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ON THE SYMPLECTIC FILLINGS OF STANDARD REAL PROJECTIVE SPACES

PAOLO GHIGGINI AND KLAUS NIEDERKRÜGER-EID

ABSTRACT. We prove, in a geometric way, that the standard contact structure on $\mathbb{R}\mathbb{P}^{2n-1}$ is not Liouville fillable for $n \geq 3$ and odd. We also prove that, for all n , semipositive fillings of those contact structures are simply connected. Finally we give yet another proof of the Eliashberg–Floer–McDuff theorem on the diffeomorphism type of the symplectically aspherical fillings of the standard contact structure on S^{2n-1} .

1. INTRODUCTION

The standard contact structure ξ on S^{2n-1} is described in coordinates by the equation

$$\xi = \ker \sum_{j=1}^n x_j dy_j - y_j dx_j.$$

Geometrically, ξ_p is the unique complex hyperplane in $T_p S^{2n-1}$ for every $p \in S^{2n-1}$. The antipodal involution of S^{2n-1} preserves ξ , and therefore induces a contact structure on $\mathbb{R}\mathbb{P}^{2n-1}$ which we still denote by ξ . The disc bundle of the line bundle $\mathcal{O}_{\mathbb{P}^{n-1}}(-2)$ on $\mathbb{C}\mathbb{P}^{n-1}$ is a strong symplectic filling of $(\mathbb{R}\mathbb{P}^{2n-1}, \xi)$. On the other hand, $\mathbb{R}\mathbb{P}^{2n-1}$ cannot be the boundary of a $2n$ -dimensional manifold with the homotopy type of an n -dimensional CW complex if $2n - 1 \geq 5$; see [2, Section 6.2]. This implies that a real projective space of dimension at least 5 does not admit any Weinstein fillable contact structure. Our main result is the following.

Theorem 1.1. *The standard contact structure on $\mathbb{R}\mathbb{P}^{2n-1}$ admits no symplectically aspherical fillings for $n > 1$ and odd. In particular, it is not Liouville fillable.*

These are the first examples of strongly but not Liouville fillable contact structures in high dimension. Examples in dimension three were given by the first author in [3] using Heegaard Floer homology. In contrast with the high dimensional situation, the standard contact structure on $\mathbb{R}\mathbb{P}^3$ is the canonical contact structure on the unit cotangent bundle of S^2 and therefore is Weinstein fillable.

After a preliminary version of our result (originally for $\mathbb{R}\mathbb{P}^5$ only) was announced, Zhou proved in [10] that $(\mathbb{R}\mathbb{P}^{2n-1}, \xi)$ is not Liouville fillable if $n \neq 2^k$. He also proves similar nonfillability results for more general links of cyclic quotient singularities. Zhou’s proof uses advanced properties of symplectic cohomology; in contrast our

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proof is more direct, as it relies on the analysis of how a certain moduli space of holomorphic spheres can break, in the spirit of McDuff's classification of symplectic fillings of $\mathbb{R}\mathbb{P}^3$ in [5].

The strategy is the following. The standard contact structure ξ on $\mathbb{R}\mathbb{P}^{2n-1}$ admits a contact form whose Reeb orbits are the fibers of the Hopf fibration $\mathbb{R}\mathbb{P}^{2n-1} \rightarrow \mathbb{C}\mathbb{P}^{n-1}$. If (W, ω) is a strong symplectic filling of $(\mathbb{R}\mathbb{P}^{2n-1}, \xi)$, by a symplectic reduction of ∂W (informally speaking, by replacing ∂W with the quotient by the Reeb flow) we obtain a closed symplectic manifold $(\overline{W}, \overline{\omega})$ with a codimension two symplectic submanifold $W_\infty \cong \mathbb{C}\mathbb{P}^{n-1}$ (corresponding to the quotient of ∂W) such that $\overline{W} \setminus W_\infty$ is symplectomorphic to $\text{int}(W)$; that is, $\overline{\omega}|_{\overline{W} \setminus W_\infty} = \omega|_{\text{int}(W)}$. The normal bundle of W_∞ is isomorphic to $\mathcal{O}_{\mathbb{P}^{n-1}}(2)$.

We fix a point and a hyperplane in W_∞ , and we consider the moduli space of holomorphic spheres in \overline{W} which are homotopic to a projective line and pass both through the point and the hypersurface. We prove, by topological considerations, that the homology class of a projective line in W_∞ is an even multiple of a homology class in \overline{W} if the compactification of that moduli space contains only nodal curves with at most two irreducible components each of which intersects W_∞ nontrivially. If n is odd this is a contradiction because the first Chern class of a line is $n + 2$; only at this step we use the hypothesis on the parity of n . This implies that there is either a nodal holomorphic sphere in \overline{W} in the homology class of a line of W_∞ with at least three connected components or a nodal holomorphic sphere with an irreducible component which is disjoint from W_∞ . Since a nodal sphere intersects W_∞ in exactly two points, in either case at least one irreducible component must lie entirely in $\text{int}(W)$, which therefore is not symplectically aspherical.

If (W, ω) is not symplectically aspherical we lose control on the compactification of the moduli space, which is not surprising, given that $(\mathbb{R}\mathbb{P}^{2n-1}, \xi)$ admits spherical fillings. However, if W is semipositive (and maybe even more generally, using some abstract perturbation scheme) we still have enough control to be able to draw conclusions about the fundamental group of W .

Theorem 1.2. *If (W, ω) is a semipositive symplectic filling of $(\mathbb{R}\mathbb{P}^{2n-1}, \xi)$, then W is simply connected.*

If we apply the same techniques to a symplectically aspherical filling of the standard contact structure on S^{2n-1} we obtain that the filling must be diffeomorphic to the ball, a result originally due to Eliashberg, Floer and McDuff. This is, at least, the third proof, after the original one in [6] and the one in [4]. The proof given here is close to the original one, but uses a different compactification of the filling and is slightly simpler.

2. THE MODULI SPACE OF LINES

2.1. The smooth stratum. by the Weinstein neighborhood theorem, W_∞ has a tubular neighborhood that is symplectomorphic to a neighborhood of the zero section in the total space of $\mathcal{O}_{\mathbb{P}^{n-1}}(2)$. Let \mathcal{J} be the space of almost complex structures on \overline{W} which are compatible with $\overline{\omega}$ and coincide near W_∞ with the natural (integrable) complex structure on $\mathcal{O}_{\mathbb{P}^{n-1}}(2)$.

For any almost complex structure $J \in \mathcal{J}$, any line $\ell \subset \mathbb{C}\mathbb{P}^{n-1} \cong W_\infty$ is a J -holomorphic sphere. Moreover,

$$(1) \quad T\overline{W}|_\ell \cong \mathcal{O}_{\mathbb{P}^1}(2) \oplus \underbrace{\mathcal{O}_{\mathbb{P}^1}(1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(1)}_{n-2} \oplus \mathcal{O}_{\mathbb{P}^1}(2)$$

as *holomorphic* vector bundle, where the first $\mathcal{O}_{\mathbb{P}^1}(2)$ summand is the tangent bundle of ℓ , the $(n-2)$ -many $\mathcal{O}_{\mathbb{P}^1}(1)$ -summands correspond to the normal bundle of ℓ in $\mathbb{C}\mathbb{P}^{n-1} \cong W_\infty$ and the last $\mathcal{O}_{\mathbb{P}^1}(2)$ -summand is the normal bundle of W_∞ in \overline{W} restricted to ℓ .

Let \mathcal{M} be the moduli space of J -holomorphic spheres in \overline{W} that are homotopic to the lines in $W_\infty \cong \mathbb{C}\mathbb{P}^{n-1}$. We fix a point $p_0 \in W_\infty$ and a hyperplane $H_\infty \cong \mathbb{C}\mathbb{P}^{n-2} \subset W_\infty$ such that $p_0 \notin H_\infty$ and denote the space of J -holomorphic spheres of \mathcal{M} that intersect both p_0 and H_∞ by $\mathcal{M}(p_0, H_\infty)$. We also consider the moduli space $\mathcal{M}_z(p_0, H_\infty)$ of J -holomorphic spheres as above with an extra free marked point z . There is a projection

$$f: \mathcal{M}_z(p_0, H_\infty) \rightarrow \mathcal{M}(p_0, H_\infty).$$

that forgets the marked point.

Lemma 2.1. *$\mathcal{M}(p_0, H_\infty)$ has expected dimension $2n-2$ and $\mathcal{M}_z(p_0, H_\infty)$ has expected dimension $2n$, where $\dim \overline{W} = 2n$.*

Proof. The decomposition (1) gives $\langle c_1(T\overline{W}), [\ell] \rangle = n+2$. The Riemann-Roch formula gives $\text{vir-dim } \mathcal{M} = 4n-2$. Passing through the codimension 4 submanifold H_∞ and through p_0 which is of codimension $2n$, we obtain, after adding two additional marked points needed to handle these constraints, that $\text{vir-dim } \mathcal{M}(p_0, H_\infty) = 2n-2$. Adding a free marked point increases the expected dimension by 2, so we finally obtain $\text{vir-dim } \mathcal{M}_z(p_0, H_\infty) = 2n$. \square

The main reason for keeping the almost complex structure integrable near W_∞ is to have positivity of intersection between W_∞ and J -holomorphic spheres. This fact makes our moduli space particularly well behaved, as the following lemma shows.

Lemma 2.2. *All J -holomorphic spheres of $\mathcal{M}(p_0, H_\infty)$ are simply covered and are either lines in W_∞ or intersect W_∞ transversely in exactly two points.*

Proof. Since the algebraic intersection between W_∞ and ℓ is 2, positivity of intersection implies that a sphere of $\mathcal{M}(p_0, H_\infty)$ is either contained in W_∞ , in which case it is a line and therefore simply covered, or it intersect W_∞ with total multiplicity two. Since the constraints force two distinct intersection points, positivity of intersection implies that they are the only ones and that they each have multiplicity one. \square

Proposition 2.3. *For a generic almost complex structure $J \in \mathcal{J}$ the moduli spaces $\mathcal{M}(p_0, H_\infty)$ and $\mathcal{M}_z(p_0, H_\infty)$ are smooth manifolds of dimension $2n-2$ and $2n$ respectively.*

Proof. Let \mathcal{M}^* be the subset of \mathcal{M} consisting of simply covered J -holomorphic spheres. By Lemma 2.2, $\mathcal{M}(p_0, H_\infty) \subset \mathcal{M}^*$. J -holomorphic spheres of \mathcal{M}^* which are contained in the neighborhood of W_∞ where J is integrable correspond to sections of $\mathcal{O}_{\mathbb{P}^{n-1}}$ and therefore admit a decomposition of the restriction of $T\overline{W}$

as in Equation (1). Since the decomposition is into positive line bundles, those spheres are Fredholm regular for every almost complex structures $J \in \mathcal{J}$ because the Cauchy-Riemann operator on a positive line bundle over $\mathbb{C}\mathbb{P}^1$ is surjective by Serre duality; see [7, Lemma 3.3.1]

All other J -holomorphic spheres of \mathcal{M}^* are Fredholm regular for a generic $J \in \mathcal{J}$ because they are simply covered and intersect the region where J is generic. Then for generic J the moduli space \mathcal{M} is a smooth manifold. The constraints cut out $\mathcal{M}(p_0, H_\infty)$ transversely for a generic J : for spheres near W_∞ this is an explicit computation, and for all other spheres of \mathcal{M}^* it follows from [7, Theorem 3.4.1] and [7, Remark 3.4.8]. Therefore $\mathcal{M}(p_0, H_\infty)$ is a submanifold of the dimension predicted by Lemma 2.1. The corresponding statements for $\mathcal{M}_z(p_0, H_\infty)$ follow from those for $\mathcal{M}(p_0, H_\infty)$. \square

2.2. The compactified moduli space. Let $\overline{\mathcal{M}}(p_0, H_\infty)$ and $\overline{\mathcal{M}}_z(p_0, H_\infty)$ be the Gromov compactifications of $\mathcal{M}(p_0, H_\infty)$ and $\mathcal{M}_z(p_0, H_\infty)$ respectively, and let

$$\bar{f}: \overline{\mathcal{M}}_z(p_0, H_\infty) \rightarrow \overline{\mathcal{M}}(p_0, H_\infty)$$

be the forgetful map. We denote $\mathcal{M}^{\text{red}}(p_0, H_\infty) = \overline{\mathcal{M}}(p_0, H_\infty) \setminus \mathcal{M}(p_0, H_\infty)$ and $\mathcal{M}_z^{\text{red}}(p_0, H_\infty) = \overline{\mathcal{M}}_z(p_0, H_\infty) \setminus \mathcal{M}_z(p_0, H_\infty)$.

Lemma 2.4. *If $\overline{W} \setminus W_\infty$ is symplectically aspherical, then every nodal sphere of $\mathcal{M}^{\text{red}}(p_0, H_\infty)$ has exactly two irreducible components, one of which intersects W_∞ only at p_0 and the other one which intersects W_∞ only at a point of H_∞ . Both components are simply covered and their intersection with W_∞ is transverse.*

Proof. Irreducible components of nodal spheres belonging to $\overline{\mathcal{M}}(p_0, H_\infty)$ are not contained in W_∞ because the homology class of a line ℓ is primitive in $\mathbb{C}\mathbb{P}^{n-1}$. Then by positivity of intersection with W_∞ a nodal sphere must intersect W_∞ in at most two points. Moreover, if $\overline{W} \setminus W_\infty$ is symplectically aspherical, every irreducible component must intersect W_∞ . This implies that there are exactly two irreducible components and their intersection with W_∞ has multiplicity one, and therefore both components are simply covered. \square

This lemma implies that we have enough topological control on the nodal curves to show that they have smooth moduli spaces.

Lemma 2.5. *The moduli space $\mathcal{M}^{\text{red}}(p_0, H_\infty)$ is a smooth manifold of dimension $n - 4$. The forgetful map*

$$f^{\text{red}}: \mathcal{M}_z^{\text{red}}(p_0, H_\infty) \rightarrow \mathcal{M}^{\text{red}}(p_0, H_\infty)$$

is a locally trivial fibration with fiber $S^2 \vee S^2$.

Proof. By Lemma 2.4 the irreducible components of the nodal spheres of the moduli space $\mathcal{M}^{\text{red}}(p_0, H_\infty)$ are simply covered and intersect the region where the almost complex structure can be made generic. Then the statement follows from [7, Theorem 6.2.6]. \square

Gluing theory usually produces only topological manifolds; in our situation however, we obtain slightly more because we have a global gauge fixing for the spheres we want to glue.

Identify the neighborhood of W_∞ in \overline{W} with a neighborhood of the 0-section of $\mathcal{O}_{\mathbb{P}^{n-1}}(2)$ as already discussed above. The hyperplane H_∞ is the 0-set of a

holomorphic section σ in $\mathcal{O}_{\mathbb{P}^{n-1}}(1)$, and it follows that σ^2 is a section of $\mathcal{O}_{\mathbb{P}^{n-1}}(2)$ that has a zero of order two along H_∞ . Multiplying σ^2 with a small constant, we can assume that its image lies in an arbitrarily small neighborhood of the 0-section. Its graph is a J -holomorphic hypersurface in \overline{W} that we will call \widetilde{W}_∞ . In particular $\widetilde{W}_\infty \cap W_\infty = H_\infty$ and $T\widetilde{W}_\infty|_{H_\infty} = TW_\infty|_{H_\infty}$. Then every sphere of $\mathcal{M}(p_0, H_\infty)$ which is not contained in W_∞ intersects \widetilde{W}_∞ in two points: one in H_∞ and one in $\widetilde{W}_\infty \setminus H_\infty$.

Let $\widetilde{\mathcal{M}}(p_0, H_\infty)$ be the open subset of $\mathcal{M}(p_0, H_\infty)$ consisting of those sphere which are not contained in W_∞ . By the discussion in the previous paragraph, we can fix a parametrization for every element in $\widetilde{\mathcal{M}}(p_0, H_\infty)$ identifying this moduli space with the set of J -holomorphic maps $u: S^2 \rightarrow \overline{W}$ whose image is homotopic to ℓ but not contained in W_∞ , and such that $u(0) = p_0$, $u(1) \in \widetilde{W}_\infty \setminus H_\infty$ and $u(\infty) \in H_\infty$.

We also denote by $\widetilde{\mathcal{M}}^{\text{red}}(p_0, H_\infty)$ the set of pairs of J -holomorphic maps (u^0, u^∞) , with $u^0, u^\infty: S^2 \rightarrow \overline{W}$, such that $u^0(0) = p_0$, $u^0(1) \in \widetilde{W}_\infty \setminus H_\infty$, $u^0(\infty) = u^\infty(0)$, $u^\infty(\infty) \in H_\infty$, $|d_\infty u^\infty| = 1$ and the image of the ‘‘connected sum map’’ $u^0 \# u^\infty: S^2 \# S^2 \cong S^2 \rightarrow \overline{W}$ is homotopic to ℓ . The group of complex numbers of modulus 1, acts on $\widetilde{\mathcal{M}}^{\text{red}}(p_0, H_\infty)$ by $\theta \cdot (u^0, u^\infty) = (u^0, u^\infty(\theta^{-1} \cdot))$ and the quotient by this action is $\mathcal{M}^{\text{red}}(p_0, H_\infty)$.

Let $\pi: S^2 \times S^2 \dashrightarrow S^2$ be the rational map $\pi(x, y) = y/x$, which is not defined at the points $(0, 0)$ and (∞, ∞) . If we make S^1 act on $S^2 \times S^2$ by $\theta \cdot (x, y) = (x, \theta y)$ and on S^2 by $\theta \cdot w = \theta^{-1} w$, then π is S^1 -equivariant. Let \mathfrak{X} be the smooth variety obtained by blowing up $S^2 \times S^2$ at $(0, 0)$ and (∞, ∞) . The action of S^1 on $S^2 \times S^2$ induces an action on \mathfrak{X} and π extend to a smooth S^1 -equivariant map $\pi: \mathfrak{X} \rightarrow S^2$.

We denote by D_ϵ the disc with center in 0 and radius ϵ in $\mathbb{C} \subset S^2$ and $\mathfrak{X}_\epsilon = \pi^{-1}(D_\epsilon)$. We define $E_\epsilon = D_\epsilon \times_{S^1} \widetilde{\mathcal{M}}^{\text{red}}(p_0, H_\infty)$ and $X_\epsilon = \mathfrak{X}_\epsilon \times_{S^1} \widetilde{\mathcal{M}}^{\text{red}}(p_0, H_\infty)$. We have bundle maps

$$\begin{array}{ccc} X_\epsilon & \xrightarrow{\quad} & E_\epsilon \\ & \searrow & \swarrow \\ & \mathcal{M}^{\text{red}}(p_0, H_\infty) & \end{array}$$

Let \dot{E}_ϵ denote E_ϵ with the zero section removed, and \dot{X}_ϵ the preimage of \dot{E}_ϵ . We can identify $\dot{E}_\epsilon \cong [\epsilon^{-1}, +\infty) \times \widetilde{\mathcal{M}}^{\text{red}}(p_0, H_\infty)$, and therefore standard gluing theory (see for example [7, Chapter 10]) yields C^1 -embeddings $\mathfrak{g}: \dot{E}_\epsilon \rightarrow \widetilde{\mathcal{M}}(p_0, H_\infty)$ and $\mathfrak{G}: \dot{X}_\epsilon \rightarrow \widetilde{\mathcal{M}}(p_0, H_\infty)$ which are moreover compatible with the forgetful maps. Combining [7, Theorem 6.2.6] with the discussion above we obtain the following structural result for the moduli spaces we are interested in.

Proposition 2.6. *The moduli spaces $\overline{\mathcal{M}}(p_0, H_\infty)$ and $\overline{\mathcal{M}}_z(p_0, H_\infty)$ are compact and orientable C^1 -manifolds and there is a C^1 -map*

$$\bar{f}: \overline{\mathcal{M}}_z(p_0, H_\infty) \rightarrow \overline{\mathcal{M}}(p_0, H_\infty)$$

which forgets the marked point.

While $\overline{\mathcal{M}}(p_0, H_\infty)$ is not a priori connected, since we have not ruled out that a J -holomorphic sphere could be homotopic to a line $\ell \subset W_\infty$ but not homotopic through J -holomorphic spheres, we can assume without loss of generality that

$\overline{\mathcal{M}}(p_0, H_\infty)$ is connected by restricting our attention to the connected component which contains a line $\ell \subset W_\infty$.

3. PROOF OF THE MAIN THEOREM

3.1. Degree of the evaluation map. Let $\text{ev}: \overline{\mathcal{M}}_z(p_0, H_\infty) \rightarrow \overline{W}$ be the evaluation map at the free marked point.

Lemma 3.1. *There is an open subset $U \subset \overline{W}$ such that every J -holomorphic sphere of $\overline{\mathcal{M}}(p_0, H_\infty)$ passing through a point of U belongs to $\mathcal{M}(p_0, H_\infty)$ and its image is contained in the neighborhood of W_∞ on which J is integrable.*

Proof. Choose a point $q_0 \in W_\infty \setminus H_\infty$ such that $q_0 \neq p_0$. The unique line ℓ_0 in W_∞ passing through p_0 and q_0 also intersects H_∞ , and therefore determines an element of $\mathcal{M}(p_0, H_\infty)$. Moreover, any sphere of $\mathcal{M}(p_0, H_\infty)$ passing through q_0 intersects W_∞ in three points, and therefore must be contained in it, so it is equal to ℓ_0 .

Since none of the nodal spheres passes through q_0 , and since $\mathcal{M}_z^{\text{red}}(p_0, H_\infty)$ is compact, there is a neighborhood U of q_0 in \overline{W} such that $\text{ev}^{-1}(U) \subset \mathcal{M}_z(p_0, H_\infty)$.

After possibly reducing the size of U , we can assume that every J -holomorphic sphere of $\mathcal{M}(p_0, H_\infty)$ passing through U is contained in the neighborhood of W_∞ on which J is integrable. Suppose on the contrary that there is a sequence $[u_n]$ of elements of $\mathcal{M}(p_0, H_\infty)$ and a sequence of points $q_n \in \overline{W}$ converging to q_0 such that the image of u_n contains q_n , but is not contained in some fixed neighborhood of W_∞ . Then by Gromov compactness there is a subsequence of $[u_n]$ converging to a (possibly nodal) J -holomorphic sphere of $\overline{\mathcal{M}}(p_0, H_\infty)$ passing through q_0 and not contained in the fixed neighborhood of W_∞ . This is a contradiction because the only element of $\overline{\mathcal{M}}(p_0, H_\infty)$ passing through q_0 is ℓ_0 , which is contained in W_∞ . \square

Lemma 3.2. *The evaluation map $\text{ev}: \overline{\mathcal{M}}_z(p_0, H_\infty) \rightarrow \overline{W}$ has degree one.*

Proof. Let U be the neighborhood defined in Lemma 3.1. We will show that $\#\text{ev}^{-1}(q) = 1$ for every $q \in U$.

Since all J -holomorphic spheres passing through U are contained in the neighborhood where J is integrable, we can pretend we are working in the total space of $\mathcal{O}_{\mathbb{P}^{n-1}}(2)$. Given $q \in U$, let \bar{q} be its projection to $\mathbb{C}\mathbb{P}^{n-1} \cong W_\infty$. Any J -holomorphic sphere of $\mathcal{M}(p_0, H_\infty)$ passing through q projects to the unique line ℓ_q in W_∞ passing through p_0 and \bar{q} . The sphere itself corresponds then to a section of $\mathcal{O}_{\mathbb{P}^{n-1}}(2)|_{\ell_q} \cong \mathcal{O}_{\mathbb{P}^1}(2)$ which vanishes at p_0 and at $p_\infty = \ell_q \cap H_\infty$. The space of sections of $\mathcal{O}_{\mathbb{P}^1}(2)$ vanishing at p_0 and p_∞ has complex dimension one, and thus there is a unique such section for any point q in the fiber of $\mathcal{O}_{\mathbb{P}^1}(2)$ over \bar{q} .

This shows that $\#\text{ev}^{-1}(q) = 1$ for every $q \in U$, and since U is open, by Sard's theorem it contains a regular value of the evaluation map. This proves that ev has degree one. \square

It is important to have a degree one map because degree one maps induce surjections in homology. More generally, we have the following lemma.

Lemma 3.3. *Let $f: X \rightarrow Y$ be a smooth map between compact oriented C^1 -manifolds of the same dimension. Assume that f has degree d , and let $S \subset Y$ be a compact, oriented submanifold of dimension k which is transverse to f .*

Then $f^{-1}(S)$ has an induced orientation and, with that orientation, we have the equality

$$f_*([f^{-1}(S)]) = d[S]$$

in $H_k(Y; \mathbb{Z})$.

Proof. A submanifold S is transverse to a map f if, for every $y \in S$ and $x \in f^{-1}(y)$ we have $T_y S \oplus d_x f(T_x X) = T_y Y$. This property implies that

- $S' = f^{-1}(S)$ is a compact submanifold of X , and
- df defines an isomorphism between the normal bundle of S' and the normal bundle of S .

The orientations of S and Y determine an orientation of the normal bundle of S . This in turn induces an orientation of the normal bundle of S' via df which, combined with the orientation of X induces the orientation of S' .

Let $f_S: S' \rightarrow S$ be the restriction of f . The condition on the normal bundles implies that the regular values of f_S are also regular values of f . If y is a regular value of f_S , then

$$\begin{aligned} \deg(f_S) &= \sum_{x \in f_S^{-1}(y)} \text{sign}(d_x f_S) \\ \deg(f) &= \sum_{x \in f^{-1}(y)} \text{sign}(d_x f). \end{aligned}$$

Since $f_S^{-1}(y) = f^{-1}(y)$ by the definition of f_S and $\text{sign}(d_x f_S) = \text{sign}(d_x f)$ because df is an orientation preserving isomorphism between the normal bundles, we obtain $\deg(f_S) = \deg(f) = d$.

Now we consider the commutative diagram

$$\begin{array}{ccc} H_k(S'; \mathbb{Z}) & \xrightarrow{(f_S)_*} & H_k(S; \mathbb{Z}) \\ \downarrow & & \downarrow \\ H_k(X; \mathbb{Z}) & \xrightarrow{f_*} & H_k(Y; \mathbb{Z}) \end{array}$$

where the vertical arrows are induced by the inclusions. The fundamental class of S' is mapped by $(f_S)_*$ to $\deg(f_S)$ times the fundamental class of S . The homology classes $[S']$ and $[S]$ are the images of the fundamental classes of S' and S in $H_k(X; \mathbb{Z})$ and $H_k(Y; \mathbb{Z})$ respectively, and therefore $f_*[S'] = \deg(f_S)[S] = d[S]$. \square

3.2. Decomposition of the line. The following lemma is a warm up which illustrates how to derive topological implications from Lemma 3.2.

Lemma 3.4. *The moduli space $\mathcal{M}_z(p_0, H_\infty)$ is not compact.*

Proof. The moduli space $\mathcal{M}_z(p_0, H_\infty)$ is an S^2 -bundle over $\mathcal{M}(p_0, H_\infty)$ with two distinguished sections $\text{ev}^{-1}(p_0)$ and $\text{ev}^{-1}(H_\infty)$. Then $\mathcal{M}_z(p_0, H_\infty) \setminus \text{ev}^{-1}(H_\infty)$ retracts onto $\text{ev}^{-1}(p_0)$. This implies that

$$\text{ev}_*: H_k(\mathcal{M}_z(p_0, H_\infty) \setminus \text{ev}^{-1}(H_\infty); \mathbb{Z}) \rightarrow H_k(\overline{W}; \mathbb{Z})$$

is trivial whenever $k > 0$.

Take an embedded sphere $\ell \subset \overline{W}$ which is homologous to a line in W_∞ but disjoint from H_∞ . It is possible to find such a sphere because H_∞ has codimension 4 in \overline{W} , but ℓ will not be holomorphic. We perturb ℓ to be transverse to the evaluation map. If $\mathcal{M}_z(p_0, H_\infty)$ is compact, $\text{ev}_*([\text{ev}^{-1}(\ell)]) = [\ell]$ by Lemma 3.3. Since $\ell \cap H_\infty = \emptyset$, we have $\text{ev}^{-1}(\ell) \subset \mathcal{M}_z(p_0, H_\infty) \setminus \text{ev}^{-1}(H_\infty)$. Then the previous paragraph implies that $[\ell] = \text{ev}_*([\text{ev}^{-1}(\ell)]) = 0$. This is a contradiction because ℓ is homologous to a symplectic sphere, and therefore is nontrivial in homology. \square

Lemma 3.4 tells us that $\mathcal{M}^{\text{red}}(p_0, H_\infty)$ is nonempty. We decompose it into connected components

$$\mathcal{M}^{\text{red}}(p_0, H_\infty) = \mathcal{M}^{(1)}(p_0, H_\infty) \sqcup \cdots \sqcup \mathcal{M}^{(N)}(p_0, H_\infty)$$

and, correspondingly, we decompose the moduli space with a free marked point into connected components

$$\mathcal{M}_z^{\text{red}}(p_0, H_\infty) = \mathcal{M}_z^{(1)}(p_0, H_\infty) \sqcup \cdots \sqcup \mathcal{M}_z^{(N)}(p_0, H_\infty).$$

Each $\mathcal{M}_z^{(i)}(p_0, H_\infty)$ is an $S^2 \vee S^2$ -bundle over $\mathcal{M}^{(i)}(p_0, H_\infty)$ with three distinguish sections: one, denoted $\mathcal{S}_0^{(i)}$, where the free marked point is mapped to p_0 , one, denoted $\mathcal{S}_\infty^{(i)}$, where the free marked point is mapped to H_∞ , and one, denoted $\mathcal{S}_n^{(i)}$, where the free marked point lies on the node.¹ Therefore we can see each $\mathcal{M}_z^{(i)}(p_0, H_\infty)$ as the union of two sphere bundles $\mathcal{N}_0^{(i)}$ and $\mathcal{N}_\infty^{(i)}$ over $\mathcal{M}^{(i)}(p_0, H_\infty)$ glued together along the section $\mathcal{S}_n^{(i)}$. An element of $\mathcal{M}_z^{(i)}(p_0, H_\infty)$ belongs to $\mathcal{N}_0^{(i)}$ when the free marked point is in the domain of the irreducible component passing through p_0 , and to $\mathcal{N}_\infty^{(i)}$ when the free marked point is in the domain of the irreducible component passing through H_∞ .

Given homology classes A and B (in the same manifold) we denote by $A \cdot B$ their intersection product. If A and B are represented by closed, oriented submanifolds which intersect transversely, $A \cdot B$ is the algebraic count of intersection points.

Lemma 3.5. *Let ℓ be a sphere in W_∞ which is disjoint from H_∞ , transverse to the evaluation map ev , and homologous to a line.*

Then, there exists at least one nodal component $\mathcal{M}_z^{(i)}(p_0, H_\infty)$ such that

$$[\text{ev}^{-1}(\ell)] \cdot [\mathcal{N}_\infty^{(i)}] \neq 0.$$

Proof. Denote by ℓ' the oriented submanifold $\text{ev}^{-1}(\ell)$ in $\mathcal{M}_z(p_0, H_\infty)$. Due to Lemma 3.3 we have $\text{ev}_*([\ell']) = [\ell]$.

If $[\ell'] \cdot [\mathcal{N}_\infty^{(i)}] = 0$, we can group the points in $\ell' \cap \mathcal{N}_\infty^{(i)}$ in pairs of opposite sign, and we can modify ℓ' by adding small 1-handles to cancel all intersection points between ℓ' and $\mathcal{N}_\infty^{(i)}$. For this, choose for any pair of intersection points that we want to cancel a path in $\mathcal{N}_\infty^{(i)}$ connecting them, and use this path as the core of the 1-handle. We can choose the paths in a way that they do not intersect each other and that they do not intersect $\mathcal{S}_\infty^{(i)}$ either.

The resulting manifold is still homologous to ℓ' . Furthermore, if the handles have a sufficiently small diameter, the resulting manifold is disjoint from $\text{ev}^{-1}(H_\infty)$ because the cores of the handles avoid $\mathcal{S}_\infty^{(i)}$.

If the intersection product $[\ell'] \cdot [\mathcal{N}_\infty^{(i)}]$ is trivial for all $i = 1, \dots, N$, the end result of these modifications is a surface $\tilde{\ell} \subset \mathcal{M}_z(p_0, H_\infty) \setminus \text{ev}^{-1}(H_\infty)$ such that $[\tilde{\ell}] = [\ell']$.

On the one hand, this yields $\text{ev}_*([\tilde{\ell}]) = [\ell]$ by Lemma 3.3, but on the other hand, since $\mathcal{M}_z(p_0, H_\infty) \setminus \text{ev}^{-1}(H_\infty)$ retracts to $\text{ev}^{-1}(p_0)$, we have that $\text{ev}_*([\tilde{\ell}]) = 0$ as in Lemma 3.4. This is a contradiction because $[\ell] \neq 0$ \square

¹Strictly speaking ghost bubbles appear in these three cases and we tacitly contract them. We ignore this technical complication as it has no topological consequence.

Lemma 3.6. *Let $\text{ev}_\infty^{(i)}: \mathcal{S}_\infty^{(i)} \rightarrow H_\infty$ denote the restriction of the evaluation map $\text{ev}: \overline{\mathcal{M}}_z(p_0, H_\infty) \rightarrow \overline{W}$ to $\mathcal{S}_\infty^{(i)}$. Representing the homology class of a line by an embedded surface ℓ in \overline{W} that is transverse to the evaluation map ev , we obtain*

$$[\mathcal{N}_\infty^{(i)}] \cdot [\text{ev}^{-1}(\ell)] = \text{deg}(\text{ev}_\infty^{(i)}).$$

Proof. Let $y \in H_\infty$ be a regular value of $\text{ev}_\infty^{(i)}$ for all $i = 1, \dots, N$, and let ℓ_0 be a line in W_∞ intersecting H_∞ transversely at y . It follows that ℓ_0 is transverse to $\text{ev}|_{\mathcal{N}_\infty^{(i)}}$ at y , that is, for every $x \in \text{ev}^{-1}(y) \cap \mathcal{N}_\infty^{(i)}$ we have

$$(2) \quad d_x \text{ev}(T_x \mathcal{N}_\infty^{(i)}) \oplus T_y \ell_0 = T_y \overline{W},$$

because the nodal J -holomorphic spheres in $\overline{\mathcal{M}}(p_0, H_\infty)$ are all transverse to W_∞ .

By construction $(\text{ev}_\infty^{(i)})^{-1}(y) = (\text{ev}|_{\mathcal{N}_\infty^{(i)}})^{-1}(\ell_0)$. If $x \in (\text{ev}|_{\mathcal{N}_\infty^{(i)}})^{-1}(\ell_0)$, we define $\text{sign}(x) = +1$ if the equality of Equation (2) preserves the orientation, and $\text{sign}(x) = -1$ otherwise. Then $\text{sign}(x) = \text{sign}(d_x \text{ev}_\infty^{(i)})$ because $d_x \text{ev}$ is complex linear in the extra direction $T_x \mathcal{N}_\infty^{(i)}/T_x \mathcal{S}_\infty^{(i)}$.

Now let ℓ be a small perturbation of ℓ_0 which is transverse to ev . By Equation (2) we can assume that the perturbation is supported away from y and that no new intersection points between $\text{ev}^{-1}(\ell)$ and $\mathcal{N}_\infty^{(i)}$ are created. Then

$$[\mathcal{N}_\infty^{(i)}] \cdot [\text{ev}^{-1}(\ell)] = \sum_{x \in (\text{ev}|_{\mathcal{N}_\infty^{(i)}})^{-1}(\ell_0)} \text{sign}(x) = \sum_{x \in (\text{ev}_\infty^{(i)})^{-1}(y)} \text{sign}(d_x \text{ev}_\infty^{(i)}) = \text{deg}(\text{ev}_\infty^{(i)}).$$

□

The pull-back of the symplectic form $\overline{\omega}$ is nontrivial on the fibers of $\mathcal{N}_*^{(i)}$ for any $i = 1, \dots, N$ and $* \in \{0, \infty\}$. Therefore, by the Leray-Hirsch Theorem (see [1, Theorem 5.11] for its cohomological form),

$$(3) \quad H_2(\mathcal{N}_*^{(i)}; \mathbb{Z}) \cong H_2(\mathcal{S}_*^{(i)}; \mathbb{Z}) \oplus H_2(S^2; \mathbb{Z}) \cong H_2(\mathcal{S}_n^{(i)}; \mathbb{Z}) \oplus H_2(S^2; \mathbb{Z}),$$

where the summand $H_2(S^2; \mathbb{Z})$ is generated by a fiber of $\mathcal{N}_*^{(i)}$.

For every i we denote by $A_0^{(i)}$ and by $A_\infty^{(i)}$ the homology classes representing the fibers of $\mathcal{N}_0^{(i)}$ and of $\mathcal{N}_\infty^{(i)}$ respectively. Then for each i we have the condition $\text{ev}_*(A_0^{(i)}) + \text{ev}_*(A_\infty^{(i)}) = [\ell]$ in $H_2(\overline{W}; \mathbb{Z})$.

Let ℓ be now a sphere in H_∞ which is homologous to a line and transverse to $\text{ev}_\infty^{(i)}: \mathcal{S}_\infty^{(i)} \rightarrow H_\infty$ for all $i = 1, \dots, N$. Then $\ell'_i := (\text{ev}_\infty^{(i)})^{-1}(\ell)$ is a smooth submanifold of $\mathcal{S}_\infty^{(i)}$ which, by Lemma 3.3, satisfies $\text{ev}_*([\ell'_i]) = (\text{ev}_\infty^{(i)})_*([\ell'_i]) = \kappa_i [\ell]$, where $\kappa_i := \text{deg}(\text{ev}_\infty^{(i)})$.

By Equation (3), there is some $d_i \in \mathbb{Z}$ and $c \in H_2(\mathcal{S}_n^{(i)}; \mathbb{Z})$ such that

$$(4) \quad [\ell'_i] = d_i A_\infty^{(i)} + c.$$

Lemma 3.7. *The map*

$$(\text{ev}|_{\mathcal{S}_n^{(i)}})_*: H_2(\mathcal{S}_n^{(i)}; \mathbb{Z}) \rightarrow H_2(\overline{W}; \mathbb{Z})$$

is trivial for every $i = 1, \dots, N$.

Proof. By Equation (3) every class $c \in H_2(\mathcal{S}_n^{(i)}; \mathbb{Z})$ can be written as the sum of a class in $H_2(\mathcal{S}_0^{(i)}; \mathbb{Z})$ and a multiple of the class of the fiber. Since $\mathcal{S}_0^{(i)}$ is mapped

to p_0 , we obtain $(\text{ev}|_{\mathcal{S}_0^{(i)}})_* = 0$, and thus $\text{ev}_*(c) = k \text{ev}_*(A_0^{(i)})$. By Lemma 2.4 $\text{ev}(\mathcal{S}_n^{(i)}) \cap W_\infty = \emptyset$ while $\text{ev}_*(A_0^{(i)}) \cdot [W_\infty] = 1$ so that

$$0 = \text{ev}_*([\mathcal{S}_n^{(i)}]) \cdot [W_\infty] = k \text{ev}_*(A_0^{(i)}) \cdot [W_\infty] = k. \quad \square$$

By Equation (4) and Lemma 3.7 we obtain that $d_i \text{ev}_*(A_\infty^{(i)}) = \kappa_i [\ell]$. Intersecting with W_∞ we obtain that $d_i = 2\kappa_i$. Now we restrict our attention to an index i for which $\kappa_i \neq 0$, which exists by Lemma 3.5. In this case we can simplify the equality

$$\langle c_1(T\overline{W}), \kappa_i[\ell] \rangle = \langle c_1(T\overline{W}), d_i \text{ev}_*(A_\infty^{(i)}) \rangle$$

and obtain

$$\langle c_1(T\overline{W}), [\ell] \rangle = 2 \langle c_1(T\overline{W}), \text{ev}_*(A_\infty^{(i)}) \rangle.$$

When n is odd this is not possible because

$$\langle c_1(T\overline{W}), [\ell] \rangle = n + 2,$$

and therefore (W, ω) is not symplectically aspherical. This proves Theorem 1.1.

4. FUNDAMENTAL GROUP OF SEMIPOSITIVE FILLINGS

In this section let (W, ω) be a semipositive filling of $(\mathbb{R}\mathbb{P}^{2n-1}, \xi)$. We recall that (W, ω) is semipositive if every class A in the image of the Hurewicz homomorphism $\pi_2(W) \rightarrow H_2(W; \mathbb{Z})$ satisfying the conditions $\langle \omega, A \rangle > 0$ and $\langle c_1(TW), A \rangle \geq 3 - n$ also satisfies $\langle c_1(TW), A \rangle \geq 0$. See [7, Definition 6.4.1].

We use the same compactification $(\overline{W}, \overline{\omega})$ and the same set of almost complex structures \mathcal{J} as in the previous sections, but now that (W, ω) does not need to be symplectically aspherical we cannot assume anymore that $\overline{\mathcal{M}}(p_0, H_\infty)$ is a manifold or that its elements have no irreducible component contained completely inside $\overline{W} \setminus W_\infty$. The irreducible components which intersect W_∞ must be simply covered because the intersections are simple, and therefore are Fredholm regular for a generic almost complex structure $J \in \mathcal{J}$, but the irreducible components which are contained in $\overline{W} \setminus W_\infty$ can be multiply covered. However according to [7, Theorem 6.6.1] the image of $\mathcal{M}^{\text{red}}(p_0, H_\infty)$ under the evaluation map is contained in the union of images of finitely many compact codimension two smooth manifolds for a generic $J \in \mathcal{J}$ because the irreducible components intersecting W_∞ are Fredholm regular and the irreducible components contained in $\overline{W} \setminus W_\infty$ are controlled by semipositivity. In particular $\overline{W} \setminus \text{ev}(\mathcal{M}^{\text{red}}(p_0, H_\infty))$ is open, dense and connected. Moreover the restriction of the evaluation map

$$\text{ev}: \overline{\mathcal{M}}(p_0, H_\infty) \setminus \text{ev}^{-1}(\text{ev}(\mathcal{M}^{\text{red}}(p_0, H_\infty))) \rightarrow \overline{W} \setminus \text{ev}(\mathcal{M}^{\text{red}}(p_0, H_\infty))$$

is proper by Gromov compactness, and therefore its degree is well defined. Then Lemma 3.2 can be rephrased as follows.

Lemma 4.1. *If (W, ω) is semipositive and $y \in \overline{W} \setminus \text{ev}(\mathcal{M}^{\text{red}}(p_0, H_\infty))$ is a regular value of ev , then*

$$\sum_{x \in \text{ev}^{-1}(y)} \text{sign}(d_x \text{ev}) = 1.$$

In particular, $\text{ev}: \overline{\mathcal{M}}_z(p_0, H_\infty) \rightarrow \overline{W}$ is surjective.

If we apply the argument of Lemma 3.4 to a one-dimensional submanifold of W we obtain the following result.

Lemma 4.2. *If (W, ω) is a semipositive symplectic filling of $(\mathbb{R}\mathbb{P}^{2n-1}, \xi)$, then \overline{W} is simply connected.*

Proof. First we prove that the map

$$(5) \quad \text{ev}_* : \pi_1(\mathcal{M}_z(p_0, H_\infty) \setminus \text{ev}^{-1}(H_\infty)) \rightarrow \pi_1(\overline{W})$$

is surjective. We choose a base point b for $\pi_1(\overline{W})$ in the neighborhood U of Lemma 3.2 and a base point \tilde{b} for $\pi_1(\mathcal{M}_z(p_0, H_\infty))$ such that $\text{ev}(\tilde{b}) = b$. By a codimension argument we can represent every element of $\pi_1(\overline{W})$ by a smooth embedding

$$\gamma : S^1 \hookrightarrow \overline{W} \setminus (W_\infty \cup \text{ev}(\mathcal{M}^{\text{red}}(p_0, H_\infty)))$$

which is transverse to the evaluation map. Using the fact that ev is a diffeomorphism of U onto its image and arguing as in point (i) of the proof of [4, Lemma 2.3] we obtain a loop $\Gamma : S^1 \rightarrow \mathcal{M}_z(p_0, H_\infty)$ such that $\Gamma(1) = \tilde{b}$ and $\text{ev}_*([\Gamma]) = [\gamma]$ in $\pi_1(\overline{W})$. This proves that the map (5) is surjective.

The retraction of $\mathcal{M}_z(p_0, H_\infty) \setminus \text{ev}^{-1}(H_\infty)$ onto $\text{ev}^{-1}(p_0)$ shows that the map (5) is trivial, since it factors through $\pi_1(p_0)$. This implies that $\pi_1(\overline{W})$ is trivial. \square

Combining this with the argument found in [2, Section 6.2] we obtain the main result of this section.

Theorem 4.3. *Any semipositive symplectic filling of $(\mathbb{R}\mathbb{P}^{2n-1}, \xi)$ is simply connected.*

Proof. Let (W, ω) be a semipositive symplectic filling of $(\mathbb{R}\mathbb{P}^{2n-1}, \xi)$. By Lemma 4.2 the compactification \overline{W} is simply connected. The Seifert–van Kampen theorem implies then that the map $\pi_1(\partial W) \rightarrow \pi_1(W)$ induced by the inclusion $\iota : \partial W \hookrightarrow W$ is surjective because $W_\infty \cong \mathbb{C}\mathbb{P}^{n-1}$ is simply connected. In particular, $\pi_1(W)$ is either trivial or isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

In the latter case ι induces an isomorphism between the fundamental groups, and thus

$$\iota^* : H^1(W; \mathbb{Z}/2\mathbb{Z}) \rightarrow H^1(\partial W; \mathbb{Z}/2\mathbb{Z})$$

is also an isomorphism. Let $\alpha \in H^1(W; \mathbb{Z}/2\mathbb{Z})$ be the nontrivial element. Then $\iota^*\alpha \in H^1(\partial W; \mathbb{Z}/2\mathbb{Z})$ is also nontrivial and, since $H^*(\mathbb{R}\mathbb{P}^{2n-1}; \mathbb{Z}/2\mathbb{Z})$ is generated as an algebra by the nontrivial element of degree one, $(\iota^*\alpha)^{2n-1}$ is the nontrivial element of $H^{2n-1}(\partial W; \mathbb{Z}/2\mathbb{Z})$.

By the naturality of the cup product $(\iota^*\alpha)^{2n-1} = \iota^*(\alpha^{2n-1})$. However

$$\iota_* : H_{2n-1}(\partial W; \mathbb{Z}/2\mathbb{Z}) \rightarrow H_{2n-1}(W; \mathbb{Z}/2\mathbb{Z})$$

is trivial, and consequently $\iota^* : H_{2n-1}(W; \mathbb{Z}/2\mathbb{Z}) \rightarrow H_{2n-1}(\partial W; \mathbb{Z}/2\mathbb{Z})$ is by duality also trivial because we are working over a field. This contradicts $\iota^*(\alpha^{2n-1}) \neq 0$ and therefore shows that W is simply connected. \square

5. YET ANOTHER PROOF OF THE ELIASHBERG-FLOER-MCDUFF THEOREM

In this section we apply the constructions of this article to the symplectic fillings of the standard contact structure ξ on S^{2n-1} . This will lead to small changes in the meaning of the notation. If (W, ω) is a symplectic filling of (S^{2n-1}, ξ) and we perform symplectic reduction of its boundary, we obtain a closed symplectic manifold $(\overline{W}, \overline{\omega})$ with a codimension two symplectic submanifold $W_\infty \cong \mathbb{C}\mathbb{P}^{n-1}$ whose normal bundle is isomorphic to $\mathcal{O}_{\mathbb{P}^{n-1}}(1)$. We choose an almost complex

structure J on \overline{W} which is integrable near W_∞ and generic elsewhere. Let $p_0 \in W_\infty$ be a point; we denote by $\mathcal{M}(p_0)$ the moduli space of J -holomorphic spheres in \overline{W} that are homotopic to a line in W_∞ and pass through p_0 . If ℓ is a line in W_∞ , then

$$T\overline{W}|_\ell \cong \mathcal{O}_{\mathbb{P}^1}(2) \oplus \underbrace{\mathcal{O}_{\mathbb{P}^1}(1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(1)}_{n-1}.$$

Since $[\ell] \cdot [W_\infty] = 1$ all elements of $\mathcal{M}(p_0)$ are simply covered, and therefore $\mathcal{M}(p_0)$ is a smooth manifold by the analog of Proposition 2.3. Let $\mathcal{M}_z(p_\infty)$ is the moduli space obtained by adding a free marked point to the elements of $\mathcal{M}(p_0)$. A Riemann-Roch calculation gives $\dim \mathcal{M}(p_0) = 2n - 2$ and $\dim \mathcal{M}_z(p_\infty) = 2n$.

Lemma 5.1. *If (W, ω) is symplectically aspherical, then $\mathcal{M}_z(p_\infty)$ is compact.*

Proof. As the algebraic intersection between a line with W_∞ is one, any nodal J -holomorphic curve representing the homology class of a line must have an irreducible component in $\overline{W} \setminus W_\infty \cong W$. \square

Lemma 3.2 still holds with the minimal necessary modifications, and therefore the evaluation map $\text{ev}: \mathcal{M}_z(p_0) \rightarrow \overline{W}$ has degree one.

Lemma 5.2. *If (W, ω) is a symplectically aspherical filling of (S^{2n-1}, ξ) , then $H_*(W; \mathbb{Z}) = 0$ for $* > 0$.*

Proof. The moduli space $\mathcal{M}_z(p_0)$ is an S^2 -bundle over $\mathcal{M}(p_0)$ and $\text{ev}^{-1}(p_0)$ is a section. Let \widetilde{W}_∞ be a J -holomorphic hypersurface of \overline{W} contained in the neighborhood of W_∞ where J is integrable and obtained as the graph of a section of $\mathcal{O}_{\mathbb{P}^{n-1}}(1)$. We choose \widetilde{W}_∞ such that $p_0 \notin \widetilde{W}_\infty$: then $\text{ev}^{-1}(\widetilde{W}_\infty)$ is a section of $\mathcal{M}_z(p_0)$ which is disjoint from $\text{ev}^{-1}(p_0)$. The map

$$(6) \quad \text{ev}_*: H_*(\mathcal{M}_z(p_0) \setminus \text{ev}^{-1}(\widetilde{W}_\infty); \mathbb{Z}) \rightarrow H_*(\overline{W} \setminus \widetilde{W}_\infty; \mathbb{Z}) \cong H_*(W; \mathbb{Z})$$

is surjective by Lemma 3.3. That lemma, strictly speaking, is about homology classes represented by submanifolds, but there are several ways to extend it to general homology classes.

On the other hand $\mathcal{M}_z(p_0) \setminus \text{ev}^{-1}(\widetilde{W}_\infty)$ retracts onto $\text{ev}^{-1}(\widetilde{W}_\infty)$, and therefore the map (6) is trivial for $* > 0$. \square

The proof of Theorem 4.3 works more or less unchanged for fillings of (S^{2n-1}, ξ) , and therefore W is simply connected. Then the h -cobordism theorem implies the following corollary.

Corollary 5.3 (Eliashberg–Floer–McDuff). *If (W, ω) is a symplectically aspherical filling of (S^{2n-1}, ξ) , then W is diffeomorphic to the ball D^{2n} .*

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