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An approach attack to the strong Goldbach's conjecture

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Abstract

This paper is about the famous strong Goldbach's conjecture which states that :

every even integer greater than two is a sum of two primes.

To investigate in the validity of this amazing statement, we present a new approach attack using the contradiction or the reduction ad absurdum. The idea used here can be summarized as follows: based on the well ordering of the set of natural numbers \mathbb{N} , we consider for every even integer $2n \geq 4$, with $n \geq 2$, the finite sequence of natural numbers $S_m(n) = (s_i(n))_{i \in \{1,2,\dots,m\}}$ defined by: $s_i(n) = 2n - p_i$, where p_i is the i -th prime number in the finite strictly ordered sequence of primes

$$P_m := p_1 = 2 < p_2 = 3 < p_3 = 5 < \dots < p_m$$

where $m = \pi(2n)$ denotes the number of primes p such that $p < 2n$. We prove in first that, if the sequence $S_m(n) = (s_i(n))_{i \in \{1,2,\dots,m\}}$ do not contains a prime number then the largest terms of the sequence exceeds the number $2n$ and this is an absurdity because $S_m(n)$ is strictly decreasing sequence of natural numbers from $2n - 2$ to 1, i.e,

$$s_1(n) = 2n - 2 > s_2(n) > \dots > s_m(n) \geq 1$$

Consequently, for any natural number $n \geq 2$, there exists at least one prime number $s_r(n) = 2n - p_r$ belonging to the sequence $S_m(n)$, and we obtain then the expected result $2n = s_r(n) + p_r$, where p_r and $s_r(n)$ are primes from the set P_m .

Key Words: Well-ordering $(\mathbb{N}, <)$, the product order $(\mathbb{N}^m, <)$, basic concepts and theorems on number theory, the indirect and inductive proofs on natural numbers. **AMS 2010:** 11PXX, 11p32, 11P05, 11B37.

1 A brief history and some results on the conjecture

Historically, from the references [6] and [7], the conjecture dating from 1742 in a letter addressed to Euler from Goldbach expresses the following fact:

Any natural number $n > 5$ is the sum of three primes.

The mathematician Euler replied that this fact is equivalent to the following statement:

Every even integer $2n \geq 4$ is the sum of two primes.

Since then, three major approaches attack to this famous conjecture emerged : "asymptotic study", "almost primes study" and finally "basis".

The first result [4], obtained in the asymptotic case is due to Hardy and Littlewood in 1923 under the consideration of Riemann hypothesis. In 1937, Vinogradov showed in [12] the same result, without using this assumption.

Theorem 1 (*asymptotic theorem*). *There exists a natural number n_0 such that every odd number $n \geq n_0$ is the sum of three primes.*

A natural number $n = \prod_{i=1}^r p_i^{e_i}$ (where each p_i is a prime) is called a k -almost prime when $\sum_{i=1}^r e_i = k$; the set of k -almost primes is denoted by P_k . The approach via almost-primes consists in showing that there exist $h, k \geq 1$

such that every sufficiently large even integer is in the set $P_h + P_k$ of sums of integers of P_h and P_k . The first result in this line of study [2], was obtained by Brun in 1919 by showing that: every sufficiently large even number is in $P_9 + P_9$. In [8], Rényi proved in 1947 that there exists an integer $k \geq 1$ such that every sufficiently large even integer is in $P_1 + P_7$. In 1950, Selberg further improved the result in [10], by showing that every sufficiently large even integers is in $P_2 + P_3$. The best result in this direction is due to Chen [3] (announcement of results in 1966, proofs in detail in 1973 and 1978) proving that:

Every sufficiently large even integer may be written as $2n = p + m$, where p is a prime and $m \in P_2$.

The "basis" approach began in 1930 with the theorem of Schnirelmann [9], proving that:

There exists a positive integer S , such that every sufficiently large even integer is the sum of at most S primes.

The best known result currently stems from the proof of the weak Goldbach conjecture by Harald Helfgott[13], which directly implies that every even number $n \geq 4$ is the sum of at most 4 primes.

For further history and progress in the Goldbach problem see, for example, the paper [6] and the book [7] from respectively R. D. James and P. Ribenboim and the paper "Goldbach's conjecture" from the wikipedia the free encyclopedia [14].

The paper is organised as the follows:

The first section is devoted to giving a brief history on the Goldbach's conjecture and the idea used in this paper to proving the conjecture. We give in the second section, the theoretical elements from the number theory essential to the paper. The third section, is the main section of the paper, where the sequences $S_m(n)$ of natural numbers are given, and we prove that these sequences contains a prime numbers for each natural number $n \geq 1$. And finally, in the last subsection 3.2, we deduce the validity of the strong Goldbach's conjecture

1.1 The idea used in this paper to prove the conjecture

To prove the conjecture, we consider for every even natural number $2n \geq 4$, with $n \geq 2$, the finite sequence of natural numbers $S_m(n) = (s_i(n))_{i \in \{1,2,\dots,m\}}$

defined by $s_i(n) = 2n - p_i$, where p_i is the i -th prime number in the finite strictly ordered sequence of primes

$$P_m := p_1 = 2 < p_2 = 3 < p_3 = 5 < \dots < p_m$$

where $m = \pi(2n)$ denotes the number of primes p such that $p < 2n$. To confirm that, for each natural number $n \geq 2$, there exists at least one prime number $s_r(n) = 2n - p_r$ belonging to the sequence $S_m(n)$, we prove by contradiction that, if each term the sequence $S_m(n)$ is a composite number, then it is impossible to write all the terms of $S_m(n)$ between the numbers 1 and $2n - 2$, and this contradict the fact that we have:

$$2n > s_1(n) = 2n - p_1 = 2n - 2 > s_2(n) = 2n - p_2 = 2n - 3 > \dots > s_i(n) = 2n - p_i > s_{i+1}(n) = 2n - p_{i+1} > \dots > s_m(n) = 2n - p_m \geq 1.$$

i.e, all the terms of sequence $S_m(n)$ are normaly between the natural numbers 1 and $2n$.

Consequently, the sequence $S_m(n)$ must contains at least one prime $s_r(n) = 2n - p_r$ of P_m . We obtain then the long-awaited result $2n = s_r(n) + p_r$, where p_r is the r -th prime number of the sequence P_m ; This result will confirms the validity of the Goldbach's statement. It is noted here that any solution on primes p, q of the equation $p + q = 2n$, for a given natural number $n \geq 2$, exists if and only if $p \in P_m$ and $q = 2n - p \in S_m(n) \cap P_m$; and the smallest of these numbers, suppose that is p , must be between 2 and n , i.e., $2 \leq p \leq n$ and the other q must be in $n \leq q < 2n$. For this reason, the sequence $S_m(n)$ of natural numbers, as defined above, are considered in this paper for proving the conjecture.

2 Preliminary and theoretical elements essential to the paper

The set of natural numbers $\mathbb{N} := 1, 2, \dots, n, \dots$, is well ordered using the usual ordering relation denoted by \leq , where any subset of \mathbb{N} contains a least element. Another way to see the well-ordering of \mathbb{N} is that any natural number n can be reached in finite counting steps by ascent (adding 1) or descent (subtracting 1) from any other natural number m ; there isn't an infinite descent on natural numbers. This significant characteristic property of the set

of natural numbers \mathbb{N} , is the key of almost results of properties of natural numbers. The concept of well-ordering is of fundamental importance in view of the mathematical induction to proving, in two steps only, the validity of a property $H(n)$ depending on natural number n . For the natural numbers a, b , we say a divides b , if there is a natural number q such that $b = aq$. In this case, we also say that b is divisible by a , or that a is divisor of b , or that a is a factor of b , or that b is a multiple of a . If a is not a divisor of b , then we write $a \nmid b$. A natural number $p > 1$ is called prime if it is not divisible by any natural number other than 1 and p . Another way of saying this is that a natural number $p > 1$ is a prime if it cannot be written as the product $p = t_1 t_2$ of two smaller natural numbers t_1, t_2 not equal to 1. A natural number $b > 1$ that is not a prime is called composite. The number 1 is considered neither prime nor composite because the factors of 1 are redundant $1 = 1 \times 1 = 1 \times 1 \times \dots \times 1$. We shall denote by

$$p_1 = 2 < p_2 = 3 < p_3 = 5 < p_4 = 7 < \dots < p_i < \dots$$

the infinite increasing sequence of primes, where p_i is the i -th prime in this sequence. Euclid's theorem ensures that there are infinitely many primes, without knowing their pattern and indication of how to determine the i -th prime number. There is no regularity in the distribution of these primes on the chain (\mathbb{N}, \leq) ; in certain situation they are twins, i.e., there exists a positive integer k such $p_{k+1} = p_k + 2$, like $p_2 = 3$ and $p_3 = 5$, $p_5 = 11$ and $p_6 = 13$ (it is not known today whether there are infinitely many twin primes), while at the same time, for any integer $k \geq 2$, the sequence of successive $k - 1$ natural numbers $k! + 2, k! + 3, k! + 4, \dots, k! + k$, are all composite, for the simple reason that, any term $k! + t$, for $2 \leq t \leq k$, is divisible by t .

The fundamental theorem of arithmetics shows that any natural number $n > 1$ can be written as the product of primes uniquely up the order. For the natural number $n \geq 2$, we denote by $\pi(n)$ the number of primes $p \leq n$, where $\pi(n)$ is called also the prime counting function, for example $\pi(4) = 2$, $\pi(5) = 3, \dots$ etc.). The fundamental theorem of primes (Tcheybecheff gave an empirical estimation around 1850, Hadamard and de Vallée-Poussin theoretical proof at the end of 19 th century) shows that, for any large natural number n , we have $\pi(n) \sim \frac{n}{\ln n}$ and then $p_n \sim n \ln n$ where \ln denotes the natural logarithm of base $e = 2, 71, \dots$. The Bertrand's postulate (1845) provides that between any natural number $n \geq 2$ and its double $2n$ there exists

at least one prime. Equivalently, this may be stated as $\pi(2n) - \pi(n) \geq 1$, for $n \geq 2$, or also in compact form: $p_{n+1} < 2p_n$ for $n \geq 1$. Recall that the notation $f(x) \sim h(x)$ for a positive real valued continuous functions $f(x), h(x)$ means that $\lim_{x \rightarrow \infty} \frac{f(x)}{h(x)} = c$, with a constant positif c , and in this case $f(x), h(x)$ are said to be asymptotically equal as x tends to infinity. The following result from [1] is useful for this paper to estimate some results for a large natural number n : The prime sums of the first n primes, denoted $\Sigma(n) = \sum_{i=1}^m p_i$, is asymptotically equivalent to $\frac{1}{2}n^2 \ln n$. For the notions in elementary number theory see for example [11] and in Algorithmic number theory see [1]. Recall that the set of natural \mathbb{N} is the disjoint union $\mathbb{N} = \{1\} \cup \text{Even} \cup \text{Oddc} \cup \text{OddPrimes}$ where *Even* is the set of the even natural numbers, *Oddc* is the set of odd composite natural numbers and *OddPrimes* is the set of odd primes. Evidently, the number 2 is the only even prime number.

Recall that a sequence of natural number is an ordered set of numbers that most often follows some rule or pattern to determine the next term in the order. formally, a sequence of real numbers is a mapping $u : \mathbb{N} \rightarrow \mathbb{R}$, we typically write it as $n \mapsto u_n$. A subsequence of the sequence u is a map $v = u \circ \theta$ where $\theta : \mathbb{N} \rightarrow \mathbb{N}$ is a strictly increasing mapping. Then $v_n = u_{(\theta(n))}$. For example, the sequence of primes P is a subsequence of the sequence of natural numbers \mathbb{N} , with the respect of the usual ordering \leq . An arithmetic sequence $u_{n+1} = u_n + d$ is in which successive terms differ by the same amount d called the reason of the sequence. Dirichlet's theorem on arithmetic progression asserts that if $\gcd(a, d) = 1$ then $an + d$ is prime for infinitely many natural number n . A series is summation of the terms of a sequence. A finite series are typically written in the following manner: $\sum_{i=1}^n a_i$. Suppose we have the following arithmetic serie $\{a + (a + d) + (a + 2d) + \dots + a + (m - 1)d\}$ then
$$\sum_{k=0}^{m-1} (a + kd) = m(a + (a + (m - 1)d))/2.$$

3 The sequences $S_m(n)$ containing a prime numbers

Before giving the sequences $S_m(n)$ used to confirm the conjecture, we begin by these simple lemmas in view of their usefulness for the rest of the paper.

Lemma 2 Any natural number $b \neq 1$ admits a prime divisor. If b is not prime, then there exists a prime p divisor of b such that $p^2 \leq b$.

Proof. By the definition of prime number, if the natural number $b \neq 1$ admits only the number b as proper divisor, then b is a prime number. If b is not prime, then it can be factored as $b = pq$ such that: $1 < p < b$ and $1 < q < b$ with p is the smallest, under the usual ordering \leq , proper factor of the number b . Since p is the smallest proper factor of b then p must be a prime otherwise, it is not then the least factor of b . As p is the least factor of b then, $p \leq q$. Multiplying both sides by p , we obtain: $pp = p^2 \leq pq = b$.

■

Lemma 3 If the odd integer $t > 1$ is not prime, then it can be factorised only in the form $t = t_1 t_2$ where t_1, t_2 are proper factors $\neq 1$, and each factor t_1 and t_2 , it is also an odd natural number greater or equal to the number 3.

Proof. If the odd integer $t > 1$ is not prime, then it is composite. Let $t = t_1 t_2$ be any possible factorization of t with t_1, t_2 are proper factors $\neq 1$. If one of these factors (or both) is an even integer, then the product $t_1 t_2 = t$ will be also an even integer, but the number t is odd. Then each of the factors t_1 and t_2 must be odd and then greater or equal to the number 3. ■

Let $m \geq 1$ be a natural number. We denote by $I_m = \{1, 2, \dots, m\}$ the set of the first m consecutive natural numbers from 1 to m . For any non empty set E , we denote by E^m the cartesian product of m copies of E . If $m \geq 2$, then $E^m = E^{m-1} \times E$. An element $(a_1, a_2, \dots, a_{m-1}, a_m) = ((a_1, \dots, a_{m-1}), a_m)$ of $E^m = E^{m-1} \times E$ is called m -tuples. It follows that two ordered m -tuples $(a_1, a_2, \dots, a_m), (b_1, b_2, \dots, b_m)$ are equal if and only if $a_1 = b_1, \dots, a_m = b_m$.

Lemma 4 Let $m \geq 1$ be a natural number. Let $a = (a_1 < a_2 \dots < a_m)$, $b = (b_1 < b_2 < \dots < b_m)$ be two strictly an increasing m -sequences of natural numbers such that for each $i \in I_m$ there exists $j \in I_m$ such that $a_i = b_j$. Then we have for each $i \in I_m : a_i = b_i$.

Proof. By induction on $i \in I_m$. For $i = 1$, if $a_1 < b_1$ then $a_1 < b_1 < b_2 \dots < b_m$, and we have $a_1 \notin b = (b_i)_{i \in I_m}$, contradiction. For the same reason, if $b_1 < a_1$, we obtain $b_1 \notin a = (a_i)_{i \in I_m}$ and then it necessary that

$a_1 = b_1$. Suppose that we have for some i with $m > i \geq 2$, $a_1 = b_1$, $a_2 = b_2, \dots, a_i = b_i$. For $i + 1$, if $a_{i+1} < b_{i+1}$, then there exists no b_j for $j \geq i + 1$ such that $a_{i+1} = b_j$ and there exists no b_j for $j \leq i$ because $a_{i+1} > a_i = b_i > a_{i-1} = b_{i-1} > \dots > a_1 = b_1$. The same argument holds if $b_{i+1} < a_{i+1}$, then it is necessary that we have $a_{i+1} = b_{i+1}$. Consequently, for each $i \in I_m : a_i = b_i$. ■

Note that the condition " $a_i = b_j$ " for some $i, j \in I_m$ ", in the lemma 3 above, is necessary and sufficient condition. It is necessary, because we can exhibit two increasing m -sequences

$$a = (a_1 < a_2 \dots < a_m), b = (b_1 < b_2 < \dots < b_m)$$

with $\sum_{i=1}^m a_i = \sum_{i=1}^m b_i$ and $a_i \neq b_i$ for each $i \in I_m$. For example, $a = (2 < 10)$ and $b = (4 < 8)$.

3.0.1 The sequences $S_m(n)$

Let $n \geq 2$ be a natural number. We consider the finite strictly increasing sequence of the first m consecutive prime numbers

$$P_m : p_1 = 2 < p_2 = 3 < p_3 = 5 < \dots < p_i \dots < p_m$$

where $m = \pi(2n)$ denotes the number of primes $p < 2n$. Let $P_m = (p_i)_{i \in I_m}$ denotes the finite successive primes strictly smaller than $2n$. The Bertrand postulat asserts that at least the prime p_m is between n and its double $2n$. For any natural number $n > 2$, we consider the finite sequence $S_m(n) = (s_i(n))_{i \in I_m}$ of natural numbers defined by: $s_i(n) = 2n - p_i$, where p_i is the i -th prime of P_m . Then we have:

$$\begin{aligned} s_1(n) &= 2n - 2, \\ s_2(n) &= 2n - 3, \\ s_3(n) &= 2n - 5, \\ &\cdot \\ &\cdot \\ &\cdot \\ s_i(n) &= 2n - p_i, \\ &\cdot \\ &\cdot \end{aligned}$$

$$s_m(n) = 2n - p_m.$$

Example 5 For $n = 10$, the finite sequence of primes less than 20 are: $p_1 = 2 < p_2 = 3 < p_3 = 5 < p_4 = 7 < p_5 = 11 < p_6 = 13 < p_7 = 17 < p_8 = 19$. Consequently, $\pi(20) = 8$ and then the sequence $S_8(n) = S_8(10) = (s_i(10))_{i \in \{1,2,\dots,8\}}$ is :

$$s_1(10) = 20 - 2 = 18, \quad s_2(10) = 20 - 3 = 17, \quad s_3(10) = 20 - 5 = 15,$$

$$s_4(10) = 20 - 7 = 13, \quad s_5(10) = 20 - 11 = 9, \quad s_6(10) = 20 - 13 = 7, \\ s_7(10) = 20 - 17 = 3, \quad s_8(10) = 20 - 19 = 1.$$

Lemma 6 For the natural number $n \geq 2$ with $m = \pi(2n)$, the finite sequence of natural numbers $S_m(n) = (s_i(n))_{i \in I_m}$ defined by $s_i(n) = 2n - p_i$, with $1 \leq i \leq m$, is strictly decreasing from $s_1(n) = 2n - 2 = \max(S_m(n))$ to $s_m(n) = 2n - p_m = \min(S_m(n)) \geq 1$, and each element $s_i(n)$ of this sequence is an odd natural number except the first term $s_1(n) = 2n - 2$ that is evidently an even number. The last term $s_m(n)$ is equal to 1 only in the case when $p_m = 2n - 1$.

Proof. Let $n > 2$ be a natural number with $m = \pi(2n)$. Since the finite sequence of primes

$$P_m : p_1 = 2 < p_2 = 3 < p_3 = 5 < \dots < p_i \dots < p_m$$

is strictly increasing, and each term $s_i(n)$ is defined by $2n - p_i$, then the sequence $S_m(n)$ is strictly decreasing from $s_1(n)$ to $s_m(n)$. In fact, we have $p_{i+1} > p_i$ and then $s_i(n) = 2n - p_i > s_{i+1}(n) = 2n - p_{i+1}$ for each i with $1 \leq i \leq m - 1$. this shows that we have:

$$s_1(n) = 2n - p_1 = 2n - 2 > s_2(n) = 2n - p_2 = 2n - 3,$$

$$s_2(n) > s_3(n) = 2n - p_3 = 2n - 5,$$

..

..

..

$$s_i(n) = 2n - p_i > s_{i+1}(n) = 2n - p_{i+1} > \dots > s_m(n) = 2n - p_m \geq 1.$$

Since for each i , with $2 \leq i \leq m$, the prime p_i is odd, then the term $s_i(n) = 2n - p_i$ is also an odd natural number. The first term $s_1(n) = 2n - 2$, is the unique even number in the sequence $S_m(n)$. The last term $s_m(n) = 2n - p_m = \min(S_m(n))$ is equal to the number 1 if and only if $p_m = 2n - 1$. In fact, if $p_m = 2n - 1$ then $s_m(n) = 2n - p_m = 2n - (2n - 1) = 1$. In the reverse case, we have $p < 2n$, $\forall p \in P_m$, and then $2n - p > 0 \iff 2n - p \geq 1$; and we have $2n - p = 1$ only in the case when $p = 2n - 1 = p_m$. In the example 4, we have this situation, as $p_8 = 19$ then $s_8(10) = 20 - 19 = 1$. ■

3.1 The smallest m -tuples of the set of the strictly increasing m -sequences of odd composite natural numbers

We denote by $Odd = \{1, 3, 5, \dots\} = \{2k + 1/k \in \mathbb{N} \cup \{0\}\}$ the set of odd natural numbers and by $Oddc$ the set of odd composite natural numbers, i.e.,

$$\begin{aligned} Oddc &= \{a \in \mathbb{N}/a = (2i + 1)(2j + 1) \text{ with } i, j \in \mathbb{N} \text{ and } 1 \leq i \leq j\} \\ &= \{9, 15, 21, 25, \dots\}. \end{aligned}$$

Evidently, we have $Oddc \subset Odd \subset \mathbb{N}$.

Let $\mathbb{N}^m = \mathbb{N} \times \mathbb{N} \times \dots \times \mathbb{N}$ be the cartesian product of m copies of the set of natural numbers \mathbb{N} . The usual ordering \leq on \mathbb{N} can be extended to the ordering \preceq on \mathbb{N}^m , called the product order, in the following natural way:

For every two m -tuples $a = (a_1, a_2, \dots, a_m)$, $b = (b_1, b_2, \dots, b_m)$ of \mathbb{N}^m , we define

$$a = (a_1, a_2, \dots, a_m) \preceq b = (b_1, b_2, \dots, b_m) \text{ if } a_i \leq b_i \text{ for each } i \in I_m.$$

The strict ordering on \mathbb{N}^m noted \prec can be also defined by:

$$a = (a_1, a_2, \dots, a_m) \prec b = (b_1, b_2, \dots, b_m) \text{ if } a_i < b_i \text{ for each } i \in I_m.$$

Evidently, the ordering \preceq on \mathbb{N}^m is not a total ordering, for example the sequences $(1, 2, 3)$, $(1, 1, 4)$ of \mathbb{N}^3 are not comparable. But (\mathbb{N}^m, \preceq) contains the least element $(1, 1, \dots, 1)$ (m terms of 1), which satisfies

$$(1, 1, \dots, 1) \preceq a = (a_1, a_2, \dots, a_m) \text{ for every } a \in \mathbb{N}^m$$

Notice that for each integer $m \geq 1$, the set \mathbb{N}^m equipped the operation of addition of m -sequences defined by :

For every two m -tuples $a = (a_1, a_2, \dots, a_m)$, $b = (b_1, b_2, \dots, b_m)$ of \mathbb{N}^m , then $a + b = (a_1 + b_1, a_2 + b_2, \dots, a_m + b_m) \in \mathbb{N}^m$ is a semigroup with the neutral element $0_m = (0, 0, \dots, 0)$ (m -zeros).

An important subset of \mathbb{N}^m , which concern this paper, is the subset of \mathbb{N}^m of the strictly increasing m -tuples (or m -sequences) of m odd composite natural numbers, which we denote, for the brevity, by $Oddci(m)$.

Then, the set $Oddci(m)$ is defined by:

$Oddci(m) =$

$$\left\{ \begin{array}{l} a = (a_1 < a_2 < \dots < a_m) \in \mathbb{N}^m / \\ a_i \text{ is an odd composite natural number, for each } i \in I_m \\ \text{and } a_i < a_{i+1} \text{ for every } 1 \leq i \leq m - 1 \end{array} \right\}$$

Evidently, we have $Oddci(m) \subset (Oddc)^m \subset \mathbb{N}^m$. Since we have $Oddci(m) \subset \mathbb{N}^m$, then the subset $Oddci(m)$ is also partially ordered with the induced ordering \preceq of \mathbb{N}^m . For example, the 4-tuples $a = (9 < 15 < 21 < 25)$, $b = (21 < 35 < 39 < 63) \in Oddci(4)$ with $a \prec b$. In this paper, the phrases "increasing m -sequence" and "increasing m -tuples" have the same significance and are used equally. For further notions of ordering see [5].

Lemma 7 *For any natural number $m \geq 1$, the set $Oddci(m) \subset \mathbb{N}^m$ contains a smallest element..*

Proof. Based on the well ordering of the set \mathbb{N} , the smallest element

$$C_m = \min Oddci(m) = (c_1 < c_2 < \dots < c_{m-1} < c_m)$$

of the set $Oddci(m)$ is equal to the m -tuples of the first m consecutif odd composite natural numbers, which can be computed recursively from $i = 1$ to $i = m$, in the following manner:

For $i = 1$, the smallest element of the set $Oddc$ is the number

$$c_1 = \min \{Oddc\} = \min \{9, 15, 21, 25, \dots\} = 9, \text{ and then } C_1 = (c_1) = (9).$$

For $i = 2$, The second element c_2 of set $Oddc$ is the number

$$c_2 = 15 = \min \{Oddc - \{9\}\},$$

$$\text{consequently, we have } C_2 = ((C_1), c_2) = ((C_1), 15) = ((c_1), 15) = (9, 15).$$

Suppose that for $i = m - 1$, we have:

$C_{m-1} = \min \text{Oddci}(m-1) = (c_1, c_2, c_3, \dots, c_{m-1}) = (9, 15, 21, \dots, c_{m-1})$,
with c_{m-1} is the $(m-1)$ -th odd composite natural number of the set Oddc
then, the m -th odd composite natural number of the set Oddc is
 $c_m = \min \{\text{Oddc} - \{c_1, c_2, c_3, \dots, c_{m-1}\}\}$, and consequently we obtain:

$$C_m = (C_{m-1}, c_m) = ((c_1, c_2, c_3, \dots, c_{m-1}), c_m) = (c_1, c_2, c_3, \dots, c_{m-1}, c_m)$$

■

Example 8 For example, for $m = 7$ we have:

$$\min \text{Oddci}(7) = (9 < 15 < 21 < 25 < 27 < 33 < 35)$$

and this element of \mathbb{N}^7 is the smallest element among the set of all the 7-tuples
of odd composite natural numbers written in increasing order.

Lemma 9 Let $(c_1, c_2, c_3, \dots, c_{m-1}, c_m)$ be the smallest element of $\text{Oddci}(m)$.
If m odd composite natural numbers $\{a_1, a_2, \dots, a_m\}$ are such that $a_1 < a_2 < \dots < a_m$
with $a_1 \geq c_1$ then $a_i \geq c_i$, for each i with $1 \leq i \leq m$. And
consequently we have $\sum_{i=1}^m a_i \geq \sum_{i=1}^m c_i$.

Proof. The proof is by induction on $i \in I_m$. For $i = 1$ we have $a_1 \geq c_1$, and
then the hypothesis is given true for $i = 1$. For $i = 2$, we have $a_2 > a_1 \geq c_1$
then $a_2 \geq c_2$ because c_2 is the first odd composite after c_1 i.e., c_2 is the
consecutive odd composite of c_1 , and then the hypothesis is true for $i = 2$.
Suppose that for some $i \geq 2$, we have $a_i \geq c_i$. Since $a_{i+1} > a_i$ and $a_i \geq c_i$
then $a_{i+1} > a_i \geq c_i$, and since the first odd composite after a_i is a_{i+1} then
 $a_{i+1} \geq c_{i+1}$. We conclude by induction on i that we have, for each $i \in I_m$,
 $a_i \geq c_i$. Consequently, we have $\sum_{i=1}^m a_i \geq \sum_{i=1}^m c_i$. ■

Unfortunately, we haven't an explicit formula which determines the m -th
odd composite natural number, but we can proceed as follows to enumerate
them in increasing order. Recall that, the odd composite numbers are the
odd positive integers of the form

$$a(i, j) = (2i + 1)(2j + 1) = 4ij + 2(i + j) + 1$$

where $(i, j) \in \mathbb{N}^2$ with $1 \leq i \leq j$ (because a is a symmetric function on i and
 j).

Equivalently, an odd composite numbers are the integers of the form

$$a(p, s) = 2(2p + s) + 1$$

where $p = ij$ and $s = i + j$ with $(i, j) \in \mathbb{N}^2$ and $1 \leq i \leq j$.

Then, we can enumerate these numbers according the increasing values of the product p and then the increasing correspondant sum s . In other words for each p , from $p = 1, 2, 3, \dots$, we compute the couple (i, j) , solution for the equation $p = ij$, with the condition $1 \leq i \leq j$, and then the correspondant sum $s = i + j$, and finally the resulting odd composite number $a(p, s)$.

For example,

For $p = ij = 1$ the unique solution is $(i, j) = (1, 1)$ and then $s = 2$ which gives $a_1 = 2(2 \times 1 + 2) + 1 = 9$.

For $p = ij = 2$ the unique solution is $(i, j) = (1, 2)$ and then $s = 3$ which gives $a_2 = 2(2 \times 2 + 3) + 1 = 15$.

For $p = ij = 3$ the unique solution is $(i, j) = (1, 3)$ and then $s = 4$ which gives $a_3 = 2(2 \times 3 + 4) + 1 = 21$.

For $p = ij = 4$ the first solution is $(i, j) = (2, 2)$ and then $s = 4$ which gives $a_4 = 2(2 \times 4 + 4) + 1 = 25$.

and the second solution is $(i, j) = (1, 4)$ and then $s = 5$ which gives $a_5 = 2(2 \times 4 + 5) + 1 = 27, \dots$, and so on.

Since we haven't a closed formula which determines the m -th odd composite natural number in function of m , we cannot also write, in function of the parametre m , all the terms of the smallest m -tuple

$$C_m = (9, 15, 21, 25, \dots, c_m) = (3 \times 3, 3 \times 5, 3 \times 7, 5 \times 5, \dots, c_m),$$

but we can assert in the following lemma 7, for each $m \geq 1$, the smallest element C_m is between the m -tuple

$$T_{2m} = (2 \times 3, 2 \times 5, \dots, 2 \times (2m + 1)) = 2 \times (3, 5, \dots, (2m + 1)) \in \mathbb{N}^m$$

and the m -tuple

$$T_{3m} = (3 \times 3, 3 \times 5, \dots, 3 \times (2m + 1)) = 3 \times (3, 5, \dots, (2m + 1)) \in \mathbb{N}^m$$

i.e, $T_{2m} \prec C_m \prec T_{3m}$.

Or equivalently, in terms of finite serie, $\sum T_{2m} < \sum C_m < \sum T_{3m}$ and we have T_{3m} is an element of $Oddci(m)$ but T_{2m} is not in $Oddci(m)$. Evidently, we have $T_{2m} \prec T_{3m}$.

It is noted here that the smallest element C of the set $Oddci$ (without a restricted m) contains in increasing order: each element of the sequence $T_3 = (3 \times (2k + 1))_{k \geq 1}$, and also each element from the sequence $T_5 = (5 \times (2k + 1))_{k \geq 2}$ which are not multiple of the number 3 and each element from the sequence $T_7 = (7 \times (2k + 1))_{k \geq 3}$ which are not multiple of the number 3 or the number 5, ... and each element from the sequence $T_{p_i} = (p_i \times (2k + 1))_{2k+1 \geq p_i}$, where p_i is the i -th prime, which are not multiple of the primes p_1 or p_2 or ...or p_{i-1} ... and so on. This process of the construction of the smallest element C shows that each element of the sequence $T_3 = (3 \times (2k + 1))_{k \geq 1}$ is also an element of C , i.e, T_3 is a subsequence of the sequence C and the first element of C is the first element of the sequence T_3 . And since the difference between two consecutive elements of T_3 is exactly 6, then each other element of C , taken from the others sequence T_5 or T_7 or...or T_{p_i} ..., is inserted in increasing order between two appropriate consecutif elements of T_3 . Then the gaps $d_i = c_{i+1} - c_i$, for each $i \in \mathbb{N}$, between two consecutive odd composite natural numbers cannot exceed the number 6, and we have $d_i \in \{2, 4, 6\}$ because the difference between any two odd natural numbers is even, then $2 \leq d_i \leq 6$.

For example, the first four gaps are:

$$d_1 = c_2 - c_1 = 15 - 9 = 6, d_2 = c_3 - c_2 = 21 - 15 = 6, d_3 = c_4 - c_3 = 25 - 21 = 4, d_4 = c_5 - c_4 = 27 - 25 = 2.$$

In general manner, we have: $c_1 = 9$, and $c_{i+1} = c_i + d_i$, with $d_i \in \{2, 4, 6\}$, for each $i \in \mathbb{N}$. The first m consecutive odd composite natural numbers satisfy the following pseudo-recurrence equations:

$$c_1 = 9,$$

$$c_2 = c_1 + d_1 = 9 + d_1, \text{ with } d_1 \in \{2, 4, 6\}, \text{ but of cours } d_1 = 6 \text{ in this case.}$$

$$c_3 = c_2 + d_2 = 9 + d_1 + d_2, \text{ with } d_1, d_2 \in \{2, 4, 6\}, \text{ but we have exactly } d_1 = 6 \text{ and } d_2 = 6 \text{ in this case.}$$

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$$c_m = c_{m-1} + d_{m-1} = c_{m-2} + d_{m-2} + d_{m-1} = 9 + d_1 + d_2 + \dots + d_{m-1},$$

with $d_i \in \{2, 4, 6\}$ for each $i \in \{1, 2, \dots, m - 1\}$.

We conclude that: for each $m \in \mathbb{N}$, the m -th odd composite natural c_m is limited between the natural numbers

$$N_m = 9 + (2 + 2 + \dots + 2) = 9 + 2(m - 1)$$

(the sum of 2 is taken $(m - 1)$ times),
and the natural number

$$N'_m = 9 + (6 + 6 + \dots + 6) = 9 + 6(m - 1)$$

i.e.,

$$9 + 2(m - 1) \leq c_m \leq 9 + 6(m - 1) \text{ for } m \in \mathbb{N}$$

Since the first four gaps are exactly $d_1 = d_2 = 6$, $d_3 = 4$ and $d_4 = 2$,
we obtain then the strict inequality : for any natural number $m \geq 4$.

$$9 + 2(m - 1) < c_m < 9 + 6(m - 1)$$

The above remarks, shows that we have the following result.

Lemma 10 *For every natural number $m \geq 4$, the smallest element $C_m = (c_1 < c_2 < \dots < c_m)$ of the subset $Oddci(m)$, is limited between $\beta_m \prec C_m \prec T_{3m}$, where $\beta_m = (9 + 2i)_{i \in \{0,1,\dots,m-1\}}$ and $T_{3m} = (3 \times (2i + 1))_{i \in \{1,\dots,m\}}$. Or equivalently, the sum $\sum_{i=1}^m c_i$ of the first m consecutif odd composite natural numbers is limited between*

$$m(m + 8) < \sum_{i=1}^m c_i < 3m(m + 2).$$

Proof. The sequence $C_m = (c_1 < c_2 < \dots < c_m)$ of the first m consecutif odd composite natural numbers can be defined recursively by:
 $c_1 = 9$ and $c_{i+1} = c_i + d_i$, for each $i \in \{1, 2, \dots, m - 1\}$, with $d_i \in \{2, 4, 6\}$.

Then:

$$\sum_{i=1}^m c_i = c_1 + c_2 + \dots + c_m = 9 + (9 + d_1) + \dots + (9 + d_1 + \dots + d_{m-1}).$$

Since for each $1 \leq i \leq m - 1$, we have $d_i \in \{2, 4, 6\}$ then $\sum_{i=1}^m c_i$ is bounded

by $S_1 < \sum_{i=1}^m c_i < S_2$ with:

$$\begin{aligned} S_1 &= 9 + (9 + 2) + (9 + 2 \times 2) \dots + (9 + 2(m - 1)) \\ &= \sum_{i=0}^{m-1} (9 + 2i) = m(m + 8) \text{ is a lower integer bound of } \sum_{i=1}^m c_i, \end{aligned}$$

and

$$S_2 = 9 + (9 + 6) + (9 + 2 \times 6) \dots + (9 + 6(m - 1)) = \sum_{i=1}^m 3 \times (2i + 1)$$

$= 3m(m+2)$ is a great integer bound of $\sum_{i=1}^m c_i$.

Notice that the above result is equivalent to say that the sequence C_m is limited between the sequence $\beta_m = (9 + 2i)_{i \in \{0,1,\dots,m-1\}}$ and the sequence $T_{3m} = (3 \times (2i + 1))_{i \in \{1,\dots,m\}}$, i.e., $\beta_m \prec C_m \prec T_{3m}$ in the product order (\mathbb{N}^m, \prec) . ■

Example 11 For example, the first ten successive odd composite numbers

$$9 \xrightarrow{+6} 15 \xrightarrow{+6} 21 \xrightarrow{+4} 25 \xrightarrow{+2} 27 \xrightarrow{+6} 33 \xrightarrow{+6} 35 \xrightarrow{+6} 39 \xrightarrow{+6} 45 \xrightarrow{+4} 49$$

are limited in the top by the first ten odd composite numbers multiple of 3 :

$$9 \xrightarrow{+6} 15 \xrightarrow{+6} 21 \xrightarrow{+6} 27 \xrightarrow{+6} 33 \xrightarrow{+6} 35 \xrightarrow{+6} 39 \xrightarrow{+6} 45 \xrightarrow{6} 51 \xrightarrow{+6} 57$$

or equivalently in the product order (\mathbb{N}^{10}, \prec) , we have:

$$(9, 15, 21, 25, 27, 33, 35, 39, 45, 49) \prec (9, 15, 21, 27, 33, 35, 39, 45, 51, 57)$$

and this example shows that the sequence of the multiple of 3 grow faster than the sequence the odd composite natural numbers since the sequence of the multiple of 3 is a sub-sequence of the sequence of odd composite natural numbers.

The following lemmas gives some lower bounds of the m -th odd composite natural number c_m .

Lemma 12 The m -th odd composite natural number c_m is greater than $2m$ with $m = \pi(2n)$.

Proof. The proof is by induction on the natural $m \geq 1$. For $m = 1$, we have and $c_1 = 9 > 2 \times 1$, consequently, the hypothesis is true for $m = 1$. Suppose that for some natural $m \geq 1$, we have $c_m > 2m$ with $m = \pi(2n)$. For $m + 1$, we have the gabs $d = c_{m+1} - c_m \in \{2, 4, 6\}$, and then $c_{m+1} - c_m \geq 2$, or equivalently $c_{m+1} \geq c_m + 2$. Since $c_m > 2m$ by the hypothesis for $m \geq 1$, we obtain then $c_{m+1} \geq c_m + 2 > 2m + 2 = 2(m + 1)$, consequently $c_{m+1} > 2(m + 1)$. We conclude by induction on m that we have $c_m > 2m$, for each $m \geq 1$. ■

Even better, we have:

Lemma 13 For each natural $n \geq 1$, we have $c_m > 2n$ with $m = \pi(2n)$ and c_m is the m -th odd composite number.

Proof. The proof is by induction on the natural $n \geq 1$. For $n = 1$, we have $m = \pi(2 \times 1) = 1$ and $c_1 = 9 > 2 \times 1 = 2$, consequently, the hypothesis is true for $n = 1$. For $n = 2$, we have $m = \pi(4) = 2$. And the assertion $c_2 = 15 > 2 \times 2 = 4$ is also true for $n = 2$. Suppose that for some natural $n \geq 1$, we have $c_m > 2n$ with $m = \pi(2n)$. For $n + 1$, we have $m' = \pi(2(n + 1)) = m + 1$ in the case when $p_{m+1} = 2n + 1$, and $\pi(2(n + 1)) = m$ in the case when $p_{m+1} \neq 2n + 1$. In the case $p_{m+1} = 2n + 1$ then $c_{m+1} > 2n + 1$ since $c_m > 2n$ by the hypothesis for n . And in the case $p_{m+1} \neq 2n + 1$, we have also $c_{m+1} > 2n + 1$ since c_m is at least equal to $2n + 1$. In either cases, we have $c_{m+1} > 2(n + 1)$ since $d = c_{m+1} - c_m \in \{2, 4, 6\}$. We conclude by induction on n that we have, for each $n \geq 1$, $c_m > 2n$ with $m = \pi(2n)$. ■

Given the importance of above lemma for the paper, we give an other proof by contradiction and we confirm the validity of the lemma for a large n .

Proof. The proof is by contradiction. Suppose that, in the contrary, we have for certain $n \geq 1$, $c_m \leq 2n$. Then, the interval $\{1, 2, \dots, 2n\}$ contains m primes + m composites + n even numbers. Since the difference between two consecutif odd numbers is 2. And last odd number in the interval is $2n - 1$, which it is a prime or an odd composite number, in either cases we have then : $1 + 2m = 2n - 1 \implies m = n - 1$. Then we have $(n - 1) + (n - 1) + n = 2n$, and this is an absurdity. Consequently, we have for each $n \geq 1$, $c_m > 2n$.. ■

For a large n , we can confirm also that we have the following:

The fundamental theorem of primes $\lim n \log n / p_n = 1$ is equivalent to: the n -th prime p_n is about $n \log n$, for large n . And this may be interpreted as asserting that, if there are n primes in an interval, then the length of the interval is roughly $n \log n$. Since we have m primes, with $m = \pi(2n)$, then $2n \sim m \log m$. Suppose that $c_m \leq 2n$, then we have $m + m = n$ (m primes + m composites = n odd numbers between 1 and $2n$). Then $2m = n \implies 2m = (m \log m) / 2$ and this implies that $\log m = 4$ which is absurd for a large n .

The following lemma asserts that, if each terms of the sequence $S_m(n) = (s_i(n))_{i \in I_m}$, is a composite natural number, then the largest term of the sequence will exceed the number $2n$, and this will contradict the fact that the greatest element of the sequence $S_m(n)$ is equal to $2n - 2$.

Lemma 14 *If each term of the sequence $S_m(n)$ is a composite natural number then at least the maximal element of this sequence will be exceeds the number $2n$.*

Proof. From the lemma 6, all the terms of sequence $S_m(n)$ are an odd numbers except the first $s_1(n) = (2n - 2)$. Since each term $s_i(n) \neq 1$ it is supposed to be a composite number, we consider then the possible factorization of each odd term $s_i(n)$ as the following form:

$$s_i(n) = p'_i(n) q_i(n)$$

such that $p'_i(n)$ is the smallest prime number dividing $s_i(n)$, (the existence of this prime factor is assured by the fundamental theorem of arithmetic or it suffices to see the Lemma 2, and $q_i(n)$ is the other proper factor). According to the lemmas 3 and 6, the factors $p'_i(n)$, $q_i(n)$ are odd ≥ 3 , for all $i \in \{2, \dots, m\}$ in the case when $p_m \neq 2n - 1$, and for all $i \in \{2, \dots, m - 1\}$ in the case when $p_m = 2n - 1$ (because in this case, we have $s_m = 2n - p_m = 2n - (2n - 1) = 1$). The term $s_1(n) = 2n - 2 = 2(n - 1)$ is the only natural even number of the sequence $S_m(n)$, which it is evidently a composite number.

Since $s_i(n) = 2n - p_i > 2n - p_{i+1} = s_{i+1}(n)$,

we have also,

$$s_i(n) = p'_i(n) q_i(n) > p'_{i+1}(n) q_{i+1}(n) = s_{i+1}(n), \text{ for each } i \in \{1, \dots, m - 1\}.$$

Consequently,

$$s_1(n) = p'_1(n) q_1(n) > s_2(n) = p'_2(n) q_2(n) > \dots > s_m(n) = p'_m(n) q_m(n) \geq 1.$$

Two cases will be considered, depending on whether $p_m \neq 2n - 1$ (in this case $s_m(n) = 2n - p_m \geq 3$ it is also a composite number) or, $p_m = 2n - 1$ (in this case $s_m(n) = 2n - p_m = 1$ it is neither prime nor a composite number).

1st case: if $p_m \neq 2n - 1$, then each term $s_i(n)$ is to be considered a composite number, and we have:

$$s_1(n) = 2n - p_1 = p'_1(n) q_1(n) \text{ with } p'_1(n) \geq 2 \text{ and } q_1(n) \geq 3, \text{ then } s_1(n) \geq 9,$$

$$s_2(n) = 2n - p_2 = p'_2(n) q_2(n) \text{ with } p'_2(n) \geq 3 \text{ and } q_2(n) \geq 3, \text{ then } s_2(n) \geq 9,$$

\ddots
 \ddots
 \ddots
 $s_m(n) = 2n - p_m = p'_m(n)q_m(n)$ with $p'_m(n) \geq 3$ and $q_m(n) \geq 3$, then $s_m(n) \geq 9$.

But, since $s_1(n) > s_2(n) > \dots > s_m(n) \geq c_1 = 9$, is strictly decreasing sequence of odd composite numbers, then certainly, $s_{m-1}(n) \geq c_2 = 15$, $s_{m-2}(n) \geq c_3 = 21$, ..., and $s_1(n) \geq c_m$ (the m -th odd composite number).

According with the lemma 13 we have the $c_m > 2n$. Consequently, we have $s_1(n) \geq c_m > 2n$.

For the **second case**, i.e., if $p_m = 2n - 1$ then $s_m(n) = 2n - p_m = 2n - (2n - 1) = 1$, and in this case we have:

$$s_1(n) > s_2(n) > \dots > s_{m-1}(n) \geq 9 = c_1,$$

and consequently, $s_1(n) \geq c_{m-1}$. But we have $c_{m-1} \geq 2(n-1)$ (from the lemma 13) and since c_{m-1} is an odd composite and the prime p_m is equal to $2n - 1$, in this case, certainly $c_{m-1} > 2n + 1$ and then $s_1(n) \geq c_{m-1} > 2n + 1$. Consequently, from the cases 1 and 2 we have $s_1(n) > 2n$, if each term of the sequence $S_m(n)$ is to be considered a composite natural number. ■

3.2 Existence of prime in the sequence $S_m(n) = (s_i(n))_{i \in I_m}$, for any natural number $n \geq 2$ with $m = \pi(2n)$

Theorem 15 For any natural number $n \geq 2$ with $m = \pi(2n)$, the finite sequence of natural numbers $S_m(n) = (s_i(n))_{i \in I_m}$ defined by $s_i(n) = 2n - p_i$, with $1 \leq i \leq m$, contains at least one prime $s_r \in P_m \cap S_m(n)$.

Proof. For each natural number $n \geq 2$, let $S_m(n) = (s_i(n))_{i \in I_m}$ be the finite sequence of natural numbers as defined in the section 3 above with $m = \pi(2n)$. The proof is by contradiction, and so we begin by assuming that the following hypothesis $H(n_0)$ is true for some natural number $n_0 \geq 2$.

The hypothesis $H(n_0)$:

"There exists a natural number $n_0 \geq 2$, such that each term $s_i(n_0) \in S_m(n_0)$, for each $i \in I_m$, is not a prime number".

This is equivalent to say :

"there exists a natural number $n_0 \geq 2$, such that each term $s_i(n_0) \in S_m(n_0)$, for each $i \in I_m$, is a composite number or equal to the natural number 1". Symbolically the hypothesis $H(n_0)$ can be written as
 $\exists (n_0 > 2) \in \mathbb{N}, \forall i \in \{1, 2, 3, \dots, m\}$: the term $s_i(n_0)$ is not a prime number"
 We recall that, the unique term $s_i(n_0)$ of $S_m(n_0)$, which can be equal to the number 1 is the last term $s_m(n_0) = 2n_0 - p_m$, in the case when $p_m = 2n_0 - 1$ (see Lemma 6). The last term $s_m(n_0) = 2n_0 - p_m$ is the unique term of the sequence $S_m(n_0)$, which is neither prime nor a composite number in the case when $p_m = 2n_0 - 1$, i.e., in the case when $s_m(n_0) = 1$.

To contradict or reject the hypothesis $H(n_0)$ for all $n_0 > 2$, (in symbolic terms this contradiction is written: $\forall (n > 2) \in \mathbb{N}, \exists i \in \{1, 2, 3, \dots, m\}$ such that $s_i(n)$ is a prime number). Using the lemma 13, if the hypothesis $H(n_0)$ is true for some natural number $n_0 \geq 2$, then $s_1(n_0) = \max S_m(n_0) \geq c_m > 2n_0$ and this is an absurdity because $s_1(n_0) = 2n_0 - 2 = \max S_m(n_0) < 2n_0$ and $s_m(n_0) = 2n_0 - p_m \geq 1$. Consequently, it is impossible to write all the elements of sequence $S_m(n_0) = (s_1(n_0) > s_2(n_0) > \dots > s_m(n_0))$ between the numbers 1 and $2n_0 - 2$ if each term of sequence it is assumed to be an odd composite number except the first term, which is evidently an even composite term. Consequently, the refutation of the argument $H(n_0)$ for all $n_0 > 2$.

Then, it is impossible to write each term of sequence $S_m(n) = (s_i(n))_{i \in I_m}$ as an odd composite number between the numbers 1 and $2n_0 - 2$, if each term of this sequence it is assumed to be a composite natural number, and consequently, there must be at least one term

$s_r(n) = 2n - p_r = p'_r(n) q_r(n)$, which is not a composite number, then necessarily that the factor $q_r(n)$ is strictly inferior to the number 3, and because $q_r(n)$ is an odd number, it can not be equal to the number 2 and then we have certainly $q_r(n) = 1$ and then $s_r(n) = 2n - p_r = p'_r(n) q_r(n) = p'_r(n) \times 1$, i.e, the prime $p'_r(n)$ is equal to the term $s_r(n) = 2n - p_r = p'_r(n)$ with $p'_r(n) \in P_m$ (where P_m is the set of primes inferior strictly to the number $2n$).

Therefore, for each $(n \geq 2) \in \mathbb{N}$, the sequence $S_m(n) = (s_i(n))_{i \in I_m}$ contains at least one term $s_r(n)$, with $r \in I_m$, which it is a prime number, i.e, $s_r(n) \in P_m \cap S_m(n)$. ■

Theorem 16 *Every even integer $2n \geq 4$, with $n \geq 2$, is a sum of two primes.*

Proof. If $n = 2$ then, $4 = 2 + 2$. If $n > 2$, we consider the finite sequence of primes $P_m := (p_1 = 2 < p_2 = 3 < \dots < p_m)$ with $m = \pi(2n)$ and let $S_m(n) = (s_i(n))_{i \in I_m}$ be the finite sequence of natural numbers defined by $s_i(n) = 2n - p_i$, where p_i is i -th prime of P_m . From the theorem 15, there exists at least one prime number $s_r(n) \in S_m(n)$. Since we have $s_r(n) = 2n - p_r$, with p_r is the r -th prime number of sequence P_m , we obtain the result $2n = p_r + s_r(n)$. It follows that the Goldbach's conjecture is effectively a theorem of number theory. ■

As a consequence of this result, given an even natural number $2n \geq 4$ with $n \geq 2$, to find the pair of primes numbers (p, s) such that $2n = p + s$, it suffices that the algorithm runs through the finite sequence $S_m(n) = (s_i(n))_{i \in I_m}$, which contains, at least one solution of this equation for each natural number $n \geq 2$.

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