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Characterizations of output controllability for LTI systems

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Abstract

The objective of this paper is to make, in a simple and rigorous way, some contributions to the notion of output controllability. We first examine, for Linear Time-Invariant systems, the notion of state to output controllability, introduced in the 60s by Bertram and Sarachik. More precisely, we extend the Hautus test, well-known in the case of state controllability, to state to output controllability, and propose a controllability Gramian matrix, allowing us to build a continuous control achieving a transfer with minimal energy. We also give two other notions of output controllability, namely output to output and globally output to output controllability. For each of these two new notions, we give necessary and sufficient conditions, in terms of Kalman rank, Hautus test and Gramian matrices. All of these results are given in the framework of continuous time and discretized time systems. These results are illustrated by several numerical examples.

Key words: LTI systems, output controllability, Hautus test, Kalman rank condition, state to output controllability Gramian, continuous and discrete time systems, minimal norm control.

1 Introduction

The notion of state controllability was first introduced in [10] by R. E. Kalman in 1960 for linear systems. The aim was to answer to whether it is possible or not to transfer a system from its initial state to a prescribed final state. This topic has been widely studied within the scientific community because of its theoretical and applicative importance. For Linear Time-Invariant (LTI) systems, several criteria, see e.g. [4, Chapter 6] or [15, Chapter 3], have been established to answer this question. In addition, when the system is state controllable, one knows how to exhibit a control of minimal norm allowing us to achieve the above-mentioned transfer, see e.g. [5, Proposition 1.13]. The notion of state to output controllability was first introduced in [2] but did not get the same infatuation. Consequently, there are few documents discussing this topic and some well-know results in the framework of state controllability, as the Hautus test, are not known from the literature. In the case of LTI discrete-time systems without direct transmission from the input to the output, J. Bertram and P. Sarachik gave in [2],

an answer to whether it is possible or not to reach a prescribed output value from an arbitrarily chosen initial state data. More precisely, they established the analogues of the state controllability criteria related to the Gramian and Kalman matrices. In the case of direct transmission from the input to the output, E. Kreindler and P. Sarachik in [11], linked the state to output controllability to the positivity of a Gramian matrix. In this paper, the authors also provided the analogue of the Kalman rank condition.

As far as we know, in the framework of LTI systems with direct transmission from the input to the output, there does not exist an explicit expression of a continuous control allowing us to perform a desired transfer. Furthermore, to the best of our knowledge, there does not exist an extension of the Hautus test for state to output controllability. Also, like in the state controllability framework, one could be interested in transferring an initial output (not an initial state) of an LTI system to a final output, both chosen arbitrarily. A glance has been brought to this question by C.-T. Chen in [3, Section 5.7] for LTI systems without direct transmission of the input to the output. However, the presented results are the ones of [11], and we will see later in this paper that it is not exactly the case. Note that for a given initial output, there is in general more than one initial state corresponding to this output. Hence, one could wonder whether it is possible or not to design a control law allowing us to steer one initial state, or all initial states, to any output target. To

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our knowledge, there is rigorously nor criteria, neither control law allowing us to perform such task.

As contributions, we place ourselves within the framework of LTI systems with direct transmission from the input to the output and present different notions of output controllability, depending on whether we want to send an initial state to a final output or an initial output to a final output. For the notion that consists in steering an initial state to a final output referred here to *State to Output Controllability* (SOC), we establish an Hautus test and new analogues of the other criteria dedicated to state controllability. We also propose a continuous control built by means of a controllability Gramian matrix allowing us to realize a desired transfer. In order to steer an initial output to a final output, we introduced two other notions of output controllability. The first one is the *Output to Output Controllability* (OOC). For this notion, the purpose is to transfer one of the initial state, corresponding to the given initial output, to the prescribed final output. The second notion is the *Globally Output to Output Controllability* (GOOC). For this notion, the goal is to send all the initial states data corresponding to the prescribed initial output to the desired final output by means of the *same* input. For these two notions of output controllability, criteria and control laws are proposed.

All these results are also given in the discrete-time framework.

This paper is organized as follows. In Section 2, we recall known results on state controllability, and introduce the SOC notion. We provide in Section 3, Theorem 3.1 gathering different criteria to ensure the SOC of the system. In this section, we also give Theorem 3.2 where a continuous control is designed to perform any desired transfer. The remaining of this section is dedicated to the illustration of these results on an example. Theorems 3.1 and 3.2 are proven in Section 4. In Section 5, we propose the analogues of Theorems 3.1 and 3.2 for discrete time LTI systems. The two other notions of output controllability are introduced in Section 6, with their characterizations. Concluding remarks and open questions are given in Section 7.

The following notation will be used in this paper. For k, l two positive integers, \mathbb{R}_k^l stands for the set of real matrices with l rows and k columns, and 0_k^l denotes the null matrix. When $k = 1$ (or $l = 1$), \mathbb{R}_k^l becomes \mathbb{R}^l (or \mathbb{R}_k) and 0_k^l becomes 0^l (or 0_k). For $F \in \mathbb{R}_k^l$, we denote respectively by $\text{rk } F$, $\text{Im } F$, $\ker F$ and F^\top , the rank, image, kernel and transpose of F . $F^\perp \in \mathbb{R}_k^k$ denotes a matrix of maximal rank such that $FF^\perp = 0_k^k$, with $\kappa = \dim \ker F$. The matrix F^\dagger stands for the Moore-Penrose inverse of F . When F has linearly independent rows (respectively columns), $F^\dagger = F^\top(FF^\top)^{-1}$ (respectively $F^\dagger = (F^\top F)^{-1}F^\top$). The vector space generated by k vectors of same dimension e_i is denoted by $\text{span}\{e_i\}_{i=1, \dots, k}$. For $Q \in \mathbb{R}_k^k$,

$Q > 0_k^k$ means that Q is positive definite, $\sigma(Q)$ is the spectrum of Q , and for $\lambda \in \mathbb{C}$, $Q_\lambda = Q - \lambda I_k$, where I_k is the identity matrix of \mathbb{R}_k^k . E^\perp denotes the orthogonal space of the vector space E . The sets \mathbb{N}^* , $\mathbb{N}_{<k}$ and $\mathbb{N}_{\leq k}^*$ stand respectively for $\mathbb{N} \setminus \{0\}$, $\{0, 1, \dots, k-1\}$ and $\{1, \dots, k\}$. $\#\Lambda$ stands for the cardinal of the set Λ . For $w \in \mathbb{R}^k$, we denote by $|w|_k$ the Euclidean norm of w . For a measurable function h defined almost everywhere (a.e.) on $[0, T]$, $T > 0$, the essential supremum of h is given by $\|h\|_\infty := \text{supess}|h|_k = \inf\{c \geq 0 \mid \mu(\{t \in [0, T] : |h(t)|_k > c\}) = 0\}$ where μ is the Lebesgue measure. We then define $\mathcal{L}^\infty([0, T]; \mathbb{R}^k)$, the set of all measurable functions defined a.e. on $[0, T]$ such that $\|h\|_\infty < +\infty$, and $L^\infty([0, T]; \mathbb{R}^k) = \{[f] : f \in \mathcal{L}^\infty([0, T]; \mathbb{R}^k)\}$ where $[f] = \{g \in \mathcal{L}^\infty([0, T]; \mathbb{R}^k) \mid g = f \text{ a.e. on } [0, T]\}$. $L^2([0, T]; \mathbb{R}^k)$ is obtained similarly by substituting in the above construction the norm $\|\cdot\|_\infty$ by the norm $\|\cdot\|_2$ defined by $\|f\|_2^2 := \int_0^T |f(t)|_k^2 dt$. In the sequel, we will simply write f instead of $[f]$, i.e., for every $f \in \mathcal{L}^p([0, T]; \mathbb{R}^k)$ ($p \in \{2, \infty\}$), we identify $[f] \in L^p([0, T]; \mathbb{R}^k)$ with its representative f . The space $H^1([0, T]; \mathbb{R}^k)$ represents the Sobolev Hilbert space of function of $L^2([0, T]; \mathbb{R}^k)$ which derivative in the distributions sense is in $L^2([0, T]; \mathbb{R}^k)$. The set of continuous function $g : [0, T] \rightarrow \mathbb{R}^k$ is denoted by $\mathcal{C}^0([0, T]; \mathbb{R}^k)$ and this space is endowed with the $L^\infty([0, T]; \mathbb{R}^k)$ norm defined above.

2 Problem Statement

Consider, for all $t \geq 0$, the LTI system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad (1a)$$

$$y(t) = Cx(t) + Du(t). \quad (1b)$$

In (1), $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^q$ are respectively the state, the input (or the control) and the output of the system at time t , where n, m and q are positive integers, and $A \in \mathbb{R}_n^n$, $B \in \mathbb{R}_n^m$, $C \in \mathbb{R}_n^q$, $D \in \mathbb{R}_m^q$.

The following notations will be used:

$$\mathcal{C}^s(A, B) = (A^{n-1}B | A^{n-2}B | \dots | AB | B) \in \mathbb{R}_{nm}^n,$$

$$\mathcal{C}^o(A, B, C) = CC^s(A, B) \in \mathbb{R}_{nm}^q,$$

$$\mathcal{C}^o(A, B, C, D) = (\mathcal{C}^o(A, B, C) | D) \in \mathbb{R}_{(n+1)m}^q.$$

2.1 Reminders on the concept of state controllability

Let \mathcal{U}_s be a set of admissible controls, that is a set of all controls for which equation (1a) (named state equation) coupled with $x(0) = x_0 \in \mathbb{R}^n$ as an initial data admits a unique solution $x_u(t, x_0)$ such that $x_u(T, x_0)$ is well-defined. The notion of state controllability can be defined as follows.

Definition 2.1 *The system (1) is said to be state controllable if for any $(x_0, x_1) \in \mathbb{R}^n \times \mathbb{R}^n$, there exist a time $T > 0$ and a control $u \in \mathcal{U}_s$ such that $x_u(T, x_0) = x_1$.*

From the above, it is clear that the notion of state controllability requires a well-defined state at time $t = T$ starting from any given x_0 . This can be ensured by taking, for instance, $\mathcal{U}_s = L^\infty([0, T]; \mathbb{R}^m)$. For all $x_0 \in \mathbb{R}^n$ and $u \in \mathcal{U}_s$, equation (1a) together with the initial condition $x(0) = x_0$ admits a unique continuous solution given by

$$x_u(t, x_0) = e^{tA}x_0 + \int_0^t e^{(t-\tau)A}Bu(\tau)d\tau. \quad (2)$$

The following theorem gathers equivalent standard criteria for state controllability, which can be found for instance in [15].

Theorem 2.1 *The following properties are equivalent.*

- (sc₀) *The system (1) is state controllable.*
- (sc₁) *There exists a time $T > 0$ such that the state endpoint map $E_T^s : \mathcal{U}_s \rightarrow \mathbb{R}^n$, defined by*

$$E_T^s(u) = \int_0^T e^{(T-\tau)A}Bu(\tau)d\tau \quad (u \in \mathcal{U}_s)$$

is surjective.

- (sc₂) $\text{rk } C^s(A, B) = n$.
- (sc₃) $\ker(B^\top) \cap \ker(A_\lambda^\top) = \{0^n\}, \forall \lambda \in \mathbb{C}$.
- (sc₄) $\text{rk}(A_\lambda|B) = n, \forall \lambda \in \mathbb{C}$.

- (sc₅) $\mathcal{G}_T^s := \int_0^T e^{\tau A}BB^\top e^{\tau A^\top}d\tau > 0_n^n$, for some $T > 0$.

Remark 2.1 *Condition (sc₂) is known as Kalman rank condition, (see e.g. [10, Corollary 5.5]), where the matrix $C^s(A, B)$ is the Kalman controllability matrix. Condition (sc₄) is Popov-Belevitch-Hautus condition (mostly known as Hautus test) and can be found in [14, Lemma 3.3.7] (see also [1, 7, 12]). The matrix \mathcal{G}_T^s defined in (sc₅) is called continuous-time state controllability Gramian at time T of system (1a) (see [10, Proposition 5.2]).*

The following technical lemmas on the state controllability will be of crucial importance for output controllability.

Lemma 2.1 ([15, Corollary 3.2]) *Let $x_0 \in \mathbb{R}^n$, the set of all states that can be reached from x_0 in time T is $\mathcal{R}_s(x_0, T) = \{e^{TA}x_0\} + E_T^s(\mathcal{U}_s)$ and we have,*

$$\mathcal{R}_s(x_0, T) = \{e^{TA}x_0\} + \text{Im } C^s(A, B). \quad (3)$$

Remark 2.2 *It is well-known, for LTI systems, that the notion of state controllability depends neither on the initial state, nor on the controllability time.*

Lemma 2.2 *Let $x_0 \in \mathbb{R}^n$ and $x_1 \in \mathcal{R}_s(x_0, T)$, then for every $u_0, u_1 \in \mathbb{R}^m$, there exists $u \in C^0([0, T]; \mathbb{R}^m)$ such that $u(0) = u_0$, $u(T) = u_1$ and $x_u(T, x_0) = x_1$.*

PROOF Consider for $v \in L^\infty([0, T]; \mathbb{R}^m)$, the LTI system $\dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t) + \tilde{B}v(t)$, where $\tilde{x} \in \mathbb{R}^{n+m}$,

$$\tilde{A} = \begin{pmatrix} A & B \\ 0_n^m & 0_m^m \end{pmatrix} \in \mathbb{R}^{n+m}, \quad \tilde{B} = \begin{pmatrix} 0_n^m \\ I_m \end{pmatrix} \in \mathbb{R}^{n+m}. \quad (4)$$

Writing $\tilde{x} = (z^\top, w^\top)^\top$, where $z \in \mathbb{R}^n$ and $w \in \mathbb{R}^m$, we infer from (3) that for all $\tilde{x}_0 = (z_0^\top, w_0^\top)^\top$, we have

$$\tilde{\mathcal{R}}_s(\tilde{x}_0, T) = \mathcal{R}_s(z_0, T) \times \mathbb{R}^m,$$

where $\tilde{\mathcal{R}}_s(\tilde{x}_0, T)$ is the set of all states (in \mathbb{R}^{n+m}) that can be reached from \tilde{x}_0 at time T when the input v runs $L^\infty([0, T]; \mathbb{R}^m)$.

Take $\tilde{x}_0 = (x_0^\top, u_0^\top)^\top$ and $\tilde{x}_1 = (x_1^\top, u_1^\top)^\top$. Since $\tilde{x}_1 \in \tilde{\mathcal{R}}_s(\tilde{x}_0, T)$, there exists a control $v \in L^\infty([0, T]; \mathbb{R}^m)$ steering \tilde{x}_0 to \tilde{x}_1 in time T , and the function $u(t) = u_0 + \int_0^t v(\tau) d\tau$ is a continuous control steering x_0 to x_1 in time T and satisfying $u(0) = u_0$ and $u(T) = u_1$. \square

2.2 State to output controllability

The first concept of output controllability discussed in this paper is the one of state to output controllability.

Let \mathcal{U}_o be a set of admissible controls for (1), that is a set of all controls for which (1) admits, for any given initial data $x_0 \in \mathbb{R}^n$, a unique solution $y_u(t, x_0)$ such that $y_u(T, x_0)$ is well-defined. The following definition of state to output controllability will be considered.

Definition 2.2 *The system (1) is said to be state to output controllable (SOC) if for all $(x_0, y_1) \in \mathbb{R}^n \times \mathbb{R}^q$ there exist a time $T > 0$ and a control $u \in \mathcal{U}_o$ such that $y_u(T, x_0) = y_1$.*

Let us, from the generic description of \mathcal{U}_o , make a more precise choice of \mathcal{U}_o . Unlike the state controllability framework, the set of admissible controls \mathcal{U}_o cannot be reduced to $L^\infty([0, T]; \mathbb{R}^m)$ in the case of state to output controllability because for a control in $L^\infty([0, T]; \mathbb{R}^m)$, evaluating $y_u(t, x_0) = Cx_u(t, x_0) + Du(t)$ at time $t = T$ does not make sense when $D \neq 0$. Indeed, if $u \in L^\infty([0, T]; \mathbb{R}^m)$, u is a representative of an equivalence class and is not, a priori, well-defined at time T . So for expression $y_u(T, x_0) = Cx_u(T, x_0) + Du(T)$ to make sense, it is more convenient to take

$$\mathcal{U}_o = \{u \in \mathcal{L}^\infty([0, T]; \mathbb{R}^m) \mid u(T) \in \mathbb{R}^m\}. \quad (5)$$

Note that $C^0([0, T]; \mathbb{R}^m) \subset \mathcal{U}_o$. Setting $\mathcal{R}_o(x_0, T)$ as the set of all output points that can be reached from x_0 at time T with admissible controls in \mathcal{U}_o , we have the following proposition.

Proposition 2.1 *Let $x_0 \in \mathbb{R}^n$. If $y_1 \in \mathbb{R}^q$ can be reached from x_0 in time T by means of a control in \mathcal{U}_o , then there exists a control in $C^0([0, T]; \mathbb{R}^m)$ steering x_0 to y_1*

in time T . Moreover, we have

$$\mathcal{R}_o(x_0, T) = C\mathcal{R}_s(x_0, T) + \text{Im } D. \quad (6)$$

PROOF For every bounded function $u : [0, T] \rightarrow \mathbb{R}^m$, we have, according to (1b), $y_u(T, x_0) = Cx_u(T, x_0) + Du(T)$ and therefore that $\mathcal{R}_o(x_0, T) \subset C\mathcal{R}_s(x_0, T) + \text{Im } D$. Our aim is then to show that any point of $C\mathcal{R}_s(x_0, T) + \text{Im } D$ can be reached with a continuous control function. This will ensure that the set of reachable outputs is the same when choosing essentially bounded inputs with finite value at time $t = T$ or continuous controls.

For every $y_1 \in C\mathcal{R}_s(x_0, T) + \text{Im } D$, there exist $x_1 \in \mathcal{R}_s(x_0, T)$ and $u_1 \in \mathbb{R}^m$ such that $y_1 = Cx_1 + Du_1$. As $x_1 \in \mathcal{R}_s(x_0, T)$, we infer from Lemma 2.2 that there exists a continuous control u such that $u(T) = u_1$, steering x_0 to x_1 at time T . Using this control as input in (1) with x_0 as initial data, we deduce that $y_u(T, x_0) = Cx_u(T, x_0) + Du(T) = Cx_1 + Du_1 = y_1$. \square

Thanks to Proposition 2.1, we choose without loss of generality $\mathcal{U}_o = \mathcal{C}^0([0, T]; \mathbb{R}^m)$ for the rest of this paper.

Remark 2.3 If system (1) is SOC then it is SOC for all time $T > 0$. Indeed, if system (1) is SOC then there exists a time $T > 0$ such that $\mathcal{R}_o(0^n, T) = \mathbb{R}^q$ and hence for all time $T > 0$ thanks to (3) and (6). We infer that $\mathcal{R}_o(x_0, T) = \mathbb{R}^q$ for all $x_0 \in \mathbb{R}^n$ and for all $T > 0$. This shows that the notion of state to output controllability depends neither on the initial data nor on the state to output controllability time.

3 State to output controllability results

In this section, we give in paragraph 3.1 one of the main results of this paper, in particular, we extend Theorem 2.1 to the context of state to output controllability (Theorem 3.1). We also give a constructive result allowing us to build a continuous control steering any initial state to any final output (Theorem 3.2). These results are illustrated on an example in paragraph 3.2.

3.1 Main results

Theorem 2.1 takes the following form in the context of state to output controllability.

Theorem 3.1 The following properties are equivalent.

- (soc₀) The system (1) is state to output controllable.
- (soc₁) There exists a time $T > 0$ such that the output endpoint map $E_T^o : \mathcal{U}_o \rightarrow \mathbb{R}^q$, defined by

$$E_T^o(u) = \int_0^T C e^{(T-\tau)A} B u(\tau) d\tau + Du(T) \quad (u \in \mathcal{U}_o)$$

is surjective.

- (soc₂) $\text{rk } C^o(A, B, C, D) = q$.

- (soc₃) $\text{rk } (C|D) = q$ and

$$\text{Im} \begin{pmatrix} C^\top \\ D^\top \end{pmatrix} \cap \left(\bigoplus_{\lambda \in \sigma(A)} E_\lambda \times \{0^m\} \right) = \{0^{n+m}\},$$

where $E_\lambda = \ker(A_\lambda^\top)^{n_\lambda} \cap \left(\bigcap_{k=0}^{n_\lambda-1} \ker B^\top (A_\lambda^\top)^k \right)$, with n_λ , the algebraic multiplicity of λ in the minimal polynomial of A .

- (soc₄) $\text{rk } (C|D) = q$ and

$$\text{rk} \begin{pmatrix} K_{\lambda_1} & 0 & \cdots & 0 & (C|D)^\perp \\ 0 & K_{\lambda_2} & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & K_{\lambda_p} & (C|D)^\perp \end{pmatrix} = (n+m)p,$$

where $p = \#\sigma(A)$, $\{\lambda_1, \lambda_2, \dots, \lambda_p\} = \sigma(A)$,

$$K_\lambda = \begin{pmatrix} M_\lambda & 0 \\ 0 & I_m \end{pmatrix} \in \mathbb{R}_{n+(n_\lambda+1)m}^{n+m} \text{ and}$$

$$M_\lambda = (A_\lambda^{n_\lambda} | A_\lambda^{n_\lambda-1} B | \cdots | A_\lambda B | B) \in \mathbb{R}_{n+n_\lambda m}^{n+n_\lambda m}.$$

- (soc₅) $\mathcal{K}_T := C\mathcal{G}_T^s C^\top + DD^\top > 0_q^q$, for some $T > 0$, where \mathcal{G}_T^s is the matrix defined in (sc₅).

- (soc₆) $\mathcal{G}_T^o := \int_0^T H_o(T, \tau) H_o(T, \tau)^\top d\tau > 0_q^q$, for some $T > 0$, where $H_o(T, \tau) = CM(T - \tau)B + D$ and $M(s) = \int_0^s e^{(s-t)A} dt$, $s \in [0, T]$.

The matrix \mathcal{G}_T^o defined in Theorem 3.1, criterion (soc₆) is called the state to output controllability Gramian at time T . The proof of this theorem is provided in paragraph 4.1. For the moment, let us make some comments.

Remark 3.1 As for Remark 2.3, in (soc₁), (soc₅) and (soc₆) one can replace the existence of $T > 0$ by the fact that T can be any positive real number.

Indeed, if $\mathcal{G}_T^o > 0_q^q$ for some time $T > 0$ then $\mathcal{G}_\tau^o > 0_q^q$ for all time $\tau > 0$. This being so because if $\tau \geq T$ then $\mathcal{G}_\tau^o > 0_q^q$ thanks to the linearity of the integral and the fact that \mathcal{G}_T^o is positive definite. For $\tau < T$, assume by contradiction that \mathcal{G}_τ^o is not positive definite. Then there exists a vector $\eta \in \mathbb{R}^q \setminus \{0^q\}$ such that $\eta^\top \mathcal{G}_\tau^o \eta = 0$. This leads to the fact that $H_o(\tau, t)^\top \eta = 0$ for all $t \in [0, \tau]$. Since the application $t \mapsto H_o(\tau, t)^\top \eta$ is analytic, we deduce from the analytic continuation theorem that $H_o(t_1, t)^\top \eta = 0$ for every $t_1 \geq \tau$ and in particular for $t_1 = T$. Hence, \mathcal{G}_T^o is not positive definite, leading to a contradiction.

Remark 3.2 Conditions (soc _{i}), for $i \in \mathbb{N}_{\leq 5}$ are extensions of conditions (sc _{i}).

Observe that when considering the state as the output ($q = n$, $C = I_n$ and $D = 0_m^n$), the notion of state to output controllability becomes the one of state controllability. In this case, conditions (soc _{i}) are simplified into (sc _{i}) for $i \in \mathbb{N}_{\leq 5}$. This is obvious for (soc₀), (soc₁), (soc₂) and (soc₅). Let us now check this

statement for (soc_3) and (soc_4) .

One can show directly that the derived version of (soc_3) is equivalent to (sc_3) . To this end, remark that $\bigoplus_{\lambda \in \sigma(A)} E_\lambda = \{0^n\}$ if and only if $E_\lambda = \{0^n\}$ for all $\lambda \in \sigma(A)$. Furthermore, $E_\lambda = \{0^n\}$ is equivalent to $\ker B^\top \cap \ker A_\lambda^\top = \{0^n\}$. Indeed, we have $\ker B^\top \cap \ker A_\lambda^\top \subset E_\lambda$ and reciprocally, if there exists $\lambda \in \sigma(A)$ such that $E_\lambda \neq \{0^n\}$, then for every $z_\lambda \in E_\lambda \setminus \{0^n\}$, there exists $k \in \mathbb{N}$ such that $(A_\lambda^k)^\top z_\lambda \neq 0_n$ and $(A_\lambda^k)^\top z_\lambda \in \ker A_\lambda^\top \cap \ker B^\top$.

To show that the simplified version of (soc_4) is equivalent to (sc_4) , it is sufficient to notice that for $C = I_n$ and $D = 0_m^n$, one can choose $(C|D)^\perp = (0_n^m, I_m)^\top$ and (soc_4) is then reduced to $\text{rk diag}(M_{\lambda_1}; \dots; M_{\lambda_p}) = np$ which is equivalent to $\text{rk } M_\lambda = n$ for all $\lambda \in \sigma(A)$. Arguing as in the previous point, we get the result by duality.

Remark 3.3 Note that (soc_3) is equivalent to:

$\text{rk}(C|D) = q$ and for every $\Lambda \subset \mathbb{C}$ such that $\#\Lambda \leq n$,

$$\text{we have } \text{Im} \begin{pmatrix} C^\top \\ D^\top \end{pmatrix} \cap \left(\bigoplus_{\lambda \in \Lambda} E_\lambda \times \{0^m\} \right) = \{0^{n+m}\}.$$

This being true because for every $\Lambda \subset \mathbb{C}$ such that $\#\Lambda \leq +\infty$, we have $\bigoplus_{\lambda \in \Lambda} E_\lambda \subset \bigoplus_{\lambda \in \sigma(A)} E_\lambda$ and $\#\sigma(A) \leq n$.

In the same way, we have state to output controllability for system (1) if and only if condition (soc_4) is satisfied for every set of $p \leq n$ distinct complex numbers. It is important for λ_i to be two by two distinct in that condition. Indeed, one can see that $\ker(K_\lambda | (C|D)^\perp)^\top = \{0^{n+m}\}$ is not equivalent to

$$\ker \begin{pmatrix} K_\lambda & 0 & (C|D)^\perp \\ 0 & K_\lambda & (C|D)^\perp \end{pmatrix}^\top = \{0^{2(n+m)}\}.$$

Remark 3.4 Condition (soc_1) is trivial because it describes the fact that there exists a time $T > 0$ such that $\mathcal{R}_o(x_0, T) = \mathbb{R}^q$ for all $x_0 \in \mathbb{R}^n$. Conditions (soc_2) and (soc_5) are well-known and can be found in [11].

Remark 3.5 For every $x_0 \in \mathbb{R}^n$, we have, $\mathcal{R}_o(x_0, T) \subset \text{Im } C + \text{Im } D$. Hence, a necessary condition to have state to output controllability is

$$\text{rk}(C|D) = q. \quad (7)$$

This rank condition is implicit in conditions (soc_1) , (soc_2) , (soc_5) and (soc_6) but appears explicitly in conditions (soc_3) and (soc_4) as it will be underlined in the proof of Theorem 3.1.

The following theorem gives, in the case of state to output controllability, a continuous control steering any initial state $x_0 \in \mathbb{R}^n$ to any output $y_1 \in \mathbb{R}^q$. Its proof is postponed in paragraph 4.2.

Theorem 3.2 Let $(x_0, y_1) \in \mathbb{R}^n \times \mathbb{R}^q$, and assume that system (1) is SOC. For every $T > 0$ and $u_0 \in \mathbb{R}^m$, the control

$$u(t) = u_0 + \int_0^t H_o(T, \tau)^\top d\tau \times (\mathcal{G}_T^o)^{-1} (y_1 - y_{u_0}(T, x_0)),$$

steers x_0 to y_1 in time T . In the above, \mathcal{G}_T^o and H_o are the matrices defined in (soc_6) and $y_{u_0}(T, x_0) = Ce^{TA}x_0 + H_o(T, 0)u_0$.

Furthermore, this control is the unique minimizer of

$$\min \frac{1}{2} \int_0^T |\dot{u}(t)|_m^2 dt \quad \left| \begin{array}{l} u \in H^1([0, T]; \mathbb{R}^m), \quad u(0) = u_0, \quad y_1 = y_u(T, x_0). \end{array} \right. \quad (8)$$

Remark 3.6 We can also, in the case of state to output controllability, use the matrix \mathcal{K}_T to compute a control steering x_0 to y_1 in time T . More precisely, we observe that the control given by

$$u(t) = \begin{cases} B^\top e^{(T-t)A^\top} C^\top (\mathcal{K}_T)^{-1} \delta_y & \text{if } t \in [0, T), \\ D^\top (\mathcal{K}_T)^{-1} \delta_y & \text{if } t = T, \end{cases}$$

with $\delta_y = y_1 - Ce^{AT}x_0$ steers x_0 to y_1 in time T .

Furthermore, this control is the unique minimizer of

$$\min \frac{1}{2} \int_0^T |u(\tau)|_m^2 d\tau + \frac{1}{2} |z|_m^2 \quad \left| \begin{array}{l} u \in L^2([0, T]; \mathbb{R}^m), \quad z \in \mathbb{R}^m, \\ y_1 - Ce^{AT}x_0 = CE_T^s(u) + Dz, \end{array} \right.$$

where z stands for the final control value $u(T)$. One can see from the above expression of u that, this control is not continuous unless $CB = D$. When $D = 0_m^q$, a continuous control can be built by minimizing $\int_0^T |u(\tau)|_m^2 d\tau$. Note that even if E_T^s was not defined on $L^2([0, T]; \mathbb{R}^m)$, it admits a trivial extension to elements of this set.

3.2 Illustration

To illustrate these results, let us consider the system (1) with $n = 3$, $m = 1$, $q = 2$, and matrices A , B , C , D defined, with α , γ , ν and δ four real numbers, by,

$$A = \begin{pmatrix} 1 & \alpha & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 & 0 \\ \gamma & 0 & \nu \end{pmatrix}, \quad D = \begin{pmatrix} 0 \\ \delta \end{pmatrix},$$

Observe that through (sc_2) , the state controllability of (1) is equivalent to $\alpha \neq 0$. The necessary condition for state to output controllability (7) is equivalent to the fact that at least one of the parameters γ , ν or δ is not null.

Let us now focus on the criteria given in Theorem 3.1. The Kalman extended rank condition (soc_2) being well-known, we first derive from it a necessary and sufficient condition on parameters α , γ , ν and δ for state to output controllability. We then check that the other criteria lead to the same conclusion.

• *Criterion (soc₂)*. The Kalman state to output controllability matrix is given by

$$\mathcal{C}^o(A, B, C, D) = \begin{pmatrix} 2 & 1 & 0 & 0 \\ \nu + \alpha\gamma & \nu & \nu & \delta \end{pmatrix}.$$

From this matrix, we observe that $\text{rk } \mathcal{C}^o(A, B, C, D)$ is strictly less than 2 if and only if $\delta = \nu = \alpha\gamma = 0$. The state to output controllability then yields if and only if the parameters α, γ, ν and δ do not satisfy

$$\delta = \nu = \alpha\gamma = 0. \quad (9)$$

• *Criterion (soc₃)*. We recall that (7) is equivalent to $\nu \neq 0$ or $\gamma \neq 0$ or $\delta \neq 0$. We have in every case $\sigma(A) = \{1\}$. Also, it can be easily checked that if $\alpha \neq 0$, the algebraic multiplicity of 1 in the minimal polynomial of A is $n_1 = 3$, and we have $E_1 = \{0^n\}$. If $\alpha = 0$ then $n_1 = 2$, and we have $E_1 = \text{span}\{(1 \ 0 \ 0)^\top\}$. We also observe that $\text{Im}(C|D)^\top = \text{span}\left\{(0 \ 1 \ 0 \ 0)^\top, (\gamma \ 0 \ \nu \ \delta)^\top\right\}$.

From the above computation, we deduce that (7) and $\text{Im}(C|D)^\top \cap (E_1 \times \{0^m\}) = \{0^n\}$ are not satisfied if and only if (7) is not fulfilled in the case $\alpha \neq 0$. In the case $\alpha = 0$, (7) and $\text{Im}(C|D)^\top \cap (E_1 \times \{0^m\}) = \{0^n\}$ are not satisfied if and only if (7) is not satisfied or $\nu = \delta = 0$ and $\gamma \neq 0$.

From the above, we infer that we do not have state to output controllability if and only if the parameters α, γ, ν and δ satisfy (9). This point shows that condition (7) is of crucial importance in criterion (soc₃). Indeed, taking $\alpha = \nu = \delta = \gamma = 0$, we have $\text{Im}(C|D)^\top \cap (E_1 \times \{0^m\}) = \{0^n\}$ but the system is not state to output controllable.

• *Criterion (soc₄)*. For this condition, we have to compute matrices $J = (C|D)^\perp$ and K_1 . To compute J , we need to distinguish the different cases whether γ, δ or ν vanishes or not, and we take:

for $\gamma \neq 0$,	for $\nu \neq 0$,	for $\delta \neq 0$,
$J = \begin{pmatrix} -\nu & -\delta \\ 0 & 0 \\ \gamma & 0 \\ 0 & \gamma \end{pmatrix},$	$J = \begin{pmatrix} \nu & 0 \\ 0 & 0 \\ -\gamma & -\delta \\ 0 & \nu \end{pmatrix},$	$J = \begin{pmatrix} \delta & 0 \\ 0 & 0 \\ 0 & \delta \\ -\gamma & -\nu \end{pmatrix}.$

Similarly, to define the matrix K_1 , we need to distinguish the case $\alpha \neq 0$ with the case $\alpha = 0$ (recall that $\sigma(A) = \{1\}$ in any case). More precisely, we have:

for $\alpha \neq 0, K_1 \in \mathbb{R}_7^4$ and	for $\alpha = 0, K_1 \in \mathbb{R}_6^4$ and
$K_1 = \begin{pmatrix} 0 & 0 & 0 & \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$	$K_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$

We can now check (soc₄).

- If $\alpha \neq 0$ then $\text{rk}(K_1|J) = 4 = (n+m)p$ in all cases $\gamma \neq 0, \nu \neq 0$ or $\delta \neq 0$. The state to output controllability is then equivalent to (7), that is one of the parameters δ, γ, ν must be not null.
- If $\alpha = 0$ and $\gamma \neq 0$, then (7) is satisfied and $\text{rk}(K_1|J) = 4$ if and only if $\nu \neq 0$ or $\delta \neq 0$.
- If $\alpha = 0$ and $\nu \neq 0$ (resp. $\delta \neq 0$), then (7) is satisfied and $\text{rk}(K_1|J) = 4$ without restriction on γ and δ (resp. ν).

From the foregoing argument, we deduce that we do not have state to output controllability if and only if the parameters α, γ, ν and δ satisfy (9).

• *Criterion (soc₅)*. We first show this condition with $T = 1$. After some computations, we get

$$\det \mathcal{K}_1 = e_1 \delta^2 + e_2 [\nu + \tilde{e} \alpha \gamma]^2 + e_3 \alpha^2 \gamma^2,$$

with $e_1 = (e^2 - 1)/4$, $e_2 = (e^4 - 6e^2 + 1)/16$, $e_3 = (e^8 - 12e^6 + 38e^4 - 28e^2 + 1)/128$, and $\tilde{e} = (e^4 - 8e^2 - 1)/(4(e^4 - 6e^2 + 1))$. Noticing that $e_i > 0$ for $i = 1, 2, 3$, we deduce that $\det \mathcal{K}_1 = 0$ if and only if (9) holds.

Reciprocally, if (9) holds, then for every $T > 0$, the matrix \mathcal{K}_T takes the form $\begin{pmatrix} \star & 0 \\ 0 & 0 \end{pmatrix}$, and hence, $\det \mathcal{K}_T = 0$

for all $T > 0$. We then deduce that the system is not SOC if and only if the parameters δ, ν, α and γ satisfy (9).

• *Criterion (soc₆)*. Let us choose again $T = 1$. We obtain by computation

$$\det \mathcal{G}_1^o = \frac{1}{f_1} \left[f_1 \delta + \frac{\eta_1 \nu + \eta_2 \alpha \gamma}{2} \right]^2 + \frac{1}{f_2} \left[f_2 \nu + \frac{\eta_3}{2} \alpha \gamma \right]^2 + f_3 \alpha^2 \gamma^2,$$

where we have set

$$\begin{aligned} f_1 &= \frac{-3e^2 + 16e - 21}{4}, \\ f_2 &= \frac{3e^6 - 40e^5 + 195e^4 - 416e^3 + 325e^2 + 40e - 75}{16(3e^2 - 16e + 21)}, \\ f_3 &= \frac{13e^6 - 168e^5 + \xi}{256(3e^2 - 16e + 21)} - \frac{\eta_3^2}{4f_2}, \\ \xi &= 745e^4 - 1072e^3 - 1193e^2 + 4696e - 3405, \\ \eta_1 &= \frac{e^2 - 6e + 9}{2}, \quad \eta_2 = \frac{e^3 - 9e^2 + 27e - 27}{8}, \\ \eta_3 &= \frac{3e^6 - 36e^5 + 149e^4 - 248e^3 + 181e^2 - 292e + 435}{32(3e^2 - 16e + 21)}. \end{aligned}$$

One can easily check that $f_i > 0$ for $i = 1, 2, 3$ and that $\det \mathcal{G}_1^o = 0$ if and only (9) holds. Reciprocally, if (9) is satisfied, then, for all $T > 0$, the matrix \mathcal{G}_T^o takes the

form $\begin{pmatrix} \star & 0 \\ 0 & 0 \end{pmatrix}$, and hence, $\det \mathcal{G}_T^o = 0$ for all $T > 0$.

The outcome of this illustration is that all the criteria of Theorem 3.1 lead to the same conclusion: the system (1) with the matrices considered in this illustration is not SOC if and only if the parameters δ , ν , α and γ satisfy (9).

Remark 3.7 From (3) and (6), if system (1) is state controllable, then it is SOC provided that (7) is fulfilled. This fact can be seen from (9). Indeed, when state controllability holds (i.e., $\alpha \neq 0$), the necessary condition (7) turns out to be sufficient for state to output controllability. But (7) is not enough to ensure state to output controllability in any case. For instance, taking $\gamma = 1$ and $\alpha = \delta = \nu = 0$, (7) holds but the system is not SOC. Also, it can be seen from this illustration that system (1) can be SOC without being state controllable. Take for instance $\alpha = 0$ and $\nu \neq 0$.

Remark 3.8 The observability and the state to output controllability of system (1) are two distinct notions. Indeed, system (1) can be SOC without being observable. This is in particular the case if we choose $\alpha = \gamma = \delta = 0$ and $\nu \neq 0$. For the notion of observability we refer to [15, Section 3.3].

We now illustrate Theorem 3.2 and Remark 3.6.

Let us pick $x_0 = (1 \ 0 \ 1)^\top$, $y_1 = (1 \ 2)^\top$ and $u_0 = 1$. We would like to design a continuous control u steering x_0 to y_1 in time $T = 1$ such that $u(0) = u_0$. In order to lighten the computations, we take $\gamma = \delta = 1$ and $\nu = \alpha = 0$. From Theorem 3.2, we infer that for every $t \in [0, 1]$,

$$u(t) = \frac{4e^2 - 12e + 12}{3e^2 - 16e + 21}(t+1)e^{1-t} + \frac{e^3 - e^2 - e - 3}{3e^2 - 16e + 21}t - \frac{4e^3 - 15e^2 + 28e - 21}{3e^2 - 16e + 21}.$$

One can also check that with this particular choice of parameters, the piecewise continuous control built in Remark 3.6 is given by

$$u(t) = \begin{cases} \frac{4(t-1)e^{1-t}}{e+1} & \text{if } t \in [0, T), \\ 2-e & \text{if } t = T. \end{cases}$$

Solving (1) with u , in both cases, as input and x_0 as the initial data, we get the time trajectories depicted on Figure 1.

4 Proof of Theorems 3.1 and 3.2

4.1 Proof of Theorem 3.1

To prove Theorem 3.1, we first state and prove a lemma allowing us to show the equivalence between state

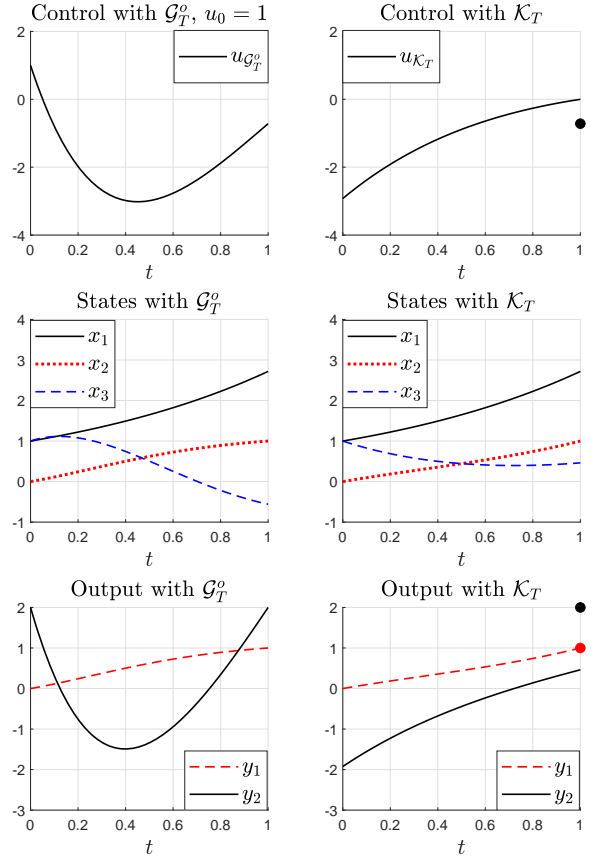


Fig. 1. Output control and trajectories given by Theorem 3.2 and Remark 3.6 with matrices A , B , C and D given in paragraph 3.2 with $\nu = \alpha = 0$, $\gamma = \delta = 1$, $x_0 = (1 \ 0 \ 1)^\top$, $y_1 = (1 \ 2)^\top$, and $T = 1$.

to output controllability of (1) and state to output controllability of a system similar to (1) without direct transmission from the input to the output. We then prove a derived version of Theorem 3.1 in the case $D = 0_m^q$ and finally show how to get the results of Theorem 3.1. As mentioned in Remark 3.4, (soc_1) is trivial and criteria (soc_2) , and (soc_5) have already been proven in [11]. We then focus our attention on the proof of conditions (soc_3) , (soc_4) and (soc_6) .

Lemma 4.1 Consider for $t \geq 0$, the system given by

$$\dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t) + \tilde{B}v(t), \quad (10a)$$

$$y(t) = \tilde{C}\tilde{x}(t), \quad (10b)$$

where $v(t) \in \mathbb{R}^m$ is the input, $\tilde{x}(t) \in \mathbb{R}^{n+m}$, $y(t) \in \mathbb{R}^q$, matrices \tilde{A} and \tilde{B} given by (4) and $\tilde{C} = (C|D)$. The state to output controllability of system (10) is equivalent to the state to output controllability of system (1).

PROOF In order to prove this lemma, we are going to compute the output accessible set of systems (1) and (10). For every $x_0 \in \mathbb{R}^n$, we deduce from (3) and (6),

$$\mathcal{R}_o(x_0, T) = \{Ce^{AT}x_0\} + \text{Im } \mathcal{C}^o(A, B, C, D). \quad (11)$$

Choose now any $u_0 \in \mathbb{R}^m$ and set $\tilde{x}_0 = (x_0^\top, u_0^\top)^\top \in \mathbb{R}^{n+m}$. Let $\tilde{\mathcal{R}}_o(\tilde{x}_0, T)$ be the set of all reachable output points from \tilde{x}_0 in time T for system (10). From equations (3) and (6), we also infer that $\tilde{\mathcal{R}}_o(\tilde{x}_0, T) = \{\tilde{C}e^{T\tilde{A}}\tilde{x}_0\} + \tilde{C} \text{Im } \mathcal{C}^s(\tilde{A}, \tilde{B})$. This expression can be re-written as

$$\tilde{\mathcal{R}}_o(\tilde{x}_0, T) = \left\{ \tilde{C}e^{T\tilde{A}}\tilde{x}_0 \right\} + \text{Im } \mathcal{C}^o(A, B, C, D). \quad (12)$$

Equations (11) and (12) show that $\mathcal{R}_o(x_0, T) = \mathbb{R}^q$ for all $x_0 \in \mathbb{R}^n$ if and only if $\tilde{\mathcal{R}}_o(\tilde{x}_0, T) = \mathbb{R}^q$ for all $\tilde{x}_0 \in \mathbb{R}^{n+m}$. \square

This lemma being proven, to show Theorem 3.1, it is enough to establish it for $D = 0_m^q$. Lemma 4.1 combined with the results of the case $D = 0_m^q$ will give the expected ones.

Lemma 4.2 *Assume that $D = 0_m^q$. The following conditions are equivalent:*

(*soc*₀⁰) *The system (1) is state to output controllable,*

(*soc*₃⁰) *rk $C = q$ and $\text{Im } C^\top \cap \bigoplus_{\lambda \in \sigma(A)} E_\lambda = \{0^n\}$,*

*where E_λ is defined in (*soc*₃).*

(*soc*₄⁰) *rk $C = q$ and, with M_λ is defined in (*soc*₄),*

$$\text{rk} \begin{pmatrix} M_{\lambda_1} & 0 & \cdots & 0 & C^\perp \\ 0 & M_{\lambda_2} & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & M_{\lambda_p} & C^\perp \end{pmatrix} = np.$$

In this lemma, we do not rewrite the corresponding conditions (*soc*₁), (*soc*₂) and (*soc*₅) in this particular case since they can be found, for instance, in [11].

PROOF (LEMMA 4.2) First notice that when $D = 0_m^q$, condition (*soc*₂) becomes $\text{rk } \mathcal{C}^o(A, B, C) = q$ and the equivalence of the controllability of system (1) with $D = 0_m^q$ and this rank condition can be found in [11].

• *Criterion (*soc*₃⁰).* Let us first define $\mathcal{W} = \text{Im } C^\top \cap \bigoplus_{\lambda \in \sigma(A)} E_\lambda$. We show that (*soc*₃⁰) is equivalent to $\text{rk } \mathcal{C}^o(A, B, C) = q$.

Assume that $\text{rk } C = q$ and $\mathcal{W} \neq \{0^n\}$ and take $z \in \mathcal{W} \setminus \{0^n\}$. Since $z \in \mathcal{W}$, there exist $\eta \in \mathbb{R}^q \setminus \{0^q\}$ and $z_\lambda \in E_\lambda$ for all $\lambda \in \sigma(A)$ such that $z = C^\top \eta = \sum_{\lambda \in \sigma(A)} z_\lambda$. From $B^\top z_\lambda = 0^m$ for all $\lambda \in \sigma(A)$, we deduce that $B^\top z = 0^m$. This implies that $B^\top C^\top \eta = 0^m$. In addition, the fact that $z_\lambda \in E_\lambda$ for $\lambda \in \sigma(A)$ implies $B^\top A^{k-1} C^\top \eta = \sum_{\lambda \in \sigma(A)} B^\top A^{k-1} z_\lambda = 0^m$ for all $k \in \mathbb{N}$. From this, we deduce that $\text{rk } \mathcal{C}^o(A, B, C) < q$.

Conversely, if $\text{rk } \mathcal{C}^o(A, B, C) < q$ then either $\text{rk } C < q$ or not. If $\text{rk } C < q$, then there is nothing to prove. If $\text{rk } C = q$, then there exists $\eta \in \mathbb{R}^q \setminus \{0^q\}$ such that

$B^\top A^{i-1} C^\top \eta = 0^m$ for all $i \in \mathbb{N}_{<n}$ and hence for all $i \in \mathbb{N}$, thanks to Cayley-Hamilton. We deduce that $C^\top \eta \in (\text{Im } C^\top \cap \mathcal{N}) \setminus \{0^n\}$ where $\mathcal{N} = \bigcap_{i \in \mathbb{N}} \ker B^\top (A^i)^\top$. Since \mathcal{N} is A^\top invariant, we have (see [6, Theorem 2.1.5])

$$\mathcal{N} = \bigoplus_{\lambda \in \sigma(A)} (\mathcal{N} \cap \ker (A_\lambda^\top)^{n_\lambda}).$$

One can see that $\mathcal{N} \cap \ker (A_\lambda^\top)^{n_\lambda} = E_\lambda$. We then deduce that $\mathcal{W} \neq \{0^n\}$. This ends the proof of the equivalence between (*soc*₃⁰) and the state to output controllability of (1) when $D = 0_m^q$.

• *Criterion (*soc*₄⁰).* We show that (*soc*₄⁰) is equivalent to (*soc*₃⁰). It can be easily seen that $E_\lambda = \ker M_\lambda^\top$. Let $z \in \mathcal{W} \setminus \{0^n\}$. Then $z \in \text{Im } C^\top$ and can be written as $z = \sum_{\lambda \in \sigma(A)} z_\lambda$, where $z_\lambda \in \ker M_\lambda$. Since $z \in \text{Im } C^\top = (\ker C)^\perp$ then for every $x \in \ker C$, $\langle z, x \rangle = 0$. Let $\kappa = \dim \ker C$. For every $x \in \ker C$, there exists $v \in \mathbb{R}^\kappa$ such that $x = C^\perp v$. The fact that $\langle z, x \rangle = 0$ for every $x \in \ker C$ is equivalent to $\langle z, C^\perp v \rangle = 0$ for all $v \in \mathbb{R}^\kappa$. This is equivalent to

$$\begin{pmatrix} z_{\lambda_1} \\ z_{\lambda_2} \\ \vdots \\ z_{\lambda_p} \end{pmatrix} \in \ker \begin{pmatrix} M_{\lambda_1}^\top & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & M_{\lambda_p}^\top \\ C^{\perp \top} & \cdots & \cdots & C^{\perp \top} \end{pmatrix}.$$

This gives the equivalence between (*soc*₃⁰) and (*soc*₄⁰). \square

We are now in position to complete the proof of Theorem 3.1.

PROOF (THEOREM 3.1) The proof of condition (*soc*₂) is straightforward considering system (10). Indeed, using Lemma 4.1, one can argue as in the case of state controllability and notice that the Kalman extended rank condition for system (10) is $\text{rk } \mathcal{C}^o(\tilde{A}, \tilde{B}, \tilde{C}) = q$. The rank condition (*soc*₂) is then recovered by noticing that $\mathcal{C}^o(\tilde{A}, \tilde{B}, \tilde{C}) = \mathcal{C}^o(A, B, C, D)$. To prove (*soc*₆), apply [11, Theorem 1] to system (10) and use, with M defined in (*soc*₆), the fact that (see e.g. [16])

$$e^{T\tilde{A}} = \begin{pmatrix} e^{TA} & M(T)B \\ 0_n^m & I_m \end{pmatrix}.$$

The only conditions requiring our attention are (*soc*₃) and (*soc*₄).

• Let us show the equivalence between the state to output controllability of system (1) and condition (*soc*₃). Applying (*soc*₃⁰) to system (10), we infer that the state

to output controllability of (10) is equivalent to

$$\text{rk } \tilde{C} = q \quad \text{and} \quad \text{Im } \tilde{C}^\top \cap \bigoplus_{\lambda \in \sigma(\tilde{A})} \tilde{E}_\lambda = \{0^{n+m}\}, \quad (13)$$

where \tilde{E}_λ is E_λ with A and B replaced by \tilde{A} and \tilde{B} . Note also that $\text{rk } \tilde{C} = \text{rk}(C|D)$ and $\sigma(\tilde{A}) = \sigma(A) \cup \{0\}$. Let us now write this condition in terms of A , B , C and D . First, one readily gets

$$\text{Im } \tilde{C}^\top = \text{Im} \begin{pmatrix} C^\top \\ D^\top \end{pmatrix} \quad \text{and} \quad \ker \tilde{B}^\top = \mathbb{R}^n \times \{0^m\}. \quad (14)$$

Let $\tilde{z}_\lambda = (\eta^\top, w^\top)^\top \in \tilde{E}_\lambda$, where $\eta \in \mathbb{R}^n$ and $w \in \mathbb{R}^m$. As $\tilde{z}_\lambda \in \tilde{E}_\lambda$, then $\tilde{z}_\lambda \in \ker \tilde{B}^\top$ (take $k = 0$ in \tilde{E}_λ). We then deduce from (14) that $w = 0^m$. For all $k \in \mathbb{N}^*$, we have

$$\left(\tilde{A}_\lambda^\top\right)^k = \begin{pmatrix} (A_\lambda^\top)^k & 0_m^n \\ \sum_{l=0}^{k-1} (-\lambda)^l B^\top (A_\lambda^\top)^{k-l-1} & (-\lambda)^k I_m \end{pmatrix}.$$

From the above, we deduce that $\tilde{z}_\lambda \in \ker \left(\tilde{A}_\lambda^\top\right)^{n_\lambda}$ if and only if

$$\eta \in \ker (A_\lambda^\top)^{n_\lambda} \quad \text{and} \quad \sum_{l=0}^{n_\lambda-1} (-\lambda)^l B^\top (A_\lambda^\top)^{n_\lambda-l-1} \eta = 0^m. \quad (15)$$

Since $\left(\tilde{A}_\lambda^\top\right)^k \tilde{z}_\lambda \in \ker \tilde{B}^\top$, $\forall k \in \mathbb{N}_{\leq n_\lambda-1}^*$, we have

$$\sum_{l=0}^{k-1} (-\lambda)^l B^\top (A_\lambda^\top)^{k-l-1} \eta = 0^m, \quad \forall k \in \mathbb{N}_{\leq n_\lambda-1}^*. \quad (16)$$

Equation (16) coupled with (15) is equivalent to

$$\eta \in \ker (A_\lambda^\top)^{n_\lambda} \quad \text{and} \quad (A_\lambda^\top)^k \eta \in \ker B^\top, \quad \forall k \in \mathbb{N}_{< n_\lambda}. \quad (17)$$

Equations (14) and (17) show that $\tilde{E}_\lambda = E_\lambda \times \{0^m\}$. Note that if $0 \notin \sigma(A)$ then $E_0 = \{0^n\}$. From $\tilde{E}_\lambda = E_\lambda \times \{0^m\}$ and the first equation of (14), we deduce (soc_3) .

• Let us prove the equivalence between (soc_3) and (soc_4) . First notice that condition (soc_3) has the same form as condition (soc_3^0) where matrix C is replaced by matrix $(C|D)$ and E_λ by $E_\lambda \times \{0^m\}$. Observing that for every $\lambda \in \sigma(A)$, $E_\lambda \times \{0^m\} = \ker(K_\lambda)^\top$ where K_λ is defined in (soc_4) , one get (soc_4) by replacing in (soc_3^0) the matrices C and M_λ respectively by $(C|D)$ and K_λ and n by $n+m$. \square

4.2 Proof of Theorem 3.2

Let $x_0 \in \mathbb{R}^n$, $y_1 \in \mathbb{R}^q$ and $u_0 \in \mathbb{R}^m$. Assume that system (1) is SOC. Then for every $T > 0$, there exists a control $u \in C^0([0, T]; \mathbb{R}^m)$ with $u(0) = u_0$ steering x_0 to y_1 in time T . There even exists an infinite number of controls doing the same transfer since, thanks to the rank Theorem, the kernel of the linear map E_T^o defined in Theorem 3.1 is of infinite dimension. To prove Theorem 3.2, we will look for the control solution of the minimization problem (8), i.e., the control whose weak time derivative is of minimal L^2 -norm. Since $u \in H^1([0, T]; \mathbb{R}^m)$ and $u(0) = u_0$, there exists a function $v \in L^2([0, T]; \mathbb{R}^m)$ such that $u(t) = u_0 + \int_0^t v(\tau) d\tau$. Replacing u by this expression in $y_u(T, x_0)$, we get,

$$y_u(T, x_0) = y_{u_0}(T, x_0) + \int_0^T C e^{(T-t)A} B \int_0^t v(s) ds dt + D \int_0^T v(t) dt.$$

Using integration by parts and Chasles relation, we have

$$\int_0^T C e^{(T-t)A} B \int_0^t v(s) ds dt = \int_0^T C M(T-t) B v(t) dt.$$

From the above, we deduce that $y_u(T, x_0) = y_{u_0}(T, x_0) + \Phi(v)$, with $\Phi(v) = \int_0^T H_o(T, t) v(t) dt$.

The minimization Problem (8) can then be rewritten as

$$\min \frac{1}{2} \int_0^T |v(t)|_m^2 dt \quad \left| \begin{array}{l} v \in L^2([0, T]; \mathbb{R}^m) \quad \text{and} \quad y_1 - y_{u_0}(T, x_0) = \Phi(v). \end{array} \right.$$

This problem being a problem of minimization of a strictly convex functional under non-empty affine constraints, it admits a unique solution in $L^2([0, T]; \mathbb{R}^m)$. Taking ψ as the Lagrange multiplier associated to the equality constraint, the Lagrangian of this optimal control problem is given by

$$\mathcal{L}(v, \psi) = \frac{1}{2} \int_0^T |v(t)|_m^2 dt + \langle \psi, y_1 - y_{u_0}(T, x_0) - \Phi(v) \rangle.$$

Deriving the first order optimality conditions, we get

$$v(t) = H_o(T, t)^\top \psi, \quad t \in [0, T],$$

where ψ is solution of $y_1 - y_{u_0}(T, x_0) = \mathcal{G}_T^o \psi$. As \mathcal{G}_T^o is invertible, thanks to the state to output controllability of system (1), we deduce that $\psi = (\mathcal{G}_T^o)^{-1} (y_1 - y_{u_0}(T, x_0))$. From what precedes, we have $v(t) = H_o(T, t)^\top (\mathcal{G}_T^o)^{-1} (y_1 - y_{u_0}(x_0, T))$, and hence,

for every $t \in [0, T]$,

$$u(t) = u_0 + \int_0^t H_o(T, \tau)^\top d\tau \times (\mathcal{G}_T^o)^{-1} (y_1 - y_{u_0}(x_0, T)).$$

This concludes the proof Theorem 3.2. \square

5 State to output controllability of linear discrete-time invariant systems

We now consider the discrete-time system defined by

$$x_{k+1} = Ax_k + Bu_k, \quad (18a)$$

$$y_k = Cx_k + Du_k, \quad (18b)$$

where $A \in \mathbb{R}^n$, $B \in \mathbb{R}^n$, $C \in \mathbb{R}^q$, $D \in \mathbb{R}^q$, $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^m$ and $y_k \in \mathbb{R}^q$. Note that, here, a set of admissible controls \mathcal{U}_d is a set of sequence $(u_i)_{i \in \mathbb{N}}$ of m -vectors. For the purpose of state controllability of discrete-time system (18a), we refer to [8, Lecture 11].

Definition 5.1 *The system (18) is said to be SOC if for all $(x_0, y_1) \in \mathbb{R}^n \times \mathbb{R}^q$ there exist $N \in \mathbb{N}$ and a sequence $(u_0, u_1, \dots, u_N) \in (\mathbb{R}^m)^{N+1}$ such that the corresponding solution of (18) with x_0 as initial data satisfies $y_N = y_1$.*

Taking $x_0 \in \mathbb{R}^n$ and an admissible control $(u_i)_{i \in \mathbb{N}}$, one can check that for any $k \in \mathbb{N}$,

$$y_k = CA^k x_0 + \sum_{i=0}^{k-1} CA^{k-1-i} Bu_i + Du_k. \quad (19)$$

From (19), we infer that the reachable set from x_0 in k iterations is given by

$$\mathcal{R}_o^d(x_0, k) = \{CA^k x_0\} + \text{Im}(CA^{k-1}B | \dots | CB | D). \quad (20)$$

The following theorem gives some criteria for the state to output controllability of discrete-time system (18).

Theorem 5.1 *The following properties are equivalent:*

(soc_0^d) *The discrete-time system (18) is state to output controllable,*

(soc_1^d) *There exists $N \in \mathbb{N}$ such that the discrete-time output endpoint map $E_N^d : (\mathbb{R}^m)^{N+1} \rightarrow \mathbb{R}^q$ defined for every $u = (u_0, u_1, \dots, u_N) \in (\mathbb{R}^m)^{N+1}$ by*

$$E_N^d(u) = \sum_{k=0}^{N-1} CA^{N-1-k} Bu_k + Du_N$$

is surjective.

(soc_2^d) $\text{rk } C^0(A, B, C, D) = q$

(soc_6^d) *There exists $N \in \mathbb{N}$ such that*

$$\mathcal{G}_N^d := \sum_{i=0}^{N-1} CA^i B B^\top (A^i)^\top C^\top + D D^\top > 0_q^q.$$

Remark 5.1 *From condition (soc_2^d) , one can deduce that (soc_3) and (soc_4) are also criteria for the state to output controllability of discrete-time system (18). Condition (soc_6^d) is the discrete-time version of (soc_6) .*

PROOF (THEOREM 5.1) The only condition that needs to be proven is (soc_6^d) . Indeed, the equivalence between (soc_0^d) , (soc_1^d) , (soc_2^d) is trivial. For (soc_6^d) , note that if (soc_2^d) is fulfilled then (soc_6^d) is also satisfied for $N = n$. Conversely, if (soc_6^d) is satisfied for some $N \in \mathbb{N}$, then $\text{rk}(CA^{N-1}B | \dots | CB | D) = q$. One concludes that (soc_2^d) holds by adding (or subtracting) matrices $CA^i B$, $i \in \{N, \dots, n-1\}$ (or $i \in \{n, \dots, N-1\}$ thanks to Cayley-Hamilton). \square

When system (18) is SOC, we define

$$N^* = \min\{N \in \mathbb{N} : (soc_1^d) \text{ holds}\}. \quad (21)$$

Remark 5.2 *From (20) and Cayley-Hamilton theorem, one can see that if a target cannot be reached from x_0 in less than n iterations where n is the state dimension, then it cannot be stricken. It follows that $N^* \leq n$.*

Note that Remark 2.3 is no longer true in discrete-time case. Indeed, if (soc_6^d) holds for some $N \in \mathbb{N}^*$ then this condition does not necessarily hold for any nonnegative integer N_1 satisfying $N_1 < N$. In other words, N^* can be positive.

The following theorem gives, in the case of state to output controllability, a control steering any $x_0 \in \mathbb{R}^n$ to any $y_1 \in \mathbb{R}^q$.

Theorem 5.2 *Assume system (18) is state to output controllable, and let N^* be the integer defined by (21). A control (u_0, u_1, \dots, u_N) steering x_0 to y_1 in $N \geq N^*$ iterations is given by*

$$(u_0^\top \ u_1^\top \ \dots \ u_N^\top)^\top = R_N^\top (\mathcal{G}_N^d)^{-1} (y_1 - CA^N x_0), \quad (22)$$

where $R_N = (CA^{N-1}B | \dots | CAB | CB | D)$.

Furthermore, this control is the unique minimizer of

$$\min \frac{1}{2} \sum_{i=0}^N |u_i|_m^2 \quad \left| \begin{array}{l} (u_0, u_1, \dots, u_N) \in (\mathbb{R}^m)^{N+1}, \\ CA^N x_0 + \sum_{i=0}^{N-1} CA^{N-1-i} Bu_i + Du_N = y_1. \end{array} \right. \quad (23)$$

PROOF Let $N \geq N^*$. From (19), one can check that

$$y_N = CA^N x_0 + R_N (u_0^\top \ u_1^\top \ \dots \ u_N^\top)^\top.$$

Using (22), it is straightforward that $y_N = y_1$. Since we are minimizing a strictly convex functional under non-empty affine constraints, problem (23) admits a

unique solution in $(\mathbb{R}^m)^{N+1}$ and one can check, using the Lagrangian method, that this solution is given by (22). \square

6 Other notions of output controllability

The notion of output controllability discussed so far, namely the state to output controllability, consists in knowing whether it is possible or not to steer an initial state $x_0 \in \mathbb{R}^n$ to a final output $y_1 \in \mathbb{R}^q$. One might be interested in steering an initial output $y_0 \in \mathbb{R}^q$ to a final output $y_1 \in \mathbb{R}^q$. Note that to any initial data $(x_0^\top, u_0^\top)^\top \in \mathbb{R}^{n+m}$, corresponds a unique output initial data $y_0 = Cx_0 + Du_0$. But for a given $y_0 \in \mathbb{R}^q$, the set of initial data $(x_0^\top, u_0^\top)^\top \in \mathbb{R}^{n+m}$ corresponding to y_0 is given by

$$\Gamma_{y_0} = \{(x_0^\top, u_0^\top)^\top \in \mathbb{R}^{n+m} : Cx_0 + Du_0 = y_0\}.$$

Observe that Γ_{y_0} is either an empty set (this holds when $y_0 \notin \text{Im}(C|D)$) or an affine space whose dimension is the one of $\ker(C|D)$. However, when (7) is satisfied, Γ_{y_0} is not empty.

In this section we will propose two other notions of output controllability. The first one, named *Output to Output Controllability* which describes the fact that for every $(y_0, y_1) \in \mathbb{R}^q \times \mathbb{R}^q$, there exists an element of Γ_{y_0} which can be transferred to the output y_1 . The second notion, named *Global Output to Output Controllability* describing the fact that for every $(y_0, y_1) \in \mathbb{R}^q \times \mathbb{R}^q$, all the elements of Γ_{y_0} can be transferred to the output y_1 with a common control. Needless to say that $\text{GOOC} \Rightarrow \text{SOC} \Rightarrow \text{OOC}$.

6.1 Output to output controllability

Let us start with the continuous-time system (1). The discrete-time case will be investigated in the last part of this paragraph. We define OOC as follows.

Definition 6.1 • *The system (1) is said to be output to output controllable (OOC) in time $T > 0$ if for all $(y_0, y_1) \in \mathbb{R}^q \times \mathbb{R}^q$, there exist $(x_0^\top, u_0^\top)^\top \in \Gamma_{y_0}$ and a control $u \in \mathcal{C}^0([0, T]; \mathbb{R}^m)$ such that $u(0) = u_0$ and the solution of (1) with u as input and x_0 as initial state data satisfies $y_u(x_0, T) = y_1$.*

• *The system (1) is said to be output to output controllable if for all $(y_0, y_1) \in \mathbb{R}^q \times \mathbb{R}^q$, there exist a time $T > 0$, $(x_0^\top, u_0^\top)^\top \in \Gamma_{y_0}$ and a control $u \in \mathcal{C}^0([0, T]; \mathbb{R}^m)$ such that $u(0) = u_0$ and the solution of (1) with u as input and x_0 as initial state data satisfies $y_u(x_0, T) = y_1$.*

Remark 6.1 *We could have introduced the notion of state to output controllability in time T in paragraph 2.1. This was not necessary since the notion of state to output controllability is time-independent. But in the present framework, we need to introduce the notion of output to output controllability in time $T > 0$. Indeed, we will see that system (1) is OOC in time $T > 0$ does not imply that*

it is OOC in any time $T > 0$. This fact will be illustrated by Example 6.1. Moreover, the OOC of system (1) does not imply its OOC in some time T . An example illustrates this fact in the Appendix section.

6.1.1 Output to output controllability in time T

In this paragraph, we propose, both in continuous and discrete time framework, criteria for the output to output controllability in time T . We also propose a control allowing us to steer an initial output towards a final output, both arbitrarily chosen.

The theorem below gives necessary and sufficient conditions for output to output controllability in time T of system (1).

Theorem 6.1 *The following properties are equivalent:*

- (ooc₀) *The system (1) is OOC in time $T > 0$.*
- (ooc₁) *The output endpoints map $E_T^{oo} : \mathbb{R}^{\kappa_{CD}} \times \mathcal{C}^0([0, T]; \mathbb{R}^m) \rightarrow \mathbb{R}^q$, defined for every $(\zeta, w) \in \mathbb{R}^{\kappa_{CD}} \times \mathcal{C}^0([0, T]; \mathbb{R}^m)$ by*

$$E_T^{oo}(\zeta, w) = (Ce^{TA}|H_o(T, 0))(C|D)^\perp \zeta + \int_0^T Ce^{(T-\tau)A} Bw(\tau) d\tau + Dw(T)$$

is surjective, where we have set $\kappa_{CD} = \dim \ker(C|D)$, and where, $H_o(T, 0)$ is given in (soc₆).

- (ooc₂) $\text{rk } \mathcal{C}^o(A, B, C, [D | (Ce^{TA}|0_m^q)(C|D)^\perp]) = q$.

Remark 6.2 *The analogues of (soc₃), (soc₄), (soc₅) and (soc₆) are obtained by replacing D by $\Delta_T = [D | (Ce^{TA}|0_m^q)(C|D)^\perp]$ and m by $m + \kappa$ with $\kappa = \dim \ker(C|\Delta_T)$.*

PROOF (THEOREM 6.1) Notice that for any $(x_0^\top, u_0^\top)^\top$ in Γ_{y_0} , there exists a vector $\zeta \in \mathbb{R}^{\kappa_{CD}}$ such that

$$(x_0^\top, u_0^\top)^\top = (C|D)^\dagger y_0 + (C|D)^\perp \zeta. \quad (24)$$

It follows that for any $(x_0^\top, u_0^\top)^\top \in \Gamma_{y_0}$, the solution of (1) evaluated at time T can be written as

$$y_u(T, x_0) = (Ce^{TA}|H_o(T, 0))(C|D)^\dagger y_0 + D(u(T) - u_0) + (Ce^{TA}|H_o(T, 0))(C|D)^\perp \zeta + \int_0^T Ce^{(T-t)A} B(u(t) - u_0) dt, \quad (25)$$

with $H_o(T, 0)$ given in (soc₆).

Let $w(t) \in \mathbb{R}^{\kappa_{CD}}$ be a continuous-time function such that $w(0) = 0^{\kappa_{CD}}$ and $w(T) = \zeta$. Set now $\hat{u}(t) = (u(t)^\top w(t)^\top)^\top \in \mathbb{R}^{m+\kappa_{CD}}$,

$$\begin{aligned} F_T &= (Ce^{TA}|H_o(T, 0))(C|D)^\dagger \in \mathbb{R}_q^q, \\ D_T &= (Ce^{TA}|H_o(T, 0))(C|D)^\perp \in \mathbb{R}_{\kappa_{CD}}^q, \\ \hat{B} &= (B \ 0_{\kappa_{CD}}^n) \in \mathbb{R}_{m+\kappa_{CD}}^n, \\ \hat{D}_T &= (D|D_T) \in \mathbb{R}_{m+\kappa_{CD}}^q, \end{aligned} \quad (26)$$

Equation (25) leads to

$$y_u(T, x_0) = F_T y_0 + \int_0^T C e^{(T-t)A} \widehat{B} \widehat{u}(t) dt + \widehat{D}_T \widehat{u}(T). \quad (27)$$

From this observation, we deduce that the output to output controllability in time T of system (1) is equivalent to the state to output controllability of

$$\dot{\widehat{x}}(t) = A \widehat{x}(t) + \widehat{B} \widehat{u}(t), \quad (28a)$$

$$\widehat{y}(t) = C \widehat{x}(t) + \widehat{D}_T \widehat{u}(t). \quad (28b)$$

Theorem 6.1 is then obtained by applying Theorem 3.1 to (28).

Set $\Delta_T = [D \mid (C e^{TA} | 0_m^q)(C|D)^\perp]$, and observe that the analogue of the Kalman extended rank condition (*soc*₂) in this case is $\text{rk}(A, B, C, \widehat{D}_T) = q$ which is equivalent to $\text{rk}(A, B, C, \Delta_T) = q$. Indeed, assuming by contradiction that $\text{rk}(A, B, C, \Delta_T) < q$, we get the existence of a vector $\eta_T \in \mathbb{R}^q \setminus \{0^q\}$ such that $(\eta_T^\top C e^{TA} | 0_m)(C|D)^\perp = 0_{\kappa_{CD}}$, $\eta_T^\top D = 0_m$ and $\eta_T^\top C A^i B = 0_m$, for all $i \in \mathbb{N}$. Since $\eta_T^\top D = 0_m$ and $\eta_T^\top C A^i B = 0_m$, for all $i \in \mathbb{N}$, we deduce that $\eta_T^\top H_o(T, 0) = 0_m$. This implies that there exists $\eta_T \in \mathbb{R}^q \setminus \{0^q\}$ such that $(\eta_T^\top C e^{TA} | \eta_T^\top H_o(T, 0))(C|D)^\perp = 0_{\kappa_{CD}}$, $\eta_T^\top D = 0_m$ and $\eta_T^\top C A^i B = 0_m$, for all $i \in \mathbb{N}$. Therefore, we have $\text{rk}(A, B, C, \widehat{D}_T) < q$. The reverse implication is obtained by arguing in the same way. \square

From equation (27), one can apply Theorem 3.2 to build a control \widehat{u} transferring any y_0 to y_1 in time T . The control u for the primal system (1) is obtained by extracting the m first components of \widehat{u} . The last κ_{CD} components being used to determine the adequate $(x_0^\top \ u_0^\top)^\top$ given by (24). One can also proceed as in Remark 3.6 and build a piecewise continuous control, having jumps at $t = 0$ and $t = T$ steering y_0 to y_1 in a time T of OOC.

Even if these two methods lead to controls that allow us to transfer any initial output y_0 to any final output y_1 in a time T of OOC, it is more natural to separate ζ from u , i.e., to look for a constant vector $\zeta \in \mathbb{R}^{\kappa_{CD}}$ and a continuous control $u \in C^0([0, T]; \mathbb{R}^m)$ that allow us to achieve the same goal. This leads to the following theorem.

Theorem 6.2 *Assume that system (1) is OOC in time $T > 0$. Then for any $(y_0, y_1) \in \mathbb{R}^q \times \mathbb{R}^q$, a control steering y_0 to y_1 in time T is given by*

$$u(t) = u_0 + \int_0^t H_o(T, \tau)^\top \psi d\tau, \quad t \in [0, T], \quad (29)$$

where ψ is given by $\psi = (\mathcal{G}_T^o + D_T D_T^\top)^{-1} (y_1 - F_T y_0)$, H_o, \mathcal{G}_T^o are defined in (*soc*₆) and F_T, D_T defined in (26).

The appropriate $(x_0^\top, u_0^\top)^\top \in \Gamma_{y_0}$ is given by

$$(x_0^\top, u_0^\top)^\top = (C|D)^\dagger y_0 + (C|D)^\perp D_T^\top \psi. \quad (30)$$

Furthermore, this control is the unique minimizer of

$$\min \left. \begin{aligned} & \frac{1}{2} \int_0^T |\dot{u}(t)|_m^2 dt + \frac{1}{2} |\zeta|_{\kappa_{CD}}^2 \\ & u \in H^1([0, T]; \mathbb{R}^m), \quad u(0) = u_0, \\ & \zeta \in \mathbb{R}^{\kappa_{CD}}, \quad y_u(T, x_0) = y_1, \\ & \text{with } (x_0, u_0) \text{ given by :} \\ & (x_0^\top, u_0^\top)^\top = (C|D)^\dagger y_0 + (C|D)^\perp \zeta. \end{aligned} \right\} \quad (31)$$

PROOF Setting, as in the proof of Theorem 3.2,

$$u(t) = u_0 + \int_0^t v(\tau) d\tau, \quad v \in L^2([0, T]; \mathbb{R}^m), \quad (32)$$

problem (31) becomes

$$\min \left. \begin{aligned} & \frac{1}{2} \int_0^T |v(t)|_m^2 dt + \frac{1}{2} |\zeta|_{\kappa_{CD}}^2 \\ & v \in L^2([0, T]; \mathbb{R}^m), \quad \zeta \in \mathbb{R}^{\kappa_{CD}}, \\ & y_1 - F_T y_0 = \Phi(v) + D_T \zeta, \end{aligned} \right\} \quad (33)$$

where Φ is defined in the proof of Theorem 3.2.

The optimization problem (33) being strictly convex under non-empty affine constraints, we have the existence and uniqueness of a solution $(v, \zeta) \in L^2([0, T]; \mathbb{R}^m) \times \mathbb{R}^{\kappa_{CD}}$ for (33). Taking $\psi \in \mathbb{R}^q$ as the Lagrange multiplier associated to the equality constraint, the Lagrangian of (33) is given by

$$\begin{aligned} \mathcal{L}(v, \zeta, \psi) = & \frac{1}{2} \int_0^T |v(t)|_m^2 dt + \frac{1}{2} |\zeta|_{\kappa_{CD}}^2 \\ & + \langle \psi, y_1 - F_T y_0 - \Phi(v) - D_T \zeta \rangle. \end{aligned}$$

The first order optimality conditions lead to

$$v(t) = H_o(T, t)^\top \psi, \quad \forall t \in [0, T] \quad \text{and} \quad \zeta = D_T^\top \psi, \quad (34)$$

where ψ is solution of $y_1 - F_T y_0 = (\mathcal{G}_T^o + D_T D_T^\top) \psi$. Thanks to the output to output controllability of system (1) in time T , $\mathcal{G}_T^o + D_T D_T^\top$ is invertible. Hence, we have $\psi = (\mathcal{G}_T^o + D_T D_T^\top)^{-1} (y_1 - F_T y_0)$. We recover v and ζ from (34), and deduce the expressions of u and (x_0, u_0) respectively from (32) and (24). \square

Let us make short example on this concept.

Example 6.1 Consider the system (1) with

$$A = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, C = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, D = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

• Condition (ooc_2) .

Choosing $(C|D)^\perp = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}^\top$, the Kalman extended matrix given in (ooc_2) becomes

$$\begin{pmatrix} 0 & 0 & 0 & 0 & \sin(T) & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{pmatrix}.$$

One can readily see that this matrix is of rank 2 if and only if $T \neq 0[\pi]$. Here, we can go further in the explanation. Indeed, for every $(y_0, y_1) \in \mathbb{R}^q \times \mathbb{R}^q$, $(x_0^\top, u_0^\top)^\top \in \Gamma_{y_0}$, and $u \in \mathcal{C}^0([0, T]; \mathbb{R}^m)$, the first component y_1 and the second component y_2 at any time T of the output of this system obey to

$$\begin{aligned} y_1(T) &= y_{0,1} \cos(T) + x_{0,1} \sin(T), \\ y_2(T) &= x_{0,3} + \int_0^T u(t) dt + u(T), \end{aligned} \quad (35)$$

where $y_{0,i}$, $i \in \{1, 2\}$ are the first and the second component of y_0 , $x_{0,i}$, $i \in \{1, 2, 3\}$ are the components of x_0 . From (35), we deduce that

$$\begin{pmatrix} \sin(T) & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_{0,1} \\ x_{0,3} \end{pmatrix} = \begin{pmatrix} y_1(T) - y_{0,1} \cos(T) \\ y_2(T) - \int_0^T u(t) dt - u(T) \end{pmatrix}.$$

Hence, to require that x_0 exist independently of the choice of the final value $y(T) = y_1$ is equivalent to require that $T \neq 0[\pi]$.

• Control computation.

Taking $y_0 = (0 \ 1)^\top$ as initial output, $y_1 = (1 \ 2)^\top$ as target and $T = \pi/2$ as output to output controllability time, we have from Theorem 6.2,

$$\begin{aligned} x_0 &= \begin{pmatrix} 1 & 0 & \frac{\pi(\pi+12)}{2(\pi^2+9\pi+12)} \end{pmatrix}^\top, \\ u_0 &= \frac{\pi^2+6\pi+24}{2(\pi^2+9\pi+12)}, \\ u(t) &= u_0 - \frac{(3\pi-12)(2+\pi-t)t}{\pi(\pi^2+9\pi+12)}, \quad \forall t \in [0, \frac{\pi}{2}]. \end{aligned}$$

Solving (1) with u , as input and $(x_0^\top, u_0^\top)^\top \in \Gamma_{y_0}$ as initial data, we get the time trajectories depicted on Figure 2.

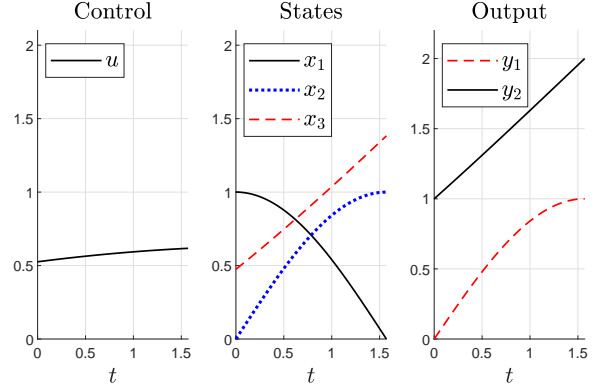


Fig. 2. Control, state and output given by Theorem 6.2 for the system considered in Example 6.1, $y_0 = (0 \ 1)^\top$, $y_1 = (1 \ 2)^\top$ and $T = \pi/2$.

6.1.2 Output to output controllability for discrete LTI systems

The notion of output to output controllability in time T applies to discrete-time system (18). Indeed, we give the following definition.

Definition 6.2 The system (18) is said to be OOC in $N \in \mathbb{N}$ iterations if for any $(y_0, y_1) \in \mathbb{R}^q \times \mathbb{R}^q$ there exist $(x_0^\top, u_0^\top)^\top \in \Gamma_{y_0}$, and controls $u_0 = u_0, u_1, \dots, u_N$ in \mathbb{R}^m such that the solution (19) of (18) with x_0 as initial state data and u_0, u_1, \dots, u_N as inputs, satisfies $y_N = y_1$.

The theorem below gives some criteria for system (18) to be output to output controllable in N iterations.

Theorem 6.3 Let $N > 0$. The following properties are equivalent.

- (ooc_0^d) System (18) is output to output controllable in N iterations.
- (ooc_1^d) The discrete-time output endpoint map $E_N^{ood} : \mathbb{R}^{\kappa_{CD}} \times (\mathbb{R}^m)^N \rightarrow \mathbb{R}^q$ defined for every $\zeta \in \mathbb{R}^{\kappa_{CD}}$ and $u = (u_1, \dots, u_N) \in (\mathbb{R}^m)^N$ by

$$E_N^{ood}(\zeta, u) = \sum_{k=1}^{N-1} CA^{N-1-k} Bu_k + \Delta_N (u_N^\top \zeta^\top)^\top,$$

with $\Delta_N = [D \mid (CA^N \mid CA^{N-1}B)(CD)^\perp]$, is surjective,

- (ooc_2^d) $\text{rk } \bar{R}_N = q$, where $\bar{R}_N = (CA^{N-2}B \mid \dots \mid CB \mid \Delta_N)$.
- (ooc_6^d) $\bar{G}_N := \sum_{i=0}^{N-2} CA^i BB^\top (A^i)^\top C^\top + \Delta_N \Delta_N^\top > 0_q^q$.

Remark 6.3 In the framework of the state to output controllability of the discrete-time system (18), the extension of the Hautus test, as it was presented in paragraph 4.1, was possible because we had an equivalence between the state to output controllability of (1) and $\text{rk } \mathcal{C}^o(A, B, C, D) = q$. In the current context, this argument no longer holds since there is no equivalence

between: there exists $N \leq n$ such that $\text{rk } \bar{R}_N = q$ and there exists $N \geq n+1$ such that $\text{rk } \bar{R}_N = q$. One can get convinced by considering the following system:

$$A = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, C = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, D = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

For this system, we have $CA^k B = 0$, for all $k \in \mathbb{N}$, and $CA^k = 0$, for all $k \geq 2$. Choosing for instance $(C|D)^\perp = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^\top$, we infer that $\Delta_1 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$. It follows

that $\text{rk } \bar{R}_1 = 2 = q$ and for $N > 1$, $\text{rk } \bar{R}_N = 1 < q = 2$.

Remark 6.4 Note that if system (18) is OOC in N iterations with $N \geq n+1$, the analogues of (soc_3) and (soc_4) are obtained by replacing the matrix D by Δ_N . **PROOF (THEOREM 6.3)** The analogue of (25) for the discrete-time system (18) is given by

$$y_N = F_N y_0 + \sum_{k=1}^{N-1} CA^{N-k-1} B u_k + \Delta_N (u_N^\top, \zeta^\top)^\top, \\ F_N = (CA^N | CA^{N-1} B) (C|D)^\dagger. \quad (36)$$

Equivalence between $(\text{oo}c_0^d)$ and $(\text{oo}c_1^d)$.

If E_N^{ood} is surjective for some $N \in \mathbb{N}$, then for every $(y_0, y_1) \in \mathbb{R}^q \times \mathbb{R}^q$, there exist $\zeta \in \mathbb{R}^{\kappa_{CD}}$ and $\bar{u} = (u_1, u_2, \dots, u_N) \in (\mathbb{R}^m)^N$ such that $E_N^{\text{ood}}(\zeta, \bar{u}) = y_1 - F_N y_0$. Using (36), it is straightforward that $y_N = y_1$ and that the system is OOC in N iterations. Conversely, if we assume that E_N^{ood} is not surjective for some $N \in \mathbb{N}$, then there exists a vector $\eta_N \in \mathbb{R}^q \setminus \{0^q\}$ such that $\eta_N^\top E_N^{\text{ood}}(\zeta, \bar{u}) = 0$ for all $\zeta \in \mathbb{R}^{\kappa_{CD}}$ and all $\bar{u} = (u_1, u_2, \dots, u_N) \in (\mathbb{R}^m)^N$. This shows that the set of all admissible outputs that can be reached from y_0 in N iterations is a subspace of $\{F_N y_0\} + \text{span}(\eta_N)^\perp$ which is a proper affine subspace of \mathbb{R}^q . We then deduce that system (18) is not OOC in N iterations. This proves equivalence between $(\text{oo}c_0^d)$ and $(\text{oo}c_1^d)$.

The equivalence between $(\text{oo}c_1^d)$ and $(\text{oo}c_2^d)$ goes the same way and the equivalence between $(\text{oo}c_2^d)$ and $(\text{oo}c_6^d)$ is straightforward since $\bar{\mathcal{G}}_N = \bar{R}_N \bar{R}_N^\top$. \square

Remark 6.5 Note that from (36), one may be tempted to say, as it was the case for the continuous-time system (1), that the output to output controllability of (18) in N iterations is equivalent to the state to output controllability of

$$\hat{x}_{k+1} = A \hat{x}_k + \hat{B} \hat{u}_k, \quad (37a)$$

$$\hat{y}_k = C \hat{x}_k + \Delta_N \hat{u}_k, \quad (37b)$$

where $\hat{u}_k = (u_k^\top \zeta^\top)^\top$. This is not true if $N < n+1$.

Indeed, observe that the state to output controllability of (37) is equivalent to $\text{rk } \mathcal{C}^o(A, \hat{B}, C, \Delta_N) = q$. This rank

condition is not equivalent to condition $(\text{oo}c_2^d)$. Indeed, consider, for example, the matrices

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}, D = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

For this system, one can see that $\ker(C|D) = \{0_1^2\}$.

We have $\text{rk } \bar{R}_2 = \text{rk}(CB|\Delta_2) = \text{rk} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} = 2 <$

$q = 3$, and $\text{rk } \mathcal{C}^o(A, \hat{B}, C, \Delta_2) = \text{rk}(CAB|\hat{C}\hat{B}|\Delta_2) = \text{rk} \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 & 1 & 0 \end{pmatrix} = 3 = q$.

Remark 6.6 Observe that contrary to the notion of state to output controllability discussed in Section 5, if N^* is the smallest integer such that $(\text{oo}c_2^d)$ holds, then we do not have necessarily $(\text{oo}c_2^d)$ for any $N > N^*$. One can get convinced by considering Example 6.1. With this example, it can be seen that system (18) is OOC in N iterations if and only if N is odd.

The following theorem, whose proof follows the same pattern as the one of Theorem 5.2 and is omitted here, gives, if the system (18) is OOC in $N \in \mathbb{N}$ iterations, a control steering y_0 to y_1 in N iterations.

Theorem 6.4 Let $N \in \mathbb{N}$ and assume that the system (18) is OOC in N iterations, i.e., $\bar{\mathcal{G}}_N > 0_q^q$. Then for every $(y_0, y_1) \in \mathbb{R}^q \times \mathbb{R}^q$, we set

$$(u_1^\top, u_2^\top, \dots, u_N^\top, \zeta^\top)^\top = \bar{R}_N^\top \psi, \quad (38) \\ (x_0^\top, u_0^\top)^\top = (C|D)^\dagger y_0 + (C|D)^\perp \zeta,$$

with $\psi = (\bar{\mathcal{G}}_N)^{-1} (y_1 - F_N y_0)$. The controls $u_0 = u_0, u_1, u_2, \dots, u_N$ given by (38) steers y_0 to y_1 in N iterations with x_0 as initial state data.

Furthermore, (u_1, \dots, u_N, ζ) is the unique minimizer of

$$\min \frac{1}{2} \sum_{i=1}^N |u_i|_m^2 + \frac{1}{2} |\zeta|_{\kappa_{CD}}^2 \\ \left| \begin{array}{l} (u_1^\top, u_2^\top, \dots, u_N^\top)^\top \in \mathbb{R}^{mN}, \quad \zeta \in \mathbb{R}^{\kappa_{CD}} \\ y_1 = F_N y_0 + \bar{R}_N (u_1^\top, u_2^\top, \dots, u_N^\top, \zeta^\top)^\top, \end{array} \right. \quad (39)$$

and (x_0, u_0) given by (24).

6.2 Globally output to output controllability

The idea behind the notion of GOOC is to build, for all $(y_0, y_1) \in \mathbb{R}^q \times \mathbb{R}^q$, a control independent of the choice of

$(x_0^\top, u_0^\top)^\top \in \Gamma_{y_0}$ sending y_0 on y_1 . In the case $D \neq 0_m^q$, it is not possible to construct a continuous control for this purpose since it has to take the value u_0 at time $t = 0$. One way to handle this problem is to look for controls that are continuous on $(0, T)$ and admit discontinuities depending on $(x_0^\top, u_0^\top)^\top$ at times $t = 0$ and $t = T$. For this purpose, we consider the following definition.

Definition 6.3 *The system (1) is GOOC if for every $(y_0, y_1) \in \mathbb{R}^q \times \mathbb{R}^q$, there exist a time $T > 0$ and a control $u \in \mathcal{C}^0([0, T]; \mathbb{R}^m)$ such that*

$$y_1 - Cx_u(T, x_0) \in \text{Im } D, \quad (40)$$

for all $x_0 \in \mathbb{R}^n$, so that there exists $u_0 \in \mathbb{R}^m$ such that $(x_0^\top, u_0^\top)^\top \in \Gamma_{y_0}$.

Let $P \in \mathbb{R}^{q \times d}$ be a full rank matrix such that $\ker P = \text{Im } D$, where $d = \text{rk } D$. From (40), we deduce that

$$Py_1 = PCe^{TA}x_0 + \int_0^T PCe^{(T-t)A}Bu(t)dt. \quad (41)$$

Taking any $(x_0^1, u_0^1)^\top$ and any $(x_0^2, u_0^2)^\top$ in Γ_{y_0} , we have $x_0^1 - x_0^2 \in \ker(PC)$ and by linearity in (41), we have $PCe^{TA}(x_0^1 - x_0^2) = 0$. From the above observation, we deduce that a necessary condition for system (1) to be GOOC is

$$A \ker(PC) \subset \ker(PC). \quad (42)$$

We deduce criteria for globally output to output controllability of system (1) by applying Theorem 3.1 with $D = 0_m^q$, C replaced by PC and adding to each item the condition (42). Also, q must be replaced by $q - d$. From the above, one can apply Theorem 3.2 to build a control u steering y_0 to y_1 in time T in the sense of globally output to output controllability or observe, from Remark 3.6, that the control given by

$$u(t) = B^\top e^{(T-t)A^\top} (PC)^\top (PC\mathcal{G}_T^s(PC)^\top)^{-1} \nu_y, \quad (43)$$

for $t \in [0, T]$, where $\nu_y = (Py_1 - PCe^{TA}(PC)^\dagger Py_0)$ and \mathcal{G}_T^s defined in (sc₂) works.

Note that if $D \neq 0_m^q$, this control does not steer, in general, exactly y_0 to y_1 in time T . However, in the case of globally output to output controllability, we can construct, as mentioned at the beginning of this section, a piecewise control sending exactly y_0 on y_1 in time T . Take for instance for any given $(x_0^\top, u_0^\top)^\top \in \Gamma_{y_0}$ the control \bar{u} satisfying

$$\begin{aligned} \bar{u}(t) &= u(t), \quad \forall t \in (0, T), \\ \bar{u}(0) &= u_0 \quad \text{and} \quad \bar{u}(T) = D^\dagger(y_1 - Cx_u(x_0, T)). \end{aligned} \quad (44)$$

Remark 6.7 *For the matrix P , one can take, for instance, the $q - d$ independent columns of the matrix $I_q - DD^\dagger$.*

Let us consider the following example.

Example 6.2 *Consider the system (1) with matrices*

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, C = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, D = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

For this example, we have $D^\dagger = (0 \ 1)$, $D^\perp = 0$ and $P = (1 \ 0)$. We deduce that $PC = (0 \ 1 \ 0)$ and $\ker(PC) = \text{span}(e_1, e_3)$. The necessary condition (42) is then satisfied. This system is GOOC since the extended Kalman matrix $(PCA^2B|PCAB|PCB)$ is equal to $(4|2|1)$, and is of rank $q - d = 1$. Let us now choose $y_0 = (0 \ 1)^\top$ as initial output, $y_1 = (1 \ 2)^\top$ as target and $T = 1$. After some computations, we get $u(t) = 4e^{2-2t}/(e^4 - 1)$, for all $t \in (0, 1)$. For $x_0 = (0 \ 0 \ 1/2)^\top$ and $u_0 = 1/2$, we have $\bar{u}(t) = u(t)$ on $(0, 1)$, $\bar{u}(0) = 1/2$ and $\bar{u}(1) = (4 - e)/2$. For $x_0 = (1 \ 0 \ 3/2)^\top$ and $u_0 = -1/2$, we get $\bar{u}(t) = u(t)$ on $(0, 1)$, $\bar{u}(0) = -1/2$ and $\bar{u}(1) = (4 - 3e)/2$. The time trajectories of this system are depicted on Figure 3.

Remark 6.8 *Note that when $D = 0_m^q$, the matrix P can be taken equals to the I_q^q matrix and there is no more discontinuity neither on the control nor on the output.*

Remark 6.9 *Observe that the globally output to output controllability of system (1) is time-independent.*

6.2.1 Globally output to output controllability for discrete-time invariant systems

For the discrete-time system (18), the notion of globally output to output controllability can be formulated as follows:

Definition 6.4 *The system (18) is GOOC if for every $(y_0, y_1) \in \mathbb{R}^q \times \mathbb{R}^q$, there exist a time $N \in \mathbb{N}$ and controls u_0, u_1, \dots, u_N in \mathbb{R}^m such that*

$$y_1 - Cx_N \in \text{Im } D,$$

for all $x_0 \in \mathbb{R}^n$, so that there exists $u_0 \in \mathbb{R}^m$ such that $(x_0^\top, u_0^\top)^\top \in \Gamma_{y_0}$, and x_N the solution at time N of (18) with x_0 as initial state and u_0, u_1, \dots, u_N as inputs.

From the discussion we had in the continuous-time case, we deduce that a necessary condition for global output to output controllability is given by (42). Also, the criteria of globally output to output controllability are obtained by applying Theorem 5.1 with $D = 0_m^q$, C replaced by PC and adding to each item the condition (42). Also, q must be replaced by $q - d$. For the control computation, let \bar{N} be the smallest integer, if it exists, such that condition (soc₆^d) is satisfied with $D = 0_m^q$, C replaced by

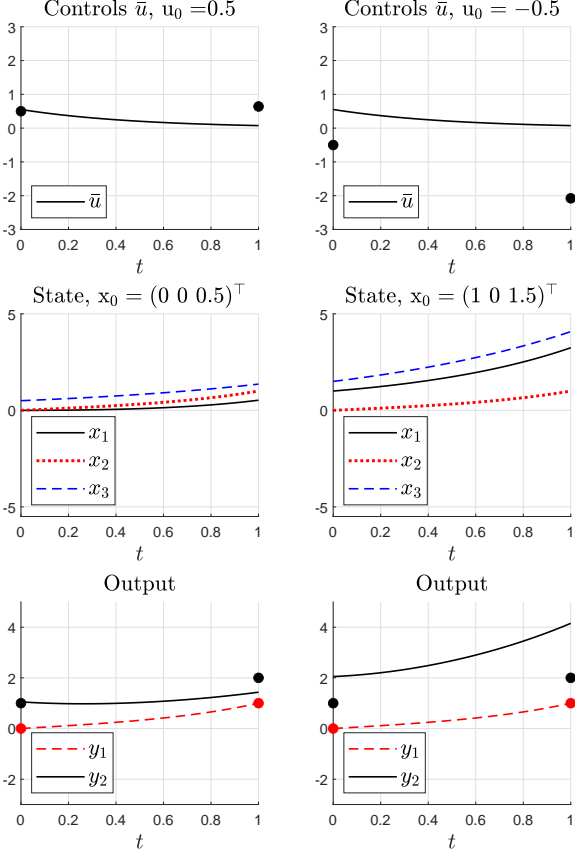


Fig. 3. Time trajectories of the controls transferring $y_0 = (0 \ 1)^T$ on $y_1 = (1 \ 2)^T$ in time $T = 1$ for the system given in Example 6.2. The control \bar{u} is computed for two different values of $(x_0^T, u_0^T)^T$ in Γ_{y_0} .

PC and q by $q - d$. From Theorem 5.2, we deduce that, for $N \geq \bar{N}$, the controls u_0, u_1, \dots, u_N in \mathbb{R}^m given by

$$\begin{aligned} (u_0^T, u_1^T, \dots, u_{N-1}^T)^T &= \tilde{R}_N^T \psi, \quad u_N \in \mathbb{R}^m, \quad \text{with} \\ \psi &= \left(\tilde{R}_N \tilde{R}_N^T \right)^{-1} (P y_1 - PC A^N (PC)^\dagger P y_0) \quad (45) \\ \text{and } \tilde{R}_N &= P(CA^{N-1}B | \dots | CAB | CB). \end{aligned}$$

steer y_0 to y_1 in N iterations in the sense of Definition 6.4. Also, for any given $(x_0^T, u_0^T)^T$ in Γ_{y_0} , the analogue of \bar{u} , constructed in (44), is given by $(\bar{u}_0, u_1, \dots, u_{N-1}, \bar{u}_N)$, where $\bar{u}_0 = u_0$ and $\bar{u}_N = D^\dagger (y_1 - Cx_N)$.

7 Conclusion

In this work, we have extended, to the output framework, the famous Popov-Hautus-Belevich criteria. Also, we propose a criterion based on the positive definiteness of a Gramian matrix, and this matrix allows us to build a continuous control to achieve any desired transfer. A characterization of two other notions of output controllability, namely, the output to output controllability and the globally output to output

controllability is also given.

Our future research will include the investigation of output controllability and its characterization for more generic classes of systems, as linear time-varying ones.

Appendix

The goal of this Appendix is to show that the OOC of system (1) does not imply its OOC in some time $T > 0$. Note that from (25), the output to output controllability of system (1) is equivalent to

$$\forall (y_0, y_1) \in \mathbb{R}^q \times \mathbb{R}^q, \text{ there exists a time } T > 0 \quad (46)$$

$$\text{such that } y_1 \in \text{Im } \mathcal{C}^o(A, B, C, \Delta_T) + \{F_T y_0\}.$$

where $\Delta_T = [D \mid (Ce^{TA} | 0_m^q)(C|D)^\perp]$, and F_T is defined in (26). It is then straightforward that the reachable set from a given y_0 , $\mathcal{R}(y_0)$, is

$$\mathcal{R}(y_0) = \bigcup_{T>0} (\text{Im } \mathcal{C}^o(A, B, C, \Delta_T) + \{F_T y_0\}), \quad (47)$$

and the OOC of system (1) is equivalent to

$$\mathcal{R}(y_0) = \mathbb{R}^q, \quad \forall y_0 \in \mathbb{R}^q. \quad (48)$$

To achieve our goal, we take $B = 0_m^n$, $D = 0_m^q$ and

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & -1 & 1 \end{pmatrix}$$

Note that for $B = 0_m^n$ and $D = 0_m^q$, criterion (48) becomes

$$\bigcup_{T>0} (\text{Im}(Ce^{TA}C^\perp) + \{Ce^{TA}C^\dagger y_0\}) = \mathbb{R}^q, \quad \forall y_0 \in \mathbb{R}^q. \quad (49)$$

We chose C^\perp as follows, and we have C^\dagger given by

$$C^\dagger = \frac{1}{8} \begin{pmatrix} 5 & 4 & 1 \\ 1 & 4 & 5 \\ -1 & 4 & 3 \\ 3 & 4 & -1 \\ 2 & 0 & 2 \end{pmatrix}, \quad C^\perp = \begin{pmatrix} -1 & 0 \\ 1 & -1 \\ -1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Setting $(v_1(T), v_2(T)) = Ce^{TA}C^\perp \in \mathbb{R}_+^3$, we get after some computations

$$v_1(T) = \begin{pmatrix} -2e^T(\sin(2T) - \sin(T)) \\ 2e^T \sin(2T) \\ 2e^T(\cos(T) - \cos(2T)) \end{pmatrix},$$

$$v_2(T) = \begin{pmatrix} e^T(\sin(2T) - \cos(2T) - 2\sin(T)) + 1 \\ e^T(\cos(2T) - \sin(2T)) - 1 \\ e^T(\cos(2T) - 2\cos(T) + \sin(2T)) + 1 \end{pmatrix}.$$

One can check that the cross product $v_1(T) \wedge v_2(T)$ vanishes if and only if $T = 0[2\pi]$. This is confirmed by Figure 4. Thus, to show that the considered system is

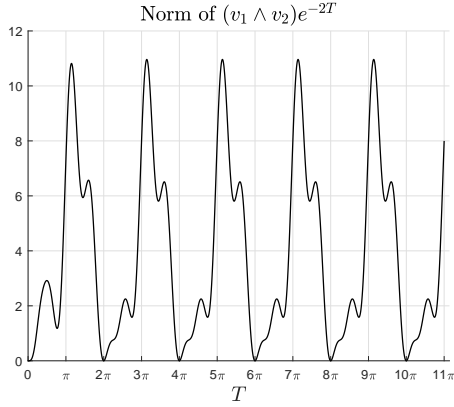


Fig. 4. Figure of the norm of $(v_1 \wedge v_2)e^{-2T}$

OOC, we have to show that for every $(y_0, y_1) \in \mathbb{R}^q \times \mathbb{R}^q$, there exists a time $T \in \mathbb{R}_+ \setminus \{2k\pi, k \in \mathbb{N}\}$ such that

$$d(T) := \det(v_1(T), v_2(T), y_1 - \bar{y}_0(T)) = 0,$$

where $\bar{y}_0(T) = Ce^{TA}C^\dagger y_0$.

Setting $y_0 = (y_{01}, y_{02}, y_{03})^\top$ and $y_1 = (y_{11}, y_{12}, y_{13})^\top$, we get after some computations

$$d(T) = -2e^{3T}(\alpha_1 h_1(T) + \alpha_2 h_2(T) + r_1(t)). \quad (50)$$

Writing $s = \sin(T)$ and $c = \cos(T)$, we have set in (50) $\alpha_1 = y_{01} + y_{02}$, $\alpha_2 = y_{02} + y_{03}$, $h_1(T) = 2s^2 - 2sc + c - 1$, $h_2(T) = s$, and $r_1(T) = e^{-2T}f_1(T) + e^{-T}f_2(T)$, with

$$f_1(T) = (y_{11} + y_{12})(1 - c - 2s^2 - 2sc) + s(y_{12} + y_{13}),$$

$$f_2(T) = y_{01} + y_{02} - y_{11} - y_{12}$$

$$+ s(2y_{01} + y_{02} - y_{03} + 2y_{11} + y_{12} - y_{13})$$

$$+ 2s^3(-y_{01} + y_{03} - y_{11} + y_{13})$$

$$+ c(y_{11} - y_{02} - y_{01} + y_{12})$$

$$+ 2cs^2(y_{01} + y_{03} - y_{11} - y_{13}).$$

In order to show that $d(\cdot)$ admits a zero on $\mathbb{R}_+ \setminus \{2k\pi, k \in \mathbb{N}\}$, we use the following lemma.

Lemma 7.1 *Let f, g be 2π -periodic real continuous functions and r a continuous real function, defined on \mathbb{R}_+ , such that*

$$\lim_{T \rightarrow +\infty} r(T) = 0. \quad (51)$$

Assume that there exist T_1, T_2, T_3, T_4 on $(0, 2\pi)$ such that $f(T_1) < 0, f(T_2) > 0, g(T_1) = g(T_2) = 0, g(T_3) < 0$ and $g(T_4) > 0$. Let $(\alpha_1, \alpha_2)^\top \in \mathbb{R}^2 \setminus \{0^2\}$ and set

$$H(T) = \alpha_1 f(T) + \alpha_2 g(T) + r(T), \quad T \in \mathbb{R}_+.$$

Then, there exist $T \in \mathbb{R}_+ \setminus \{2k\pi, k \in \mathbb{N}\}$ such that $H(T) = 0$.

PROOF • If $\alpha_1 \neq 0$, without loss of generality, we can also assume that $\alpha_1 = 1$ and $T_1 < T_2$. Since $g(T_1) = g(T_2) = 0$, we have, for every $k \in \mathbb{N}$, and every $i \in \{1, 2\}$, $H(T_i + 2k\pi) = f(T_i) + r(T_i + 2k\pi)$.

Since by assumption $f(T_1) < 0, f(T_2) > 0$ and $r(T)$ goes to zero as $T \rightarrow \infty$, there exists $k \in \mathbb{N}$, large enough such that $H(T_1 + 2k\pi) < 0 < H(T_2 + 2k\pi)$. It follows from the intermediate value theorem that H vanishes on $(T_1 + 2k\pi, T_2 + 2k\pi)$.

• If $\alpha_1 = 0$ and $\alpha_2 \neq 0$, one get the result by replacing T_1 and T_2 respectively by T_3 and T_4 in the preceding arguments. \square

Taking $f = h_1, g = h_2$ and $r = r_1$ in Lemma 7.1, we have that these functions satisfy the assumptions of this lemma (see e.g. Figure 5). Hence, unless $\alpha_1 = \alpha_2 = 0$, $d(\cdot)$ vanishes on $\mathbb{R}_+ \setminus \{2k\pi, k \in \mathbb{N}\}$. Let us now assume that $\alpha_1 = \alpha_2 = 0$, i.e.,

$$y_{01} + y_{02} = y_{02} + y_{03} = 0. \quad (52)$$

Substituting y_{02} by $-y_{01}$ and y_{03} by y_{01} in (50), we obtain

$$d(T) = -2e^{2T} \left(\tilde{f}_2(T) + r_2(T) \right), \quad (53)$$

where

$$\tilde{f}_2(T) = -y_{11} - y_{12} + s(2y_{11} + y_{12} - y_{13})$$

$$+ 2s^3(y_{13} - y_{11}) + c(y_{11} + y_{12})$$

$$+ 2cs^2(2y_{01} - y_{11} - y_{13}),$$

$$r_2(T) = e^{-T}f_1(T).$$

Observe that \tilde{f}_2 is a 2π -periodic and derivable function that satisfies, $\tilde{f}_2(0) = 0$ and $\tilde{f}_2'(0) = 2y_{11} + y_{12} - y_{13}$. It follows that if $\tilde{f}_2'(0) \neq 0$, there exist T_1 and T_2 in $(0, 2\pi)$ such that $\tilde{f}_2(T_1) < 0$ and $\tilde{f}_2(T_2) > 0$. Applying Lemma 7.1 with $\alpha_1 = 1, \alpha_2 = 0, f = \tilde{f}_2, g = 0$ and $r = r_2$, we deduce that $d(\cdot)$ vanishes on $\mathbb{R}_+ \setminus \{2k\pi, k \in \mathbb{N}\}$. Assume that (52) holds and that $\tilde{f}_2'(0) = 0$, i.e.,

$$y_{02} = -y_{01}, y_{03} = y_{01} \text{ and } y_{13} = 2y_{11} + y_{12}. \quad (55)$$

Substituting y_{13} by $2y_{11} + y_{12}$ in (54), we get

$$\tilde{f}_2(T) = (y_{11} + y_{12})h_3(T) + (y_{01} - y_{11})h_4(T),$$

where we have set $h_3(T) = 2s^3 - 2cs^2 + c - 1$ and $h_4(T) = -4cs^2$. Replacing \tilde{f}_2 by this expression in (53), (53) takes the form of (50) and again, by Lemma 7.1 (it can be checked that h_3 and h_4 satisfy the assumptions of Lemma 7.1, see e.g. Figure 5), we see that $d(\cdot)$ admits a zero on $\mathbb{R}_+ \setminus \{2k\pi, k \in \mathbb{N}\}$ unless

$$y_{11} + y_{12} = y_{01} - y_{11} = 0. \quad (56)$$

Finally, if we assume that (55) and (56) hold, then $d(T) = 0$ for all $T > 0$.

All in all, we have shown that for every $(y_0, y_1) \in \mathbb{R}^q \times \mathbb{R}^q$, there exists a time $T \in \mathbb{R}_+ \setminus \{2k\pi, k \in \mathbb{N}\}$ such that $d(T) = 0$.

From the above, we deduce that the considered system is OOC. But, for every $T > 0$, this system is not OOC in time T . Indeed, for every $T > 0$, $\text{rk}(Ce^{TA}C^\perp) \leq 2 < 3 = q$.

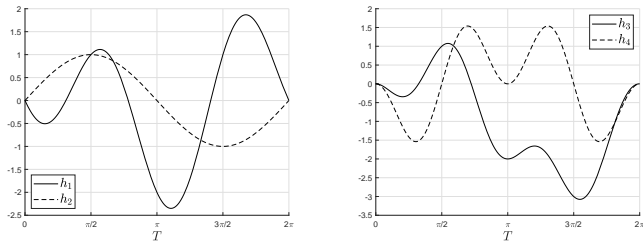


Fig. 5. On the first figure, we have the curves of the functions h_1 and h_2 and on the second figure, the curves of h_3 and h_4 .

Remark 7.1 We have shown that for continuous time systems, OOC does not in general implies OOC in some time $T > 0$. For discrete time systems, we do not know if OOC implies its OOC in N iterations. Note that for the continuous time systems, the key argument is the intermediate value Theorem. This Theorem cannot be used any more in the discrete time case.

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