

$$\sum_{k=0}^n \binom{n}{k} b_k c_{n-k} = 0.$$

This yields the recursive relation (because  $b_0 = 1$ )

$$c_n = - \sum_{k=1}^n \binom{n}{k} b_k c_{n-k}, \quad (n \geq 1).$$

It follows that, for all  $n \geq 0$ ,  $c_n$  is an integer.

Let  $n$  be a positive integer such that, for every positive integer  $m < n$  and every prime  $p \leq m$ ,  $p$  divides  $c_m$ . Consider a prime number  $p \leq n$  and an integer  $k$  in  $\{1, \dots, n\}$ . If  $p \leq k$  then  $p$  divides  $b_k$ . If  $p \leq n - k$ , then  $p$  divides  $c_{n-k}$ . If  $p > \max(k, n - k)$ , then  $p$  divides  $\binom{n}{k}$  because  $p$  divides  $n!$  but neither  $k!$  nor  $(n - k)!$ . In all cases,  $p$  divides  $\binom{n}{k} b_k c_{n-k}$ , so that  $p$  divides  $c_n$ . By strong induction on  $n$ , it follows that, for every integer  $n \geq 0$  and every prime  $p \leq n$ ,  $p$  divides  $c_n$  (this property holds trivially if  $n = 0$  or  $1$ ). By Theorem 1(i), the sequence  $(u_n)_{n \geq 0} \in \mathbb{Z}^{\mathbb{N}}$  is a primary pseudo-polynomial. Notice that so far we had no need to assume that  $f_a$  is a  $G$ -function.

We now complete the proof of Theorem 6 when  $f_a$  is also a  $G$ -function. We first observe that  $F_b(z)$  is an  $E$ -function: indeed,  $f_b(x)$  is a  $G$ -function because

$$f_b(x) := \sum_{n=0}^{\infty} b_n x^n = \frac{1}{1+x} f_a\left(\frac{x}{1+x}\right)$$

and  $f_a(x)$  is a  $G$ -function. <sup>(3)</sup> Let us assume that  $\limsup_n |u_n|^{1/n} < e$ . By the Perelli-Zannier Theorem quoted in the Introduction,  $f_u(x) := \sum_{n=0}^{\infty} u_n x^n$  is a  $G$ -function. From the equation  $c_n := \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} u_k$  ( $\forall n \geq 0$ ), we deduce that

$$f_c(x) := \sum_{n=0}^{\infty} c_n x^n = \frac{1}{1+x} f_u\left(\frac{x}{1+x}\right)$$

is also a  $G$ -function. Hence  $F_c(x) := 1/F_b(z)$  is also an  $E$ -function. Therefore,  $F_b(x)$  is a unit of the ring of  $E$ -functions, *i.e.* it is of the form  $\beta e^{\alpha x}$  with  $\alpha \in \overline{\mathbb{Q}}$  and  $\beta \in \overline{\mathbb{Q}}^*$  by [1, p. 717] (see also Footnote 4). Therefore,  $b_n = \beta \alpha^n$  for all  $n \geq 0$ , with the usual convention that  $\alpha^0 = 1$  if  $\alpha = 0$ . Since all primes  $p \leq n$  divide  $b_n$ , we deduce that  $\alpha = 0$  is the only possibility. Hence  $1 = b_0 = \beta$  and  $b_n = 0$  for all  $n \geq 0$ , so that  $a_n = 1$  for all  $n \geq 0$ . It now remains to observe the following facts: “ $a_n = 1$  for all  $n \geq 0$ ” is equivalent to “ $b_0 = 1$  and  $b_n = 0$  for all  $n \geq 1$ ” which is equivalent to “ $c_0 = 1$  and  $c_n = 0$  for all  $n \geq 1$ ”, which in turn is equivalent to “ $u_n = 1$  for all  $n \geq 0$ ”.  $\square$

To conclude, let us explain how to obtain the asymptotic behavior of  $u_n$  as  $n \rightarrow +\infty$ . Since  $F_u(x) := \sum_{n=0}^{\infty} \frac{u_n}{n!} x^n = e^x F_c(x) = e^x / F_b(x)$ , the asymptotic behavior of  $u_n/n!$  is determined by the zeroes of smallest modulus of  $F_b(x)$ , when it has at least one. Notice that an  $E$ -function  $F(x)$  with no zero in  $\mathbb{C}$  must be of the form  $\beta e^{\alpha x}$  with  $\alpha \in \overline{\mathbb{Q}}, \beta \in \overline{\mathbb{Q}}^*$ . Indeed, an  $E$ -function is an entire function satisfying  $|F(x)| \ll e^{\rho|x|}$  for some  $\rho > 0$ , so

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<sup>3</sup>Given a  $G$ -function  $f(x)$  and  $\alpha(x)$  an algebraic function over  $\overline{\mathbb{Q}}(x)$  regular at  $x = 0$ ,  $f(x\alpha(x))$  is a  $G$ -function.

that Hadamard’s factorization theorem yields  $F(x) = \beta x^m e^{\alpha x} \prod_{j \geq 1} \left(1 - \frac{x}{x_j}\right) e^{x/x_j}$  where the  $x_j$ ’s are the zeroes of  $F(x)$ ,  $m \in \mathbb{N}$  and  $\alpha, \beta \in \mathbb{C}$ .<sup>(4)</sup> Therefore, a “no zero” assumption implies that  $F(x) = \beta e^{\alpha x}$  with  $\beta \neq 0$ , and since  $F(x) \in \overline{\mathbb{Q}}[[x]]$ ,  $\alpha, \beta$  must be in  $\overline{\mathbb{Q}}$ . Coming back to  $F_b(x)$ , we have seen during the proof of Theorem 6 that this case implies that  $\alpha = 0, \beta = 1$ , hence that  $f_a(x) = \frac{1}{1-x}$ .

Therefore, assuming that  $f_a(x)$  is a  $G$ -function different of  $\frac{1}{1-x}$ , the  $E$ -function  $F_b(x)$  has at least one zero. Let  $x_1, \dots, x_m$  denote the zeroes of  $F_b(x)$  of the same modulus which is the smallest amongst all modulus of the zeroes. Classical transfer theorems in [3, Chapter VI] enable to deduce the asymptotic behavior of  $u_n$ . For instance, if the  $x_j$ ’s are simple zeroes of  $F_b(x)$ , then

$$u_n = n! \sum_{j=1}^m \frac{e^{x_j}}{F_b'(x_j)} \frac{1 + o(1)}{x_j^n}.$$

This is coherent with (6.1) above (where  $F_b(x) = 1 + x$ ) because we can rewrite it as

$$u_n = (-1)^n n! \sum_{k=0}^n \frac{(-1)^{n-k}}{(n-k)!} \sim \frac{(-1)^n}{e} n!, \quad n \rightarrow +\infty.$$

Because of the different arguments of the  $x_j$ ’s, oscillations can occur. In presence of zeroes of  $F_b(x)$  of higher multiplicities, similar but more complicated expressions can be given. Finally, even though  $F_b(x)/e^x$  is an  $E$ -function, we don’t expect  $e^x/F_b(x)$  to be  $D$ -finite in general,<sup>(5)</sup> but this is obviously the case if  $F_b(x)$  is a polynomial.

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<sup>4</sup>This argument also explains the characterization of the units of the ring of  $E$ -functions: we simply have to observe that if an  $E$ -function  $F(x)$  is a unit, *i.e.* that  $1/F(x)$  is an  $E$ -function, then it does not vanish anywhere on  $\mathbb{C}$  because an  $E$ -function is an entire function.

<sup>5</sup>A classical result of Harris-Sibuya [7] ensures that if  $y$  and  $1/y$  are both holonomic, then  $y'/y$  is an algebraic function.

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