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Exact Moment Representation in Polynomial Optimization

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In memory of Carlo Casolo

Abstract

We investigate the problem of representation of moment sequences by measures in Polynomial Optimization Problems, consisting in finding the infimum f^* of a real polynomial f on a real semialgebraic set S defined by a quadratic module Q . We analyse the exactness of Moment Matrix (MoM) relaxations, dual to the Sum of Squares (SoS) relaxations, which are hierarchies of convex cones introduced by Lasserre to approximate measures and positive polynomials. We show that the dual of the MoM relaxation coincides with the SoS relaxation extended with the real radical of the support of the associated quadratic module Q . We deduce that the real radical of the support of Q is generated by the annihilator of a generic element of the truncated moment cone, for a sufficiently high order of the MoM relaxation. We prove the exactness of MoM relaxation when regularity conditions, known as Boundary Hessian Conditions, hold on the minimizers. This implies that MoM exactness holds generically. We show that MoM exactness holds when the support of Q is zero-dimensional. When the set of minimizers is finite, we describe a MoM relaxation which involves f^* , show its MoM exactness and propose a practical algorithm to achieve MoM exactness. We prove that if the real variety of polar points is finite then the MoM relaxation extended with the polar constraints is exact. Effective numerical computations illustrate this MoM exactness property.

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1 Introduction

Let $f, g_1, \dots, g_s \in \mathbb{R}[X_1, \dots, X_n]$ be polynomials in the indeterminates X_1, \dots, X_n with real coefficients. The goal of Polynomial Optimization is to find:

$$f^* := \inf \left\{ f(x) \in \mathbb{R} \mid x \in \mathbb{R}^n, g_i(x) \geq 0 \text{ for } i = 1, \dots, s \right\}, \quad (1)$$

that is the infimum f^* of the *objective function* f on the *basic semialgebraic set* $S := \{x \in \mathbb{R}^n \mid g_i(x) \geq 0 \text{ for } i = 1, \dots, s\}$. It is a general problem, which appears in many contexts (e.g. real solution of polynomial equations, ...) and with many applications. To cite a few of them: in combinatorics, network optimization design, control, ... See for instance [Las10].

To solve this NP hard problem, Lasserre [Las01] proposed to use two hierarchies of finite dimensional convex cones depending on an order $d \in \mathbb{N}$ and he proved, for Archimedean quadratic modules, the convergence when $d \rightarrow \infty$ of the optima associated to these hierarchies to the minimum f^* of f on S . The first hierarchy replaces non-negative polynomials by Sums of Squares (SoS) and non-negative polynomials on S by polynomials of degree $\leq d$ in the truncated quadratic module $\mathcal{Q}_d(\mathbf{g})$ generated by $\mathbf{g} = \{g_1, \dots, g_s\}$.

The second and dual hierarchy replaces positive measures by linear functionals $\in \mathcal{L}_d(\mathbf{g})$ which are non-negative on the polynomials of the truncated quadratic module $\mathcal{Q}_d(\mathbf{g})$. We will describe more precisely these constructions in section 2.1.

This approach has many interesting properties (see e.g. [Las15], [Lau09], [Mar08]). It was proposed with the aim to recover the infimum f^* and, if this infimum is reached, the minimizer set $\{\xi \in S \mid f(\xi) = f^*\}$.

To tackle this challenge, one can first address the finite convergence problem, that is when the value f^* can be obtained at a given order of the relaxation(s). The second problem is the exactness of the relaxations, which is the main topic of this paper. The Sum of Squares (SoS) exactness is when the non-negative polynomial $f - f^*$ belongs to the truncated quadratic module $\mathcal{Q}_{2d}(\mathbf{g})$ for some $d \in \mathbb{N}$. The Moment Matrix (MoM) exactness is when an optimal linear functional $\sigma^* \in \mathcal{L}_{2d}(\mathbf{g})$ for f is coming from a positive measure supported on S for some $d \in \mathbb{N}$. We are going to investigate in details this MoM exactness property.

Several works have been developed over the last decades to tackle these problems. [Par02] showed that if the complex variety $\mathcal{V}_{\mathbb{C}}(I)$ defined by an ideal I generated by real polynomials is finite and I is radical, then $f - f^*$ has a representation as a sum of squares modulo I and the SoS relaxation is exact. [Lau07] showed the finite convergence property if the complex variety $\mathcal{V}_{\mathbb{C}}(I)$ is finite, and a moment sequence representation property, if moreover the ideal I is radical. [Nie13c] showed that if the semialgebraic set S is finite, then the finite convergence property holds for a finitely generated preordering defining S . [LLR08] proved that if S is finite, the value f^* and the minimizers can be recovered from moment matrices associated to the truncated preordering defining S . In [Las+13], the kernel of moment matrices is used to compute a border basis of the real radical ideal $\sqrt[\mathbb{R}]{I}$ when $S = \mathcal{V}_{\mathbb{R}}(I)$ is finite. [Sch05a] proved that $f - f^*$ is in the quadratic module \mathcal{Q} defining S modulo $(f - f^*)^2$ if and only if $f - f^* \in \mathcal{Q}$ and then the SoS relaxation is exact. [Mar06], [Mar09] proved that under some regularity conditions on the minimizers, known as Boundary Hessian Conditions (BHC), $f - f^*$ is in the quadratic module and the SoS exactness property holds. [NDS06], [DNP07] showed that adding gradient constraints when $S = \mathbb{R}^n$ or KKT constraints when S is a general basic semialgebraic set, the SoS exactness property holds when the corresponding Jacobian ideal is radical. [Nie13a] showed that adding the Jacobian constraints, the finite convergence property holds under some regularity assumption on the complex variety associated to these constraints and on the compactness of S . In [Nie14], it is shown that BHC imply finite convergence and that BHC are generic. [KS19] showed the SoS exactness property if the quadratic module defining S is Archimedean and some strict concavity properties of f at the finite minimizers are satisfied.

Though many works focussed on the SoS relaxation and on the representation of positive polynomials with sums of squares, the MoM relaxation has been much less studied. It has interesting features, that deserve a deeper exploration: the convex cones $\mathcal{L}_d(\mathbf{g})$ of truncated non-negative linear functionals are closed; finite convergence can be decided by flat extension tests on moment matrices [CF98], [LM09]; finite minimizers can be extracted from moment matrices [HL05], [Mou18]. On the other hand, exact SoS relaxations can provide certificates of positivity, which is also interesting from a theoretical and practical point of view. But no method exists to recover the minimizers when only finite convergence holds or from SoS representations of $f - f^*$. Properties of representation of the (truncated) moment sequence minimizers as measures and moment matrices have been used so far to recover the minimizers.

For the sake of completeness of Lasserre relaxation approach and its practical impact, it is then natural to ask the following question:

Are the (truncated) moment sequence minimizers of some high enough order relaxation of a Polynomial Optimization Problem coming from measures ?

The answer is known to be negative for some pathological problems (see for instance Examples 2.6, 2.7, 2.8, 2.9, 2.10 for cases of different natures), but it remained open for regular problems, for more than a decade.

In this paper, we address this question for regular problems, investigate truncated moment relaxation from a new perspective, and develop a theoretical and computational study of truncated positive linear functionals. We analyse in details the properties of moment relaxations and present new results on the representation of moments of positive linear functionals as moments of measures.

We first show in Theorem 3.9 that the dual of the MoM relaxation $(\mathcal{L}_{2d}(\mathbf{g}))_{d \in \mathbb{N}}$ is the SoS relaxation associated to the quadratic module $(\mathcal{Q}_{2d}(\mathbf{g}))_{d \in \mathbb{N}}$ extended with the real radical of the support of \mathcal{Q} . This yields the real radical ideal $\sqrt[\mathbb{R}]{\text{supp } \mathcal{Q}}$ as the ideal generated by the annihilator of a generic element $\sigma \in \mathcal{L}_{2d}(\mathbf{g})$ for d sufficiently large (see Theorem 3.16).

Our key ingredient on exact moment representations is Theorem 3.19. When the quotient by the support of \mathcal{Q} is of dimension zero, we prove that the linear functionals in $\mathcal{L}_d(\mathbf{g})$ truncated in a degree greater than twice the regularity of the points in S coincide with the measures supported on S , that is the convex hull of the evaluations at the points of S . Moreover, the ideal generated by the annihilator of a generic element in $\mathcal{L}_d(\mathbf{g})$ is the vanishing ideal of S .

The main result Theorem 4.4 shows that MoM exactness holds for regular problems. We prove that when the Boundary Hessian Conditions are satisfied, the MoM relaxation is exact. This generalizes the results on finite convergence and SoS exactness proved in [Mar06], [Mar09], and [Nie14]. It also shows that MoM exactness holds generically (Corollary 4.5).

We apply these results to Polynomial Optimization Problems, showing in Theorem 5.1 that when the set S is finite, the MoM relaxation is exact if the quotient by the support of the quadratic module \mathcal{Q} is of dimension zero. This generalizes the results of [LLR08] on semi-definite moment representations.

When the set of minimizers is finite, we describe a MoM relaxation which involves f^* , show its MoM exactness (Corollary 5.2) and propose a practical algorithm to achieve MoM exactness using approximate numerical computation. In Theorem 5.5, we prove that if the real variety of polar points is finite then the relaxation extended with Jacobian constraints is MoM exact. This generalizes the results of finite convergence and SoS exactness of the KKT and Jacobian relaxations under regularity conditions, proved in [NDS06], [DNP07], [Nie13a].

The paper is structured as follows. In the next sections of the introduction, we define the algebraic objects that we will use and recall their main properties. In Section 2, we describe in details the notions of finite convergence and exactness for the Sum of Squares (SoS) and Moment Matrix (MoM) relaxations. We give several examples showing how these notions are related. In Section 3, we recall the properties of full moment sequences (Section 3.1), investigate the truncated moment sequence properties (Section 3.2) and prove the moment representation property for a quadratic

module with a zero-dimensional support. In Section 4 we prove that exactness holds for regular, generic problems. In Section 5, we investigate polynomial optimization problems where the moment relaxation can be made exact, on finite semialgebraic sets (Section 5.1), for Polynomial Optimization Problems with a finite number of minimizers (Section 5.2) and adding constraints from the polar ideal (Section 5.3). Examples of Polynomial Optimization Problems and numerical experimentations with the Julia package `MomentTools.jl` are presented in Section 6 to illustrate these developments.

1.1 Polynomials

We provide the basic definitions on real polynomials and refer to [Mar08] for more details. Let $\mathbb{R}[\mathbf{X}] := \mathbb{R}[X_1, \dots, X_n]$ be the \mathbb{R} -algebra of polynomials in n indeterminates X_1, \dots, X_n . Let $\Sigma^2 = \Sigma^2[\mathbf{X}] := \left\{ f \in \mathbb{R}[\mathbf{X}] \mid \exists r \in \mathbb{N}, g_i \in \mathbb{R}[\mathbf{X}]: f = g_1^2 + \dots + g_r^2 \right\}$ be the convex cone of *Sum of Squares polynomials* (SoS). If $A \subset \mathbb{R}[\mathbf{X}]$, $A_d := \{f \in A \mid \deg f \leq d\}$. In particular $\mathbb{R}[\mathbf{X}]_d$ is the vector space of polynomials of degree $\leq d$.

We denote $(h_1, \dots, h_r) \subset \mathbb{R}[\mathbf{X}]$ the ideal generated by $h_1, \dots, h_r \in \mathbb{R}[\mathbf{X}]$. $Q \subset \mathbb{R}[\mathbf{X}]$ is called *quadratic module* if $1 \in Q$, $\Sigma^2 \cdot Q \subset Q$ and $Q + Q \subset Q$. If in addition $Q \cdot Q \subset Q$, Q is *preordering*. For $Q \subset \mathbb{R}[\mathbf{X}]$, we define $\text{supp } Q := Q \cap -Q$. If Q is a quadratic module then $\text{supp } Q$ is an ideal.

We say that a quadratic module Q is *finitely generated* (f.g.) if $\exists g_1, \dots, g_l \in \mathbb{R}[\mathbf{X}]: Q = \mathcal{Q}(g_1, \dots, g_l) := \Sigma^2 + \Sigma^2 \cdot g_1 + \dots + \Sigma^2 \cdot g_l$ (it is the smallest quadratic module containing g_1, \dots, g_l). We say that a preordering O is *finitely generated* if $\exists g_1, \dots, g_l \in \mathbb{R}[\mathbf{X}]: O = \mathcal{O}(g_1, \dots, g_l) := \mathcal{Q}(\prod_{j \in J} g_j \mid J \subset \{1, \dots, l\})$ (it is the smallest preordering containing g_1, \dots, g_l).

For $G \subset \mathbb{R}[\mathbf{X}]$, let $\mathcal{Q}_t(G) := \left\{ s_0 + \sum_{j=1}^r s_j g_j \in \mathbb{R}[\mathbf{X}]_t \mid r \in \mathbb{N}, g_j \in G, s_0 \in \Sigma_t^2, s_j \in \Sigma_{t-\deg g_j}^2 \right\}$ and $\langle G \rangle_t := \left\{ \sum_{i=1}^r f_i h_i \in \mathbb{R}[\mathbf{X}]_t \mid r \in \mathbb{N}, h_i \in G, f_i \in \mathbb{R}[\mathbf{X}]_{t-\deg h_i} \right\}$.

For a sequence of polynomials $\mathbf{g} := g_1, \dots, g_s$ we define $\Pi \mathbf{g} := \prod_{j \in J} g_j: J \subset \{1, \dots, s\}$ and $\pm \mathbf{g} := g_1, -g_1, \dots, g_r, -g_r$. Observe that $\mathcal{Q}_t(\mathbf{g}, \pm \mathbf{h}) = \mathcal{Q}_t(\mathbf{g}) + \langle \mathbf{h} \rangle_{2\lfloor \frac{t}{2} \rfloor}$ and $\mathcal{Q}_t(\Pi(\mathbf{g}, \pm \mathbf{h})) = \mathcal{Q}_t(\Pi \mathbf{g}, \pm \mathbf{h})$. Notice that $\langle \mathbf{h} \rangle_t \subset (\mathbf{h})_t$ and $\mathcal{Q}_t(\mathbf{g}) \subset \mathcal{Q}_t(\mathbf{g}, \pm \mathbf{h})$, but (unluckily) these inclusions are strict in general. Finally if $A \subset \mathbb{R}[\mathbf{X}]$ we define $\mathcal{S}(A) := \{x \in \mathbb{R}^n \mid f(x) \geq 0 \forall f \in A\}$. In particular we denote $\mathcal{S}(\mathbf{g}) = \{x \in \mathbb{R}^n \mid g(x) \geq 0 \forall g \in \mathbf{g}\}$ (the *basic semialgebraic set defined by \mathbf{g}*). If $Q = \mathcal{Q}(\mathbf{g})$, notice that $\mathcal{S}(\mathbf{g}) = \mathcal{S}(\Pi \mathbf{g}) = \mathcal{S}(Q)$. We denote by $\text{Pos}(S) = \{f \in \mathbb{R}[\mathbf{X}]: \forall x \in S, f(x) \geq 0\}$ the cone of positive polynomials on S .

1.2 Linear functionals

We describe the dual algebraic objects and refer to [Mou18] for more details. For $\sigma \in (\mathbb{R}[\mathbf{X}])^* = \{\sigma: \mathbb{R}[\mathbf{X}] \rightarrow \mathbb{R} \mid \sigma \text{ is linear}\}$, we denote $\langle \sigma \mid f \rangle = \sigma(f)$ the application of σ to $f \in \mathbb{R}[\mathbf{X}]$. Recall that $(\mathbb{R}[\mathbf{X}])^* \cong \mathbb{R}[[\mathbf{Y}]] := \mathbb{R}[[Y_1, \dots, Y_n]]$, with the isomorphism given by:

$$(\mathbb{R}[\mathbf{X}])^* \ni \sigma \mapsto \sum_{\alpha \in \mathbb{N}^n} \langle \sigma \mid \mathbf{X}^\alpha \rangle \frac{\mathbf{Y}^\alpha}{\alpha!} \in \mathbb{R}[[\mathbf{Y}]],$$

where $\left\{ \frac{\mathbf{Y}^\alpha}{\alpha!} \right\}$ is the dual basis to $\{\mathbf{X}^\alpha\}$, i.e. $\langle \mathbf{Y}^\alpha \mid \mathbf{X}^\beta \rangle = \alpha! \delta_{\alpha, \beta}$. With this basis we can also identify $\sigma \in (\mathbb{R}[\mathbf{X}])^*$ with its sequence of coefficients $(\sigma_\alpha)_\alpha$, where $\sigma_\alpha := \langle \sigma \mid \mathbf{X}^\alpha \rangle$. We will consider Borel measures with support included in $S \subset \mathbb{R}^n$, denoted as $\mathcal{M}(S)$, as linear functionals, i.e. $\mathcal{M}(S) \subset (\mathbb{R}[\mathbf{X}])^*$. In this case the sequence $(\mu_\alpha)_\alpha$ associated with a measure μ is the sequence of *moments*: $\mu_\alpha = \int X^\alpha d\mu$. Moreover $\mathcal{M}^{(1)}(S)$ will denote the Borel *probability* measures supported on S . We recall a version of Haviland's theorem [Mar08, th. 3.1.2]: if $\sigma \in (\mathbb{R}[\mathbf{X}])^*$, then $\sigma \in \mathcal{M}(S)$ if and only if $\forall f \in \text{Pos}(S)$, $\langle \sigma \mid f \rangle \geq 0$. In particular we are interested in *evaluations*: if $\xi \in \mathbb{R}^n$ then $\mathbf{e}_\xi(f) = \langle \mathbf{e}_\xi \mid f \rangle = \int f d\mathbf{e}_\xi = f(\xi)$ for all $f \in \mathbb{R}[\mathbf{X}]$.

If $\sigma \in (\mathbb{R}[\mathbf{X}])^*$ and $g \in \mathbb{R}[\mathbf{X}]$, we define the *convolution of g and σ* as $g \star \sigma := \sigma \circ m_g \in (\mathbb{R}[\mathbf{X}])^*$ (i.e. $\langle g \star \sigma \mid f \rangle = \langle \sigma \mid gf \rangle \forall f$) and the *Hankel operator* $H_\sigma: \mathbb{R}[\mathbf{X}] \rightarrow (\mathbb{R}[\mathbf{X}])^*$, $g \mapsto g \star \sigma$. If $\sigma = (\sigma_\alpha)_\alpha$ and $g = \sum_\alpha g_\alpha X^\alpha$ then $g \star \sigma = (\sum_\beta g_\beta \sigma_{\alpha+\beta})_\alpha$. Notice that $g \star \sigma = 0 \iff H_{g \star \sigma} = 0$. We denote by $\text{Ann}(\sigma)$ the annihilator of σ w.r.t. \star , that is $\text{Ann}(\sigma) = \ker H_\sigma = \{p \in \mathbb{R}[\mathbf{X}] \mid p \star \sigma = 0\}$.

With respect to the basis $\{\mathbf{X}^\alpha\}$ and $\{\frac{\mathbf{Y}^\alpha}{\alpha!}\}$:

- the matrix H_σ is $H_\sigma = (\sigma_{\alpha+\beta})_{\alpha,\beta}$ (the so called *moment matrix*);
- the matrix $H_{g\star\sigma}$ is $H_{g\star\sigma} = ((g\star\sigma)_{\alpha+\beta})_{\alpha,\beta} = (\sum_\gamma g_\gamma \sigma_{\alpha+\beta+\gamma})_{\alpha,\beta}$ (the so called *localizing matrix*).

We say that σ is *positive semidefinite* (psd) $\iff H_\sigma$ is psd, i.e. $\langle H_\sigma(f)|f \rangle = \langle \sigma|f^2 \rangle \geq 0 \forall f \in \mathbb{R}[\mathbf{X}]$ (see [Sch17] or [Mar08] for basic properties of psd matrices).

If $\sigma \in (\mathbb{R}[\mathbf{X}])^*$ then $\sigma^{[t]} \in (\mathbb{R}[\mathbf{X}]_t)^*$ denotes its restriction to $\mathbb{R}[\mathbf{X}]_t$ (and same for $\sigma \in (\mathbb{R}[\mathbf{X}]_r)^*$, $r \geq t$); moreover if $B \subset (\mathbb{R}[\mathbf{X}])^*$ then $B^{[t]} := \{\sigma^{[t]} \in (\mathbb{R}[\mathbf{X}]_t)^* \mid \sigma \in B\}$ (and same for $B \subset (\mathbb{R}[\mathbf{X}]_r)^*$, $r \geq t$).

If $\sigma \in (\mathbb{R}[\mathbf{X}]_t)^*$ and $g \in \mathbb{R}[\mathbf{X}]_t$, then $g\star\sigma := \sigma \circ m_g \in (\mathbb{R}[\mathbf{X}]_{t-\deg g})^*$. If $\sigma \in (\mathbb{R}[\mathbf{X}])^*$ (or $\sigma \in (\mathbb{R}[\mathbf{X}]_r)^*$, $r \geq 2t$), then we define $H_\sigma^t: \mathbb{R}[\mathbf{X}]_t \rightarrow (\mathbb{R}[\mathbf{X}]_t)^*$, $g \mapsto (g\star\sigma)^{[t]}$. We have $(g\star\sigma)^{[t]} = 0 \iff H_\sigma^t(g) = 0$: in analogy with the infinite dimensional setting, we define $\text{Ann}_d(\sigma) = \ker H_\sigma^d$.

Notice that, if $s \leq t$, we can identify the matrix of H_σ^s with the submatrix of H_σ^t indexed by monomials of degree $\leq t$.

Let $A \subset \mathbb{R}[\mathbf{X}]$ (resp. $A \subset \mathbb{R}[\mathbf{X}]_t$). We define $A^\perp := \{\sigma \in (\mathbb{R}[\mathbf{X}])^* \mid \langle \sigma|f \rangle = 0 \forall f \in A\}$ (resp. $A^\perp := \{\sigma \in (\mathbb{R}[\mathbf{X}]_t)^* \mid \langle \sigma|f \rangle = 0 \forall f \in A\}$). Notice that $\sigma \in \langle \mathbf{h} \rangle_t^\perp$ (resp. $(\mathbf{h})^\perp$) if and only if $(h\star\sigma)^{[t-\deg h]} = 0 \forall h \in \mathbf{h}$ (resp. $h\star\sigma = 0 \forall h \in \mathbf{h}$).

For $G \subset \mathbb{R}[\mathbf{X}]_t$ we define:

$$\mathcal{L}_t(G) = \{\sigma \in (\mathbb{R}[\mathbf{X}]_t)^* \mid \forall q \in \mathcal{Q}_t(G) \langle \sigma|q \rangle \geq 0\}$$

Equivalently $\sigma \in \mathcal{L}_t(G)$ if and only if $\langle \sigma|s \rangle \geq 0 \forall s \in \Sigma_t^2$ and $\langle \sigma|sf \rangle \geq 0 \forall f \in G, \forall s \in \Sigma^2: \deg fs \leq t$.

For the non-truncated version we write $\mathcal{L}(A)$. Notice that if $Q = \mathcal{Q}(\mathbf{g})$ then $\mathcal{L}(\mathbf{g}) = \mathcal{L}(Q)$ (resp. $\mathcal{L}_t(\mathbf{g}) = \mathcal{L}_t(\mathcal{Q}_t(\mathbf{g}))$) is the *dual convex cone* to Q (resp. to $\mathcal{Q}_t(\mathbf{g})$), see [Mar08, sec. 3.6]: $\mathcal{L}(\mathbf{g}) = Q^\vee$ and $\mathcal{L}_t(\mathbf{g}) = \mathcal{Q}_t(\mathbf{g})^\vee$. We give to $\mathbb{R}[\mathbf{X}]$ and $(\mathbb{R}[\mathbf{X}])^*$ the locally convex topology defined as follows. If $V = \mathbb{R}[\mathbf{X}]$ or $V = (\mathbb{R}[\mathbf{X}])^*$ and $W \subset V$ is a finitely dimensional vector subspace, W is equipped with the Euclidean topology. We define $U \subset V$ open if and only if $U \cap W$ is open in W for every finitely dimensional vector subspace W . By conic duality: $\overline{Q} = \mathcal{L}(\mathbf{g})^\vee$ and $\overline{\mathcal{Q}_t(\mathbf{g})} = \mathcal{L}_t(\mathbf{g})^\vee$. If $A \subset V$, we denote by $\text{cone}(A)$ the convex cone generated by A , by $\text{conv}(A)$ its convex hull and by $\langle A \rangle$ its linear span.

1.3 Nullstellensatz and Positivstellensatz

We refer to [BCR98] and [Mar08] for real algebra and geometry. An ideal I is called *real* (or *real radical*) if $a_1^2 + \dots + a_s^2 \in I \implies a_i \in I \forall i$. We define the *real radical* of an ideal I as:

$$\sqrt[\mathbb{R}]{I} := \{f \in \mathbb{R}[\mathbf{X}] \mid \exists h \in \mathbb{N}, s \in \Sigma^2 \quad f^{2h} + s \in I\} \quad (2)$$

$$= \{f \in \mathbb{R}[\mathbf{X}] \mid \exists k \in \mathbb{N}, s \in \Sigma^2 \quad f^{2^k} + s \in I\}. \quad (3)$$

Definition (2) is the classical one, and it is equivalent to (3), that will be more convenient in the paper. The real radical of I is the smallest real ideal containing I . If Q is a quadratic module and I is an ideal, we say that I is Q -convex if $\forall g_1, g_2 \in Q, g_1 + g_2 \in I \implies g_1, g_2 \in I$. Then I is a real ideal if and only if I is radical and Σ^2 -convex.

Minimal primes lying over $\text{supp } Q$ are Q -convex (see [Mar08, prop. 2.1.7]), and thus Σ^2 -convex. Prime ideals are radical, then minimal primes lying over $\text{supp } Q$ are real. Then $\sqrt{\text{supp } Q} = \sqrt[\mathbb{R}]{\text{supp } Q}$: we will write $\sqrt[\mathbb{R}]{\text{supp } Q}$ instead of $\sqrt{\text{supp } Q}$ to stress that it is a real ideal.

If $I \subset \mathbb{R}[\mathbf{X}]$ is an ideal, we denote by $\mathcal{V}(I)$ its (complex) variety, and we define $\mathcal{V}_{\mathbb{R}}(I) := \mathcal{V}(I) \cap \mathbb{R}^n$. Moreover, if $S \subset \mathbb{R}^n$ we denote $\mathcal{I}(S)$ its (real) vanishing ideal.

We recall the description of the Zariski closure of basic semialgebraic sets.

Theorem 1.1 (Real Nullstellensatz, [Mar08, th. 2.2.1], [BCR98, cor. 4.4.3]). *Let $S = \mathcal{S}(\mathbf{g})$ be a basic semialgebraic set. Then $\mathcal{I}(S) = \sqrt[\mathbb{R}]{\text{supp } \mathcal{O}(\mathbf{g})}$. In other words, $f = 0$ on $S \iff \exists h \in \mathbb{N}: -f^{2^h} \in \mathcal{O}(\mathbf{g})$.*

In particular, $\mathcal{S}(\mathbf{g})$ is empty if and only if $-1 \in \mathcal{O}(\mathbf{g})$ (and thus $\mathbb{R}[\mathbf{X}] = \mathcal{O}(\mathbf{g})$), and if I is an ideal then $\mathcal{I}(\mathcal{V}_{\mathbb{R}}(I)) = \sqrt{\text{supp}(\Sigma^2 + I)} = \sqrt[n]{I}$.

If $S \subset \mathbb{R}^n$, we denote by $\text{Pos}(S)$ the convex cone of non-negative polynomials on S : $\text{Pos}(S) := \{f \in \mathbb{R}[\mathbf{X}] \mid f(x) \geq 0 \forall x \in S\}$.

We say that a quadratic module Q is *Archimedean* if $\exists 0 \leq r \in \mathbb{R}: r - \|\mathbf{X}\|^2 \in Q$. Notice that if Q is Archimedean then $\mathcal{S}(Q)$ is compact. The viceversa is true for preorderings, see Theorem 1.2 and also [Sch05c].

When S is compact, one can obtain an Archimedean quadratic module from $Q(\mathbf{g})$ by adding a generator $g_M = M - \|\mathbf{X}\|^2 \geq 0$, for M big enough or by adding all the products of the g_i 's, replacing the generators \mathbf{g} by $\Pi\mathbf{g}$.

The importance of Archimedean quadratic modules is illustrated by Schmüdgen/Putinar's characterization of strictly positive polynomials, and their solution of the moment problem (see theorem 3.1).

Theorem 1.2 (Schmüdgen / Putinar Positivstellensatz,[Sch91] [Put93]). *Let $\mathcal{S}(\mathbf{g})$ be a basic semialgebraic set.*

- If $\mathcal{S}(\mathbf{g})$ is compact then $\mathcal{O}(\mathbf{g})$ is Archimedean;
- If $Q(\mathbf{g})$ is Archimedean, then $f > 0$ on $\mathcal{S}(\mathbf{g})$ implies $f \in Q(\mathbf{g})$.

As a corollary one can prove that, if Q is Archimedean, then $\overline{Q} = \text{Pos}(S)$.

1.4 Finite varieties, interpolation polynomials and bases

Now we move to interpolator polynomials, a tool which will be often used in the proofs.

Consider a finite set of points $\Xi = \{\xi_1, \dots, \xi_r\} \in \mathbb{C}^n$. It is well known that it admits a family of interpolator polynomials. Such a family $(u_i) \subset \mathbb{C}[\mathbf{X}]$ is by definition such that $u_i(\xi_j) = \delta_{i,j}$. The minimal degree $\iota(\Xi)$ of a family of interpolator polynomials is called the *interpolation degree* of Ξ .

Let $\mathcal{I}(\Xi) = \{p \in \mathbb{C}[X] \mid p(\xi_i) = 0 \forall i \in 1, \dots, r\}$ be the complex vanishing ideal of the points Ξ . The Castelnuovo-Mumford *regularity* of an ideal I (resp. Ξ) is $\max_i(\deg S_i - i)$ where S_i is the i^{th} module of syzygies in a minimal resolution of I (resp. $\mathcal{I}(\Xi)$). Let denote it by $\rho(I)$ (resp. $\rho(\Xi)$).

Since a family of interpolator polynomials (p_i) is a basis of $\mathbb{C}[\mathbf{X}]/\mathcal{I}(\Xi)$, the ideal $\mathcal{I}(\Xi)$ is generated in degree $\leq \iota(\Xi) + 1$ and $\rho(\Xi) \leq \iota(\Xi) + 1$. A classical result [Eis05, th. 4.1] relates the interpolation degree of Ξ with its regularity, and the minimal degree of a basis of $\mathbb{C}[\mathbf{X}]/\mathcal{I}(\Xi)$. This result can be stated as follows, for real points $\Xi \subset \mathbb{R}^n$:

Proposition 1.3. *Let $\Xi = \{\xi_1, \dots, \xi_r\} \subset \mathbb{R}^n$ with regularity $\rho(\Xi)$. Then $\iota(\Xi) = \rho(\Xi) - 1$, the minimal degree of a basis of $\mathbb{R}[\mathbf{X}]/\mathcal{I}(\Xi)$ is $\rho(\Xi) - 1$ and there exists interpolator polynomials $u_1, \dots, u_r \in \mathbb{R}[\mathbf{X}]_{\rho(\Xi)-1}$.*

We say that $\mathbf{h} = \{h_1, \dots, h_s\}$ is a *graded basis* of an ideal I if for all $p \in I$, there exists $q_i \in \mathbb{R}[\mathbf{X}]$ with $\deg(q_i) \leq \deg(p) - \deg(h_i)$ such that $p = \sum_{i=1}^s h_i q_i$. Equivalently, we have for all $t \in \mathbb{N}$, $\langle \mathbf{h} \rangle_t = I_t$.

For $p \in \mathbb{R}[\mathbf{X}]$ and I an ideal of $\mathbb{R}[\mathbf{X}]$, let $\lambda_{\deg}(p)$ be its homogeneous component of highest degree, that we call the *initial* of p , and let $\lambda_{\deg}(I) = (\{\lambda_{\deg}(p) \mid p \in I\})$ be the initial of I . A family $\mathbf{h} = (h_1, \dots, h_s)$ is a graded basis of the ideal $I = (h_1, \dots, h_s)$ iff $\lambda_{\deg}(I) = (\lambda_{\deg}(h_1), \dots, \lambda_{\deg}(h_s))$. For more properties of graded bases, also known as *H-bases*, see e.g. [Mac16].

A graded basis of an ideal $I = (\mathbf{h})$ can be computed as a Grobner basis using a monomial ordering \prec , which refines the degree ordering (see e.g. [CLO15]). It can also be computed as a border basis for a monomial basis of least degree of $\mathbb{R}[\mathbf{X}]/I$ (see e.g. [MT05]).

The degree of a graded basis of an ideal I is bounded by its regularity $\rho(I)$ (see e.g. [BS87]).

For a set of points $\Xi = \{\xi_1, \dots, \xi_r\}$, the ideal $\mathcal{I}(\Xi)$ has a graded (resp. Grobner, resp. border) basis of degree equal to the regularity $\rho(\Xi)$. The minimal degree of a monomial basis B of $\mathbb{R}[\mathbf{X}]/\mathcal{I}(\Xi)$ is $\iota(\Xi) = \rho(\Xi) - 1$. Such a basis B can be chosen so that it is stable by monomial division.

Proposition 1.4. Let $\Xi = \{\xi_1, \dots, \xi_r\} \subset \mathbb{R}^n$, $I = \mathcal{I}(\Xi)$ its real vanishing ideal and let $\rho = \rho(\Xi)$ the regularity of Ξ . For $t \geq \rho - 1$, $\sigma \in I_t^\perp$ if and only if $\sigma \in \langle \mathbf{e}_{\xi_1}^{[t]}, \dots, \mathbf{e}_{\xi_r}^{[t]} \rangle$. Moreover if $t \geq 2(\rho - 1)$ and $\sigma \in \mathcal{L}_t(I_t)$ then $\sigma \in \text{cone}(\mathbf{e}_{\xi_1}^{[t]}, \dots, \mathbf{e}_{\xi_r}^{[t]})$.

Proof. Let $u_1, \dots, u_r \in \mathbb{R}[\mathbf{X}]_t$ be interpolation polynomials of degree $\leq \rho - 1 \leq t$ (Proposition 1.3). Consider the sequence of vector space maps:

$$\begin{aligned} 0 \rightarrow I_t \rightarrow \mathbb{R}[\mathbf{X}]_t &\xrightarrow{\psi} \langle u_1, \dots, u_r \rangle \rightarrow 0 \\ p &\mapsto \sum_{i=1}^r p(\xi_i) u_i, \end{aligned}$$

which is exact since $\ker \psi = \{p \in \mathbb{R}[\mathbf{X}]_t \mid p(\xi_i) = 0\} = I_t$. Therefore we have $\mathbb{R}[\mathbf{X}]_t = \langle u_1, \dots, u_r \rangle \oplus I_t$.

Let $\sigma \in I_t^\perp$. Then $\tilde{\sigma} = \sigma - \sum_{i=1}^r \langle \sigma | u_i \rangle \mathbf{e}_{\xi_i}^{[t]} \in I_t^\perp$ is such that $\langle \tilde{\sigma} | u_i \rangle = 0$ for $i = 1, \dots, r$. Thus, $\tilde{\sigma} \in \langle u_1, \dots, u_r \rangle^\perp \cap I_t^\perp = (\langle u_1, \dots, u_r \rangle \oplus I_t)^\perp = \mathbb{R}[\mathbf{X}]_t^\perp$, i.e. $\tilde{\sigma} = 0$ showing that $I_t^\perp \subset \langle \mathbf{e}_{\xi_1}^{[t]}, \dots, \mathbf{e}_{\xi_r}^{[t]} \rangle$. The reverse inclusion is direct since I_t is the space of polynomials of degree $\leq t$ vanishing at ξ_i for $i = 1, \dots, r$.

Assume that $t \geq 2(\rho - 1)$ and that $\sigma \in \mathcal{L}_t(I_t)$. Then $\sigma \in I_t^\perp$ and $\langle \sigma | p^2 \rangle \geq 0$ for any $p \in \mathbb{R}[\mathbf{X}]_{\lfloor \frac{t}{2} \rfloor}$. By the previous analysis,

$$\sigma = \sum_{i=1}^r \omega_i \mathbf{e}_{\xi_i}^{[t]}$$

As $\langle \sigma | u_i^2 \rangle = \omega_i \geq 0$ for $i = 1, \dots, r$, we deduce that $\sigma \in \text{cone}(\mathbf{e}_{\xi_1}^{[t]}, \dots, \mathbf{e}_{\xi_r}^{[t]})$. \square

2 Finite Convergence and Exactness

We describe now the Lasserre SoS and MoM relaxations [Las01], and we define the *exactness* property. Hereafter we assume that the minimum f^* of the objective function f is always attained on S , that is: $S^{\min} := \{x \in S \mid f(x) = f^*\} \neq \emptyset$.

2.1 Polynomial optimization relaxations

We define the *SoS relaxation of order d* of problem (1) as $\mathcal{Q}_{2d}(\mathbf{g})$ and the supremum:

$$f_{\text{SoS},d}^* := \sup \{ \lambda \in \mathbb{R} \mid f - \lambda \in \mathcal{Q}_{2d}(\mathbf{g}) \}. \quad (4)$$

When necessary we will replace \mathbf{g} by $\Pi \mathbf{g}$ (that is $\mathcal{Q}(\mathbf{g})$ by $\mathcal{O}(\mathbf{g})$).

We want to define the dual approximation of the polynomial optimization problem. We are interested in an affine hyperplane section of the cone $\mathcal{L}_d(\mathbf{g})$:

$$\mathcal{L}_d^{(1)}(\mathbf{g}) := \{ \sigma \in \mathcal{L}_d(\mathbf{g}) \mid \langle \sigma | 1 \rangle = 1 \}.$$

We will use the notation $\mathcal{L}^{(1)}(\mathbf{g})$ in the infinite dimensional case. The convex sets $\mathcal{L}_d^{(1)}(\mathbf{g})$ are spectrahedra: they are defined by the Linear Matrix Inequalities $H_\sigma^{\lfloor \frac{d}{2} \rfloor} \geq 0$, $H_{g_1 \star \sigma}^{N_1} \geq 0$, \dots , $H_{g_r \star \sigma}^{N_r} \geq 0$, where $N_i = d - \lfloor \frac{\text{deg } g_i}{2} \rfloor$. The convex set $\mathcal{L}_d^{(1)}(\mathbf{g})$ is also called the state space of $(\mathbb{R}[\mathbf{X}]_d, \mathcal{Q}_d(\mathbf{g}), 1)$ in [KS19]. The pure states are the extreme points of this convex set.

With this notation we define the *MoM relaxation of order d* of problem (1) as $\mathcal{L}_{2d}(\mathbf{g})$ and the infimum:

$$f_{\text{MoM},d}^* := \inf \{ \langle \sigma | f \rangle \in \mathbb{R} \mid \sigma \in \mathcal{L}_{2d}^{(1)}(\mathbf{g}) \}. \quad (5)$$

When necessary we will replace \mathbf{g} by $\Pi \mathbf{g}$ (that is $\mathcal{Q}(\mathbf{g})$ by $\mathcal{O}(\mathbf{g})$). We are interested, in particular, in the linear functionals that realize the minimum. We easily verify that $f_{\text{SoS},d}^* \leq f_{\text{MoM},d}^* \leq f^*$. When $S^{\min} := \{\xi \in S \mid f(\xi) = f^*\} \neq \emptyset$, the infimum $f_{\text{MoM},d}^*$ is reached since $\mathcal{L}_d^{(1)}(\mathbf{g})$ is closed.

Definition 2.1. Let $f \in \mathbb{R}[\mathbf{X}]$ and f^* denote its minimum on $\mathcal{S}(\mathbf{g})$. We define the set of *functional minimizers* as:

$$\mathcal{L}_{2d}^{\min}(\mathbf{g}) := \left\{ \sigma \in \mathcal{L}_{2d}^{(1)}(\mathbf{g}) \mid \langle \sigma | f \rangle = f^* \right\}.$$

Notice that $\mathcal{L}_d(\mathbf{g})$ is the cone over $\mathcal{L}_d^{(1)}(\mathbf{g})$, since for $\sigma \in \mathcal{L}_d(\mathbf{g})$ we have $\langle \sigma | 1 \rangle = 0 \Rightarrow \sigma = 0$ (see [Las+13, lem. 3.12]), and $\sigma \in \mathcal{L}_d(\mathbf{g}) \neq 0$ implies $\frac{1}{\langle \sigma | 1 \rangle} \sigma \in \mathcal{L}_d^{(1)}(\mathbf{g})$.

We now introduce two convergence properties that will be central in the article.

Definition 2.2 (Finite Convergence). We say that the SoS relaxation $(\mathcal{Q}_{2d}(\mathbf{g}))_{d \in \mathbb{N}}$ (resp. the MoM relaxation $(\mathcal{L}_{2d}(\mathbf{g}))_{d \in \mathbb{N}}$) has the *Finite Convergence* property for f if $\exists k \in \mathbb{N}$ such that for every $d \geq k$, $f_{\text{SoS},d}^* = f^*$ (resp. $f_{\text{MoM},d}^* = f^*$).

Notice that if the SoS relaxation has finite convergence then the MoM relaxation has finite convergence too, since $f_{\text{SoS},d}^* \leq f_{\text{MoM},d}^* \leq f^*$.

Definition 2.3 (SoS Exactness). We say that the SoS relaxation $(\mathcal{Q}_{2d}(\mathbf{g}))_{d \in \mathbb{N}}$ is *exact* for f if it has the finite convergence property and for all d big enough, we have $f - f^* \in \mathcal{Q}_{2d}(\mathbf{g})$ (in other words $\text{sup} = \text{max}$ in the definition of $f_{\text{SoS},d}^*$).

For the moment relaxation we can ask the (stronger) property that every truncated functional minimizer is coming from a measure:

Definition 2.4 (MoM Exactness). We say that the MoM relaxation $(\mathcal{L}_{2d}(\mathbf{g}))_{d \in \mathbb{N}}$ is *exact* for f on the basic closed semialgebraic set S if:

- it has the finite convergence property;
- for every $k \in \mathbb{N}$ big enough, for $d = d(k) \in \mathbb{N}$ big enough, every truncated functional minimizer is coming from a probability measure supported on S , i.e. $\mathcal{L}_{2d}^{\min}(\mathbf{g})^{[k]} \subset \mathcal{M}^{(1)}(S)^{[k]}$.

If not specified, S will be the semialgebraic set $S = \mathcal{S}(\mathbf{g})$ defined by \mathbf{g} .

MoM exactness may be considered as a particular instance of the so called *Moment Problem* (i.e. asking if $\sigma \in \mathbb{R}[\mathbf{X}]^*$ is coming from a measure) or of the *Strong Moment Problem* (i.e. asking that the measure has a specified support). More precisely, MoM exactness can be considered as a *Truncated Strong Moment Property* (since we are considering functionals restricted to polynomials up to a certain degree).

In practice, to check the finite convergence, one tests the flat extension or the flat truncation property of moment matrices (see [CF98], [LM09], [Nie13b]). Notice that flat truncation certifies MoM exactness, and not only finite convergence. As the flat truncation property implies MoM exactness, we will deduce from Theorem 4.4 that the flat truncation property holds at some finite order of the MoM relaxation (and not only asymptotically) for regular Polynomial Optimization Problems.

We recall results of strong duality, i.e. cases when we know that $f_{\text{SoS},d}^* = f_{\text{MoM},d}^*$, that will be using. See also Proposition 3.13.

Theorem 2.5 (Strong duality). Let $Q = \mathcal{Q}(\mathbf{g})$ be a quadratic module and f the objective function. Then:

- (i) if $\text{supp } Q = 0$ then $\forall d$: $f_{\text{SoS},d}^*$ is attained (i.e. $f - f_{\text{SoS},d}^* \in \mathcal{Q}_d(\mathbf{g})$) and $f_{\text{SoS},d}^* = f_{\text{MoM},d}^*$ [Mar08, prop. 10.5.1];
- (ii) if $r^2 - \|\mathbf{X}\|^2 \in \mathbf{g}$ then $f_{\text{SoS},d}^* = f_{\text{MoM},d}^*$ for all d [JH16].

Remark. [JH16] applies when the ball constraint $r^2 - \|\mathbf{X}\|^2$ appears explicitly in the description of S . But if we consider a problem with MoM finite convergence and such that $\mathcal{Q}(\mathbf{g})$ is Archimedean, then we can use [JH16] to prove that we have also SoS finite convergence. Indeed, if $\mathcal{Q}(\mathbf{g})$ is Archimedean there exists r, d such that $r^2 - \|\mathbf{X}\|^2 \in \mathcal{Q}_{2d}(\mathbf{g})$. This means that $\mathcal{Q}_{2t}(\mathbf{g}, r^2 - \|\mathbf{X}\|^2) \subset \mathcal{Q}_{2d+2t}(\mathbf{g})$. If we define:

- $f_{\text{SoS},d}^* = \sup \{ \lambda \in \mathbb{R} \mid f - \lambda \in \mathcal{Q}_{2d}(\mathbf{g}) \}$
- $f_{\text{SoS},d}' = \sup \{ \lambda \in \mathbb{R} \mid f - \lambda \in \mathcal{Q}_{2d}(\mathbf{g}, r^2 - \|X\|^2) \}$

and $f_{\text{MoM},d}^*, f_{\text{MoM},d}'$ the corresponding MoM relaxations, then:

$$f_{\text{MoM},t}^* \leq f_{\text{MoM},t}' = f_{\text{SoS},t}' \leq f_{\text{SoS},d+t}^* \leq f^*.$$

Then finite convergence of the MoM relaxation implies finite convergence of the SoS one (we thank Monique Laurent for her useful comments).

We recall that we are assuming $S^{\min} \neq \emptyset$ (in particular f^* is finite: otherwise it may happen that $f_{\text{SoS},d}^* = -\infty$). Notice that if strong duality holds, then SoS finite convergence is equivalent to MoM finite convergence.

2.2 Examples and counterexamples

In this section, we give examples showing how these notions are (not) related.

No finite convergence. The first example shows that SoS and MoM relaxations for polynomial optimization on algebraic curves do not have necessarily the finite convergence property.

Example 2.6 ([Sch00]). Let $\mathcal{C} \subset \mathbb{R}^n$ be a smooth connected curve of genus ≥ 1 , with only real points at infinity. Let $\mathbf{h} = \{h_1, \dots, h_s\} \subset \mathbb{R}[\mathbf{X}]$ be a graded basis of $I = \mathcal{I}(\mathcal{C}) = (\mathbf{h})$. Then there exists $f \in \mathbb{R}[\mathbf{X}]$ such that the SoS relaxation $\mathcal{Q}_{2d}(\pm \mathbf{h})$ and the MoM relaxation $\mathcal{L}_{2d}(\pm \mathbf{h})$ have no finite convergence and are not exact.

Indeed by [Sch00, Theorem 3.2], there exists $f \in \mathbb{R}[\mathbf{X}]$ such that $f \geq 0$ on $\mathcal{C} = \mathcal{S}(\pm \mathbf{h})$, which is not a sum of squares in $\mathbb{R}[\mathcal{C}] = \mathbb{R}[\mathbf{X}]/I$. Consequently, $f \notin \Sigma^2[\mathbf{X}] + I = \mathcal{Q}(\pm \mathbf{h})$. As $f \geq 0$ on \mathcal{C} , its infimum f^* is non-negative and we also have $f - f^* \notin \mathcal{Q}(\pm \mathbf{h})$.

Using Proposition 3.13 we deduce that $\mathcal{Q}_d(\pm \mathbf{h})$ is closed, that there is no duality gap and that the supremum $f_{\text{SoS},d}^*$ is reached. Thus if the SoS relaxation has finite convergence then $f - f^* \in \mathcal{Q}_{2d}(\pm \mathbf{h})$ for some $d \in \mathbb{N}$. This is a contradiction, showing that the SoS and the MoM relaxations have no finite convergence and cannot be SoS exact for f .

In dimension 2, there are also cases where the SoS and MoM relaxations cannot have finite convergence or be exact.

Example 2.7 ([Mar08]). Let $g_1 = X_1^3 - X_2^2, g_2 = 1 - X_2$. Then $S = \mathcal{S}(\mathbf{g})$ is a compact semialgebraic set of dimension 2 and $\mathcal{O}(\mathbf{g})$ is Archimedean. We have $f = X_1 \geq 0$ on S but $X_1 \notin \mathcal{O}(\mathbf{g})$ (see [Mar08, Example 9.4.6(3)]). The infimum of f on S is $f^* = 0$. Assume that we have MoM finite convergence. By Theorem 2.5 and remark below, $\mathcal{Q}_d(\Pi \mathbf{g})$ is closed, the supremum $f_{\text{SoS},d}^*$ is reached and strong duality holds: $f_{\text{SoS},d}^* = f_{\text{MoM},d}^* = f^* = 0$ for $d \in \mathbb{N}$ big enough. Then $f - f^* = f \in \mathcal{O}(\mathbf{g})$: but this is a contradiction. Therefore, the relaxations $\mathcal{Q}_{2d}(\Pi \mathbf{g})$ and $\mathcal{L}_{2d}(\Pi \mathbf{g})$ cannot have finite convergence and thus cannot be exact for $f = X_1$.

The next example shows that non-finite convergence and non-exactness always happen in dimension ≥ 3 .

Example 2.8. Let $n \geq 3$. Let Q be an Archimedean quadratic module generated by $g_1, \dots, g_s \in \mathbb{R}[\mathbf{X}]$ such that $\mathcal{S}(Q) \subset \mathbb{R}^n$ is of dimension $m \geq 3$ and let \mathbf{h} be a graded basis of $\sqrt[m]{\text{supp } Q}$ (in particular $\mathbf{h} = 0$ if $\text{supp } Q = 0$ or if $m = n$, i.e. $\mathcal{S}(Q)$ is of maximal dimension), then there exists $f \in \mathbb{R}[\mathbf{X}]$ such that the SoS relaxation $(\mathcal{Q}_{2d}(\mathbf{g}, \pm \mathbf{h}))_{d \in \mathbb{N}}$ and MoM relaxation $(\mathcal{L}_{2d}(\mathbf{g}, \pm \mathbf{h}))_{d \in \mathbb{N}}$ do not have the finite convergence property (and thus are not exact).

Indeed by Proposition 3.13 $f_{\text{SoS},d}^* = f_{\text{MoM},d}^*$ for d big enough and the supremum $f_{\text{SoS},d}^*$ is reached. By [Sch00, Prop. 6.1] for $m \geq 3$, $\text{Pos}(\mathcal{S}(Q)) = \text{Pos}(\mathcal{S}(Q + (\mathbf{h}))) \supsetneq Q + (\mathbf{h})$. So let $f \in \text{Pos}(\mathcal{S}(Q)) \setminus Q + (\mathbf{h})$ and let f^* be its minimum on $\mathcal{S}(Q)$. Suppose that $f - f^* \in Q + (\mathbf{h})$, then $f \in Q + (\mathbf{h}) + f^* = Q + (\mathbf{h})$, a contradiction. Then the SoS and the MoM relaxations do not have the finite convergence property (and they are not exact).

SoS exactness, no MoM exactness.

Example 2.9. We want to find the global minimum of $f = X_1^2 \in \mathbb{R}[X_1, \dots, X_n] = \mathbb{R}[\mathbf{X}]$ for $n \geq 3$. Let $d \geq 2$, $\mathbf{X}' = (X_2, \dots, X_n)$ and $\bar{\sigma} \in \mathcal{L}_d(\Sigma^2[\mathbf{X}'])$ such that $\bar{\sigma} \notin \mathcal{M}(\mathbb{R}^{n-1})^{[d]}$. Such a linear functional exists because when $n > 2$ there are non-negative polynomials in $\mathbb{R}[\mathbf{X}']$ which are not sum of squares, such as the Motzkin polynomial (see [Rez96]). As $\mathcal{Q}_d(\Sigma^2[\mathbf{X}'])$ is closed, such a polynomial can be separated from $\mathcal{Q}_d(\Sigma^2[\mathbf{X}'])$ by a linear functional $\bar{\sigma} \in \mathcal{L}_d(\Sigma^2[\mathbf{X}'])$, which cannot be the truncation of a measure (i.e. $\Sigma^2[\mathbf{X}']$ does not have the truncated moment property). Define $\sigma : h \mapsto \langle \sigma | h \rangle = \langle \bar{\sigma} | h(0, X_2, \dots, X_n) \rangle$. We have $\sigma \in \mathcal{L}(\Sigma^2[\mathbf{X}])$ since $\bar{\sigma} \in \mathcal{L}(\Sigma^2[\mathbf{X}'])$. Obviously $\langle \sigma | f \rangle = 0 = f^*$ (the minimum of X_1^2), $f - f^* = X_1^2 \in \Sigma^2$ and the SoS relaxation is exact. Since σ is coming from a measure if and only if $\bar{\sigma}$ is coming from a measure, the MoM relaxation cannot be exact.

The previous example generalizes easily to quadratic modules Q with $\text{supp}(Q) \neq \{0\}$, which do not have the (truncated) moment property, i.e. there exists $\sigma \in \mathcal{L}_d(Q)$ such that $\sigma \notin \mathcal{M}(S(Q))^{[d]}$. Taking $f = h^2$ with $h \in \text{supp}(Q)$, $h \neq 0$, we have $\langle \sigma | f \rangle = 0 = f^*$ and the MoM relaxation cannot be exact since $\sigma \notin \mathcal{M}(S(Q))^{[d]}$, while the SoS relaxation is exact ($f - f^* = h^2 \in Q$).

Example 2.9 is an example where the number of minimizers of f on S is infinite. We show that this can happen also when the minimizers are finite (and even when S is finite!).

Example 2.10 ([Sch05a, ex. 3.2], [Sch05b, rem. 3.15], Example 3.6). We want to minimize the constant function $f = 1$ on the origin $S = S(Q) = \{0\}$, where $Q = Q(1 - X^2 - Y^2, -XY, X - Y, Y - X^2) \subset \mathbb{R}[X, Y]$. In this case $\text{supp } Q = \sqrt[\mathbb{R}]{\text{supp } Q} = (0)$. Notice that the SoS relaxation is exact and the MoM relaxations has finite convergence, since f is a square. Now suppose that the MoM relaxation is exact, i.e. $\mathcal{L}_{2d}^{\min}(\mathbf{g})^{[2k]} = \mathcal{L}_{2d}^{(1)}(\mathbf{g})^{[2k]} \subset \mathcal{M}^{(1)}(S)^{[2k]} = \{\mathbf{e}_0^{[2k]}\}$ for some d, k big enough. Then for $\sigma^* \in \mathcal{L}_{2d}(\mathbf{g})$ generic we have $(\text{Ann}_k(\sigma^*)) = (\text{Ann}_k(\mathbf{e}_0)) = (X, Y)$. But from Theorem 3.16 we know that for d, k big enough $(\text{Ann}_k(\sigma^*)) = \sqrt[\mathbb{R}]{\text{supp } Q} = (0)$, a contradiction. Then the MoM relaxation is not exact.

SoS finite convergence, MoM exactness.

Example 2.11. Let $f = (X^4Y^2 + X^2Y^4 + Z^6 - 2X^2Y^2Z^2) + X^8 + Y^8 + Z^8 \in \mathbb{R}[X, Y, Z]$. We want to optimize f over the gradient variety $\mathcal{V}_{\mathbb{R}}\left(\frac{\partial f}{\partial X}, \frac{\partial f}{\partial Y}, \frac{\partial f}{\partial Z}\right)$ which is zero dimensional (see [NDS06]). By Theorem 5.1 the MoM relaxation is exact, and by Theorem 2.5 and remark below the SoS has the finite convergence property (notice that $\mathcal{Q}(\pm \frac{\partial f}{\partial X}, \pm \frac{\partial f}{\partial Y}, \pm \frac{\partial f}{\partial Z}) = \mathcal{O}(\pm \frac{\partial f}{\partial X}, \pm \frac{\partial f}{\partial Y}, \pm \frac{\partial f}{\partial Z})$ is Archimedean since $\mathcal{V}_{\mathbb{R}}\left(\frac{\partial f}{\partial X}, \frac{\partial f}{\partial Y}, \frac{\partial f}{\partial Z}\right)$ is compact). But the SoS relaxation is not exact, as shown in [NDS06].

Example 2.12. Let $f = X_1$. We want to find its value at the origin, defined by $\|\mathbf{X}\|^2 = 0$. As proved in [Nie13c] there is finite convergence but not exactness for the SoS relaxation. By Theorem 5.1 the MoM relaxation is exact.

We summarize the previous examples in Table 1 in terms of the properties of finite convergence (SoS f.c. and MoM f.c.) exactness (SoS ex. and MoM ex.) and the dimension m of the semialgebraic set S .

3 Geometry of Moment Representations

We give a description of the moment linear functionals in the full dimensional and truncated case. In particular, in Section 3.1 we describe known properties of quadratic modules and their duals (see for instance [Mar08, ch. 3]) and we focus on generic elements (Definition 3.2), infinite dimensional equivalents of the finite dimensional ones (see for instance [Las+13]). The finite dimensional (*truncated*) case is described in Section 3.2, where the main technical tools of the paper are developed. In particular we analyse the extended quadratic module associated with the MoM relaxations (Theorem 3.9), and consequently prove properties of the cones $\mathcal{L}_d(\mathbf{g})$ (Lemma 3.10) and of their generic elements (Theorem 3.16). Finally we apply these results to the zero dimensional case (Theorem 3.19).

Table 1: Summary of convergence results.

Expl.	SoS f. c.	SoS ex.	MoM f. c.	MoM ex.	m
2.6	NO	NO	NO	NO	1
2.7	NO	NO	NO	NO	2
2.8	NO	NO	NO	NO	≥ 3
2.9	YES	YES	YES	NO	≥ 3
2.10	YES	YES	YES	NO	0
2.11	YES	NO	YES	YES	0
2.12	YES	NO	YES	YES	0

3.1 Infinite moment representations

With our setting, the classical moment problem can be stated as follows: given $\sigma \in \mathbb{R}[\mathbf{X}]^*$, when there exists $\mu \in \mathcal{M}(\mathbb{R}^n)$ such that:

$$\forall f \in \mathbb{R}[\mathbf{X}] \quad \langle \sigma | f \rangle = \int f \, d\mu.$$

Haviland's theorem (see [Mar08, th. 3.1.2] and [Sch17, th. 1.12]) says that this happens if and only if σ is positive on positive polynomials. Since checking this is a computationally hard task, then it is interesting to find (proper) subsets of positive polynomials that have the same property, chosen in such a way that checking this conditions is easy. Important results in this direction are theorems of Schmüdgen and Putinar.

Theorem 3.1 ([Sch91],[Put93]). *Let Q be an Archimedean finitely generated quadratic module and $S = \mathcal{S}(Q)$. Then $\mathcal{L}(Q) = \mathcal{M}(S) = \text{cone}(\mathbf{e}_\xi : \xi \in S)$.*

This theorem solves the moment problem in the Archimedean (compact) case. Notice that $\mathcal{M}(\mathcal{S}(Q))$ depends only on $S = \mathcal{S}(Q)$ and not on the generators of Q . In particular, if Q and Q' are Archimedean and $\mathcal{S}(Q) = \mathcal{S}(Q')$ then $\mathcal{L}(Q) = \mathcal{L}(Q')$.

If we have a *generic* measure $\mu \in \mathcal{M}(S)$, i.e. which is nonzero on any nonzero polynomial on S , obviously its support is equal to S : $\text{supp } \mu = S$. We want to generalize this property to linear functionals which are not necessary coming from measures. In particular we want to recover informations about the semialgebraic set $S = \mathcal{S}(\mathbf{g})$ from linear functionals $\sigma \in \mathcal{L}(\mathbf{g})$. We are interested in *generic* elements $\sigma^* \in \mathcal{L}(Q)$, that we characterize in terms of the annihilator, i.e. the kernel of the Hankel operator (see also Proposition 3.15).

Definition 3.2. We say that $\sigma^* \in \mathcal{L}(Q)$ is *generic* if $\text{Ann}(\sigma^*) \subset \text{Ann}(\sigma) \forall \sigma \in \mathcal{L}(Q)$.

Proposition 3.3. *Let I be an ideal of $\mathbb{R}[\mathbf{X}]$ and $\sigma^* \in \mathcal{L}(I)$ be generic. Then $\text{Ann}(\sigma^*) = \sqrt[\mathbb{R}]{I}$.*

Proof. Notice that if $x \in \mathcal{V}_{\mathbb{R}}(I)$ then $\mathbf{e}_x \in \mathcal{L}(I)$. Moreover $\text{Ann}(\mathbf{e}_x) = \mathcal{I}(x)$. This implies:

$$\text{Ann}(\sigma^*) \subset \bigcap_{x \in \mathcal{V}_{\mathbb{R}}(I)} \text{Ann}(\mathbf{e}_x) = \bigcap_{x \in \mathcal{V}_{\mathbb{R}}(I)} \mathcal{I}(x) = \mathcal{I}(\mathcal{V}_{\mathbb{R}}(I)) = \sqrt[\mathbb{R}]{I},$$

where the last equality is the Real Nullstellenatz, Theorem 1.1.

By definition, $I \subset \text{Ann}(\sigma^*)$. Since $\text{Ann}(\sigma^*)$ is a real radical ideal (see [Las+13, prop. 3.13]) we have $\sqrt[\mathbb{R}]{I} \subset \text{Ann}(\sigma^*)$, which proves that $\text{Ann}(\sigma^*) = \sqrt[\mathbb{R}]{I}$. \square

Proposition 3.3 generalizes to quadratic modules as follows.

Proposition 3.4. *Let Q be a quadratic module, $S = \mathcal{S}(Q)$ and $\sigma^* \in \mathcal{L}(Q)$ be generic. Then $\sqrt[\mathbb{R}]{\text{supp } Q} \subset \text{Ann}(\sigma^*) \subset \mathcal{I}(S)$. Moreover if Q is Archimedean then $\text{Ann}(\sigma^*) = \mathcal{I}(S)$.*

Proof. As in the proof of Proposition 3.3, we get:

$$\text{Ann}(\sigma^*) \subset \bigcap_{x \in S} \text{Ann}(\mathbf{e}_x) = \bigcap_{x \in S} \mathcal{I}(x) = \mathcal{I}(S).$$

Now observe that $\text{supp } Q \subset \text{Ann}(\sigma^*)$ by definition. Since $\text{Ann}(\sigma^*)$ is a real radical ideal (see [Las+13, prop. 3.13]), then $\sqrt[\mathbb{R}]{\text{supp } Q} \subset \text{Ann}(\sigma^*)$.

For the second part, if Q is Archimedean, then by Theorem 3.1 $\mathcal{L}(Q) = \mathcal{M}(S)$. In particular σ^* is a measure $\mu \in \mathcal{M}(S)$ supported on S : $\forall f \in \mathbb{R}[\mathbf{X}], \langle \sigma^* | f \rangle = \int f d\mu$. Let $h \in \mathcal{I}(S)$ and $f \in \mathbb{R}[\mathbf{X}]$. Then:

$$\langle \sigma^* | fh \rangle = \int fh d\mu = \int 0 d\mu = 0,$$

i.e. $h \in \text{Ann}(\sigma^*)$, which proves the reverse inclusion. \square

Now we describe $\mathcal{L}(Q)$ without the Archimedean hypothesis (compare with Theorem 3.1).

Lemma 3.5. *Let Q be a quadratic module. Then $\mathcal{L}(Q) = \mathcal{L}(\sqrt[\mathbb{R}]{\text{supp } Q} + Q)$. In particular for any ideal $I \subset \sqrt[\mathbb{R}]{\text{supp } Q}$ we have $\mathcal{L}(Q) = \mathcal{L}(\sqrt{I} + Q)$.*

Proof. Since $\sqrt{\text{supp } Q} \subset \overline{Q}$ (see [Mar08, th. 4.1.2]), we have $\sqrt{\text{supp } Q} + Q \subset \overline{Q} + Q = \overline{Q}$. Then:

$$\mathcal{L}(Q) = \mathcal{L}(\overline{Q}) \subset \mathcal{L}(\sqrt{\text{supp } Q} + Q) \subset \mathcal{L}(Q).$$

Since $\sqrt[\mathbb{R}]{\text{supp } Q} = \sqrt{\text{supp } Q}$ (see Section 1.3) we have $\mathcal{L}(\sqrt[\mathbb{R}]{\text{supp } Q} + Q) = \mathcal{L}(\sqrt{\text{supp } Q} + Q) = \mathcal{L}(Q)$. \square

Remark. Lemma 3.5 shows that, even if the semialgebraic set is not compact, we can replace any ideal in the description of the semialgebraic set with its real radical. In particular, since $\mathcal{I}(\mathcal{S}(\mathbf{g})) = \sqrt[\mathbb{R}]{\mathcal{O}(\mathbf{g})}$ (by Theorem 1.1), we have $\mathcal{L}(\mathcal{O}(\mathbf{g})) = \mathcal{L}(\mathcal{O}(\mathbf{g}) + \mathcal{I}(\mathcal{S}(\mathbf{g})))$.

The inclusion $Q + \sqrt{\text{supp } Q} \subset \overline{Q}$ can be strict, as shown by the following example.

Example 3.6 ([Sch05a, ex. 3.2], [Sch05b, rem. 3.15], Example 3.6). Let $Q = \mathcal{Q}(1 - X^2 - Y^2, -XY, X - Y, Y - X^2) \subset \mathbb{R}[X, Y]$. Notice that $S = \mathcal{S}(Q) = \{\mathbf{0}\}$ and that, since Q is Archimedean, $\overline{Q} = \text{Pos}(\{\mathbf{0}\})$. In this case $\text{supp } Q = (0)$ and $\mathcal{I}(S) = \text{supp } \overline{Q} = (X, Y)$, and thus $Q + \sqrt{\text{supp } Q} \subsetneq \overline{Q}$.

3.2 Truncated moment representations

Now we prove the corresponding results in the truncated case. For a finitely generated quadratic module $Q = \mathcal{Q}(\mathbf{g}) \subset \mathbb{R}[\mathbf{X}]$, we denote $Q_{[k]} = \mathcal{Q}_k(\mathbf{g})$. Notice that $\overline{Q_{[k]}}^\vee = Q_{[k]}^\vee = \mathcal{L}_d(\mathbf{g})$, and thus the following definition is natural and important for the study of the MoM relaxations.

Definition 3.7. Let $Q = \mathcal{Q}(\mathbf{g})$ be a finitely generated quadratic module. We define $\tilde{Q} = \bigcup_d \overline{Q_{[d]}} = \bigcup_d \overline{Q_{[d]}}$.

Notice that \tilde{Q} depends a priori on the generators \mathbf{g} of Q : we will prove that \tilde{Q} is a finitely generated quadratic module and that it does not depend on the particular choice of generators. Moreover notice that $Q \subset \tilde{Q} = \bigcup_d \overline{Q_{[d]}} \subset \bigcup_d Q_{[d]} = \overline{Q}$, but these inclusions may be strict as we will see.

Lemma 3.8. *Let $Q = \mathcal{Q}(\mathbf{g})$ and $J = \sqrt[\mathbb{R}]{\text{supp } Q}$. Then for every $d \in \mathbb{N}$ there exists $k \geq d$ such that $J_d \subset \overline{Q_{[k]}}$.*

Proof. Let m be big enough such that $\forall f \in J = \sqrt[\mathbb{R}]{\text{supp } Q} = \sqrt{\text{supp } Q}$ we have: $f^{2^m} \in \text{supp } Q$ (if $\sqrt{J} = (h_1, \dots, h_t)$ and $h_i^{a_i} \in I$, we can take m such that $2^m \geq a_1 + \dots + a_t$). Let $f \in J_d$ with $\deg f \leq d$. Then $f^{2^m} \in \text{supp } Q_{[k']} \subset Q_{[k']}$ for $k' \in \mathbb{N}$ big enough. Using the identity [Sch05b, remark 2.2]:

$$m - a = \left(1 - \frac{a}{2}\right)^2 + \left(1 - \frac{a^2}{8}\right)^2 + \left(1 - \frac{a^4}{128}\right)^2 + \dots + \left(1 - \frac{a^{2^{m-1}}}{2^{2^m-1}}\right)^2 - \frac{a^{2^m}}{2^{2^{m+1}-2}},$$

substituting a by $-\frac{mf}{\varepsilon}$ and multiplying by $\frac{\varepsilon}{m}$, we have that $\forall \varepsilon > 0, f + \varepsilon \in Q_{[k]}$ for $k = \max\{k', 2^m d\}$ (the degree of the representation of $f + \varepsilon$ does not depend on ε). This implies that $f \in \overline{Q_{[k]}}$. \square

We can now describe \widetilde{Q} .

Theorem 3.9. *Let $Q = Q(\mathbf{g})$ be a finitely generated quadratic module and let $J = \sqrt[\mathbb{R}]{\text{supp } Q}$. Then $\widetilde{Q} = \cup_{d \in \mathbb{N}} \overline{Q_{[d]}} = Q + J$ and $\text{supp } \widetilde{Q} = J$. In particular, \widetilde{Q} is a finitely generated quadratic module and does not depend on the particular choice of generators of Q .*

Proof. By [Mar08, lemma 4.1.4] $Q_{[d]} + J_d$ is closed in $\mathbb{R}[\mathbf{X}]_d$, thus $\overline{Q_{[d]}} \subset Q_{[d]} + J_d$. Taking unions we prove that $\widetilde{Q} \subset Q + J$.

Conversely by Lemma 3.8 for $d \in \mathbb{N}$ and $k \geq d \in \mathbb{N}$ big enough, $J_d \subset \overline{Q_{[k]}}$. Then, we have $Q_{[d]} + J_d \subset Q_{[k]} + \overline{Q_{[k]}} \subset \overline{Q_{[k]}} + \overline{Q_{[k]}} \subset \overline{Q_{[k]}}$. Taking unions on both sides gives $Q + J \subset \widetilde{Q}$.

Finally $\text{supp } \widetilde{Q} = \text{supp}(Q + J) = J$ by [Sch05b, lemma 3.16]. \square

Remark. We proved that $\widetilde{Q} = Q + \sqrt[\mathbb{R}]{\text{supp } Q}$, and thus in Example 3.6 we have that $\widetilde{Q} \subsetneq \overline{Q}$. We also have $\text{supp } \widetilde{Q} = \sqrt[\mathbb{R}]{\text{supp } Q}$ so that if $\text{supp } Q$ is not real radical then $Q \subsetneq \widetilde{Q}$. Example 2.12 is such a case where $\text{supp } Q \neq \sqrt[\mathbb{R}]{\text{supp } Q}$. We notice that, by Theorem 3.9 and [Sch05b, th. 3.17], if Q is stable¹ then $\widetilde{Q} = \overline{Q}$.

Theorem 3.9 suggests the idea that, when we consider the MoM relaxation, we are extending the quadratic module $Q(\mathbf{g})$ to $Q(\mathbf{g}, \pm \mathbf{h})$, where \mathbf{h} are generators of $\sqrt[\mathbb{R}]{\text{supp } Q(\mathbf{g})}$. We specify this idea in Lemma 3.10, Proposition 3.13 Theorem 3.16.

We describe relations between the truncated parts of $\mathcal{L}_d(\mathbf{g})$.

Lemma 3.10. *Let $J = \sqrt[\mathbb{R}]{\text{supp } Q(\mathbf{g})}$. If $(\mathbf{h}) \subset J$, $\deg \mathbf{h} \leq d$, then $\exists k \geq d$:*

$$\mathcal{L}_k(\mathbf{g})^{[d]} \subset \mathcal{L}_d(\mathbf{g}, \pm \mathbf{h}) \subset \mathcal{L}_d(\mathbf{g}).$$

In particular $\mathcal{L}_k(\mathbf{g})^{[d]} \subset \mathcal{L}_d(\pm \mathbf{h})$.

Proof. By Lemma 3.8, $\langle \mathbf{h} \rangle_d \subset (\mathbf{h})_d \subset \overline{Q_k(\mathbf{g})}$ for some $k \geq d$. Let $h \in \mathbf{h}$ and $f \in \mathbb{R}[\mathbf{X}]_{d - \deg h}$. Then $\pm fh \in \overline{Q_k(\mathbf{g})}$, and for $\sigma \in \mathcal{L}_k(\mathbf{g})$, we have $\langle \sigma^{[d]} | fh \rangle = \langle \sigma | fh \rangle = 0$, i.e. $\mathcal{L}_k(\mathbf{g})^{[d]} \subset \mathcal{L}_d(\mathbf{g}, \pm \mathbf{h})$. The other inclusion $\mathcal{L}_d(\mathbf{g}, \pm \mathbf{h}) \subset \mathcal{L}_d(\mathbf{g})$ follows by definition. \square

Remark. Lemma 3.10 says that the MoM relaxation $(\mathcal{L}_{2d}(\mathbf{g}))_{d \in \mathbb{N}}$ is equivalent to the MoM relaxation $(\mathcal{L}_{2d}(\mathbf{g}, \pm \mathbf{h}))_{d \in \mathbb{N}}$, where $(\mathbf{h}) = \sqrt[\mathbb{R}]{\text{supp } Q(\mathbf{g})}$.

Lemma 3.10 is an algebraic result, in the sense that $\text{supp } Q(\mathbf{g})$ may be unrelated to the geometry $\mathcal{S}(\mathbf{g})$ that it defines. If some additional conditions hold (namely if we have only equalities, or a preordering, or a small dimension), it can however provide geometric characterizations that will be useful in Section 5.

Corollary 3.11. *Suppose that $\mathcal{S}(\mathbf{g}) \subset \mathcal{V}_{\mathbb{R}}(\mathbf{h})$. Then for every $t_0 \geq \deg \mathbf{h}$ there exists $t_1 \geq t_0$ such that:*

$$\mathcal{L}_{t_1}(\Pi \mathbf{g})^{[t_0]} \subset \mathcal{L}_{t_0}(\pm \mathbf{h}).$$

In particular this holds when $(\mathbf{h}) = \mathcal{I}(\mathcal{S}(\mathbf{g}))$.

Proof. $\mathcal{S}(\mathbf{g}) \subset \mathcal{V}_{\mathbb{R}}(\mathbf{h})$ if and only if $\sqrt[\mathbb{R}]{(\mathbf{h})} \subset \mathcal{I}(\mathcal{S}(\mathbf{g})) = \sqrt[\mathbb{R}]{\text{supp } Q(\Pi \mathbf{g})}$ by Theorem 1.1. Then we can apply Lemma 3.10. \square

Corollary 3.12. *Let $Q = Q(\mathbf{g})$. Suppose that $\mathcal{S}(\mathbf{g}) \subset \mathcal{V}_{\mathbb{R}}(\mathbf{h})$ and $\dim \frac{\mathbb{R}[\mathbf{X}]}{\text{supp } Q} \leq 1$. Then for every $t_0 \geq \deg \mathbf{h}$ there exists $t_1 \geq t_0$ such that:*

$$\mathcal{L}_{t_1}(\mathbf{g})^{[t_0]} \subset \mathcal{L}_{t_0}(\pm \mathbf{h}).$$

In particular this holds when $(\mathbf{h}) = \mathcal{I}(\mathcal{S}(\mathbf{g}))$.

¹ $Q(\mathbf{g})$ is stable if $\forall d \in \mathbb{N}$ there exists $k \in \mathbb{N}$ such that $Q(\mathbf{g}) \cap \mathbb{R}[\mathbf{X}]_d = Q_k(\mathbf{g}) \cap \mathbb{R}[\mathbf{X}]_d$.

Proof. We prove it as Corollary 3.11, using [Mar08, cor. 7.4.2 (3)] instead of Theorem 1.1. \square

We mention a strong duality result, that is useful to produce examples and counterexamples for exactness and finite convergence. It is essentially the result in [Mar03]. This generalizes the condition $\text{supp } Q = 0$ in Theorem 2.5. We conjecture that a similar result holds more generally when Q is reduced (i.e. $\text{supp } Q = \sqrt[\mathbb{R}]{\text{supp } Q}$) without adding the generators of the radical of the support.

Proposition 3.13. *Let $Q = \mathcal{Q}(\mathbf{g})$ be a finitely generated quadratic module, and let \mathbf{h} be a graded basis of $\sqrt[\mathbb{R}]{\text{supp } Q}$. Then for any d we have $\mathcal{Q}_d(\mathbf{g}, \pm \mathbf{h}) = \overline{\mathcal{Q}_d(\mathbf{g}, \pm \mathbf{h})}$ is closed. Moreover, if we consider the extended relaxations $\mathcal{Q}_{2d}(\mathbf{g}, \pm \mathbf{h})$ and $\mathcal{L}_{2d}(\mathbf{g}, \pm \mathbf{h})$, then for any $f \in \mathbb{R}[\mathbf{X}]$ such that $f^* > -\infty$ we have that $f_{\text{SoS},d}^*$ is attained (i.e. $f - f_{\text{SoS},d}^* \in \mathcal{Q}_{2d}(\mathbf{g}, \pm \mathbf{h})$) and there is no duality gap: $f_{\text{SoS},d}^* = f_{\text{MoM},d}^*$.*

Proof. By [Mar08, lemma 4.1.4], $\mathcal{Q}_d(\mathbf{g}, \pm \mathbf{h}) = \mathcal{Q}_d(\mathbf{g}) + I_d$ is closed. Therefore we have $\mathcal{L}_d(\mathbf{g}, \pm \mathbf{h})^\vee = (\mathcal{Q}_d(\mathbf{g}, \pm \mathbf{h}))^{\vee\vee} = \overline{\mathcal{Q}_d(\mathbf{g}, \pm \mathbf{h})} = \mathcal{Q}_d(\mathbf{g}, \pm \mathbf{h})$, from which we deduce that there is not duality gap, by classical convexity arguments, as follows.

If $f \in \mathbb{R}[\mathbf{X}]$ such that $f^* > -\infty$, then $\{\lambda \in \mathbb{R} \mid f - \lambda \in \mathcal{Q}_{2d}(\mathbf{g}, \pm \mathbf{h})\}$ is bounded from above. Since $\mathcal{Q}_{2d}(\mathbf{g}, \pm \mathbf{h})$ is closed $f_{\text{SoS},d}^* = \sup\{\lambda \in \mathbb{R} \mid f - \lambda \in \mathcal{Q}_{2d}(\mathbf{g}, \pm \mathbf{h})\}$ is attained. If $f_{\text{SoS},d}^* < f_{\text{MoM},d}^*$, then $f - f_{\text{MoM},d}^* \notin \mathcal{Q}_{2d}(\mathbf{g}, \pm \mathbf{h})$. Thus there exists a separating functional $\sigma \in \mathcal{L}_{2d}^{(1)}(\mathbf{g}, \pm \mathbf{h})$ such that $\langle \sigma \mid f - f_{\text{MoM},d}^* \rangle < 0$, which implies that $\langle \sigma \mid f \rangle < f_{\text{MoM},d}^*$ in contradiction with the definition of $f_{\text{MoM},d}^*$. Consequently, $f_{\text{SoS},d}^* = f_{\text{MoM},d}^*$. \square

With the characterization of \widetilde{Q} we can now describe the kernel of Hankel operators associated to truncated moment sequences, in analogy to the infinite dimensional case analyzed in Proposition 3.4. First we recall the definition of genericity in the truncated setting and equivalent characterizations.

Definition 3.14. We say that $\sigma^* \in \mathcal{L}_k(\mathbf{g})$ is *generic* if $\text{rank } H_{\sigma^*}^k = \max\{\text{rank } H_{\eta}^k \mid \eta \in \mathcal{L}_k(\mathbf{g})\}$.

This genericity can be characterized as follows, see [Las+13, prop. 4.7].

Proposition 3.15. *Let $\sigma \in \mathcal{L}_{2k}(\mathbf{g})$. The following are equivalent:*

- (i) σ is generic;
- (ii) $\text{Ann}_k(\sigma) \subset \text{Ann}_k(\eta) \forall \eta \in \mathcal{L}_{2k}(\mathbf{g})$;
- (iii) $\forall d \leq k$, we have: $\text{rank } H_{\sigma}^d = \max\{\text{rank } H_{\eta}^d \mid \eta \in \mathcal{L}_{2k}(\mathbf{g})\}$.

Remark. By Proposition 3.15 notice that $\forall d \leq k$, if $\sigma^* \in \mathcal{L}_{2k}(\mathbf{g})$ is generic then $(\sigma^*)^{[2d]}$ is generic in $\mathcal{L}_{2k}(\mathbf{g})^{[2d]}$. In particular, $\text{Ann}_d(\sigma^*) \subset \text{Ann}_d(\eta) \forall \eta \in \mathcal{L}_k(\mathbf{g})$.

Notice that the linear functionals in the relative interior of $\mathcal{L}_k(\mathbf{g})$ are generic. If we use an SDP solver based on interior point method we will (approximately) get a moment sequence in the relative interior of the face $\mathcal{L}_{2d}(\mathbf{g}) \cap \{\langle \sigma \mid f \rangle = f_{\text{MoM},d}^*\}$, which is then generic. We will use generic linear functionals to recover the minimizers when we have exactness, see Theorem 4.4 and Theorem 5.1.

We are now ready to describe the kernel of generic elements.

Theorem 3.16. *Let $Q = \mathcal{Q}(\mathbf{g})$ and $J = \sqrt[\mathbb{R}]{\text{supp } Q}$. Then for all $d, t \in \mathbb{N}$ big enough and for $\sigma^* \in \mathcal{L}_d(\mathbf{g})$ generic, we have $J = (\text{Ann}_t(\sigma^*))$.*

Proof. Let $t \in \mathbb{N}$ such that J is generated in degree $\leq t$, by the graded basis $\mathbf{h} = \{h_1, \dots, h_s\}$. From Lemma 3.8 we deduce that there exists $d \in \mathbb{N}$ such that $J_{2t} \subset \overline{\mathcal{Q}_d(\mathbf{g})}$. Let $\sigma^* \in \mathcal{L}_d(\mathbf{g})$ generic.

We first prove that $J \subset (\text{Ann}_t(\sigma^*))$. By Proposition 3.15 we have $\text{Ann}_t(\sigma^*) = \bigcap_{\sigma \in \mathcal{L}_d(\mathbf{g})} \text{Ann}_t(\sigma)$. Then it is enough to prove that $J_t \subset \text{Ann}_t(\sigma)$ for all $\sigma \in \mathcal{L}_d(\mathbf{g})$.

By Lemma 3.10 $\mathcal{L}_d(\mathbf{g})^{[2t]} \subset \mathcal{L}_{2t}(\pm \mathbf{h}) \subset \langle \mathbf{h} \rangle_{2t}^\perp$. Then $\forall f \in J_t \subset \langle \mathbf{h} \rangle_t$, $\forall p \in \mathbb{R}[\mathbf{X}]_t$, $\forall \sigma \in \mathcal{L}_d(\mathbf{g})$, we have $f p \in \langle \mathbf{h} \rangle_{2t}$ and $\langle \sigma^{[2t]} | f p \rangle = 0$. This shows that $H_\sigma^t(f)(p) = \langle (f \star \sigma)^{[t]} | p \rangle = \langle \sigma | f p \rangle = 0$, i.e. $f \in \text{Ann}_t(\sigma)$.

Conversely, we show that $(\text{Ann}_t(\sigma^*)) \subset J$ for σ^* generic in $\mathcal{L}_d(\mathbf{g})$. Since $J = \text{supp } \widetilde{Q} = \text{supp } \bigcup_j \overline{Q_j(\mathbf{g})}$ (by Theorem 3.9) it is enough to prove that $\text{Ann}_t(\sigma^*) \subset \text{supp } \overline{Q_d(\mathbf{g})} = \text{supp } \mathcal{L}_d(\mathbf{g})^\vee$.

Let $f \in \text{Ann}_t(\sigma^*) = \bigcap_{\sigma \in \mathcal{L}_t(\mathbf{g})} \text{Ann}_t(\sigma)$ (we use again Proposition 3.15) and let $\sigma \in \mathcal{L}_d(\mathbf{g})$. Then $\langle \sigma | f \rangle = \langle (f \star \sigma)^{[t]} | 1 \rangle = H_\sigma^t(f)(1) = 0$. In particular $f \in \mathcal{L}_d(\mathbf{g})^\vee$. We prove that $-f \in \mathcal{L}_d(\mathbf{g})^\vee$ in the same way. Then $f \in \text{supp } \overline{Q_d(\mathbf{g})}$, which proves that $\text{Ann}_t(\sigma^*) \subset \text{supp } \widetilde{Q} = J$. \square

Theorem 3.16 is important as it shows the possibilities and the limits of the Lasserre MoM relaxations. For instance we cannot expect exactness of the MoM relaxation $\mathcal{L}_{2d}(\mathbf{g})$ for any objective function f (i.e. $\mathcal{L}_{2d}(\mathbf{g})^{[k]} \subset \mathcal{M}(S)^{[k]}$) if $\sqrt[\mathbb{R}]{\text{supp } Q} \neq \mathcal{I}(S)$: see Example 2.10.

The geometric corollary of this theorem is the following:

Corollary 3.17. *Let $O = \mathcal{O}(\mathbf{g})$ and $S = \mathcal{S}(\mathbf{g})$. Then there exists $d, t \in \mathbb{N}$ such that for $\sigma^* \in \mathcal{L}_d(\Pi \mathbf{g})$ generic, we have $\mathcal{I}(S) = (\text{Ann}_t(\sigma^*))$.*

Proof. Apply Theorem 3.16 and Theorem 1.1. \square

We deduce another corollary, giving degree bounds for the case of a graded basis of a real radical ideal.

Corollary 3.18. *Let $I = \sqrt[\mathbb{R}]{I}$ be a real radical ideal and \mathbf{h} be a graded basis of I . Let $\rho = \rho(I)$ denote its regularity. Then for all $d \geq \rho$ and $\sigma^* \in \mathcal{L}_{2d}(\pm \mathbf{h})$ generic, we have $(\text{Ann}_d(\sigma^*)) = I$.*

Proof. We can find a graded basis \mathbf{h} of degree $\leq \rho$, see Section 1.4. Let $d \geq \rho$ and $\sigma^* \in \mathcal{L}_{2d}(\pm \mathbf{h})$ generic. Then for all i we have $(h_i \star \sigma^*)^{[d]} = 0$, i.e. $h_i \in \text{Ann}_d(\sigma^*)$ and thus $I \subset (\text{Ann}_d(\sigma^*))$.

Conversely, from Theorem 3.16 and its proof we see that $(\text{Ann}_d(\sigma^*)) \subset I$ for all d . \square

The following theorem is central in the paper: when $\text{supp}(Q)$ is a zero-dimensional ideal (and thus S is finite), we can completely describe our positive linear functionals with evaluations.

Theorem 3.19. *Suppose that $\dim \frac{\mathbb{R}[\mathbf{X}]}{\text{supp } Q(\mathbf{g})} = 0$. Then, $S = \mathcal{S}(\mathbf{g}) = \{\xi_1, \dots, \xi_r\}$ is non-empty and finite and there exists $d \in \mathbb{N}$ such that $\forall k \in \mathbb{N}$:*

$$\mathcal{L}_{d+k}(\mathbf{g})^{[2(\rho-1)+k]} = \text{cone}(\mathbf{e}_{\xi_1}, \dots, \mathbf{e}_{\xi_r})^{[2(\rho-1)+k]}.$$

where $\rho = \rho(\xi_1, \dots, \xi_r)$ is the regularity of S . Moreover, if $d \geq t \geq \rho(\xi_1, \dots, \xi_r)$, d big enough, and $\sigma^* \in \mathcal{L}_{2d}(\mathbf{g})$ is generic, then $(\text{Ann}_t(\sigma^*)) = \mathcal{I}(\xi_1, \dots, \xi_r) = \mathcal{I}(S)$ is the vanishing ideal of S .

Proof. Let $I = \text{supp } Q(\mathbf{g})$ and $J = \sqrt[\mathbb{R}]{\text{supp } Q(\mathbf{g})} = \sqrt{\text{supp } Q(\mathbf{g})}$. Since $\dim \frac{\mathbb{R}[\mathbf{X}]}{I} = \dim \frac{\mathbb{R}[\mathbf{X}]}{J} = 0$ we have $\mathcal{I}(S(\mathbf{g})) = \sqrt[\mathbb{R}]{\text{supp } Q(\mathbf{g})} = J$ by [Mar08, cor. 7.4.2 (3)]. Then $\mathcal{V}_{\mathbb{R}}(J) = \mathcal{V}_{\mathbb{R}}(\mathcal{I}(S(\mathbf{g}))) = \mathcal{S}(\mathbf{g}) = \{\xi_1, \dots, \xi_r\}$ is finite.

We choose a graded basis \mathbf{h} of J with $\deg \mathbf{h} \leq \rho = \rho(\xi_1, \dots, \xi_r)$, see Section 1.4. By Corollary 3.12 and Proposition 1.4, there exists $d \in \mathbb{N}$ big enough such that for every $k \in \mathbb{N}$:

$$\mathcal{L}_{d+k}(\mathbf{g})^{[2(\rho-1)+k]} \subset \mathcal{L}_{2(\rho-1)+k}(\pm \mathbf{h}) = \mathcal{L}_{2(\rho-1)+k}(J_{2(\rho-1)+k}) = \text{cone}(\mathbf{e}_{\xi_1}, \dots, \mathbf{e}_{\xi_r})^{[2(\rho-1)+k]}.$$

Since the converse inclusion is obvious, we prove that $\mathcal{L}_{d+k}(\mathbf{g})^{[2(\rho-1)+k]} = \text{cone}(\mathbf{e}_{\xi_1}, \dots, \mathbf{e}_{\xi_r})^{[2(\rho-1)+k]}$.

The second part follows from $\mathcal{I}(S(\mathbf{g})) = \sqrt[\mathbb{R}]{\text{supp } Q(\mathbf{g})}$ and Corollary 3.18. \square

Remark. Notice that there exist examples with $\mathcal{S}(\mathbf{g})$ finite and $\dim \frac{\mathbb{R}[\mathbf{X}]}{\text{supp } Q(\mathbf{g})} > 1$, see Example 3.6. However the hypothesis:

- (i) $\dim \frac{\mathbb{R}[\mathbf{X}]}{\text{supp } Q(\mathbf{g})} = 0$; and

(ii) $\mathcal{S}(\mathbf{g})$ is finite and $\dim \frac{\mathbb{R}[\mathbf{X}]}{\text{supp } \mathcal{Q}(\mathbf{g})} \leq 1$

are equivalent: (i) \Rightarrow (ii) is shown in the proof of Theorem 3.19, while (ii) \Rightarrow (i) follows from $\mathcal{I}(\mathcal{S}(\mathbf{g})) = \sqrt[\mathbb{R}]{\text{supp } \mathcal{Q}(\mathbf{g})}$ (see [Mar08, cor. 7.4.2 (3)]).

As special cases, we have that the conclusions of Theorem 3.19 hold for:

- $\mathcal{Q}(\mathbf{g}) = \mathcal{Q}(\pm \mathbf{h}) = \mathcal{I}(\mathbf{h}) + \Sigma^2$ with $\mathbf{h} = (h_1, \dots, h_s) \subset \mathbb{R}[\mathbf{X}]$ and $\mathcal{V}_{\mathbb{R}}(\mathbf{h}) = \{\xi_1, \dots, \xi_r\}$ non-empty and finite. In this case, $(\text{Ann}_t(\sigma^*)) = \mathcal{I}(\xi_1, \dots, \xi_r) = \sqrt[\mathbb{R}]{(\mathbf{h})}$ is the real radical of (\mathbf{h}) .
- $\mathcal{Q}(\Pi \mathbf{g}) = \mathcal{O}(\mathbf{g})$ is a preordering with $S = \mathcal{S}(\mathbf{g}) = \{\xi_1, \dots, \xi_r\}$ non-empty and finite, since by Theorem 1.1, $\dim \frac{\mathbb{R}[\mathbf{X}]}{\text{supp } \mathcal{O}(\mathbf{g})} = \dim \frac{\mathbb{R}[\mathbf{X}]}{\sqrt{\text{supp } \mathcal{O}(\mathbf{g})}} = \dim \frac{\mathbb{R}[\mathbf{X}]}{\mathcal{I}(\mathcal{S}(\mathbf{g}))} = 0$

Related results were obtained in [LLR08] and [Las+13], where they focus on the case of equations \mathbf{h} defining a finite real variety. They prove that, for degree big enough and for every positive linear functional $\sigma \in \mathcal{L}_{2d}(\mathbf{g})$, the flat truncation property holds for H_{σ}^t , and that $\sigma^{[2t]}$ is a conic linear combination of evaluations at the points of $\mathcal{V}_{\mathbb{R}}(\mathbf{h})$. In [LLR08, rem. 4.9] it is also mentioned that the same can be proved for a preordering defining a finite semialgebraic set.

Theorem 3.19 generalizes these results, as shown by the following example where the quadratic module that is not a preordering whose support is zero dimensional.

Example 3.20 ([Mar08, ex. 7.4.5 (1)]). Let $Q = \mathcal{Q}(X, Y, 1 - X, 1 - Y, -X^4, -Y^4) \subset \mathbb{R}[X, Y]$. In this case $\text{supp } Q$ is zero dimensional and Q is not a preordering. Theorem 3.19 applies in this case, but the results of [LLR08], [Las+13] do not apply.

4 Exactness for regular polynomial optimisation problems

In this section, we address the main question: Is the moment relaxation exact for regular polynomial optimisation problems? The regularity conditions that we consider are the Boundary Hessian Conditions (BHC). These are conditions on the minimizers of a polynomial f on a basic semialgebraic set S introduced by Marshall in [Mar06] and [Mar09], which as particular cases of the so called local-global principle. Under these conditions, global properties of polynomials (e.g. $f \in Q$) can be deduced from local properties (e.g. checking the BHC at the minimizers of f on $\mathcal{S}(Q)$). We refer to [Sch05a], [Sch06] and [Mar08, ch. 9] for more details. We introduce BHC conditions following [Nie14].

Definition 4.1 (Boundary Hessian Conditions). Consider a POP with inequality constraints $g = \{g_1, \dots, g_r\}$, equality constraints $\mathbf{h} = \{h_1, \dots, h_s\}$ and objective function f . Let $V = \mathcal{V}(\mathbf{h}) \subset \mathbb{R}^n$ and suppose that $Q = \mathcal{Q}(\mathbf{g}, \pm \mathbf{h})$ is Archimedean. We say that the *Boundary Hessian Conditions* hold at a minimizer point $\xi \in S(\mathbf{g}, \pm \mathbf{h})$ of f if ξ is a smooth point of V and:

- we can choose $g_{i_1} = t_1, \dots, g_{i_k} = t_k$ that are part of a regular system of parameters t_1, \dots, t_m , $m \geq k$, for V at ξ and for some neighbourhood U of ξ we have $\mathcal{S}(g_{i_1}, \dots, g_{i_k}, \pm \mathbf{h}) \cap U = \mathcal{S}(\mathbf{g}, \pm \mathbf{h}) \cap U$;
- On V , locally at ξ we have that $\nabla f = a_1 \nabla t_1 + \dots + a_m \nabla t_m$, where a_i are strictly positive real numbers;
- On V , locally at ξ we have that $\text{Hess}(f)(0, \dots, 0, t_{k+1}, \dots, t_m)$ is positive definite in t_{k+1}, \dots, t_m .

These conditions are related to standard conditions in optimization at a point $\xi \in S$ (see e.g. [Ber99]). Hereafter, the active constraints at $\xi \in S$ are the constraints g_{i_1}, \dots, g_{i_m} such that $g_{i_j}(\xi) = 0$. To simplify the description of these conditions, we consider a constraint $\pm g(x) \geq 0$ as a single (equality) constraint. Therefore an equality constraint defining the set S is an active constraint at a point $\xi \in S$.

- *Constraint Qualification Condition (CQC)*: for the active constraints g_{i_1}, \dots, g_{i_l} at ξ , the gradients $\nabla g_{i_1}(\xi), \dots, \nabla g_{i_l}(\xi)$ are linearly independent.

- *Strict Complementary Condition (SCC)*: for the active constraints g_1, \dots, g_m at ξ , there exist $a_1, \dots, a_m \in \mathbb{R}$ with $a_i > 0$ if g_i is not an equality constraint such that $\nabla f = a_1 \nabla g_1(\xi) + \dots + a_m \nabla g_m(\xi)$.
- *Second Order Sufficiency Condition (SOSC)*: for $L(x) = f(x) - \sum_{j=1}^m a_j g_j$ with $a_i > 0$ if g_i is not an equality constraint, we have $\forall v \in \langle \nabla g_1(\xi), \dots, \nabla g_m(\xi) \rangle^\perp, v \neq 0, v^t \nabla^2 L(\xi) v > 0$.

If these conditions are satisfied, then the BHC conditions are satisfied with the active sign constraints at ξ as regular parameters $t_1 = g_1, \dots, t_k = g_k$, see [Nie14].

Notice that when BHC hold, the minimizers are non-singular, isolated points and thus finite.

It is proved in [Mar06] that if BHC holds at every minimizer of f on $\mathcal{S}(\mathbf{g})$ then $f - f^* \in \mathcal{Q}(\mathbf{g})$, which implies that the SoS relaxation is exact. [Nie14] proved that the BHC at every minimizer of f , which hold generically, implies the SoS finite convergence property.

In this section, we prove that, if the BHC hold, then the MoM relaxation is exact. We need some preliminary lemmas.

Lemma 4.2. *Let $p, g \in \mathbb{R}[\mathbf{X}]$, $k \geq \deg p + \deg g$ and $d \geq 2k + \deg g$. If $\sigma = \sigma^{[d]} \in \mathcal{L}_d(\mathbf{g})$ then: $\langle \sigma^{[d]} | p^2 g \rangle = 0$ implies $pg \in \text{Ann}_k(\sigma)$.*

Proof. Let $h \in \mathbb{R}[\mathbf{X}]_k$ and $\sigma = \sigma^{[d]} \in \mathcal{L}_d(\mathbf{g})$. Since $\sigma^{[d]}$ is positive on $\mathcal{Q}_d(\mathbf{g})$ and $\langle \sigma^{[d]} | p^2 g \rangle = 0$, then $\forall t \in \mathbb{R}$:

$$0 \leq \langle \sigma^{[d]} | (p + th)^2 g \rangle = t^2 \langle \sigma^{[d]} | h^2 g \rangle + 2t \langle \sigma^{[2d]} | phg \rangle.$$

As a function of t the last expression is non-negative, and equal to 0 for $t = 0$. Then $t = 0$ must be a double root, and thus $\langle \sigma^{[d]} | phg \rangle = \langle (fg \star \sigma)^{[k]} | h \rangle = 0$ for all $h \in \mathbb{R}[\mathbf{X}]_k$. But this means $pg \in H_G^k$. \square

Lemma 4.3. *Let $f \in \mathcal{Q}_l(\mathbf{g})$. Then for k and $d \geq 2k$ big enough, if $\sigma \in \mathcal{L}_d(\mathbf{g})$ then: $\langle \sigma | f \rangle = 0$ if and only if $f \in \text{Ann}_k(\sigma)$.*

Proof. The *if* part is obvious.

For the *only if* part, we set $g_0 = 1$ for notation convenience. Since $f \in \mathcal{Q}_l(\mathbf{g})$, then $f = \sum_i s_i g_i$, with $s_i = \sum_j p_{i,j}^2$ and $\deg s_i g_i \leq l$. Let $d \geq \max_{i,j} \{2 \deg(p_{i,j}) + \deg(g_i)\}$ and $\sigma \in \mathcal{L}_d(\mathbf{g})$. By hypothesis:

$$0 = \langle \sigma^{[d]} | f \rangle = \sum_{i,j} \langle \sigma^{[d]} | p_{i,j}^2 g_i \rangle,$$

which implies $\langle \sigma^{[d]} | p_{i,j}^2 g_i \rangle = 0$ for all i and j . Let $k_{i,j}$ and $d_{i,j}$ be given from Lemma 4.2 (applied to $p_{i,j}$ and g_i). Let $k' \geq \max_{i,j} \{k_{i,j}\}$. Then $p_{i,j} g_i \in \text{Ann}_{k'}(\sigma)$ for all i and j which implies that $p_{i,j}^2 g_i \in \text{Ann}_{k' - \deg p_{i,j}}(\sigma)$. Letting $k = \min_{i,j} \{k' - \deg p_{i,j}\}$, we finally get $p_{i,j}^2 g_i \in \text{Ann}_k(\sigma)$ for all i and j , and $f = \sum_{i,j} p_{i,j}^2 g_i \in \text{Ann}_k(\sigma)$. \square

Theorem 4.4. *Let $f \in \mathbb{R}[\mathbf{X}]$, $Q = \mathcal{Q}(\mathbf{g})$ be an Archimedean finitely generated quadratic module and assume that the BHC hold at every minimizer of f on $S = \mathcal{S}(\mathbf{g})$. Then the moment relaxation $(\mathcal{L}_{2d}(\mathbf{g}))_{d \in \mathbb{N}}$ is exact. For $t \in \mathbb{N}$ and $d, e \geq t$ big enough:*

$$\mathcal{L}_{2d}^{\min}(\mathbf{g})^{[2t]} = \mathcal{L}_{2e}(\mathbf{g}, \pm(f - f^*))^{[2t]} = \text{conv}(\mathbf{e}_{\xi_1}, \dots, \mathbf{e}_{\xi_r})^{[2t]}.$$

where $\{\xi_1, \dots, \xi_r\}$ is the finite set of minimizers of f on S . Moreover, if $d \geq t \geq \rho(\xi_1, \dots, \xi_r)$, d big enough, and $\sigma^* \in \mathcal{L}_{2d}^{\min}(\mathbf{g})$ is generic, then $(\text{Ann}_t(\sigma^*)) = \mathcal{I}(\xi_1, \dots, \xi_r)$ is the vanishing ideal of the minimizers of f on S .

Proof. We can assume without loss of generality that $f^* = 0$. For d, e big enough, if $\sigma \in \mathcal{L}_{2d}^{\min}(\mathbf{g})$ then $f \in \text{Ann}_e(\sigma)$ by Lemma 4.3. This implies that $\mathcal{L}_{2d}^{\min}(\mathbf{g})^{[2e]} \subset \mathcal{L}_{2e}(\mathbf{g}, \pm f)$. Since the BHC hold, we know that $\dim \frac{\mathbb{R}[\mathbf{X}]}{\text{supp}(Q+(f))} = 0$ (see the proof of [Mar06, th. 2.3]). By Theorem 3.19 applied

to $\mathcal{L}_{2e}(\mathbf{g}, \pm f)$, we have $\mathcal{L}_{2e}(\mathbf{g}, \pm f)^{[2t]} = \text{conv}(\mathbf{e}_{\xi_1}, \dots, \mathbf{e}_{\xi_r})^{[2t]}$ for $t \in \mathbb{N}$ and e big enough. Since $\text{conv}(\mathbf{e}_{\xi_1}, \dots, \mathbf{e}_{\xi_r})^{[2t]} \subset \mathcal{L}_{2d}^{\min}(\mathbf{g})^{[2t]}$ by definition, we proved the first part: up to restriction, functional minimizers are coming from convex sums of evaluations at the minimizers of f .

We prove that for $d \geq t \geq \rho(\xi_1, \dots, \xi_r)$ big enough and $\sigma^* \in \mathcal{L}_{2d}^{\min}(\mathbf{g})$ generic, $(\text{Ann}_t(\sigma^*)) = \mathcal{I}(\xi_1, \dots, \xi_r)$ by applying Theorem 3.19 to $\mathcal{L}_{2d}^{\min}(\mathbf{g})^{[2t]} = \mathcal{L}_{2e}(\mathbf{g}, \pm(f - f^*))^{[2t]} = \text{conv}(\mathbf{e}_{\xi_1}, \dots, \mathbf{e}_{\xi_r})^{[2t]}$ for $e \in \mathbb{N}$ big enough. \square

Remark. The conclusions of Theorem 4.4 remain valid if

- $f - f^* \in Q$ (i.e. SoS exactness);
- $\dim \frac{\mathbb{R}[\mathbf{X}]}{\text{supp}(Q + (f - f^*))} = 0$

using Lemma 4.3 and the proof of Theorem 4.4.

We show now that moment exactness holds *generically*. For polynomials $f \in \mathbb{R}[\mathbf{X}]_d$ and $g_1 \in \mathbb{R}[\mathbf{X}]_{d_1}, \dots, g_s \in \mathbb{R}[\mathbf{X}]_{d_s}$, we say that a property holds generically (or that the property holds for generic f, g_1, \dots, g_s) if there exists finitely many nonzero polynomials ϕ_1, \dots, ϕ_l in the coefficients of polynomials in $\mathbb{R}[\mathbf{X}]_d$ and $\mathbb{R}[\mathbf{X}]_{d_1}, \dots, \mathbb{R}[\mathbf{X}]_{d_s}$ such that, when $\phi_1(f, \mathbf{g}) \neq 0, \dots, \phi_l(f, \mathbf{g}) \neq 0$, the property holds.

Corollary 4.5. *For $f \in \mathbb{R}[\mathbf{X}]_d$ and $g_1 \in \mathbb{R}[\mathbf{X}]_{d_1}, \dots, g_s \in \mathbb{R}[\mathbf{X}]_{d_s}$ generic, the moment relaxation $(\mathcal{L}_{2d}(\mathbf{g}))_{d \in \mathbb{N}}$ is exact.*

Proof. By [Nie14, th. 1.2] BHC hold generically. We apply Theorem 4.4 to conclude. \square

5 Exact Moment Relaxations

In this section, we consider the Polynomial Optimization Problem of minimizing $f \in \mathbb{R}[\mathbf{X}]$ on a basic semialgebraic set $S = \mathcal{S}(\mathbf{g})$ where $\mathbf{g} = \{g_1, \dots, g_s\} \subset \mathbb{R}[\mathbf{X}]$. We describe cases, where the Moment Relaxation can be made exact, even if the given problem is not a regular Polynomial Optimization Problem.

5.1 Finite semialgebraic set

We assume that $S = \mathcal{S}(\mathbf{g}) = \{\xi_1, \dots, \xi_r\} \subset \mathbb{R}^n$ is non-empty and finite.

Theorem 5.1. *Let f^* denote the infimum of f on $S = \mathcal{S}(\mathbf{g})$ and let $Q = Q(\mathbf{g})$. Suppose that $\dim \frac{\mathbb{R}[\mathbf{X}]}{\text{supp} Q} = 0$. Then the moment relaxation $(\mathcal{L}_{2d}(\mathbf{g}))_{d \in \mathbb{N}}$ is exact. For $t \in \mathbb{N}$ and $d \geq t$ big enough,*

$$\mathcal{L}_{2d}^{\min}(\mathbf{g})^{[2t]} = \text{conv}(\mathbf{e}_{\xi_1}, \dots, \mathbf{e}_{\xi_r})^{[2t]},$$

where $\{\xi_1, \dots, \xi_r\} \subset \mathbb{R}^n$ is the finite set of minimizers of f on S . Moreover, if $d \geq t \geq \rho = \rho(\xi_1, \dots, \xi_r)$, d big enough, and $\sigma \in \mathcal{L}_{2d}^{\min}(\mathbf{g})$ is generic, then $(\text{Ann}_t(\sigma)) = \mathcal{I}(\xi_1, \dots, \xi_r)$ is the vanishing ideal of the minimizers $\{\xi_1, \dots, \xi_r\}$ of f on S .

Proof. By Theorem 3.19 for $d \geq t$ big enough, $\mathcal{L}_{2d}(\mathbf{g})^{[2t]} = \text{cone}(\mathbf{e}_{\xi_1}, \dots, \mathbf{e}_{\xi_r})^{[2t]}$. Then:

$$\mathcal{L}_{2d}^{\min}(\mathbf{g})^{[2t]} = \{\sigma \in \mathcal{L}_{2d}^{(1)}(\mathbf{g})^{[2t]} \mid \langle \sigma | f \rangle = f^*\} = \{\sigma \in \text{conv}(\mathbf{e}_{\xi_1}, \dots, \mathbf{e}_{\xi_r})^{[2t]} \mid \langle \sigma | f \rangle = f^*\} = \text{conv}(\mathbf{e}_{\xi_1}, \dots, \mathbf{e}_{\xi_r})^{[2t]}.$$

For the second part we apply Proposition 1.4 and Corollary 3.18 to a graded basis \mathbf{h} of $\mathcal{I}(\xi_1, \dots, \xi_r)$. \square

As corollaries, we have that the conclusions of Theorem 5.1 hold for:

- $Q(\Pi\mathbf{g}) = \mathcal{O}(\mathbf{g})$ a preordering such that $S = \mathcal{S}(\mathbf{g}) = \mathcal{S}(\Pi\mathbf{g})$ is finite, since by Theorem 1.1, $\dim \frac{\mathbb{R}[\mathbf{X}]}{\text{supp} \mathcal{O}(\mathbf{g})} = \dim \frac{\mathbb{R}[\mathbf{X}]}{\sqrt{\text{supp} \mathcal{O}(\mathbf{g})}} = \dim \frac{\mathbb{R}[\mathbf{X}]}{\mathcal{I}(\mathcal{S}(\mathbf{g}))} = 0$.

- $Q(\mathbf{g}, \pm \mathbf{h})$ with $\mathcal{V}_{\mathbb{R}}(\mathbf{h})$ is finite, since for $Q = Q(\mathbf{g}, \pm \mathbf{h})$,

$$\dim \frac{\mathbb{R}[\mathbf{X}]}{\text{supp } Q} = \dim \frac{\mathbb{R}[\mathbf{X}]}{\sqrt{\text{supp } Q}} = \dim \frac{\mathbb{R}[\mathbf{X}]}{\sqrt[\mathbb{R}]{\text{supp } Q}} \leq \dim \frac{\mathbb{R}[\mathbf{X}]}{\sqrt[\mathbb{R}]{\mathbf{h}}} = 0.$$

Notice that, even if the SoS relaxation has the finite convergence property and the MoM relaxation is exact, it may not be SoS exact for real finite varieties as shown in Example 2.11 and Example 2.12.

5.2 Finite minimizers

In this section, we consider Polynomial Optimization Problems for which the non-empty set of minimizers is finite and we propose a strategy to recover them.

If the set of minimizers is non-empty and finite, and we know the minimum f^* of f on $S = \mathcal{S}(\mathbf{g})$, by adding the equation $f - f^*$ to the definition of the truncated quadratic module, we obtain a quadratic module $Q' = Q(\mathbf{g}, \pm(f - f^*))$, which defines the finite set $\mathcal{S}(Q')$ of minimizers of f on S . We can then apply the results of Section 5.1 to the relaxation $(\mathcal{L}_{2d}(\mathbf{g}, \pm(f - f^*)))_d$ or $(\mathcal{L}_{2d}(\Pi\mathbf{g}, \pm(f - f^*)))_d$.

Corollary 5.2. *Let $f \in \mathbb{R}[\mathbf{X}]$, $Q = Q(\mathbf{g})$ be a finitely generated quadratic module. Assume that the minimizers of f on $S = \mathcal{S}(\mathbf{g})$ are finite: $\{x \in S \mid f(x) = f^*\} = \{\xi_1, \dots, \xi_r\}$. Then for any $t \in \mathbb{N}$ and $d \geq t$ big enough:*

$$\mathcal{L}_{2d}(\Pi\mathbf{g}, \pm(f - f^*))^{[2t]} = \text{conv}(\mathbf{e}_{\xi_1}, \dots, \mathbf{e}_{\xi_r})^{[2t]}.$$

Moreover, if $d \geq t \geq \rho = \rho(\xi_1, \dots, \xi_r)$ and $\sigma \in \mathcal{L}_{2d}^{\min}(\mathbf{g}, \pm(f - f^*))$ is generic, then $(\text{Ann}_t(\sigma)) = \mathcal{I}(\xi_1, \dots, \xi_r)$ is the vanishing ideal of the minimizers of f on S .

In practice, the minimum f^* is usually not known. Since the computation of moment minimizers $\sigma^* \in \mathcal{L}_d^{\min}(\mathbf{g}, \pm(f - f^*))$ is based on numerical Semi-Definite Program (SDP) solvers, we can replace f^* by an approximate value, taking for instance $f_{\text{MoM},d}^* = \inf \{ \langle \sigma | f \rangle \in \mathbb{R} \mid \sigma \in \mathcal{L}_{2d}^{(1)}(\mathbf{g}) \} \leq f^*$ for $d \in \mathbb{N}$. Notice that if $v < f^*$ then $\mathcal{L}_{2d}^{(1)}(\mathbf{g}, \pm(f - v))$ is empty since $\mathcal{S}(\Pi\mathbf{g}, \pm(f - v))$ is empty. If v is not close to f^* , the SDP solvers can detect the feasibility/infeasibility of the relaxation, that is if $\mathcal{L}_d(\Pi\mathbf{g}, \pm(f - v))$ is empty or not.

Notice also that by Theorem 1.2 $\mathcal{O}(\mathbf{g})$ is Archimedean if the semialgebraic set is finite. If also $Q(\mathbf{g})$ is Archimedean, since the SDP solvers perform approximate numerical computations, and since in this case $\mathcal{L}(Q(\mathbf{g})) = \mathcal{L}(\mathcal{O}(\mathbf{g})) = \mathcal{M}(\mathcal{S}(\mathbf{g}))$, we can also replace the relaxation associated to the preordering by the relaxation associated to the quadratic module (the convergence to moments of measures should be studied in details in the Archimedean case). This leads to Algorithm 1 for the computation of finite minimizers.

Algorithm 1: FINITE MINIMIZERS

input : $d \in \mathbb{N}$, $f, g_1, \dots, g_r \in \mathbb{R}[\mathbf{X}]_d$ such that f has a finite set of minimizers on $S = \mathcal{S}(\mathbf{g})$.

output: The minimizers $\{\xi_1, \dots, \xi_r\}$ of f on S and $f^* = \inf_{x \in S} f(x)$.

$k = \lceil \frac{d}{2} \rceil$

repeat

Compute $f_{\text{MoM},k}^* = \inf \{ \langle \sigma | f \rangle \in \mathbb{R} \mid \sigma \in \mathcal{L}_{2k}^{(1)}(\mathbf{g}) \}$.

Compute a generic element $\sigma^* \in \mathcal{L}_{2k}^{(1)}(\mathbf{g}, \pm(f - f_{\text{MoM},k}^*))$

Extract of the minimizers ξ_1, \dots, ξ_r from $H_{\sigma^*}^t$ for $t \leq k$ big enough.

$k = k + 1$

until minimizer extraction success

return the minimizers $\{\xi_1, \dots, \xi_r\}$ and $f^* = \langle \sigma^* | f \rangle$

Each loop of this algorithm requires two calls to SDP solvers. The first one is to compute $f_{\text{MoM},k}^*$ on the convex set $\mathcal{L}_{2k}^{(1)}(\mathbf{g})$. The second one is to compute an interior or generic point σ^* of $\mathcal{L}_{2k}^{(1)}(\mathbf{g}, \pm(f - f_{\text{MoM},k}^*))$, using an interior point SDP solver.

The extraction of minimizers from the Hankel matrix $H_{\sigma^*}^t$ is based on the algorithm of polynomial-exponential decomposition of series described in [Mou18]. It involves numerical linear algebra functions such as SVD, eigenvalue and eigenvector computation. It provides an approximation of the linear functional σ^* as a weighted sum of evaluations $\sigma^* \approx \sum_{i=1}^r \omega_i \mathbf{e}_{\xi_i}$. We consider that the minimizer extraction succeeds when such an approximation of σ^* is obtained within a given threshold.

If the set of minimizers is finite and the moment relaxation $(\mathcal{L}_{2d}(\mathbf{g}))_{d \in \mathbb{N}}$ is exact for f then this algorithm terminates. When this is not the case, it shall also terminate using approximate computation. Indeed, increasing the degree k , we obtain better approximations of f^* and of a generic element of $\mathcal{L}_{2d}(\Pi \mathbf{g}, \pm(f - f^*))^{[2t]} = \text{conv}(\mathbf{e}_{\xi_1}, \dots, \mathbf{e}_{\xi_r})^{[2t]}$. When a sufficiently good approximation of a generic element of $\text{conv}(\mathbf{e}_{\xi_1}, \dots, \mathbf{e}_{\xi_r})^{[2t]}$ is obtained, the minimizer extraction succeeds and Algorithm 1 outputs an approximation of the minimizers $\{\xi_1, \dots, \xi_r\}$ and the minimum f^* . We will illustrate it in Example 6.1.

5.3 Gradient, KKT and Polar ideals

Another approach which has been investigated to make the relaxations exact, is to add equality constraints satisfied by the minimizers (and independent of the minimum f^*) to a Polynomial Optimization Program.

For global optimization we can consider the gradient equations (see [NDS06]): obviously $\nabla f(x^*) = \mathbf{0}$ for all the minimizers x^* of f on $S = \mathbb{R}^n$. For constrained optimization we can consider Karush–Kuhn–Tucker (KKT) constraints, adding new variables (see [DNP07]) or projecting them to the variables \mathbf{X} (Jacobian equations, see [Nie13a]). We shortly describe them.

Let $g_1, \dots, g_r, h_1, \dots, h_s \in \mathbb{R}[\mathbf{X}]$ defining $S = \mathcal{S}(\mathbf{g}, \pm \mathbf{h})$, and let $f \in \mathbb{R}[\mathbf{X}]$ be the objective function. Let $\Lambda = (\Lambda_1, \dots, \Lambda_r)$ and $\Gamma = (\Gamma_1, \dots, \Gamma_s)$ be variables representing the *Lagrange multipliers* associated with \mathbf{g} and \mathbf{h} . The *KKT constraints* associated to the optimization problem $\min f(x)$: $x \in \mathcal{S}(\mathbf{g}, \pm \mathbf{h})$ are:

$$\begin{cases} \frac{\partial f}{\partial X_i} - \sum_{k=1}^r \Lambda_k \frac{\partial g_k}{\partial X_i} - \sum_{j=1}^s \Gamma_j \frac{\partial h_j}{\partial X_i} = 0 & \forall i \\ \Lambda_k g_k = 0, \quad h_j = 0, \quad g_k \geq 0 & \forall j, k, \end{cases} \quad (6)$$

where the polynomials belong to $\mathbb{R}[\mathbf{X}, \Gamma, \Lambda]$. These are sufficient but not necessary conditions for $x^* \in S$ being a minimizer.

For $x \in S$, we say that g_i is an *active constraint at x* if $g_i(x) = 0$. Let $x^* \in S$ and g_{i_1}, \dots, g_{i_k} be the active constraints at x^* . The KKT constraints are necessary if the Constraint Qualification Condition (CQC) holds, that is, if $\nabla h_1(x^*), \dots, \nabla h_s(x^*), \nabla g_{i_1}(x^*), \dots, \nabla g_{i_k}(x^*)$ are linearly independent at the minimizer $x^* \in S$ (also called Linear Independence Constraint Qualification in [NW06, th. 12.1]). We cannot avoid the CQC hypothesis: for example if $f = X_1 \in \mathbb{R}[X_1]$ and $g_1 = X_1^3 \in \mathbb{R}[X_1]$, then $x^* = 0$ is a minimizer, but the KKT equations are not satisfied at $x^* = 0$.

To avoid this problem we define the *polar ideal*. Observe from eq. (6) that, if KKT constraints are satisfied at x and

- if g_i is not an active constraint at x , then $\Lambda_i = 0$;
- if g_{i_1}, \dots, g_{i_k} are the active constraints at x , then the gradients $\nabla f(x), \nabla h_1(x), \dots, \nabla h_r(x), \nabla g_{i_1}(x), \dots, \nabla g_{i_k}(x)$ are linearly dependent.

Definition 5.3. For $f, g_1, \dots, g_r, h_1, \dots, h_s \in \mathbb{R}[\mathbf{X}]$ as before, the *polar ideal* is defined as follows:

$$J := (\mathbf{h}) + \prod_{\{a_1, \dots, a_k\} \subset \{1, \dots, r\}} \left((g_{a_1}, \dots, g_{a_k}) + \left(\text{rank Jac}(f, \mathbf{h}, g_{a_1}, \dots, g_{a_k}) \right) < s + k + 1 \right).$$

where $((\text{rank Jac}(f, \mathbf{h}, g_{a_1}, \dots, g_{a_k}) < l)$ is the ideal generated by the $l \times l$ minors of the Jacobian matrix $\text{Jac}(f, \mathbf{h}, g_{a_1}, \dots, g_{a_k})$. The generators of J besides \mathbf{h} are the product of active constraints and the generators of rank ideals.

In this definition, we could replace the product of ideals by their intersection and the $l \times l$ minors of the Jacobian matrices by polynomials defining the same varieties.

We prove that every minimizer belongs to $\mathcal{V}_{\mathbb{R}}(J)$.

Lemma 5.4. *Let x^* be a minimizer of f on $S = \mathcal{S}(\mathbf{g}, \pm \mathbf{h})$. Then $x^* \in \mathcal{V}_{\mathbb{R}}(J)$.*

Proof. Since $x^* \in S$, then $x^* \in \mathcal{V}_{\mathbb{R}}(\mathbf{h})$.

If the CQC hold at x^* , then x^* is a KKT point (see [NW06, th. 12.1]) and $\nabla f(x) = \sum_j \gamma_j \nabla \mathbf{h}_j(x) + \sum_j \lambda_j^2 \nabla \mathbf{g}_j(x)$ for some γ_j and λ_j in \mathbb{R} . As $\lambda_k = 0$ if g_k is not an active constraint, we have that

$$\nabla f(x^*), \nabla h_1(x^*), \dots, \nabla h_r(x^*), \nabla g_{i_1}(x^*), \dots, \nabla g_{i_k}(x^*)$$

are linearly dependent, where g_{i_1}, \dots, g_{i_k} are the active constraints at x^* . Thus $x^* \in \mathcal{V}_{\mathbb{R}}(g_{a_1}, \dots, g_{a_k})$ and $(\text{rank Jac}(f, h_1, \dots, h_s, g_{a_1}, \dots, g_{a_k})(x^*)) < s + k + 1$. This implies $x^* \in \mathcal{V}_{\mathbb{R}}(J)$.

If the CQC do not hold at x^* and g_{i_1}, \dots, g_{i_k} are the active constraints, then the gradients $\nabla h_1(x^*), \dots, \nabla h_s(x^*)$ and $\nabla g_{i_1}(x^*), \dots, \nabla g_{i_k}(x^*)$ are linearly dependent. This implies that $\nabla f(x^*), \nabla h_1(x^*), \dots, \nabla h_s(x^*)$ and $\nabla g_{i_1}(x^*), \dots, \nabla g_{i_k}(x^*)$ are also linearly dependent, and we conclude as in the previous case. \square

Theorem 5.5. *Let $Q = \mathcal{Q}(\mathbf{g}, \pm \mathbf{h})$ and $J = (\mathbf{h}')$ be the polar ideal, where $\mathbf{h}' \subset \mathbb{R}[\mathbf{X}]$ is a finite set of generators. If $\mathcal{V}_{\mathbb{R}}(J)$ is finite then the moment relaxation $(\mathcal{L}_{2d}(\mathbf{g}, \pm \mathbf{h}'))_{d \in \mathbb{N}}$ is exact.*

Proof. Minimizers belongs to $\mathcal{V}_{\mathbb{R}}(J)$ by Lemma 5.4. Then MoM exactness follows from Theorem 5.1. \square

The assumption in [NDS06], [DNP07] and [Nie13a] for finite convergence and SoS exactness are smoothness conditions or radicality assumptions on the associated complex variety. Our condition for MoM exactness is of a different nature, since it is on the finiteness of the real polar variety (see Example 6.4).

Notice that by taking equations \mathbf{h}' such that $(\mathbf{h}') = \sqrt[\mathbb{R}]{J}$ instead of generators of J , we have the same MoM relaxation (by Lemma 3.10 and following remark). Then the SoS exactness property under the (real) radicality assumption implies SoS exactness for the extended relaxation $(\mathcal{Q}_{2d}(\mathbf{g}, \mathbf{h}'))_{d \in \mathbb{N}}$.

6 Examples

We give some examples where we compute the minimum and the minimizers for some POP, which MoM relaxation is exact. Computations were performed with the Julia package `MomentTools.jl`² using the SDP solver Mosek, based on an interior point method.

Example 6.1 (Motzkin polynomial). We find the global minimizers of the bivariate Motzkin polynomial $f = x^4 y^2 + x^2 y^4 - 3x^2 y^2 + 1$. This is an example of a (globally) positive polynomial which is not sum of squares (and then the SoS relaxation cannot be exact). Its minimum is $f^* = 0$ and the four minimizers are $(\pm 1, \pm 1) \in \mathbb{R}^2$ (see [Rez96]).

```
v0, M = minimize(f, [], [], X, 4, Mosek.Optimizer)
```

Here $f_{\text{MoM},4}^* \approx v_0 = -1.23437 \cdot 10^{-10}$, but we cannot recover the minimizers: exactness does not hold. We add the constraint $f - f_{\text{MoM},4}^* = 0$ to find them, i.e. use $\mathcal{L}_d(\pm(f - f_{\text{MoM},4}^*))$.

²<https://gitlab.inria.fr/AlgebraicGeometricModeling/MomentTools.jl>

```
v1, M = minimize(f, [f-v0], [], X, 4, Mosek.Optimizer)
```

Here the new optimum if $v_1 \approx 1.84908 \cdot 10^{-10}$. In this case the approximation of the minimum is of the same order as before, but we can recover the minimizers by Corollary 5.2:

```
w, Xi = get_measure(M)
```

We obtain the following approximation of the 4 minimizers:

$$\begin{aligned} \xi_1 &= (1.0000009448913, 1.00000094519956) & \xi_2 &= (1.00000094499890, -1.00000094499890) \\ \xi_3 &= (-1.0000009448913, 1.00000094499890) & \xi_4 &= (-1.000000945184, -1.00000094519956). \end{aligned}$$

Example 6.2 (Robinson form). We find the minimizers of the Robinson form $f = x^6 + y^6 + z^6 + 3x^2y^2z^2 - x^4(y^2 + z^2) - y^4(x^2 + z^2) - z^4(x^2 + y^2)$ on the unit sphere $h = x^2 + y^2 + z^2 - 1$. The Robinson polynomial has minimum $f^* = 0$ (globally and on the unit sphere), and the minimizers on $\mathcal{V}_{\mathbb{R}}(h)$ are:

$$\frac{\sqrt{3}}{3}(\pm 1, \pm 1, \pm 1), \frac{\sqrt{2}}{2}(0, \pm 1, \pm 1), \frac{\sqrt{2}}{2}(\pm 1, 0, \pm 1), \frac{\sqrt{2}}{2}(\pm 1, \pm 1, 0).$$

BHC are satisfied at every minimizer (see [Nie14, ex. 3.2]) and we can recover the minimizers by Theorem 4.4.

```
v, M = minimize(f, [h], [], X, 5, Mosek.Optimizer)
w, Xi = get_measure(M)
```

Here $f_{\text{MoM},5}^* \approx v = -1.27211 \cdot 10^{-7}$ and the minimizers with positive coordinates are (all the twenty minimizers are found):

$$\begin{array}{llll} \xi_x: & 0.577351068999 & 8.812477930640 \cdot 10^{-12} & 0.707107158043 & 0.707107157553 \\ \xi_y: & 0.577351069076 & 0.707107158048 & 1.271729446125 \cdot 10^{-13} & 0.707107157555 \\ \xi_z: & 0.577351066102 & 0.707107158048 & 0.707107158042 & 2.478771201340 \cdot 10^{-9} \end{array}$$

Example 6.3 (Gradient ideal). We compute the minimizers of Example 2.12. Let $f = (X^4Y^2 + X^2Y^4 + Z^6 - 2X^2Y^2Z^2) + X^8 + Y^8 + Z^8 \in \mathbb{R}[X, Y, Z]$. We want to minimize f over the gradient variety $\mathcal{V}_{\mathbb{R}}\left(\frac{\partial f}{\partial X}, \frac{\partial f}{\partial Y}, \frac{\partial f}{\partial Z}\right)$.

```
v, M = minimize(f, differentiate(f,X), [], X, 4, Mosek.Optimizer)
w, Xi = get_measure(M, 2.e-2)
```

The approximation of the minimum $f^* = 0$ is $v = -1.6279 \cdot 10^{-9}$, and the decomposition with a threshold of $2 \cdot 10^{-2}$ gives the following numerical approximation of the minimizer (the origin):

$$\xi = (2.976731510689691 \cdot 10^{-17}; -9.515032317137384 \cdot 10^{-19}; 3.763401209219283 \cdot 10^{-18}).$$

Example 6.4 (Singular minimizer). We minimize $f = x$ on the compact semialgebraic set $S = \mathcal{S}(x^3 - y^2, 1 - x^2 - y^2)$. The only minimizer is the origin, which is a singular point of the boundary of S . Thus BHC do not hold. The regularity conditions for the Jacobian and KKT constraints are not satisfied, but the real polar variety is finite. Adding the polar constraints, we have an exact MoM relaxation. We can recover an approximation of the minimizer from the MoM relaxation of order 5:

```
v, M = polar_minimize(f, [], [x^3-y^2, 1-x^2-y^2], X, 5, Mosek.Optimizer)
w, Xi = get_measure(M, 2.e-3)
```

The approximation of the minimum $f^* = 0$ is $v = -0.0045$, and the decomposition of the moment sequence with a threshold of $2 \cdot 10^{-3}$ gives the following approximation of the minimizer (the origin):

$$\xi = (-0.004514367348787526, 2.1341684460860045 \cdot 10^{-21}).$$

The error of approximation on the minimizer is of the same order than the error on the minimum f^* .

References

- [BCR98] Jacek Bochnak, Michel Coste, and Marie-Francoise Roy. *Real Algebraic Geometry*. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge / A Series of Modern Surveys in Mathematics. Berlin Heidelberg: Springer-Verlag, 1998. ISBN: 978-3-540-64663-1.
- [Ber99] Dimitri P. Bertsekas. *Nonlinear Programming*. en. Athena Scientific, 1999. ISBN: 978-1-886529-00-7.
- [BS87] David Bayer and Michael Stillman. “A criterion for detectingm-regularity”. *Inventiones Mathematicae* 87.1 (1987), pp. 1–11.
- [CF98] Raúl E. Curto and Lawrence A. Fialkow. *Flat Extensions of Positive Moment Matrices: Recursively Generated Relations*. American Mathematical Soc., 1998. 73 pp. ISBN: 978-0-8218-0869-6.
- [CLO15] David A. Cox, John B. Little, and Donal O’Shea. *Ideals, varieties, and algorithms: an introduction to computational algebraic geometry and commutative algebra*. Fourth edition. Undergraduate texts in mathematics. Cham Heidelberg New York Dordrecht London: Springer, 2015. ISBN: 978-3-319-16720-6 978-3-319-16721-3.
- [DNP07] James Demmel, Jiawang Nie, and Victoria Powers. “Representations of positive polynomials on noncompact semialgebraic sets via KKT ideals”. *Journal of Pure and Applied Algebra* 209.1 (2007), pp. 189–200.
- [Eis05] David Eisenbud. *The Geometry of Syzygies: A Second Course in Algebraic Geometry and Commutative Algebra*. Graduate Texts in Mathematics. New York: Springer-Verlag, 2005. ISBN: 978-0-387-22215-8.
- [HL05] Didier Henrion and Jean bernard Lasserre. “Detecting global optimality and extracting solutions in GloptiPoly”. *Chapter in D. Henrion, A. Garulli (Editors). Positive polynomials in control. Lecture Notes in Control and Information Sciences*. Springer Verlag, 2005.
- [JH16] Cédric Jozz and Didier Henrion. “Strong duality in Lasserre’s hierarchy for polynomial optimization”. *Optimization Letters* 10.1 (2016), pp. 3–10.
- [KS19] Tom-Lukas Kriel and Markus Schweighofer. “On the Exactness of Lasserre Relaxations and Pure States Over Real Closed Fields”. *Foundations of Computational Mathematics* 19.6 (2019), pp. 1223–1263.
- [Las01] Jean B. Lasserre. “Global Optimization with Polynomials and the Problem of Moments”. *SIAM Journal on Optimization* 11.3 (2001), pp. 796–817.
- [Las10] Jean-Bernard Lasserre. *Moments, positive polynomials and their applications*. Imperial College Press optimization series v. 1. London : Signapore ; Hackensack, NJ: Imperial College Press ; Distributed by World Scientific Publishing Co, 2010. ISBN: 978-1-84816-445-1.

- [Las+13] Jean-Bernard Lasserre, Monique Laurent, Bernard Mourrain, Philipp Rostalski, and Philippe Trébuchet. “Moment matrices, border bases and real radical computation”. *Journal of Symbolic Computation* 51 (2013), pp. 63–85.
- [Las15] Jean Bernard Lasserre. *An Introduction to Polynomial and Semi-Algebraic Optimization*. Cambridge: Cambridge University Press, 2015. ISBN: 978-1-107-44722-6.
- [Lau07] Monique Laurent. “Semidefinite representations for finite varieties”. *Mathematical Programming* 109.1 (2007), pp. 1–26.
- [Lau09] Monique Laurent. “Sums of squares, moment matrices and optimization over polynomials”. *Emerging applications of algebraic geometry*. Vol. 149. IMA Volumes in Mathematics and Its Applications. Springer, 2009, pp. 157–270.
- [LLR08] Jean Bernard Lasserre, Monique Laurent, and Philipp Rostalski. “Semidefinite Characterization and Computation of Zero-Dimensional Real Radical Ideals”. *Foundations of Computational Mathematics* 8.5 (2008), pp. 607–647.
- [LM09] Monique Laurent and Bernard Mourrain. “A Generalized Flat Extension Theorem for Moment Matrices”. *Archiv der Mathematik* 93.1 (2009), pp. 87–98.
- [Mac16] Francis S. Macaulay. *The Algebraic Theory of Modular Systems*. Cambridge University Press, 1916. 148 pp. ISBN: 978-0-521-45562-6.
- [Mar03] M. Marshall. “Optimization of Polynomial Functions”. en. *Canadian Mathematical Bulletin* 46.4 (2003), pp. 575–587.
- [Mar06] Murray Marshall. “Representations of Non-Negative Polynomials Having Finitely Many Zeros”. *Annales de la faculté des sciences de Toulouse Mathématiques* 15.3 (2006), pp. 599–609.
- [Mar08] Murray Marshall. *Positive Polynomials and Sums of Squares*. American Mathematical Soc., 2008. ISBN: 978-0-8218-7527-8.
- [Mar09] M. Marshall. “Representations of Non-Negative Polynomials, Degree Bounds and Applications to Optimization”. *Canadian Journal of Mathematics* 61.1 (2009), pp. 205–221.
- [Mou18] Bernard Mourrain. “Polynomial–Exponential Decomposition From Moments”. *Foundations of Computational Mathematics* 18.6 (2018), pp. 1435–1492.
- [MT05] B. Mourrain and P. Trébuchet. “Generalized normal forms and polynomials system solving”. *ISSAC: Proceedings of the ACM SIGSAM International Symposium on Symbolic and Algebraic Computation*. Ed. by M. Kauers. 2005, pp. 253–260.
- [NDS06] Jiawang Nie, James Demmel, and Bernd Sturmfels. “Minimizing Polynomials via Sum of Squares over the Gradient Ideal”. *Mathematical Programming* 106.3 (2006), pp. 587–606.
- [Nie13a] Jiawang Nie. “An exact Jacobian SDP relaxation for polynomial optimization”. *Mathematical Programming* 137.1-2 (2013), pp. 225–255.
- [Nie13b] Jiawang Nie. “Certifying convergence of Lasserre’s hierarchy via flat truncation”. en. *Mathematical Programming* 142.1 (2013), pp. 485–510.
- [Nie13c] Jiawang Nie. “Polynomial Optimization with Real Varieties”. *SIAM Journal on Optimization* 23.3 (2013), pp. 1634–1646.
- [Nie14] Jiawang Nie. “Optimality conditions and finite convergence of Lasserre’s hierarchy”. *Mathematical Programming* 146.1-2 (2014), pp. 97–121.
- [NW06] Jorge Nocedal and S. Wright. *Numerical Optimization*. 2nd ed. Springer Series in Operations Research and Financial Engineering. New York: Springer-Verlag, 2006. ISBN: 978-0-387-30303-1.
- [Par02] Pablo A. Parrilo. *An Explicit Construction of Distinguished Representations of Polynomials Nonnegative Over Finite Sets*. 2002.

- [Put93] Mihai Putinar. “Positive Polynomials on Compact Semi-algebraic Sets”. *Indiana University Mathematics Journal* 42.3 (1993), pp. 969–984.
- [Rez96] Bruce Reznick. “Some Concrete Aspects Of Hilbert’s 17th Problem”. In *Contemporary Mathematics*. American Mathematical Society, 1996, pp. 251–272.
- [Sch00] Claus Scheiderer. “Sums of squares of regular functions on real algebraic varieties”. *Transactions of the American Mathematical Society* 352.3 (2000), pp. 1039–1069.
- [Sch05a] Claus Scheiderer. “Distinguished representations of non-negative polynomials”. *Journal of Algebra* 289.2 (2005), pp. 558–573.
- [Sch05b] Claus Scheiderer. “Non-existence of degree bounds for weighted sums of squares representations”. *Journal of Complexity* 21.6 (2005), pp. 823–844.
- [Sch05c] Markus Schweighofer. “Optimization of Polynomials on Compact Semialgebraic Sets”. *SIAM Journal on Optimization* 15.3 (2005), pp. 805–825.
- [Sch06] Claus Scheiderer. “Sums of squares on real algebraic surfaces”. *manuscripta mathematica* 119.4 (2006), pp. 395–410.
- [Sch17] Konrad Schmüdgen. *The Moment Problem*. Graduate Texts in Mathematics. Springer International Publishing, 2017. ISBN: 978-3-319-64545-2.
- [Sch91] Konrad Schmüdgen. “TheK-moment problem for compact semi-algebraic sets”. *Mathematische Annalen* 289.1 (1991), pp. 203–206.