Exact Moment Representation in Polynomial Optimization

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In memory of Carlo Casolo

Abstract

We investigate the problem of representation of moment sequences by measures in Polynomial Optimization Problems, consisting in finding the infimum f^* of a real polynomial f on a real semialgebraic set S defined by a quadratic module $\mathcal{Q}$. We analyse the exactness of Moment Matrix (MoM) relaxations, dual to the Sum of Squares (SoS) relaxations, which are hierarchies of convex cones introduced by Lasserre to approximate measures and positive polynomials. We show that the MoM relaxation coincides with the dual of the SoS relaxation extended with the real radical of the support of the associated quadratic module $\mathcal{Q}$. We prove that the vanishing ideal of the semialgebraic set S is generated by the kernel of the Hankel operator associated to a generic element of the truncated moment cone for a sufficiently high order of the MoM relaxation. When the quadratic module $\mathcal{Q}$ is Archimedean, we show the convergence, in Hausdorff distance, of the convex sets of the MoM relaxations to the convex set of probability measures supported on S truncated in a given degree. We prove the exactness of MoM relaxation when S is finite and when regularity conditions, known as Boundary Hessian Conditions, hold on the minimizers. This implies that MoM exactness holds generically. When the set of minimizers is finite, we describe a MoM relaxation which involves f^*, show its MoM exactness and propose a practical algorithm to achieve MoM exactness. We prove that if the real variety of polar points is finite then the MoM relaxation extended with the polar constraints is exact. Effective numerical computations illustrate this MoM exactness property.

Contents

1 Introduction

Let $f, g_1, \ldots, g_s \in \mathbb{R}[X_1, \ldots, X_n]$ be polynomials in the indeterminates $X_1, \ldots, X_n$ with real coefficients. The goal of Polynomial Optimization is to find:

$$f^* := \inf \left\{ f(x) \in \mathbb{R} \mid x \in \mathbb{R}^n,\ g_i(x) \geq 0 \ \text{ for } i = 1, \ldots, s \right\}. \tag{1}$$

that is the infimum f^* of the *objective function* f on the *basic semialgebraic set* $S := \{x \in \mathbb{R}^n \mid\ g_i(x) \geq 0 \ \text{ for } i = 1, \ldots, s\}$. It is a general problem, which appears in many contexts (e.g. real solution of polynomial equations, ...) and with many applications. To cite a few of them: in combinatorics, network optimization design, control, ... See for instance [Las10].

To solve this NP hard problem, Lasserre [Las01] proposed to use two hierarchies of finite dimensional convex cones depending on an order $d \in \mathbb{N}$ and he proved, for Archimedean quadratic modules, the convergence when $d \to \infty$ of the optima associated to these hierarchies to the minimum f^* of f on S. The first hierarchy replaces non-negative polynomials by Sums of Squares (SoS) and non-negative polynomials on S by polynomials of degree $\leq d$ in the truncated quadratic module $\mathcal{Q}_d(\mathbf{g})$ generated by $\mathbf{g} = \{g_1, \ldots, g_s\}$.

The second and dual hierarchy replaces positive measures by linear functionals $\in \mathcal{L}_d(\mathbf{g})$ which are non-negative on the polynomials of the truncated quadratic module $\mathcal{Q}_d(\mathbf{g})$. We will describe more precisely these constructions in section 2.1.

This approach has many interesting properties (see e.g. [Las15], [Lau09], [Mar08]). But it also raises a challenging question, of practical importance: *Can the solution of* (1) *be recovered at a finite order of these convex relaxations ?* The aim is to recover the infimum f^* and, if this infimum is reached, the minimizer set $\{\xi \in S \mid f(\xi) = f^*\}$.

To answer this question, one can first address the finite convergence problem, that is when the value f^* can be obtained at a given order of the relaxation(s). The second problem is the exactness of the relaxations, which is the main topic of this paper. The Sum of Squares (SoS) exactness is when the non-negative polynomial $f - f^*$ belongs to the truncated quadratic module $\mathcal{Q}_{2d}(\mathbf{g})$ for some $d \in \mathbb{N}$. The Moment Matrix (MoM) exactness is when an optimal linear functional $\sigma^* \in \mathcal{L}_{2d}(\mathbf{g})$ for f is coming from a positive measure supported on S for some $d \in \mathbb{N}$. We are going to investigate in details this MoM exactness property.

Several works have been developed over the last decades to tackle these problems. [Par02] showed that if the complex variety $\mathcal{V}_{\mathbb{C}}(I)$ defined by an ideal I generated by real polynomials is finite and I is radical, then $f - f^*$ has a representation as a sum of squares modulo I and the SoS relaxation is exact. [Lau07] showed the finite convergence property if the complex variety $\mathcal{V}_{\mathbb{C}}(I)$ is finite, and a moment sequence representation property, if moreover the ideal I is radical. [Nie13b] showed that if the semialgebraic set S is finite, then the finite convergence property holds for a finitely generated preordering defining S. [LLR08] proved that if S is finite, the value f^* and the minimizers can be recovered from moment matrices associated to the truncated preordering defining S. In [Las+13], the kernel of moment matrices is used to compute a border basis of the real radical ideal $\sqrt[\mathbb{R}]{I}$ when $S = \mathcal{V}_{\mathbb{R}}(I)$ is finite. [Sch05a] proved that $f - f^*$ is in the quadratic module $\mathcal{Q}$ defining S modulo $(f - f^*)^2$ if and only if $f - f^* \in \mathcal{Q}$ and then the SoS relaxation is exact. [Mar06], [Mar09] proved that under some regularity conditions on the minimizers, known as Boundary Hessian Conditions (BHC), $f - f^*$ is in the quadratic module and the SoS exactness property holds. [NDS06], [DNP07] showed that adding gradient constraints when $S = \mathbb{R}^n$ or KKT constraints when S is a general basic semialgebraic set, the SoS exactness property holds when

the corresponding Jacobian ideal is radical. [Nie13a] showed that adding the Jacobian constraints, the finite convergence property holds under some regularity assumption on the complex variety associated to these constraints and on the compactness of S. In [Nie14], it is shown that BHC imply finite convergence and that BHC are generic. [KS19] showed the SoS exactness property if the quadratic module defining S is Archmedian and some strict concavity properties of f at the finite minimizers are satisfied.

Though many works focussed on the SoS relaxation and on the representation of positive polynomials with sums of squares, the MoM relaxation has been much less studied. It has interesting features, that deserve a deeper exploration: the convex cones $\mathcal{L}_d(\mathbf{g})$ of truncated non-negative linear functionals are closed; finite convergence can be decided by flat extension tests on moment matrices [CF98], [LM09]; finite minimizers can be extracted from moment matrices [HL05], [Mou18]. On the other hand, exact SoS relaxations can provide certificates of positivity, which is also interesting from a theoretical and practical point of view.

In this paper, we investigate the truncated moment relaxation from a new perspective, developing a theoretical and computational study of truncated positive linear functionals. We analyse in details the properties of moment relaxations and present new results on the representation of moments of positive linear functionals as moments of measures.

We first show in Theorem 3.9 that the MoM relaxation $(\mathcal{L}_{2d}(\mathbf{g}))_{d\in\mathbb{N}}$ dual to the SoS relaxation $(\mathcal{Q}_{2d}(\mathbf{g}))_{d\in\mathbb{N}}$ is the same as the one associated to the quadratic module $\mathcal{Q}$ extended with the real radical of the support of $\mathcal{Q}$. This yields the vanishing ideal of S as the ideal generated by the kernel of the Hankel operator H_σ associated to a generic element $\sigma \in \mathcal{L}_{2d}(\mathbf{g})$ for d sufficiently large (see Theorem 3.16).

The second result concerns the convergence of truncated moment sequences. When the quadratic module $\mathcal{Q}$ defining S is Archimedean, the optima of the SoS and MoM relaxations converge to the minimum f^* of f on S [Las01]. We generalize this result in Theorem 3.19, showing that the convex sets $\mathcal{L}_d^{(1)}(\mathbf{g})$ of linear functionals σ non-negative on $\mathcal{Q}_d(\mathbf{g})$ such that $\sigma(1) = 1$ truncated in degree t converge, in Hausdorff distance, to the probability measures supported on S truncated in degree t, when $d \to \infty$.

Our main result on exact moment representations is given in Theorem 3.23. When S is finite and the quotient by the support of $\mathcal{Q}$ is of dimension zero, we prove that the linear functionals in $\mathcal{L}_d(\mathbf{g})$ truncated in a degree greater than twice the regularity of the points in S coincide with the measures supported on S, that is the convex hull of the evaluations at the points of S. Moreover, the ideal generated by the kernel of the Hankel operator of a generic element in $\mathcal{L}_d(\mathbf{g})$ is the vanishing ideal of S.

We apply these results to Polynomial Optimization Problems, showing in Theorem 4.1 that when the set S is finite, the MoM relaxation is exact if the quotient by the support of the quadratic module $\mathcal{Q}$ is of dimension zero. This generalizes the results of [LLR08] on semi-definite moment representations.

The main result on exactness is Theorem 4.8. We prove that when the Boundary Hessian Conditions are satisfied, the MoM relaxation is exact. This generalizes the results on SoS exactness proved in [Mar06], [Mar09]. It also shows that MoM exactness holds generically (Corollary 4.9).

When the set of minimizers is finite, we describe a MoM relaxation which involves f^*, show its MoM exactness (Corollary 4.10) and propose a practical algorithm to achieve MoM exactness using approximate numerical computation. In Theorem 4.13, we prove that if the real variety of polar points is finite then the relaxation extended with Jacobian constraints is MoM exact. This generalizes the results of finite convergence and SoS exactness of the KKT and Jacobian relaxations under regularity conditions, proved in [NDS06], [DNP07], [Nie13a].

The paper is structured as follows. In the next sections of the introduction, we define the algebraic objects that we will use and recall their main properties. In Section 2, we describe in details the notions of finite convergence and exactness for the Sum of Squares (SoS) and Moment Matrix (MoM) relaxations. We give several examples showing how these notions are related. In Section 3, we recall the properties of full moment sequences (Section 3.1), investigate the trun-

cated moment sequence properties (Section 3.2), analyse the convergence of truncated moment sequences (Section 3.3) and prove the moment representation property for a finite semialgebraic set S (Section 3.4). Finally in Section 3.5 we describe the hierarchies of truncated moments and kernels of generic elements. In Section 4, we apply these results to polynomial optimization problems on finite semialgebraic sets (Section 4.1), show that the MoM exactness property holds if the Boundary Hessian conditions are satisfied (Section 4.2) and investigate Polynomial Optimization Problems with a finite number of minimizers from a moment representation point of view (Section 4.3). In Section 4.4, we define polar ideals and analyse the MoM exactness property of the relaxation extended with these polar constraints. Examples of Polynomial Optimization Problems and numerical experimentations with the Julia package `MomentTools.jl` are presented in Section 4.5.

1.1 Polynomials

We provide the basic definitions on real polynomials and refer to [Mar08] for more details. Let $\mathbb{R}[\mathbf{X}] := \mathbb{R}[X_1,\dots,X_n]$ be the $\mathbb{R}$-algebra of polynomials in n indeterminates $X_1,\dots,X_n$. Let $\Sigma^2 = \Sigma^2[\mathbf{X}] := \left\{ f \in \mathbb{R}[\mathbf{X}] \mid \exists r \in \mathbb{N},\ g_i \in \mathbb{R}[\mathbf{X}] \colon f = g_1^2 + \cdots + g_r^2 \right\}$ be the convex cone of *Sum of Squares polynomials* (SoS). If $A \subset \mathbb{R}[\mathbf{X}]$, $A_d := \{ f \in A \mid \deg f \leq d \}$. In particular $\mathbb{R}[\mathbf{X}]_d$ is the vector space of polynomials of degree $\leq d$.

We denote $(h_1,\dots,h_r) \subset \mathbb{R}[\mathbf{X}]$ the ideal generated by $h_1,\dots,h_r \in \mathbb{R}[\mathbf{X}]$. $Q \subset \mathbb{R}[\mathbf{X}]$ is called *quadratic module* if $1 \in Q$, $\Sigma^2 \cdot Q \subset Q$ and $Q + Q \subset Q$. If in addition $Q \cdot Q \subset Q$, Q is *preordering*. For $Q \subset \mathbb{R}[\mathbf{X}]$, we define $\operatorname{supp} Q := Q \cap -Q$. If Q is a quadratic module then $\operatorname{supp} Q$ is an ideal.

We say that a quadratic module Q is *finitely generated* (f.g.) if $\exists g_1 \dots g_l \in \mathbb{R}[\mathbf{X}] \colon Q = \mathcal{Q}(g_1,\dots,g_l) := \Sigma^2 + \Sigma^2 \cdot g_1 + \cdots + \Sigma^2 \cdot g_l$ (it is the smallest quadratic module containing $g_1,\dots,g_l$). We say that a preordering O is *finitely generated* if $\exists g_1,\dots,g_l \in \mathbb{R}[\mathbf{X}] \colon O = \mathcal{O}(g_1,\dots,g_l) := \mathcal{Q}(\prod_{j\in J} g_j \mid J \subset \{1,\dots,l\})$ (it is the smallest preordering containing $g_1,\dots,g_l$).

For $G \subset \mathbb{R}[\mathbf{X}]$, let $\mathcal{Q}_t(G) := \left\{ s_0 + \sum_{j=1}^r s_j g_j \in \mathbb{R}[\mathbf{X}]_t \mid r \in \mathbb{N},\ g_j \in G,\ s_0 \in \Sigma^2_t,\ s_j \in \Sigma^2_{t-\deg g_j} \right\}$ and $\langle G \rangle_t := \left\{ \sum_{i=1}^r f_i h_i \in \mathbb{R}[\mathbf{X}]_t \mid r \in \mathbb{N},\ h_i \in G,\ f_i \in \mathbb{R}[\mathbf{X}]_{t-\deg h_i} \right\}$.

For a sequence of polynomials $\mathbf{g} := g_1,\dots,g_s$ we define $\Pi\mathbf{g} := \prod_{j\in J} g_j \colon J \subset \{1,\dots,t\}$ and $\pm\mathbf{g} := g_1,-g_1,\dots,g_r,-g_r$. Observe that $\mathcal{Q}_t(\mathbf{g},\pm\mathbf{h}) = \mathcal{Q}_t(\mathbf{g}) + \langle \mathbf{h} \rangle_{2\lfloor \frac{t}{2} \rfloor}$ and $\mathcal{Q}_t(\mathbf{\Pi}(\mathbf{g},\pm\mathbf{h})) = \mathcal{Q}_t(\mathbf{\Pi}\mathbf{g},\pm\mathbf{h})$. Notice that $\langle \mathbf{h} \rangle_t \subset (\mathbf{h})_t$ and $\mathcal{Q}_t(\mathbf{g}) \subset \mathcal{Q}(\mathbf{g})_t$, but (unluckily) these inclusions are strict in general. Finally if $A \subset \mathbb{R}[\mathbf{X}]$ we define $\mathcal{S}(A) := \left\{ x \in \mathbb{R}^n \mid f(x) \geq 0\ \forall f \in A \right\}$. In particular we denote $\mathcal{S}(\mathbf{g}) = \left\{ x \in \mathbb{R}^n \mid g(x) \geq 0\ \forall g \in \mathbf{g} \right\}$ (the *basic semialgebraic set defined by* $\mathbf{g}$). If $Q = \mathcal{Q}(\mathbf{g})$, notice that $\mathcal{S}(\mathbf{g}) = \mathcal{S}(\Pi\mathbf{g}) = \mathcal{S}(Q)$. We denote by $\operatorname{Pos}(S) = \{ f \in \mathbb{R}[\mathbf{X}] : \forall x \in S, f(x) \geq 0 \}$ the cone of positive polynomials on S.

1.2 Linear functionals

We describe the dual algebraic objects and refer to [Mou18] for more details. For $\sigma \in (\mathbb{R}[\mathbf{X}])^* = \{ \sigma \colon \mathbb{R}[\mathbf{X}] \to \mathbb{R} \mid \sigma \text{ is linear} \}$, we denote $\langle \sigma | f \rangle = \sigma(f)$ the application of σ to $f \in \mathbb{R}[\mathbf{X}]$. Recall that $(\mathbb{R}[\mathbf{X}])^* \cong \mathbb{R}[[\mathbf{Y}]] := \mathbb{R}[[Y_1,\dots,Y_n]]$, with the isomorphism given by:

$$(\mathbb{R}[\mathbf{X}])^* \ni \sigma \mapsto \sum_{\alpha \in \mathbb{N}^n} \langle \sigma | X^\alpha \rangle \frac{Y^\alpha}{\alpha!} \in \mathbb{R}[[\mathbf{Y}]],$$

where $\{\frac{Y^\alpha}{\alpha!}\}$ is the dual basis to $\{X^\alpha\}$, i.e. $\left\langle Y^\alpha \middle| X^\beta \right\rangle = \alpha! \, \delta_{\alpha,\beta}$. With this basis we can also identify $\sigma \in (\mathbb{R}[\mathbf{X}])^*$ with its sequence of coefficients $(\sigma_\alpha)_\alpha$, where $\sigma_\alpha := \langle \sigma | X^\alpha \rangle$. We will consider Borel measures with support included in $S \subset \mathbb{R}^n$, denoted as $\mathcal{M}(S)$, as linear fuctionals, i.e. $\mathcal{M}(S) \subset (\mathbb{R}[\mathbf{X}])^*$. In this case the sequence $(\mu_\alpha)_\alpha$ associated with a measure μ is the sequence of *moments*: $\mu_\alpha = \int X^\alpha \, \mathrm{d}\mu$. Moreover $\mathcal{M}^{(1)}(S)$ will denote the Borel *probability* measures supported on S. We recall a version of Haviland's theorem [Mar08, th. 3.1.2]: if $\sigma \in (\mathbb{R}[\mathbf{X}])^*$, then $\sigma \in \mathcal{M}(S)$ if and only if $\forall f \in \operatorname{Pos}(S)$, $\langle \sigma | f \rangle \geq 0$. In particular we are interested in *evaluations*: if $\xi \in \mathbb{R}^n$ then $\mathbf{e}_\xi(f) = \left\langle \mathbf{e}_\xi \middle| f \right\rangle = \int f \, \mathrm{d}\mathbf{e}_\xi = f(\xi)$ for all $f \in \mathbb{R}[\mathbf{X}]$.

If $\sigma \in (\mathbb{R}[\mathbf{X}])^*$ and $g \in \mathbb{R}[\mathbf{X}]$, we define the *convolution of g and σ* as $g \star \sigma := \sigma \circ m_g \in (\mathbb{R}[\mathbf{X}])^*$ (i.e. $\langle g \star \sigma | f \rangle = \langle \sigma | gf \rangle \ \forall f$) and the *Hankel operator* $H_\sigma \colon \mathbb{R}[\mathbf{X}] \to (\mathbb{R}[\mathbf{X}])^*$, $g \mapsto g \star \sigma$. If $\sigma = (\sigma_\alpha)_\alpha$ and $g = \sum_\alpha g_\alpha X^\alpha$ then $g \star \sigma = (\sum_\beta g_\beta \sigma_{\alpha+\beta})_\alpha$; the matrix H_σ in the basis $\{X^\alpha\}$ and $\{\frac{Y^\alpha}{\alpha!}\}$ is $H_\sigma = (\sigma_{\alpha+\beta})_{\alpha,\beta}$. Notice that $g \star \sigma = 0 \iff H_{g\star\sigma} = 0$.

We say that σ is *positive semidefinite* (psd) $\iff H_\sigma$ is psd, i.e. $\langle H_\sigma(f) | f \rangle = \left\langle \sigma \middle| f^2 \right\rangle \geq 0 \ \forall f \in \mathbb{R}[\mathbf{X}]$ (see [Sch17] or [Mar08] for basic properties of psd matrices).

If $\sigma \in (\mathbb{R}[\mathbf{X}])^*$ then $\sigma^{[t]} \in (\mathbb{R}[\mathbf{X}]_t)^*$ denotes its restriction to $\mathbb{R}[\mathbf{X}]_t$ (and same for $\sigma \in (\mathbb{R}[\mathbf{X}]_r)^*$, $r \geq t$); moreover if $B \subset (\mathbb{R}[\mathbf{X}])^*$ then $B^{[t]} := \{\sigma^{[t]} \in (\mathbb{R}[\mathbf{X}]_t)^* \mid \sigma \in B\}$ (and same for $B \subset (\mathbb{R}[\mathbf{X}]_r)^*$, $r \geq t$).

If $\sigma \in (\mathbb{R}[\mathbf{X}]_t)^*$ and $g \in \mathbb{R}[\mathbf{X}]_t$, then $g \star \sigma := \sigma \circ m_g \in (\mathbb{R}[\mathbf{X}]_{t-\deg g})^*$. If $\sigma \in (\mathbb{R}[\mathbf{X}])^*$ (or $\sigma \in (\mathbb{R}[\mathbf{X}]_r)^*$, $r \geq 2t$), then we define $H_\sigma^t \colon \mathbb{R}[\mathbf{X}]_t \to (\mathbb{R}[\mathbf{X}]_t)^*$, $g \mapsto (g \star \sigma)^{[t]}$. We have $(g \star \sigma)^{[2t]} = 0 \iff H^t_{g\star\sigma} = 0$. Notice that, if $s \leq t$, we can identify the matrix of H_σ^s with the submatrix of H_σ^t indexed by monomials of degree $\leq t$.

Let $A \subset \mathbb{R}[\mathbf{X}]$ (resp. $A \subset \mathbb{R}[\mathbf{X}]_t$). We define $A^\perp := \left\{\sigma \in (\mathbb{R}[\mathbf{X}])^* \mid \langle \sigma | f \rangle = 0 \ \forall f \in A\right\}$ (resp. $A^\perp := \left\{\sigma \in (\mathbb{R}[\mathbf{X}]_t)^* \mid \langle \sigma | f \rangle = 0 \ \forall f \in A\right\}$). Notice that $\sigma \in \langle \mathbf{h} \rangle_t^\perp$ (resp. $(\mathbf{h})^\perp$) if and only if $(h \star \sigma)^{[t-\deg h]} = 0 \ \forall h \in \mathbf{h}$ (resp. $h \star \sigma = 0 \ \forall h \in \mathbf{h}$).

For $G \subset \mathbb{R}[\mathbf{X}]_t$ we define:

$$\mathcal{L}_t(G) = \{\sigma \in (\mathbb{R}[\mathbf{X}]_t)^* \mid \forall q \in \mathcal{Q}_t(G) \ \langle \sigma | q \rangle \geq 0\}$$

Equivalently $\sigma \in \mathcal{L}_t(G)$ if and only if $\langle \sigma | s \rangle \geq 0 \ \forall s \in \Sigma_t^2$ and $\langle \sigma | sf \rangle \geq 0 \ \forall f \in G, \forall s \in \Sigma^2 \colon \deg fs \leq t$.

For the non truncated version we write $\mathcal{L}(A)$. Notice that if $Q = \mathcal{Q}(\mathbf{g})$ then $\mathcal{L}(\mathbf{g}) = \mathcal{L}(Q)$ (resp. $\mathcal{L}_t(\mathbf{g}) = \mathcal{L}_t(\mathcal{Q}_t(\mathbf{g}))$) is the *dual convex cone* to Q (resp. to $\mathcal{Q}_t(\mathbf{g})$), see [Mar08, sec. 3.6]: $\mathcal{L}(\mathbf{g}) = Q^\vee$ and $\mathcal{L}_t(\mathbf{g}) = \mathcal{Q}_t(\mathbf{g})^\vee$. We give to $\mathbb{R}[\mathbf{X}]$ and $(\mathbb{R}[\mathbf{X}])^*$ the locally convex topology defined as follows. If $V = \mathbb{R}[\mathbf{X}]$ or $V = (\mathbb{R}[\mathbf{X}])^*$ and $W \subset V$ is a finitely dimensional vector subspace, W is equipped with the Euclidean topology. We define $U \subset V$ open if and only if $U \cap W$ is open in W for every finitely dimensional vector subspace W. By conic duality: $\overline{Q} = \mathcal{L}(\mathbf{g})^\vee$ and $\overline{\mathcal{Q}_t(\mathbf{g})} = \mathcal{L}_t(\mathbf{g})^\vee$. If $A \subset V$, we denote by $\mathrm{cone}(A)$ the convex cone generated by A, by $\mathrm{conv}(A)$ its convex hull and by $\langle A \rangle$ its linear span.

1.3 Nullstellensatz and Positivestellensatz

We refer to [BCR98] and [Mar08] for real algebra and geometry. An ideal I is called *real* (or *real radical*) if $a_1^2 + \cdots + a_s^2 \in I \Rightarrow a_i \in I \ \forall i$. We define the *real radical* of an ideal I as:

$$\sqrt[\mathbb{R}]{I} := \{f \in \mathbb{R}[\mathbf{X}] \mid \exists h \in \mathbb{N}, \ s \in \Sigma^2 \quad f^{2h} + s \in I\} \tag{2}$$

$$= \{f \in \mathbb{R}[\mathbf{X}] \mid \exists k \in \mathbb{N}, \ s \in \Sigma^2 \quad f^{2^k} + s \in I\}. \tag{3}$$

Definition (2) is the classical one, and it is equivalent to (3), that will be more convenient in the paper. The real radical of I is the smallest real ideal containing I. If Q is a quadratic module and I is an ideal, we say that I is Q-convex if $\forall g_1, g_2 \in Q, \ g_1 + g_2 \in I \Rightarrow g_1, g_2 \in I$. Then I is a real ideal if and only if I is radical and Σ^2-convex.

Minimal primes lying over $\mathrm{supp}\, Q$ are Q-convex (see [Mar08, prop. 2.1.7]), and thus Σ^2-convex. Prime ideals are radical, then minimal primes lying over $\mathrm{supp}\, Q$ are real. Then $\sqrt{\mathrm{supp}\, \overline{Q}} = \sqrt[\mathbb{R}]{\mathrm{supp}\, Q}$.

If $I \subset \mathbb{R}[\mathbf{X}]$ is an ideal, we denote by $\mathcal{V}(I)$ its (complex) variety, and we define $\mathcal{V}_\mathbb{R}(I) := \mathcal{V}(I) \cap \mathbb{R}^n$. Moreover, if $S \subset \mathbb{R}^n$ we denote $\mathcal{I}(S)$ its (real) vanishing ideal.

We recall the description of the Zariski closure of basic semialgebraic sets.

Theorem 1.1 (Real Nullstellensatz, [Mar08, th. 2.2.1], [BCR98, cor. 4.4.3])**.** *Let $S = \mathcal{S}(\mathbf{g})$ be a basic semialgebraic set. Then $\mathcal{I}(S) = \sqrt[\mathbb{R}]{\mathrm{supp}\, \mathcal{O}(\mathbf{g})}$. In other words, $f = 0$ on $S \iff \exists h \in \mathbb{N} \colon -f^{2^h} \in \mathcal{O}(\mathbf{g})$.*

In particular, $\mathcal{S}(\mathbf{g})$ is empty if and only if $-1 \in \mathcal{O}(\mathbf{g})$ (and thus $\mathbb{R}[\mathbf{X}] = \mathcal{O}(\mathbf{g})$), and if I is an ideal then $\mathcal{I}(\mathcal{V}_\mathbb{R}(I)) = \sqrt{\mathrm{supp}(\Sigma^2 + I)} = \sqrt[\mathbb{R}]{I}$.

If $S \subset \mathbb{R}^n$, we denote by $\mathrm{Pos}(S)$ the convex cone of non-negative polinomials on S: $\mathrm{Pos}(S) := \{f \in \mathbb{R}[\mathbf{X}] \mid f(x) \geq 0 \ \forall x \in S\}$.

We say that a quadratic module Q is *Archimedean* if $\exists\ 0 \leq r \in \mathbb{R}$: $r - \|\mathbf{X}\|^2 \in Q$. Notice that if Q is Archimedean then $\mathcal{S}(Q)$ is compact. By [Wö98] (see also [Mar08, th. 6.1.1]) for a finitely generated preordering $O = \mathcal{O}(\mathbf{g})$, $\mathcal{S}(\mathbf{g})$ is compact if and only if O is Archimedean.

When S is compact, one can obtain an Archimedean quadratic module from $Q(\mathbf{g})$ by adding a generator $g_M = M - \|\mathbf{X}\|^2 \geq 0$, for M big enough or by adding all the products of the g_i's, replacing the generators $\mathbf{g}$ by $\Pi\mathbf{g}$.

The importance of Archimedean quadratic modules is illustrated by Schmüdgen/Putinar's characterization of strictly positive polynomials, and their solution of the moment problem (see theorem 3.1).

Theorem 1.2 (Schmüdgen / Putinar Positivestellensatz,[Sch91] [Put93])**.** *Let $\mathcal{S}(\mathbf{g})$ be a basic semi-algebraic set and suppose that $\mathcal{Q}(\mathbf{g})$ Archimedean. If $f > 0$ on $\mathcal{S}(\mathbf{g})$ then $f \in \mathcal{Q}(\mathbf{g})$.*

As a corollary one can prove that, if Q is Archimedean, then $\overline{Q} = \mathrm{Pos}(S)$.

1.4 Finite varieties, interpolation polynomials and bases

Now we move to interpolator polynomials, a tool which will be often used in the proofs.

Consider a finite set of points $\Xi = \{\xi_1, \ldots, \xi_r\} \in \mathbb{C}^n$. It is well known that it admits a family of interpolator polynomials. Such a family $(u_i) \subset \mathbb{C}[\mathbf{X}]$ is by definition such that $u_i(\xi_j) = \delta_{i,j}$. The minimal degree $\iota(\Xi)$ of a family of interpolator polynomials is called the *interpolation degree* of Ξ.

Let $\mathcal{I}(\Xi) = \{p \in \mathbb{C}[X] \mid p(\xi_i) = 0\ \forall i \in 1, \ldots, r\}$ be the complex vanishing ideal of the points Ξ. The Catelnuovo-Mumford *regularity* of an ideal I (resp. Ξ) is $\max_i(\deg S_i - i)$ where S_i is the i^{th} module of syzygies in a minimal resolution of I (resp. $\mathcal{I}(\Xi)$). Let denote it by $\rho(I)$ (resp. $\rho(\Xi)$).

Since a family of interpolator polynomials (p_i) is a basis of $\mathbb{C}[\mathbf{X}]/\mathcal{I}(\Xi)$, the ideal $\mathcal{I}(\Xi)$ is generated in degree $\leq \iota(\Xi) + 1$ and $\rho(\Xi) \leq \iota(\Xi) + 1$. A classical result [Eis05, th. 4.1] relates the interpolation degree of Ξ with its regularity, and the minimal degree of a basis of $\mathbb{C}[\mathbf{X}]/\mathcal{I}(\Xi)$. This result can be stated as follows, for real points $\Xi \subset \mathbb{R}^n$:

Proposition 1.3. *Let $\Xi = \{\xi_1, \ldots, \xi_r\} \subset \mathbb{R}^r$ with regularity $\rho(\Xi)$. Then $\iota(\Xi) = \rho(\Xi) - 1$, the minimal degree of a basis of $\mathbb{R}[\mathbf{X}]/\mathcal{I}(\Xi)$ is $\rho(\Xi) - 1$ and there exists interpolator polynomials $u_1, \ldots, u_r \in \mathbb{R}[\mathbf{X}]_{\rho(\Xi)-1}$.*

We say that $\mathbf{h} = \{h_1, \ldots, h_s\}$ is a *graded basis* of an ideal I if for all $p \in I$, there exists $q_i \in \mathbb{R}[\mathbf{X}]$ with $\deg(q_i) \leq \deg(p) - \deg(h_i)$ such that $p = \sum_{i=1}^{s} h_i q_i$. Equivalently, we have for all $t \in \mathbb{N}$, $\langle \mathbf{h} \rangle_t = I_t$.

For $p \in \mathbb{R}[\mathbf{X}]$ and I an ideal of $\mathbb{R}[\mathbf{X}]$, let $\lambda_{\deg}(p)$ be its homogeneous component of highest degree, that we call the initial of p, and let $\lambda_{\deg}(I) = (\{\lambda_{\deg}(p) \mid p \in I\})$ be the initial of I. A family $\mathbf{h} = (h_1, \ldots, h_s)$ is a graded basis of the ideal $I = (h_1, \ldots, h_s)$ iff $\lambda_{\deg}(I) = (\lambda_{\deg}(h_1), \ldots, \lambda_{\deg}(h_s))$. For more properties of graded bases, also known as H-bases, see e.g. [Mac16].

A graded basis of an ideal $I = (\mathbf{h})$ can be computed as a Grobner basis using a monomial ordering $\prec$, which refines the degree ordering (see e.g. [CLO15]). It can also be computed as a border basis for a monomial basis of least degree of $\mathbb{R}[\mathbf{X}]/I$ (see e.g. [MT05]).

The degree of a graded basis of an ideal I is bounded by its regularity $\rho(I)$ (see e.g. [BS87]).

For a set of points $\Xi = \{\xi_1, \ldots, \xi_r\}$, the ideal $\mathcal{I}(\Xi)$ has a graded (resp. Grobner, resp. border) basis of degree equal to the regularity $\rho(\Xi)$. The minimal degree of a monomial basis B of $\mathbb{R}[\mathbf{X}]/\mathcal{I}(\Xi)$ is $\iota(\Xi) = \rho(\Xi) - 1$. Such a basis B can be chosen so that it is stable by monomial division.

Proposition 1.4. *Let $\Xi = \{\xi_1, \ldots, \xi_r\} \subset \mathbb{R}^n$, $I = \mathcal{I}(\Xi)$ its real vanishing ideal and let $\rho = \rho(\Xi)$ the regularity of Ξ. For $t \geq \rho - 1$, $\sigma \in I_t^\perp$ if and only if $\sigma \in \langle \mathbf{e}_{\xi_1}^{[t]}, \ldots, \mathbf{e}_{\xi_r}^{[t]} \rangle$. Moreover if $t \geq 2(\rho - 1)$ and $\sigma \in \mathcal{L}_t(I_t)$ then $\sigma \in \mathrm{cone}(\mathbf{e}_{\xi_1}^{[t]}, \ldots, \mathbf{e}_{\xi_r}^{[t]})$.*

Proof. Let $u_1,\dots,u_r \in \mathbb{R}[\mathbf{X}]_t$ be interpolation polynomials of degree $\leq \rho - 1 \leq t$ (Proposition 1.3). Consider the sequence of vector space maps:

$$\begin{array}{rcl} 0 \to I_t \to \mathbb{R}[\mathbf{X}]_t & \xrightarrow{\psi} & \langle u_1,\dots,u_r\rangle \to 0 \\ p & \mapsto & \displaystyle\sum_{i=1}^{r} p(\xi_i)u_i, \end{array}$$

which is exact since $\ker\psi = \{p \in \mathbb{R}[\mathbf{X}]_t \mid p(\xi_i) = 0\} = I_t$. Therefore we have $\mathbb{R}[\mathbf{X}]_t = \langle u_1,\dots,u_r\rangle \oplus I_t$.

Let $\sigma \in I_t^\perp$. Then $\tilde{\sigma} = \sigma - \sum_{i=1}^r \langle\sigma|u_i\rangle \mathbf{e}_{\xi_i}^{[t]} \in I_t^\perp$ is such that $\langle\tilde{\sigma}|u_i\rangle = 0$ for $i = 1,\dots,r$. Thus, $\tilde{\sigma} \in \langle u_1,\dots,u_r\rangle^\perp \cap I_t^\perp = (\langle u_1,\dots,u_r\rangle \oplus I_t)^\perp = \mathbb{R}[\mathbf{X}]_t^\perp$, i.e. $\tilde{\sigma} = 0$ showing that $I_t^\perp \subset \langle \mathbf{e}_{\xi_1}^{[t]},\dots,\mathbf{e}_{\xi_r}^{[t]}\rangle$. The reverse inclusion is direct since I_t is the space of polynomials of degree $\leq t$ vanishing at ξ_i for $i = 1,\dots,r$.

Assume that $t \geq 2(\rho-1)$ and that $\sigma \in \mathcal{L}_t(I_t)$. Then $\sigma \in I_t^\perp$ and $\left\langle \sigma \middle| p^2 \right\rangle \geq 0$ for any $p \in \mathbb{R}[\mathbf{X}]_{\lfloor \frac{t}{2} \rfloor}$. By the previous analysis,

$$\sigma = \sum_{i=1}^{r} \omega_i \mathbf{e}_{\xi_i}^{[t]}$$

As $\left\langle \sigma \middle| u_i^2 \right\rangle = \omega_i \geq 0$ for $i = 1,\dots,r$, we deduce that $\sigma \in \mathrm{cone}(\mathbf{e}_{\xi_1}^{[t]},\dots,\mathbf{e}_{\xi_r}^{[t]})$. □

2 Finite Convergence and Exactness

We describe now the Lasserre SoS and MoM relaxations [Las01], and we define the *exactness* property. Hereafter we assume that the minimum f^* of the objective function f is always attained on S, that is: $S^{\min} := \{x \in S \mid f(x) = f^*\} \neq \emptyset$.

2.1 Polynomial optimization relaxations

We define the *SoS relaxation of order d* of problem (1) as $\mathcal{Q}_{2d}(\mathbf{g})$ and the supremum:

$$f_{\mathrm{SoS},d}^* := \sup\left\{\lambda \in \mathbb{R} \mid f - \lambda \in \mathcal{Q}_{2d}(\mathbf{g})\right\}. \tag{4}$$

When necessary we will replace $\mathbf{g}$ by $\Pi\mathbf{g}$ (that is $\mathcal{Q}(\mathbf{g})$ by $\mathcal{O}(\mathbf{g})$).

We want to define the dual approximation of the polynomial optimization problem. We are interested in an affine hyperplane section of the cone $\mathcal{L}_d(\mathbf{g})$:

$$\mathcal{L}_d^{(1)}(\mathbf{g}) := \left\{\sigma \in \mathcal{L}_d(\mathbf{g}) \mid \langle\sigma|1\rangle = 1\right\}.$$

We will use the notation $\mathcal{L}^{(1)}(\mathbf{g})$ in the infinite dimensional case. The convex set $\mathcal{L}_d^{(1)}(\mathbf{g})$ is also called the state space of $(\mathbb{R}[\mathbf{X}]_d, \mathcal{Q}_d(\mathbf{g}), 1)$ (see [KS19]). The pure states are the extreme points of this convex set.

With this notation we define the *MoM relaxation of order d* of problem (1) as $\mathcal{L}_{2d}(\mathbf{g})$ and the infimum:

$$f_{\mathrm{MoM},d}^* := \inf\left\{\langle\sigma|f\rangle \in \mathbb{R} \mid \sigma \in \mathcal{L}_{2d}^{(1)}(\mathbf{g})\right\}. \tag{5}$$

When necessary we will replace $\mathbf{g}$ by $\Pi\mathbf{g}$ (that is $\mathcal{Q}(\mathbf{g})$ by $\mathcal{O}(\mathbf{g})$). We are interested, in particular, in the linear functionals that realize the minimum. We easily verify that $f_{\mathrm{SoS},d}^* \leq f_{\mathrm{MoM},d}^* \leq f^*$. When $S^{\min} \neq \emptyset$, the infimum $f_{\mathrm{MoM},d}^*$ is reached since $\mathcal{L}_d^{(1)}(\mathbf{g})$ is closed.

Definition 2.1. Let $f \in \mathbb{R}[\mathbf{X}]$ and f^* denote its minimum on $\mathcal{S}(\mathbf{g})$. We define the set of *functional minimizers* as:

$$\mathcal{L}_{2d}^{\min}(\mathbf{g}) := \left\{\sigma \in \mathcal{L}_{2d}^{(1)}(\mathbf{g}) \mid \langle\sigma|f\rangle = f^*\right\}.$$

Notice that $\mathcal{L}_d(\mathbf{g})$ is the cone over $\mathcal{L}_d^{(1)}(\mathbf{g})$, since for $\sigma \in \mathcal{L}_d(\mathbf{g})$ we have $\langle \sigma|1\rangle = 0 \Rightarrow \sigma = 0$ (see [Las+13, lem. 3.12]), and $\sigma \in \mathcal{L}_d(\mathbf{g}) \neq 0$ implies $\frac{1}{\langle \sigma|1\rangle}\sigma \in \mathcal{L}_d^{(1)}(\mathbf{g})$ We introduce two convergence properties that will be central in the article.

Definition 2.2 (Finite Convergence)**.** We say that the SoS relaxation $(\mathcal{Q}_{2d}(\mathbf{g}))_{d\in\mathbb{N}}$ (resp. the MoM relaxation $\mathcal{L}_{2d}(\mathbf{g}))_{d\in\mathbb{N}}$ has the *Finite Convergence* property for f if $\exists k \in \mathbb{N}$ such that for every $d \geq k$, $f^*_{\mathrm{SoS},d} = f^*$ (resp. $f^*_{\mathrm{MoM},d} = f^*$).

Notice that if the SoS relaxation has finite convergence then the MoM relaxation has finite convergence too, since $f^*_{\mathrm{SoS},d} \leq f^*_{\mathrm{MoM},d} \leq f^*$.

Definition 2.3 (SoS Exactness)**.** We say that the SoS relaxation $(\mathcal{Q}_{2d}(\mathbf{g}))_{d\in\mathbb{N}}$ is *exact* for f if it has the finite convergence property and for all d big enough, we have $f - f^* \in \mathcal{Q}_{2d}(\mathbf{g})$ (in other words sup = max in the definition of $f^*_{\mathrm{SoS},d}$).

For the moment relaxation we can ask the (stronger) property that every truncated functional minimizer is coming from a measure:

Definition 2.4 (MoM Exactness)**.** We say that the MoM relaxation $(\mathcal{L}_{2d}(\mathbf{g}))_{d\in\mathbb{N}}$ is *exact* for f on the basic closed semialgebraic set S if:

- it has the finite convergence property;
- for every $k \in \mathbb{N}$ big enough, for $d = d(k) \in \mathbb{N}$ big enough, every truncated functional minimizer is coming from a probability measure supported on S, i.e. $\mathcal{L}_{2d}^{\min}(\mathbf{g})^{[k]} \subset \mathcal{M}^{(1)}(S)^{[k]}$.

If not specified, S will be the semialgebraic set $S = \mathcal{S}(\mathbf{g})$ defined by $\mathbf{g}$.

MoM exactness may be considered as a particular instance of the so called *Moment Problem* (i.e. asking if $\sigma \in \mathbb{R}[\mathbf{X}]^*$ is coming from a measure) or of the *Strong Moment Problem* (i.e. asking that the measure has a specified support). More precisely, MoM exactness can be considered as a *Truncated Strong Moment Property* (since we are considering functionals restricted to polynomials up to a certain degree).

We recall results of strong duality, i.e. cases when we know that $f^*_{\mathrm{SoS},d} = f^*_{\mathrm{MoM},d}$, that will be using. See also Proposition 3.10.

Theorem 2.5 (Strong duality)**.** *Let $Q = \mathcal{Q}(\mathbf{g})$ be a quadratic module and f the objective function. Then:*

- *if* $\operatorname{supp} Q = 0$ *then* $\forall d$*:* $f^*_{\mathrm{SoS},d}$ *is attained (i.e.* $f - f^*_{\mathrm{SoS},d} \in \mathcal{Q}_d(\mathbf{g})$*) and* $f^*_{\mathrm{SoS},d} = f^*_{\mathrm{MoM},d}$*[Mar08, prop. 10.5.1];*
- *if there exists* $0 \leq r \in \mathbb{R}$*:* $r - \|\mathbf{X}\|^2 \in \mathcal{Q}_d(\mathbf{g})$ *then* $f^*_{\mathrm{SoS},d} = f^*_{\mathrm{MoM},d}$ *[JH16].*

We recall that we are assuming $S^{\min} \neq \emptyset$ (in particular f^* is finite: otherwise it may happen that $f^*_{\mathrm{SoS},d} = -\infty$). Notice that if strong duality holds, then SoS finite convergence is equivalent to MoM finite convergence.

2.2 Examples and counterexamples

In this section, we give examples showing how these notions are (not) related.

No finite convergence. The first example shows that SoS relaxations for polynomial optimization on algebraic curves do not have necessarily the finite convergence property.

Example 2.6 ([Sch00])**.** Let $\mathcal{C} \subset \mathbb{R}^n$ be a smooth connected curve of genus ≥ 1, with only real points at infinity. Let $\mathbf{h} = \{h_1, \ldots, h_s\} \subset \mathbb{R}[\mathbf{X}]$ such that $I = \mathcal{I}(\mathcal{C}) = (\mathbf{h})$. Then there exists $f \in \mathbb{R}[\mathbf{X}]$ such that the SoS relaxation $\mathcal{Q}_{2d}(\pm\mathbf{h})$ and the MoM relaxation $\mathcal{L}_{2d}(\pm\mathbf{h})$ have no finite convergence and are not exact.

Indeed by [Sch00, Theorem 3.2], there exists $f \in \mathbb{R}[\mathbf{X}]$ such that $f \geq 0$ on $\mathcal{C} = \mathcal{S}(\pm\mathbf{h})$, which is not a sum of squares in $\mathbb{R}[\mathcal{C}] = \mathbb{R}[\mathbf{X}]/I$. Consequently, $f \notin \Sigma^2[\mathbf{X}] + I = \mathcal{Q}(\pm\mathbf{h})$. As $f \geq 0$ on $\mathcal{C}$, its infimum f^* is non-negative and we also have $f - f^* \notin \mathcal{Q}(\pm\mathbf{h})$.

As $I = \operatorname{supp}\mathcal{Q}(\pm\mathbf{h})$ is real radical, using Proposition 3.10 we deduce that $\mathcal{Q}_d(\pm\mathbf{h})$ is closed, that there is no duality gap and that the supremum $f^*_{\mathrm{SoS},d}$ is reached. Thus if the SoS relaxation has finite convergence then $f - f^* \in \mathcal{Q}_{2d}(\pm\mathbf{h})$ for some $d \in \mathbb{N}$. This is a contradiction, showing that the SoS and the MoM relaxations have no finite convergence and cannot be SoS exact for f.

In dimension 2, there are also cases where the SoS and MoM relaxations cannot have finite convergence or be exact.

Example 2.7 ([Mar08])**.** Let $g_1 = X_1^3 - X_2^2, g_2 = 1 - X_2$. Then $S = \mathcal{S}(\mathbf{g})$ is a compact semialgebraic set of dimension 2 and $\mathcal{O}(\mathbf{g})$ is Archimedean. We have $f = X_1 \geq 0$ on S but $X_1 \notin \mathcal{O}(\mathbf{g})$ (see [Mar08, Example 9.4.6(3)]). The infimum of f on S is $f^* = 0$. By theorem 2.5, $\mathcal{Q}_d(\Pi\mathbf{g})$ is closed, the supremum $f^*_{\mathrm{SoS},d}$ is reached and strong duality holds. Assume that $f^*_{\mathrm{SoS},d} = f^*_{\mathrm{MoM},d} = f^* = 0$ for $d \in \mathbb{N}$ big enough, then $f - f^* = f \in \mathcal{O}(\mathbf{g})$: but this is a contraction. Therefore, the relaxations $\mathcal{Q}_{2d}(\Pi\mathbf{g})$ and $\mathcal{L}_{2d}(\Pi\mathbf{g})$ cannot have finite convergence and be thus cannot be exact for $f = X_1$.

The next example shows that non-finite convergence and non-exactnesss always happen in dimension ≥ 3.

Example 2.8. Let $n \geq 3$. Let Q be an Archimedean quadratic module generated by $g_1, \ldots, g_s \in \mathbb{R}[\mathbf{X}]$ such that $\mathcal{S}(Q) \subset \mathbb{R}^n$ is of dimension $m \geq 3$. If Q is reduced, i.e. if $\operatorname{supp} Q = \sqrt[\mathbb{R}]{\operatorname{supp} Q}$ (in particular this happens if $\operatorname{supp} Q = 0$ or if $m = n$, i.e. $\mathcal{S}(Q)$ is of maximal dimension), then there exists $f \in \mathbb{R}[\mathbf{X}]$ such that the SoS relaxation $(\mathcal{Q}_{2d}(\mathbf{g}))_{d\in\mathbb{N}}$ and MoM relaxation $(\mathcal{L}_{2d}(\mathbf{g}))_{d\in\mathbb{N}}$ do not have the finite convergence property (and thus are not exact).

Indeed by Proposition 3.10 $f^*_{\mathrm{SoS},d} = f^*_{\mathrm{MoM},d}$ for d big enough and the supremum $f^*_{\mathrm{SoS},d}$ is reached. By [Sch00, Prop. 6.1] for $m \geq 3$, $\operatorname{Pos}(\mathcal{S}(Q)) \supsetneq Q$. So let $f \in \operatorname{Pos}(\mathcal{S}(Q)) \setminus Q$ and let f^* be its minimum on $\mathcal{S}(Q)$. Suppose that $f - f^* \in Q$, then $f \in Q + f^* = Q$, a contradiction. Then the SoS and the MoM relaxations do not have the finite convergence property (and they are not exact).

Remark. The reduceness condition in Example 2.8 is not restrictive: if Q is a quadratic module then $Q + \sqrt[\mathbb{R}]{\operatorname{supp} Q}$ is reduced (see [Sch05b, lemma 3.16]) and $\mathcal{S}(Q) = \mathcal{S}(Q + \sqrt[\mathbb{R}]{\operatorname{supp} Q})$.

SoS exactness, no MoM exactness.

Example 2.9. We want to find the global minimum of $f = X_1^2 \in \mathbb{R}[X_1, \ldots, X_n] = \mathbb{R}[\mathbf{X}]$ for $n \geq 3$. Let $d \geq 2$, $\mathbf{X}' = (X_2, \ldots, X_n)$ and $\overline{\sigma} \in \mathcal{L}_d(\Sigma^2[\mathbf{X}'])$ such that $\overline{\sigma} \notin \mathcal{M}(\mathbb{R}^{n-1})^{[d]}$. Such a linear functional exists because when $n > 2$ there are non-negative polynomials in $\mathbb{R}[\mathbf{X}']$ which are not sum of squares, such as the Motzkin polynomial (see [Rez96]). As $\mathcal{Q}_d(\Sigma^2[\mathbf{X}'])$ is closed, such a polynomial can be separated from $\mathcal{Q}_d(\Sigma^2[\mathbf{X}'])$ by a linear functional $\overline{\sigma}\mathcal{L}_d(\Sigma^2[\mathbf{X}'])$, which cannot be the truncation of a measure (i.e. $\Sigma^2[\mathbf{X}']$ does not have the truncated moment property). Define $\sigma : h \mapsto \langle\sigma|h\rangle = \langle\overline{\sigma}|h(0, X_2, \ldots, X_n)\rangle$. We have $\sigma \in \mathcal{L}(\Sigma^2[\mathbf{X}])$ since $\overline{\sigma} \in \mathcal{L}(\Sigma^2[\mathbf{X}'])$. Obviously $\langle\sigma|f\rangle = 0 = f^*$ (the minimum of X_n^2), $f - f^* = X_1^2 \in \Sigma^2$ and the SoS relaxation is exact. Since σ is coming from a measure if and only if $\overline{\sigma}$ is coming from a measure, the MoM relaxation cannot be exact.

The previous example generalizes easily to quadratic modules Q with $\operatorname{supp}(Q) \neq \{0\}$, which do not have the (truncated) moment property, i.e. there exists $\sigma \in \mathcal{L}_d(Q)$ such that $\sigma \notin \mathcal{M}(\mathcal{S}(Q))^{[d]}$. Taking $f = h^2$ with $h \in \operatorname{supp}(Q)$, $h \neq 0$, we have $\langle\sigma|f\rangle = 0 = f^*$ and the MoM relaxation cannot be exact since $\sigma \notin \mathcal{M}(\mathcal{S}(Q))^{[d]}$, while the SoS relaxation is exact ($f - f^* = h^2 \in Q$).

SoS finite convergence, MoM exactness.

Example 2.10. Let $f = (X^4Y^2 + X^2Y^4 + Z^6 - 2X^2Y^2Z^2) + X^8 + Y^8 + Z^8 \in \mathbb{R}[X, Y, Z]$. We want to optimize f over the gradient variety $\mathcal{V}_{\mathbb{R}}\left(\frac{\partial f}{\partial X}, \frac{\partial f}{\partial Y}, \frac{\partial f}{\partial Z}\right)$ which is zero dimensional (see [NDS06]). By Theorem 4.1 the MoM relaxation is exact, and by Corollary 4.4 the SoS has the finite convergence property. But the SoS relaxation is not exact, as shown in [NDS06].

Table 1: Summary of convergence results.

Expl.	SoS f. c.	SoS ex.	MoM f. c.	MoM ex.	m
2.6	NO	NO	NO	NO	1
2.7	NO	NO	NO	NO	2
2.8	NO	NO	NO	NO	≥ 3
2.9	YES	YES	YES	NO	≥ 3
2.10	YES	NO	YES	YES	0
2.11	YES	NO	YES	YES	0

Example 2.11. Let $f = X_1$. We want to find its value at the origin, defined by $\|\mathbf{X}\|^2 = 0$. As proved in [Nie13b] there is finite convergence but not exactness for the SoS relaxation. By Theorem 4.1 the MoM relaxation is exact.

We summarize the previous examples in Table 1 in terms of the properties of finite convergence (SoS f.c. and MoM f.c.) exactness (SoS ex. and MoM ex.) and the dimension m of the semialgebraic set S.

3 Geometry of Moment Representations

We give a description of the moment linear functionals in the full dimensional and truncated case.

3.1 Infinite moment representations

With our setting, the classical moment problem can be stated as follows: given $\sigma \in \mathbb{R}[\mathbf{X}]^*$, when there exists $\mu \in \mathcal{M}(\mathbb{R}^n)$ such that:

$$\forall f \in \mathbb{R}[\mathbf{X}] \ \langle \sigma | f \rangle = \int f \, \mathrm{d}\mu.$$

Haviland's theorem (see [Mar08, th. 3.1.2] and [Sch17, th. 1.12]) says that this happens if and only if σ is positive on positive polynomials. Since checking this is a computationally hard task, then it is interesting to find (proper) subsets of positive polynomials that have the same property, chosen in such a way that checking this conditions is easy. Important results in this direction are theorems of Schmüdgen and Putinar.

Theorem 3.1 ([Sch91],[Put93])**.** *Let Q be an Archimedean finitely generated quadratic module and $S = \mathcal{S}(Q)$. Then $\mathcal{L}(Q) = \mathcal{M}(S) = \overline{\mathrm{cone}(\mathbf{e}_\xi : \xi \in S)}$.*

This theorem solves the moment problem in the Archimedean (compact) case. Notice that $\mathcal{M}(\mathcal{S}(Q))$ depends only on $S = \mathcal{S}(Q)$ and not on the generators of Q. In particular, if Q and Q' are Archimedean and $\mathcal{S}(Q) = \mathcal{S}(Q')$ then $\mathcal{L}(Q) = \mathcal{L}(Q')$.

If we have a *generic* measure $\mu \in \mathcal{M}(S)$, i.e. which is nonzero on any nonzero polynomial on S, obviously its support is equal to S: $\operatorname{supp} \mu = S$. We want to generalize this property to linear functionals which are not necessary coming from measures. In particular we want to recover informations about the semialgebraic set $S = \mathcal{S}(\mathbf{g})$ from linear functionals $\sigma \in \mathcal{L}(\mathbf{g})$. We are interested in *generic* elements $\sigma^* \in \mathcal{L}(Q)$, that we characterize in terms of the the kernel of the Hankel operator (see also Proposition 3.15).

Definition 3.2. We say that $\sigma^* \in \mathcal{L}(Q)$ is *generic* if $\ker H_{\sigma^*} \subset \ker H_\sigma \ \forall \sigma \in \mathcal{L}(Q)$.

Proposition 3.3. *Let I be an ideal of $\mathbb{R}[\mathbf{X}]$ and $\sigma^* \in \mathcal{L}(I)$ be generic. Then* $\ker H_{\sigma^*} = \sqrt[\mathbb{R}]{I}$.

Proof. Notice that if $x \in \mathcal{V}_{\mathbb{R}}(I)$ then $\mathbf{e}_x \in \mathcal{L}(I)$. Moreover $\ker H_{\mathbf{e}_x} = \mathcal{I}(x)$. This implies:

$$\ker H_{\sigma^*} \subset \bigcap_{x \in \mathcal{V}_{\mathbb{R}}(I)} \ker H_{\mathbf{e}_x} = \bigcap_{x \in \mathcal{V}_{\mathbb{R}}(I)} \mathcal{I}(x) = \mathcal{I}(\mathcal{V}_{\mathbb{R}}(I)) = \sqrt[\mathbb{R}]{I},$$

where the last equality is the Real Nullstellenstatz, Theorem 1.1.

By definition, $I \subset \ker H_{\sigma^*}$. Since $\ker H_{\sigma^*}$ is a real radical ideal (see [Las+13, prop. 3.13]) we have $\sqrt[\mathbb{R}]{I} \subset \ker H_{\sigma^*} =$, which proves that $\ker H_{\sigma^*} = \sqrt[\mathbb{R}]{I}$. □

Proposition 3.3 generalizes to quadratic modules as follows.

Proposition 3.4. *Let Q be a quadratic module, $S = \mathcal{S}(Q)$ and $\sigma^* \in \mathcal{L}(Q)$ be generic. Then $\sqrt{\operatorname{supp} Q} \subset \ker H_{\sigma^*} \subset \mathcal{I}(S)$. Moreover if Q is Archimedean then $\ker H_{\sigma^*} = \mathcal{I}(S)$.*

Proof. As in the proof of Proposition 3.3, we get:

$$\ker H_{\sigma^*} \subset \bigcap_{x \in S} \ker H_{\mathbf{e}_x} = \bigcap_{x \in S} \mathcal{I}(x) = \mathcal{I}(S).$$

Now observe that $\operatorname{supp} Q \subset \ker H_{\sigma^*}$ by definition. Since $\ker H_{\sigma^*}$ is a real radical ideal (see [Las+13, prop. 3.13]), then $\sqrt[\mathbb{R}]{\operatorname{supp} Q} \subset \ker H_{\sigma^*}$.

For the second part, if Q is Archimedean, then by Theorem 3.1 $\mathcal{L}(Q) = \mathcal{M}(S)$. In particular σ^* is a measure $\mu \in \mathcal{M}(S)$ supported on S: $\forall f \in \mathbb{R}[\mathbf{X}]$, $\langle \sigma^* | f \rangle = \int f \, \mathrm{d}\mu$. Let $h \in \mathcal{I}(S)$ and $f \in \mathbb{R}[\mathbf{X}]$. Then:

$$\langle \sigma^* | fh \rangle = \int fh \, \mathrm{d}\mu = \int 0 \, \mathrm{d}\mu = 0,$$

i.e. $h \in \ker H_{\sigma^*}$, which proves the reverse inclusion. □

Now we describe $\mathcal{L}(Q)$ without the Archimedean hypothesis (compare with Theorem 3.1).

Lemma 3.5. *Let Q be a quadratic module. Then $\mathcal{L}(Q) = \mathcal{L}(\sqrt[\mathbb{R}]{\operatorname{supp} Q} + Q)$. In particular for any ideal $I \subset \sqrt[\mathbb{R}]{\operatorname{supp} Q}$ we have $\mathcal{L}(Q) = \mathcal{L}(\sqrt[\mathbb{R}]{I} + Q)$.*

Proof. Since $\sqrt{\operatorname{supp} Q} \subset \overline{Q}$ (see [Mar08, th. 4.1.2]), we have $\sqrt{\operatorname{supp} Q} + Q \subset \overline{Q} + Q = \overline{Q}$. Then:

$$\mathcal{L}(Q) = \mathcal{L}(\overline{Q}) \subset \mathcal{L}(\sqrt{\operatorname{supp} Q} + Q) \subset \mathcal{L}(Q).$$

Since $\sqrt[\mathbb{R}]{\operatorname{supp} Q} = \sqrt{\operatorname{supp} Q}$ (see Section 1.3) we have $\mathcal{L}(\sqrt[\mathbb{R}]{\operatorname{supp} Q} + Q) = \mathcal{L}(\sqrt{\operatorname{supp} Q} + Q) = \mathcal{L}(Q)$. □

Remark. Lemma 3.5 shows that, even if the semialgebraic set is not compact, we can replace any ideal in the description of the semialgebraic set with its real radical. In particular, since $\mathcal{I}(\mathcal{S}(\mathbf{g})) = \sqrt[\mathbb{R}]{\mathcal{O}(\mathbf{g})}$ (by Theorem 1.1), we have $\mathcal{L}(\mathcal{O}(\mathbf{g})) = \mathcal{L}(\mathcal{O}(\mathbf{g}) + \mathcal{I}(\mathcal{S}(\mathbf{g})))$.

The inclusion $Q + \sqrt{\operatorname{supp} Q} \subset \overline{Q}$ can be strict, as shown by the following example.

Example 3.6 ([Sch05a, ex. 3.2], [Sch05b, rem. 3.15])**.** Let $Q = \mathcal{Q}(1 - X^2 - Y^2, -XY, X - Y, Y - X^2) \subset \mathbb{R}[X, Y]$. Notice that $S = \mathcal{S}(Q) = \{\mathbf{0}\}$ and that, since Q is Archimedean, $\overline{Q} = \operatorname{Pos}(\{\mathbf{0}\})$. In this case $\operatorname{supp} Q = (0)$ and $\mathcal{I}(S) = \operatorname{supp} \overline{Q} = (X, Y)$, and thus $Q + \sqrt{\operatorname{supp} Q} \subsetneq \overline{Q}$.

3.2 Truncated moment representations

Now we prove the corresponding results in the truncated case. For a finitely generated quadratic module $Q = \mathcal{Q}(\mathbf{g}) \subset \mathbb{R}[\mathbf{X}]$, we denote $Q_{[k]} = \mathcal{Q}_k(\mathbf{g})$.

Definition 3.7. Let $Q = \mathcal{Q}(\mathbf{g})$ be a finitely generated quadratic module. We define $\widetilde{Q} = \bigcup_d \overline{\mathcal{Q}_d(\mathbf{g})} = \bigcup_d \overline{Q_{[d]}}$.

Notice that $\widetilde{Q}$ depends a priori on the generators $\mathbf{g}$ of Q: we will prove that $\widetilde{Q}$ is a finitely generated quadratic module and that it does not depend on the particular choice of generators. Moreover notice that $Q \subset \widetilde{Q} = \bigcup_d \overline{Q_{[d]}} \subset \overline{\bigcup_d Q_{[d]}} = \overline{Q}$, but these inclusions may be strict as we will see.

Lemma 3.8. *Let $Q = \mathcal{Q}(\mathbf{g})$ and $J = \sqrt[\mathbb{R}]{\operatorname{supp} Q}$. Then for every $d \in \mathbb{N}$ there exists $k \geq d$ such that $J_d \subset \overline{\mathcal{Q}_{[k]}}$.*

Proof. Let m be big enough such that $\forall f \in J = \sqrt[\mathbb{R}]{\operatorname{supp} Q} = \sqrt{\operatorname{supp} Q}$ we have: $f^{2^m} \in \operatorname{supp} Q$ (if $\sqrt{J} = (h_1, \dots, h_t)$ and $h_i^{a_i} \in I$, we can take m such that $2^m \geq a_1 + \cdots + a_t$). Let $f \in J_d$ with $\deg f \leq d$. Then $f^{2^m} \in \operatorname{supp} \mathcal{Q}_{[k']} \subset Q_{[k']}$ for $k' \in \mathbb{N}$ big enough. Using the identity [Sch05b, remark 2.2]:

$$m - a = (1 - \frac{a}{2})^2 + (1 - \frac{a^2}{8})^2 + (1 - \frac{a^4}{128})^2 + \cdots + (1 - \frac{a^{2^{m-1}}}{2^{2^m-1}})^2 - \frac{a^{2^m}}{2^{2^{m+1}-2}},$$

substituting a by $-\frac{mf}{\varepsilon}$ and multiplying by $\frac{\varepsilon}{m}$, we have that $\forall \varepsilon > 0$, $f + \varepsilon \in Q_{[k]}$ for $k = \max\{k', 2^m d\}$ (the degree of the representation of $f + \varepsilon$ does not depend on ε). This implies that $f \in \overline{\mathcal{Q}_{[k]}}$. □

We can now prove the main result of this section.

Theorem 3.9. *Let $Q = \mathcal{Q}(\mathbf{g})$ be a finitely generated quadratic module and let $J = \sqrt[\mathbb{R}]{\operatorname{supp} Q}$. Then $\widetilde{Q} = \cup_{d \in \mathbb{N}} \overline{Q_{[d]}} = Q + J$ and $\operatorname{supp} \widetilde{Q} = J$. In particular, $\widetilde{Q}$ is a finitely generated quadratic module and does not depend on the particular choice of generators of Q.*

Proof. By [Mar08, lemma 4.1.4] $Q_{[d]} + J_d$ is closed in $\mathbb{R}[\mathbf{X}]_d$, thus $\overline{Q_{[d]}} \subset Q_{[d]} + J_d$. Taking unions we prove that $\widetilde{Q} \subset Q + J$.

Conversely by Lemma 3.8 for $d \in \mathbb{N}$ and $k \geq d \in \mathbb{N}$ big enough, $J_d \subset \overline{Q_{[k]}}$. Then, we have $Q_{[d]} + J_d \subset Q_{[k]} + \overline{Q_{[k]}} \subset \overline{Q_{[k]}} + \overline{Q_{[k]}} \subset \overline{Q_{[k]}}$. Taking unions on both sides gives $Q + J \subset \widetilde{Q}$.

Finally $\operatorname{supp} \widetilde{Q} = \operatorname{supp}(Q + J) = J$ by [Sch05b, lemma 3.16]. □

Remark. We proved that $\widetilde{Q} = Q + \sqrt[\mathbb{R}]{\operatorname{supp} Q}$, and thus in Example 3.6 we have that $\widetilde{Q} \subsetneq \overline{Q}$. We also have $\operatorname{supp} \widetilde{Q} = \sqrt[\mathbb{R}]{\operatorname{supp} Q}$ so that if $\operatorname{supp} Q$ is not real radical then $Q \subsetneq \widetilde{Q}$. Example 2.11 is such a case where $\operatorname{supp} Q \neq \sqrt[\mathbb{R}]{\operatorname{supp} Q}$. We notice that, by Theorem 3.9 and [Sch05b, th. 3.17], if Q is *stable*[1] then $\widetilde{Q} = \overline{Q}$.

As a consequence we have no duality gap when $\operatorname{supp} Q$ is real radical. This generalizes the condition $\operatorname{supp} Q = 0$ in Theorem 2.5.

Proposition 3.10. *Let $Q = \mathcal{Q}(\mathbf{g})$ be a finitely generated quadratic module. If $\sqrt[\mathbb{R}]{\operatorname{supp} Q} = \operatorname{supp} Q$ then for any $d \in \mathbb{N}$, $\mathcal{Q}_d(\mathbf{g}) = \overline{\mathcal{Q}_d(\mathbf{g})}$ is closed and $\widetilde{Q} = Q$. Moreover for any $f \in \mathbb{R}[\mathbf{X}]$ such that $f^* > -\infty$ we have that $f^*_{\mathrm{SoS},d}$ is attained (i.e. $f - f^*_{\mathrm{SoS},d} \in \mathcal{Q}_{2d}(\mathbf{g})$) and there is no duality gap: $f^*_{\mathrm{SoS},d} = f^*_{\mathrm{MoM},d}$.*

Proof. Let $I = \operatorname{supp} Q$. By hypothesis, $\sqrt{I} = I$ so that $\widetilde{Q} = Q$. By [Mar08, lemma 4.1.4], $Q_{[d]} + I_d$ is closed. As $\operatorname{supp} Q_{[d]} \subset I_d$ is a closed finite-dimensional subspace of I_d, we also have that $Q_{[d]} + \operatorname{supp} Q_{[d]} = Q_{[d]}$ is closed. Therefore we have $\mathcal{L}_d(\mathbf{g})^\vee = Q_{[d]}^{\vee\vee} = \overline{Q_{[d]}} = Q_{[d]}$, from which we deduce that there is not duality gap, by classical convexity arguments, as follows.

If $f \in \mathbb{R}[\mathbf{X}]$ such that $f^* > -\infty$, then $\left\{ \lambda \in \mathbb{R} \mid f - \lambda \in \mathcal{Q}_{2d}(\mathbf{g}) \right\}$ is bounded from above. Since $\mathcal{Q}_{2d}(\mathbf{g})$ is closed $f^*_{\mathrm{SoS},d} = \sup\left\{ \lambda \in \mathbb{R} \mid f - \lambda \in \mathcal{Q}_{2d}(\mathbf{g}) \right\}$ is attained. If $f^*_{\mathrm{SoS},d} < f^*_{\mathrm{MoM},d}$, then $f - f^*_{\mathrm{MoM},d} \notin Q_{[d]}$. Thus there exists a separating functional $\sigma \in \mathcal{L}_d^{(1)}(\mathbf{g})$ such that $\left\langle \sigma \middle| f - f^*_{\mathrm{MoM},d} \right\rangle < 0$, which implies that $\langle \sigma | f \rangle < f^*_{\mathrm{MoM},d}$ in contradiction with the definition of $f^*_{\mathrm{MoM},d}$. Consequently, $f^*_{\mathrm{SoS},d} = f^*_{\mathrm{MoM},d}$. □

[1] $\mathcal{Q}(\mathbf{g})$ is stable if $\forall d \in \mathbb{N}$ there exists $k \in \mathbb{N}$ such that $\mathcal{Q}(\mathbf{g}) \cap \mathbb{R}[\mathbf{X}]_d = \mathcal{Q}_k(\mathbf{g}) \cap \mathbb{R}[\mathbf{X}]_d$.

We describe now relations between the truncated parts of $\mathcal{L}_d(\mathbf{g})$.

Lemma 3.11. *Let* $J = \sqrt[\mathbb{R}]{\operatorname{supp}\mathcal{Q}(\mathbf{g})}$. *If* $(\mathbf{h}) \subset J$, $\deg \mathbf{h} \leq d$, *then* $\exists k \geq d$:

$$\mathcal{L}_k(\mathbf{g})^{[d]} \subset \mathcal{L}_d(\mathbf{g}, \pm\mathbf{h}) \subset \mathcal{L}_d(\mathbf{g}).$$

In particular $\mathcal{L}_k(\mathbf{g})^{[d]} \subset \mathcal{L}_d(\pm\mathbf{h})$.

Proof. By Lemma 3.8, $\langle \mathbf{h} \rangle_d \subset (\mathbf{h})_d \subset \overline{\mathcal{Q}_k(\mathbf{g})}$ for some $k \geq d$. Let $h \in \mathbf{h}$ and $f \in \mathbb{R}[\mathbf{X}]_{d-\deg h}$. Then $\pm f h \in \overline{\mathcal{Q}_k(\mathbf{g})}$, and for $\sigma \in \mathcal{L}_k(\mathbf{g})$, we have $\left\langle \sigma^{[d]} \middle| f h \right\rangle = \langle \sigma | f h \rangle = 0$, i.e. $\mathcal{L}_k(\mathbf{g})^{[d]} \subset \mathcal{L}_d(\mathbf{g}, \pm\mathbf{h})$. The other inclusion $\mathcal{L}_d(\mathbf{g}, \pm\mathbf{h}) \subset \mathcal{L}_d(\mathbf{g})$ follows by definition. □

Remark. Lemma 3.11 says that the MoM relaxation $(\mathcal{L}_{2d}(\mathbf{g}))_{d\in\mathbb{N}}$ is equivalent to the MoM relaxation $(\mathcal{L}_{2d}(\mathbf{g}, \pm\mathbf{h}))_{d\in\mathbb{N}}$, where $(\mathbf{h}) = \sqrt[\mathbb{R}]{\operatorname{supp}\mathcal{Q}(\mathbf{g})}$. Since $\operatorname{supp}(\mathcal{Q}(\mathbf{g}) + (\mathbf{h})) = \operatorname{supp}\widetilde{Q} = (\mathbf{h})$ is a real radical ideal, we can apply Proposition 3.10 to it: then the MoM relaxation $(\mathcal{L}_{2d}(\mathbf{g}))_{d\in\mathbb{N}}$ is equivalent to the SoS relaxation $(\mathcal{Q}_{2d}(\mathbf{g}, \pm\mathbf{h}))_{d\in\mathbb{N}}$.

Lemma 3.11 is an algebraic result, in the sense that $\operatorname{supp}\mathcal{Q}(\mathbf{g})$ may be unrelated to the geometry $\mathcal{S}(\mathbf{g})$ that it defines. If some additional conditions hold (namely if we have only equalities, or a preordering, or a small dimension), it can however provide geometric characterizations that will be useful in Section 4.

Corollary 3.12. *Suppose that* $\mathcal{S}(\mathbf{g}) \subset \mathcal{V}_{\mathbb{R}}(\mathbf{h})$. *Then for every* $t_0 \geq \deg \mathbf{h}$ *there exists* $t_1 \geq t_0$ *such that:*

$$\mathcal{L}_{t_1}(\Pi\mathbf{g})^{[t_0]} \subset \mathcal{L}_{t_0}(\pm\mathbf{h}).$$

In particular this holds when $(\mathbf{h}) = \mathcal{I}(\mathcal{S}(\mathbf{g}))$.

Proof. $\mathcal{S}(\mathbf{g}) \subset \mathcal{V}_{\mathbb{R}}(\mathbf{h})$ if and only if $\sqrt[\mathbb{R}]{(\mathbf{h})} \subset \mathcal{I}(\mathcal{S}(\mathbf{g})) = \sqrt[\mathbb{R}]{\operatorname{supp}\mathcal{Q}(\Pi\mathbf{g})}$ by Theorem 1.1. Then we can apply Lemma 3.11. □

Corollary 3.13. *Let* $Q = \mathcal{Q}(\mathbf{g})$. *Suppose that* $\mathcal{S}(\mathbf{g}) \subset \mathcal{V}_{\mathbb{R}}(\mathbf{h})$ *and* $\dim \frac{\mathbb{R}[\mathbf{X}]}{\operatorname{supp} Q} \leq 1$. *Then for every* $t_0 \geq \deg \mathbf{h}$ *there exists* $t_1 \geq t_0$ *such that:*

$$\mathcal{L}_{t_1}(\mathbf{g})^{[t_0]} \subset \mathcal{L}_{t_0}(\pm\mathbf{h}).$$

In particular this holds when $(\mathbf{h}) = \mathcal{I}(\mathcal{S}(\mathbf{g}))$.

Proof. We prove it as Corollary 3.12, using [Mar08, cor. 7.4.2 (3)] instead of Theorem 1.1. □

With the characterization of $\widetilde{Q}$ we can now describe the kernel of Hankel operators associated to truncated moment sequences, in analogy to the infinite dimensional case analyzed in Proposition 3.4. First we recall the definition of genericity in the truncated setting and equivalent characterizations.

Definition 3.14. We say that $\sigma^* \in \mathcal{L}_k(\mathbf{g})$ is *generic* if $\operatorname{rank} H^k_{\sigma^*} = \max\{\operatorname{rank} H^k_\eta \mid \eta \in \mathcal{L}_k(\mathbf{g})\}$.

This genericity can be characterized as follows, see [Las+13, prop. 4.7].

Proposition 3.15. *Let* $\sigma \in \mathcal{L}_{2k}(\mathbf{g})$. *The following are equivalent:*

(i) σ *is generic;*

(ii) $\ker H^k_\sigma \subset \ker H^k_\eta \ \forall \eta \in \mathcal{L}_{2k}(\mathbf{g})$;

(iii) $\forall d \leq k$, *we have:* $\operatorname{rank} H^d_\sigma = \max\{\operatorname{rank} H^d_\eta \mid \eta \in \mathcal{L}_{2k}(\mathbf{g})\}$.

Remark. By Proposition 3.15 notice that $\forall d \leq k$, if $\sigma^* \in \mathcal{L}_{2k}(\mathbf{g})$ is generic then $(\sigma^*)^{[2d]}$ is generic in $\mathcal{L}_{2k}(\mathbf{g})^{[2d]}$. In particular, $\ker H^d_{\sigma^*} \subset \ker H^d_\eta \ \forall \eta \in \mathcal{L}_k(\mathbf{g})$.

We are now ready to describe the kernel of generic elements.

Theorem 3.16. *Let $Q = \mathcal{Q}(\mathbf{g})$ and $J = \sqrt[\mathbb{R}]{\operatorname{supp} Q}$. Then there exists $d, t \in \mathbb{N}$ such that for $\sigma^* \in \mathcal{L}_d(\mathbf{g})$ generic, we have $J = (\ker H^t_{\sigma^*})$.*

Proof. Let $t \in \mathbb{N}$ such that J is generated in degree $\leq t$, by the graded basis $\mathbf{h} = \{h_1, \dots, h_s\}$. From Lemma 3.8 we deduce that there exists $d \in \mathbb{N}$ such that $J_{2t} \subset \overline{\mathcal{Q}_d(\mathbf{g})}$. Let $\sigma^* \in \mathcal{L}_d(\mathbf{g})$ generic.

We first prove that $J \subset (\ker H^t_{\sigma^*})$. By Proposition 3.15 we have $\ker H^t_{\sigma^*} = \bigcap_{\sigma \in \mathcal{L}_d(\mathbf{g})} \ker H^t_\sigma$. Then it is enough to prove that $J_t \subset \ker H^t_\sigma$ for all $\sigma \in \mathcal{L}_d(\mathbf{g})$.

By Lemma 3.11 $\mathcal{L}_d(\mathbf{g})^{[2t]} \subset \mathcal{L}_{2t}(\pm\mathbf{h}) \subset \langle \mathbf{h} \rangle_{2t}^\perp$. Then $\forall f \in J_t = \langle \mathbf{h} \rangle_t$, $\forall p \in \mathbb{R}[\mathbf{X}]_t$, $\forall \sigma \in \mathcal{L}_d(\mathbf{g})$, we have $fp \in \langle \mathbf{h} \rangle_{2t}$ and $\left\langle \sigma^{[2t]} \middle| fp \right\rangle = 0$. This shows that $H^t_\sigma(f)(p) = \left\langle (f \star \sigma)^{[t]} \middle| p \right\rangle = \langle \sigma | fp \rangle = 0$, i.e. $f \in \ker H^t_\sigma$.

Conversely, we show that $(\ker H^t_{\sigma^*}) \subset J$ for σ^* generic in $\mathcal{L}_d(\mathbf{g})$. Since $J = \operatorname{supp} \widetilde{Q} = \operatorname{supp} \bigcup_j \overline{\mathcal{Q}_j(\mathbf{g})}$ (by Theorem 3.9) it is enough to prove that $\ker H^t_{\sigma^*} \subset \operatorname{supp} \overline{\mathcal{Q}_d(\mathbf{g})} = \operatorname{supp} \mathcal{L}_d(\mathbf{g})^\vee$.

Let $f \in \ker H^t_{\sigma^*} = \bigcap_{\sigma \in \mathcal{L}_k(\mathbf{g})} \ker H^t_\sigma$ (we use again Proposition 3.15) and let $\sigma \in \mathcal{L}_d(\mathbf{g})$. Then $\langle \sigma | f \rangle = \left\langle (f \star \sigma)^{[t]} \middle| 1 \right\rangle = H^t_\sigma(f)(1) = 0$. In particular $f \in \mathcal{L}_d(\mathbf{g})^\vee$. We prove that $-f \in \mathcal{L}_d(\mathbf{g})^\vee$ in the same way. Then $f \in \operatorname{supp} \overline{\mathcal{Q}_d(\mathbf{g})}$, which proves that $\ker H^t_{\sigma^*} \subset \operatorname{supp} \widetilde{Q} = J$. □

As a corollary of this theorem we have the following result: there exists $d, t \in \mathbb{N}$ such that $\sqrt[\mathbb{R}]{(\mathbf{h})} = (\ker H^t_{\sigma^*})$ for $\sigma^* \in \mathcal{L}_d(\pm\mathbf{h})$ generic. The particular cases of zero dimensional ideals were investigated in [Lau07], [LLR08], [Las+13].

The geometric corollary of this theorem is the following:

Corollary 3.17. *Let $O = \mathcal{O}(\mathbf{g})$ and $S = \mathcal{S}(\mathbf{g})$. Then there exists $d, t \in \mathbb{N}$ such that for $\sigma^* \in \mathcal{L}_d(\Pi\mathbf{g})$ generic, we have $\mathcal{I}(S) = (\ker H^t_{\sigma^*})$.*

Proof. Apply Theorem 3.16 and Theorem 1.1. □

3.3 Convergence of moments

Lasserre [Las01] proved that, if $\mathcal{Q}(\mathbf{g})$ is an Archimedian quadratic module, then $\lim_{d \to \infty} f^*_{\mathrm{MoM},d} = f^*$, i.e. the minimum of the truncated moment relaxation is close to the evaluation f^* of f at the minimizers. We will show that this happens because truncated linear funcionals are indeed close to measures.

We first recall a compactness property for measures with compact support.

Lemma 3.18. *Let $S \subset \mathbb{R}^n$ be compact. Then $\mathcal{M}^{(1)}(S)^{[k]}$ is compact.*

Proof. Every truncated linear functional coming from a measure is coming from (finite) sums of evaluations, see [Sch17, th. 1.24]. Then $\mathcal{M}^{(1)}(S)^{[k]}$ is the image of S under a continuous map, so that it is compact. □

Theorem 3.19. *Let $Q = \mathcal{Q}(\mathbf{g})$ be an Archimedean quadratic module and $S = \mathcal{S}(\mathbf{g})$. Then $\forall d$:*

$$\bigcap_{k=d}^{\infty} \mathcal{L}_k(\mathbf{g})^{[d]} = \mathcal{M}(S)^{[d]}.$$

Moreover:

$$\lim_{k \to \infty} \mathrm{d_H}(\mathcal{L}_k^{(1)}(\mathbf{g})^{[d]}, \mathcal{M}^{(1)}(S)^{[d]}) = 0,$$

where $\mathrm{d_H}$ denotes the Hausdorff distance.

Proof. Since $\mathcal{L}_k(\mathbf{g})^{[d]} = \operatorname{cone}(\mathcal{L}_k^{(1)}(\mathbf{g})^{[d]})$ and $\mathcal{M}(\mathcal{S}(\mathbf{g}))^{[d]} = \operatorname{cone}(\mathcal{M}^{(1)}(S)^{[d]})$, it is enough to prove:

$$\bigcap_{k=d}^{\infty} \mathcal{L}_k^{(1)}(\mathbf{g})^{[d]} = \mathcal{M}^{(1)}(S)^{[d]}.$$

The inclusion $\bigcap_{k=d}^{\infty} \mathcal{L}_k^{(1)}(\mathbf{g})^{[d]} \supset \mathcal{M}^{(1)}(S)^{[d]}$ is direct.

Conversely, suppose that $\tau \in \mathcal{L}_k^{(1)}(\mathbf{g})^{[d]} \setminus \mathcal{M}^{(1)}(S)^{[d]}$. We want to prove that $\exists h \in \mathbb{N}\colon\ \tau \notin \mathcal{L}_h^{(1)}(\mathbf{g})^{[d]}$.

By [Sch17, th. 17.6], as τ is not coming from a measure supported on S, there exists $f \in \mathrm{Pos}(S)_d$ such that $\langle \tau | f \rangle < 0$. Let $\varepsilon > 0$ such that $\langle \tau | f \rangle < -\varepsilon$. Now, $f + \varepsilon > 0$ on $\mathcal{S}(\mathbf{g})$ and by Theorem 1.2 there exists $h \geq d$ such that $f + \varepsilon \in \mathcal{Q}_h(\mathbf{g})$. Then $\forall \sigma \in \mathcal{L}_h^{(1)}(\mathbf{g})$:

$$\left\langle \sigma^{[d]} \middle| f + \varepsilon \right\rangle = \langle \sigma | f + \varepsilon \rangle \geq 0 \Rightarrow \left\langle \sigma^{[d]} \middle| f \right\rangle \geq -\varepsilon,$$

and thus $\sigma^{[d]} \neq \tau$, i.e. $\tau \notin \mathcal{L}_h^{(1)}(\mathbf{g})^{[d]}$, which is a contradiction. We deduce the reverse inclusion which concludes the proof of the first point.

For the second part, we proceed by contradiction. If the distance is not going to zero, then $\forall k \in \mathbb{N}\ \exists \tau_k \in \mathcal{L}_k^{(1)}(\mathbf{g})^{[d]}$, with $\mathrm{d_H}(\tau_k, \mathcal{M}^{(1)}(S)^{[d]}) = \varepsilon > 0$ (using the fact that $\mathcal{L}_k^{(1)}(\mathbf{g})^{[d]}$ is convex). Since $\mathcal{M}^{(1)}(S)^{[d]}$ is compact (see Lemma 3.18), the set of points at distance ε from $\mathcal{M}^{(1)}(S)^{[d]}$ is compact too. Then up to restricting to a subsequence, we can assume that τ_k has limit $\tau \in (\mathbb{R}[\mathbf{X}]_d)^*$, and $\mathrm{d_H}(\tau, \mathcal{M}^{(1)}(S)^{[d]}) = \varepsilon$ by continuity. But since $\mathcal{L}_k^{(1)}(\mathbf{g})^{[d]}$ are closed and $\tau_k \in \mathcal{L}_h^{(1)}(\mathbf{g})^{[d]}$ for $k \geq h$, then $\tau \in \bigcap_{k=d}^{\infty} \mathcal{L}_k^{(1)}(\mathbf{g})^{[d]} = \mathcal{M}^{(1)}(S)^{[d]}$, which is a contradiction to $\mathrm{d_H}(\tau, \mathcal{M}^{(1)}(S)^{[d]}) = \varepsilon > 0$. □

Remark. Notice that since $\mathcal{M}^{(1)}(S)^{[d]}$ is compact, this proves that $\mathcal{L}_k^{(1)}(\mathbf{g})^{[d]}$ is also compact.

As $\mathcal{L}_k^{(1)}(\mathbf{g})^{[d]} \supset \mathcal{L}_k^{(1)}(\Pi\mathbf{g})^{[d]} \supset \mathcal{M}^{(1)}(S)^{[d]}$, we deduce the following result.

Corollary 3.20. *If $\mathcal{Q}(\mathbf{g})$ is Archimedean then for $d \in \mathbb{N}$, $\lim_{k\to\infty} \mathrm{d_H}(\mathcal{L}_k^{(1)}(\mathbf{g})^{[d]}, \mathcal{L}_k^{(1)}(\Pi\mathbf{g})^{[d]}) = 0$.*

From a computational point of view Corollary 3.20 says that, in the Archimedean case, working with $\mathcal{L}_d(\mathbf{g})$ yields good numerical approximation of $\mathcal{L}_d(\Pi\mathbf{g})$.

It would be interesting to investigate the rate of convergence in Theorem 3.19. A possible approach (on the polynomial SoS side) could be applying results from [NS07], where the degree of Putinar representation is analyzed.

In Section 3.4 we will show that, in the case of a finite semialgebraic set, we need only a finite number of steps in the intersection of Theorem 3.19.

3.4 Moment representation of finite semialgebraic sets

In the case of a finite semialgebraic set we have an easy characterization of positive functionals. Theorem 3.1 reads as follows: if Q is Archimedean and $\mathcal{S}(Q) = \{\xi_1, \ldots, \xi_r\}$ is finite then $\mathcal{L}(Q) = \mathcal{L}(\mathcal{I}(\mathcal{S}(Q))) = \mathrm{cone}(\mathbf{e}_{\xi_1}, \ldots, \mathbf{e}_{\xi_r})$.

We want to prove that this holds also for truncated positive functionals. We recall an auxiliary lemma.

Lemma 3.21. *Let I be a real radical ideal such that $|\mathcal{V}_{\mathbb{R}}(I)| < +\infty$. Then $\mathcal{V}_{\mathbb{C}}(I) = \mathcal{V}_{\mathbb{R}}(I)$.*

Proof. Let $y \in \mathcal{V}_{\mathbb{C}}(I) \setminus \mathcal{V}_{\mathbb{R}}(I)$ and let $\bar{y}$ be its conjugate. Since $|\mathcal{V}_{\mathbb{R}}(I)| < +\infty$ we can consider the interpolator polynomial $u_y \in \mathbb{C}[\mathbf{X}]$ such that:

$$u_y(x) = \begin{cases} 1 & x = y \\ 0 & x \in \mathcal{V}_{\mathbb{R}}(I) \cup \{\bar{y}\}. \end{cases}$$

Since $\overline{u_y} = u_{\bar{y}}$, we have $u_y + u_{\bar{y}} \in \mathbb{R}[\mathbf{X}]$ and $u_y + u_{\bar{y}}$ vanishes on $\mathcal{V}_{\mathbb{R}}(I)$, i.e. $u_y + u_{\bar{y}} \in \mathcal{I}(\mathcal{V}_{\mathbb{R}}(I)) = I$. But $(u_y + u_{\bar{y}})(y) = 1$, then $y \notin \mathcal{V}_{\mathbb{C}}(I)$. □

We prove a strong moment property for ideals whose associated real variety is finite.

Theorem 3.22. *Let $I = (\mathbf{h})$ be a real radical ideal with finite variety: $\mathcal{V}(I) = \{\xi_1, \ldots, \xi_r\} \subset \mathbb{R}^n$. Let $\rho = \rho(\xi_1, \ldots, \xi_r)$ be the regularity of the points. Then there exits $d \in \mathbb{N}$ such that $\forall k \in \mathbb{N}$*

- $\dim \operatorname{cone}(\mathbf{e}_{\xi_1},\dots,\mathbf{e}_{\xi_r})^{[\rho-1+k]} = r$;
- $\mathcal{L}_{d+k}(\pm\mathbf{h})^{[2(\rho-1)+k]} = \operatorname{cone}(\mathbf{e}_{\xi_1},\dots,\mathbf{e}_{\xi_r})^{[2(\rho-1)+k]}$.

Proof. We start to prove the first point. Let $\Xi = \{\xi_1,\dots,\xi_r\} = \mathcal{V}(I)$ and $\rho = \rho(\Xi)$ be the regularity, $t \geq \rho - 1$ and $u_1,\dots,u_r \in \mathbb{R}[\mathbf{X}]_t$ interpolator polynomials of degree $< \rho$ (see Proposition 1.3). Suppose that $a_1\mathbf{e}_{\xi_1}^{[t]} + \cdots + a_r\mathbf{e}_{\xi_r}^{[t]} = 0$. Then for any $i \in \{1,\dots,r\}$:

$$a_i = \left\langle a_i\mathbf{e}_{\xi_i}^{[t]} \middle| u_i \right\rangle = -\left\langle \sum_{j\neq i} a_j\mathbf{e}_{\xi_j}^{[t]} \middle| u_i \right\rangle = 0.$$

For the second part, the inclusion $\operatorname{cone}(\mathbf{e}_{\xi_1},\dots,\mathbf{e}_{\xi_r})^{[2(\rho-1)+k]} \subset \mathcal{L}_{d+k}(\pm\mathbf{h})^{[2(\rho-1)+k]}$ is obvious. Let us take $d \geq 2(\rho-1)$ big enough such that $\langle\mathbf{h}\rangle_d$ contains a graded basis $\mathbf{h}'$ of degree ρ of $I = \mathcal{I}(\Xi)$. Then we have $I_{2(\rho-1)+k} = \langle\mathbf{h}'\rangle_{2(\rho-1)+k} \subset \langle\mathbf{h}\rangle_{d+k}$ and $\mathcal{L}_{d+k}(\pm\mathbf{h})^{[2(\rho-1)+k]} \subset \mathcal{L}_{2(\rho-1)+k}(\pm\mathbf{h}') = \mathcal{L}_{2(\rho-1)+k}(I_{2(\rho-1)+k})$. By Proposition 1.4 for $k \in \mathbb{N}$, $\mathcal{L}_{2(\rho-1)+k}(I_{2(\rho-1)+k}) = \operatorname{cone}(\mathbf{e}_{\xi_1},\dots,\mathbf{e}_{\xi_r})^{[2(\rho-1)+k]}$, which proves the reverse inclusion and $\mathcal{L}_{d+k}(\pm\mathbf{h})^{[2(\rho-1)+k]} = \operatorname{cone}(\mathbf{e}_{\xi_1},\dots,\mathbf{e}_{\xi_r})^{[2(\rho-1)+k]}$. □

Now we prove a theorem that is central in the paper. We generalize Theorem 3.22 to the case of inequalities: finite semialgebraic sets enjoy a truncated strong moment property.

Theorem 3.23. *Suppose that* $\dim \frac{\mathbb{R}[\mathbf{X}]}{\operatorname{supp}\mathcal{Q}(\mathbf{g})} = 0$. *Then,* $S = \mathcal{S}(\mathbf{g}) = \{\xi_1,\dots,\xi_r\}$ *is non-empty and finite and there exists* $d \in \mathbb{N}$ *such that* $\forall k \in \mathbb{N}$*:*

$$\mathcal{L}_{d+k}(\mathbf{g})^{[2(\rho-1)+k]} = \operatorname{cone}(\mathbf{e}_{\xi_1},\dots,\mathbf{e}_{\xi_r})^{[2(\rho-1)+k]}.$$

where $\rho = \rho(\xi_1,\dots,\xi_r)$ *is the regularity of* S.

Proof. Let $I = \operatorname{supp}\mathcal{Q}(\mathbf{g})$ and $J = \sqrt[\mathbb{R}]{\operatorname{supp}\mathcal{Q}(\mathbf{g})} = \sqrt{\operatorname{supp}\mathcal{Q}(\mathbf{g})}$. Since $\dim \frac{\mathbb{R}[\mathbf{X}]}{J} = \dim \frac{\mathbb{R}[\mathbf{X}]}{I} = 0$ we have $\mathcal{I}(\mathcal{S}(\mathbf{g})) = \sqrt[\mathbb{R}]{\operatorname{supp}\mathcal{Q}(\mathbf{g})} = J$ by [Mar08, cor. 7.4.2 (3)]. Then $\dim \frac{\mathbb{R}[\mathbf{X}]}{\mathcal{I}(\mathcal{S}(\mathbf{g}))} = 0$ and $\mathcal{V}_{\mathbb{R}}(J) = \mathcal{V}_{\mathbb{R}}(\mathcal{I}(\mathcal{S}(\mathbf{g}))) = \mathcal{S}(\mathbf{g}) = \{\xi_1,\dots,\xi_r\}$ is finite.

We choose a graded basis $\mathbf{h}$ of J with $\deg\mathbf{h} \leq \rho = \rho(\xi_1,\dots,\xi_r)$, see Section 1.4. By Corollary 3.13 and Proposition 1.4, there exists $d \in \mathbb{N}$ big enough such that for every $k \in \mathbb{N}$:

$$\mathcal{L}_{d+k}(\mathbf{g})^{[2(\rho-1)+k]} \subset \mathcal{L}_{2(\rho-1)+k}(\pm\mathbf{h}) = \mathcal{L}_{2(\rho-1)+k}(I_{2(\rho-1)+k}) = \operatorname{cone}(\mathbf{e}_{\xi_1},\dots,\mathbf{e}_{\xi_r})^{[2(\rho-1)+k]}.$$

Since the converse inclusion is obvious, we prove that $\mathcal{L}_{d+k}(\mathbf{g})^{[2(\rho-1)+k]} = \operatorname{cone}(\mathbf{e}_{\xi_1},\dots,\mathbf{e}_{\xi_r})^{[2(\rho-1)+k]}$. □

Remark. Notice that there exist examples with $\mathcal{S}(\mathbf{g})$ finite and $\dim \frac{\mathbb{R}[\mathbf{X}]}{\operatorname{supp}\mathcal{Q}(\mathbf{g})} > 1$, see Example 3.6. However the hypothesis:

(i) $\dim \frac{\mathbb{R}[\mathbf{X}]}{\operatorname{supp}\mathcal{Q}(\mathbf{g})} = 0$; and

(ii) $\mathcal{S}(\mathbf{g})$ is finite and $\dim \frac{\mathbb{R}[\mathbf{X}]}{\operatorname{supp}\mathcal{Q}(\mathbf{g})} \leq 1$

are equivalent: (i) $\Rightarrow$ (ii) is shown in the proof of Theorem 3.23, while (ii) $\Rightarrow$ (i) follows from $\mathcal{I}(\mathcal{S}(\mathbf{g})) = \sqrt[\mathbb{R}]{\operatorname{supp}\mathcal{Q}(\mathbf{g})}$ (see [Mar08, cor. 7.4.2 (3)]).

Corollary 3.24. *Suppose that* $\mathcal{S}(\mathbf{g}) = \{\xi_1,\dots,\xi_r\}$ *is non-empty and finite and let* $\rho = \rho(\xi_1,\dots,\xi_r)$ *be the regularity. Then there exists* $d \in \mathbb{N}$ *such that* $\forall k \in \mathbb{N}$*:*

$$\mathcal{L}_{d+k}(\Pi\mathbf{g})^{[2(\rho-1)+k]} = \operatorname{cone}(\mathbf{e}_{\xi_1},\dots,\mathbf{e}_{\xi_r})^{[2(\rho-1)+k]}.$$

Proof. We combine Theorem 3.23 and Theorem 1.1 replacing $\mathbf{g}$ by $\Pi\mathbf{g}$. □

Related results were obtained in [LLR08] and [Las+13]. In particular Theorem 3.23 generalize their setting, i.e. the case when the description of the quadratic module defines an ideal with finite real variety. Theorem 3.23 gives a complete description of the truncated moment sequences in terms of convex sums of evaluations at the points of the semialgebraic set.

3.5 The truncated moment hierarchy

We summarize the relations between the truncated moment cones that we have seen. Let $Q = \mathcal{Q}(\mathbf{g})$, $S = \mathcal{S}(\mathbf{g})$ and let $\mathbf{h}$ be a (graded) basis of $J = \sqrt[\mathbb{R}]{\operatorname{supp} Q}$. For d, t big enough, we have the following inclusions:

$$\mathcal{L}_t(\mathbf{g}, \pm\mathbf{h}) \underset{(1)}{\supseteq} \mathcal{L}_d(\mathbf{g})^{[t]} \underset{(2)}{\supseteq} \mathcal{L}_{d+1}(\mathbf{g})^{[t]} \supset \ldots \supset \bigcap_{d=t}^{\infty} \mathcal{L}_d(\mathbf{g})^{[t]} \underset{(3)}{\supseteq} \mathcal{L}(Q)^{[t]} \underset{(4)}{\supseteq} \mathcal{M}(S)^{[t]}.$$

All these inclusions are obvious, except for (1) which is Lemma 3.11. In this section we analyzed cases when these inclusions are equalities.

(1) is an equality if we can extend degree-t positive functionals on $\mathcal{Q}_t(\mathbf{g})$ to degree-d positive functionals on $\mathcal{Q}_d(\mathbf{g})$, i.e. if $\mathcal{L}_t(\mathbf{g}) = \mathcal{L}_d(\mathbf{g})^{[t]}$.

(2) is an equality if Q is stable: if $\mathcal{Q}_t(\mathbf{g}) = (\mathcal{Q}_d(\mathbf{g}))_t$ then $\mathcal{L}_t(\mathbf{g}) = \mathcal{L}_d(\mathbf{g})^{[t]}$ (see [Mar08, ch. 4]).

(3) and (4) are equalities if Q is Archimedan, see Theorem 3.19.

(4) is an equality if $\mathcal{L}(Q)$ has the strong moment property (and this is the case when Q is Archimedean, see Theorem 1.2). We cannot deduce that (3) is an equality with just strong moment property hypothesis.

Theorem 3.23 says that, if J defines a finite real variety, then all these inclusions are equalities. Morever notice that, if Q is a reduced, Archimedean quadratic module with $\dim S \geq 3$, then we cannot have finite convergence in general (see Example 2.8) and inclusions (2) are always proper: (3) and (4) are equalities since Q is Archimedean, but there exists $\sigma \in \mathcal{L}_d(\mathbf{g})^{[t]} \setminus \bigcap_{e=t}^{\infty} \mathcal{L}_e(\mathbf{g})^{[t]}$ for all d.

The situation is simplier if we consider generic linear funcionals and kernels of Hankel operators. If $\tau_d^* \in \mathcal{L}_d(\mathbf{g})$ and $\sigma^* \in \mathcal{L}(Q)$ are generic, then:

$$J \underset{(a)}{=} (\ker H^t_{\tau_d^*}) \underset{(b)}{=} (\ker H^t_{\tau_{d+1}^*}) = \ldots \underset{(c)}{\subset} (\ker H^t_{\sigma^*}) \underset{(d)}{\subset} \mathcal{I}(S).$$

(a) and (b) are Theorem 3.16. (d) is an equality if Q is Archimedean, see Proposition 3.4. Inclusion (c) can be proper, even in the Archimedean case: in Example 3.6 $(\ker H^t_{\tau_d^*}) = (0)$ but $(\ker H^t_{\sigma^*}) = (X, Y)$.

4 Applications to Polynomial Optimization

In this section, we consider the Polynomial Optimization Problem of minimizing $f \in \mathbb{R}[\mathbf{X}]$ on a basic semialgebraic set $S = \mathcal{S}(\mathbf{g})$ where $\mathbf{g} = \{g_1, \ldots, g_s\} \subset \mathbb{R}[\mathbf{X}]$.

4.1 Finite semialgebraic set

Theorem 4.1. *Let f^* denote the infimum of f on $S = \mathcal{S}(\mathbf{g})$ and let $Q = \mathcal{Q}(\mathbf{g})$. Suppose that $\dim \frac{\mathbb{R}[\mathbf{X}]}{\operatorname{supp} Q} = 0$. Then the moment relaxation $(\mathcal{L}_{2d}(\mathbf{g}))_{d\in\mathbb{N}}$ is exact. For $t \in \mathbb{N}$ and $d \geq \frac{t}{2}$ big enough,*

$$\mathcal{L}_{2d}^{\min}(\mathbf{g})^{[t]} = \operatorname{conv}(\mathbf{e}_{\xi_1}, \ldots, \mathbf{e}_{\xi_l})^{[t]},$$

where $\{\xi_1, \ldots, \xi_l\} \subset \mathbb{R}^n$ is the finite set of minimizers of f on S. Moreover, if $d \geq t \geq \rho = \rho(\xi_1, \ldots, \xi_l)$ and $\sigma \in \mathcal{L}_{2d}^{\min}(\mathbf{g})$ is generic, then $(\ker H^t_\sigma) = \mathcal{I}(\xi_1, \ldots, \xi_l)$ is the vanishing ideal of the minimizers $\{\xi_1, \ldots, \xi_l\}$ of f on S.

Proof. By Theorem 3.23) for $t \geq \deg f$ and $d \geq \frac{t}{2}$, $\mathcal{L}_{2d}(\Pi\mathbf{g})^{[t]} = \operatorname{cone}(\mathbf{e}_{\xi_1}, \ldots, \mathbf{e}_{\xi_r})^{[t]}$. Assume that the minimizers of f on $S = \{\xi_l, \ldots, \xi_r\}$ are $\{\xi_l, \ldots, \xi_l\}$ $(l \leq r)$ so that $f(\xi_i) > f^*$ if $l < i \leq r$. Then for all $\sigma \in \mathcal{L}_{2d}^{\min}(\mathbf{g})$, $\langle \sigma | f \rangle = \left\langle \sigma^{[t]} \middle| f \right\rangle = f^* = \sum_{i=1}^r w_i f(\xi_i)$ is a convex sum of evaluations of f at the points of the semialgebraic set S with $\sum_{i=1}^r w_i = \left\langle \sigma^{[t]} \middle| f \right\rangle = 1$. We deduce that $\sum_{i=1}^r w_i (f(\xi_i) - f^*) = 0$ so that

if $f(\xi_i) > f^*$ then $w_i = 0$. Consequently $\sigma^{[t]} \in \text{cone}(\mathbf{e}_{\xi_i}, \ldots, \mathbf{e}_{\xi_l})^{[t]}$ and the first part of the theorem follows.

Let us choose $t \geq \rho$ and $d \geq t$ big enough. We have shown that if $\sigma \in \mathcal{L}_{2d}^{\min}(\mathbf{g})$ then $\sigma^{[2t]} = \sum_{i=1}^{l} \omega_i \mathbf{e}_{\xi_i}$ with $\omega_i \geq 0$. Let $(u_i)_{i=1,\ldots,r}$ be a family of interpolation polynomials at the points $\xi_l, \ldots, \xi_l$ of degree $\leq \rho - 1$ (Proposition 1.3). As $\left\langle \sigma^{[2t]} \middle| u_i^2 \right\rangle = \omega_i$, $u_i \in \ker H_\sigma^t$ if and only if $\omega_i = 0$. Therefore a generic element $\sigma \in \mathcal{L}_{2d}^{\min}(\mathbf{g})$ is such that $\omega_i > 0$, since the kernel of H_σ^t is included in all the other kernels.

For $p \in I = \mathcal{I}(\xi_l, \ldots, \xi_l)$ with $\deg(p) \leq t$, we have $\left\langle \sigma^{[2t]} \middle| p^2 \right\rangle = \sum_{i=1}^{l} \omega_i p^2(\xi_i) = 0$ so that $p \in \ker H_\sigma^t$. Conversely, for $p \in \ker H_\sigma^t$, $\left\langle \sigma^{[2t]} \middle| p^2 \right\rangle = \sum_{i=1}^{l} \omega_i p^2(\xi_i) = 0$, which implies that $p(\xi_i) = 0$ and $p \in I_t$. This shows that $\ker H_\sigma^t = I_t$. As I is generated in degree $\rho \leq t$, we have proved the second part of theorem: $(\ker H_\sigma^t) = I$. □

Corollary 4.2. *With the same notation, if $S = \mathcal{S}(\mathbf{g})$ is finite then the moment relaxation $(\mathcal{L}_{2d}(\Pi\mathbf{g}))_{d\in\mathbb{N}}$ is exact. For $t \in \mathbb{N}$ and $d \geq \frac{t}{2}$ big enough,*

$$\mathcal{L}_{2d}^{\min}(\Pi\mathbf{g})^{[t]} = \text{conv}(\mathbf{e}_{\xi_1}, \ldots, \mathbf{e}_{\xi_l})^{[t]}.$$

where $\{\xi_1, \ldots, \xi_l\}$ is the finite set of minimizers of f on S. Moreover, if $d \geq t \geq \rho = \rho(\xi_1, \ldots, \xi_l)$ and $\sigma \in \mathcal{L}_{2d}^{\min}(\Pi\mathbf{g})$ is generic, then $(\ker H_\sigma^t) = \mathcal{I}(\xi_1, \ldots, \xi_l)$ is the vanishing ideal of the minimizers $\{\xi_1, \ldots, \xi_l\}$ of f on S.

Proof. It is a consequence of Theorem 4.1 and Theorem 1.1, since if $\mathcal{S}(\mathbf{g})$ is finite then $\dim \frac{\mathbb{R}[\mathbf{X}]}{\text{supp}\,\mathcal{O}(\mathbf{g})} = \dim \frac{\mathbb{R}[\mathbf{X}]}{\sqrt{\text{supp}\,\mathcal{O}(\mathbf{g})}} = \dim \frac{\mathbb{R}[\mathbf{X}]}{\mathcal{I}(\mathcal{S}(\mathbf{g}))} = 0$. □

We consider now the case of finite semialgebraic set $S = \mathcal{S}(\mathbf{g}, \pm\mathbf{h})$ defined with equations $\mathbf{h}$ with an associated finite real variety: $|\mathcal{V}_{\mathbb{R}}(\mathbf{h})| < \infty$.

Corollary 4.3. *If $\mathcal{V}_{\mathbb{R}}(\mathbf{h})$ is finite then the moment relaxation $(\mathcal{L}_{2d}(\mathbf{g}, \pm\mathbf{h}))_{d\in\mathbb{N}}$ is exact. For $t \in \mathbb{N}$ and $d \geq \frac{t}{2}$ big enough,*

$$\mathcal{L}_{2d}^{\min}(\mathbf{g}, \pm\mathbf{h})^{[t]} = \text{conv}(\mathbf{e}_{\xi_1}, \ldots, \mathbf{e}_{\xi_l})^{[t]}.$$

where $\{\xi_1, \ldots, \xi_l\}$ is the finite set of minimizers of f on $S = \mathcal{S}(\mathbf{g}, \pm\mathbf{h})$. Moreover, if $d \geq t \geq \rho = \rho(\xi_1, \ldots, \xi_l)$ and $\sigma \in \mathcal{L}_{2d}^{\min}(\mathbf{g}, \pm\mathbf{h})$ is generic, then $(\ker H_\sigma^t) = \mathcal{I}(\xi_1, \ldots, \xi_l)$ is the vanishing ideal of the minimizers $\{\xi_1, \ldots, \xi_l\}$ of f on S.

Proof. Let $Q = \mathcal{Q}(\mathbf{g}, \pm\mathbf{h})$. If $|\mathcal{V}_{\mathbb{R}}(\mathbf{h})| < \infty$ then $\dim \frac{\mathbb{R}[\mathbf{X}]}{\sqrt[\mathbb{R}]{(\mathbf{h})}} = 0$. Since $\sqrt{\text{supp}\,Q} = \sqrt[\mathbb{R}]{\text{supp}\,Q}$ (see Section 1.3) and $(\mathbf{h}) \subset \text{supp}\,Q$, we have:

$$\dim \frac{\mathbb{R}[\mathbf{X}]}{\text{supp}\,Q} = \dim \frac{\mathbb{R}[\mathbf{X}]}{\sqrt{\text{supp}\,Q}} = \dim \frac{\mathbb{R}[\mathbf{X}]}{\sqrt[\mathbb{R}]{\text{supp}\,Q}} \leq \dim \frac{\mathbb{R}[\mathbf{X}]}{\sqrt[\mathbb{R}]{(\mathbf{h})}} = 0.$$

Then the relaxation is exact by Theorem 4.1. □

Applying strong duality we can also deduce finite convergence for the SoS relaxation.

Corollary 4.4. *Under the assumptions of Theorem 4.1 (resp. Corollary 4.2, resp. Corollary 4.3) the SoS relaxation:*

$$\begin{aligned} f_{\text{SoS},d}^* &= \sup\left\{ \lambda \in \mathbb{R} \mid f - \lambda \in \mathcal{Q}_{2d}(\mathbf{g}) \right\} \\ (\textit{resp. } f_{\text{SoS},d}^* &= \sup\left\{ \lambda \in \mathbb{R} \mid f - \lambda \in \mathcal{Q}_{2d}(\mathbf{\Pi g}) \right\}, \\ \textit{resp. } f_{\text{SoS},d}^* &= \sup\left\{ \lambda \in \mathbb{R} \mid f - \lambda \in \mathcal{Q}_{2d}(\mathbf{g}, \pm\mathbf{h}) \right\}) \end{aligned}$$

has the finite convergence property.

Proof. For the quadratic module cases, $\mathcal{Q}(\mathbf{g})$ is Archimedean by [Mar08, cor. 7.4.3]. Then by strong duality Theorem 2.5 the result follows.

For the preordering case, $\mathcal{O}(\mathbf{g})$ is Archimedean by [Wö98] (see also [Mar08, th. 6.1.1]). Then by strong duality Theorem 2.5 the result follows. □

Notice that, even if the SoS relaxation has the finite convergence property, it may not be SoS exact as shown in Example 2.10 and Example 2.11.

4.2 Boundary Hessian Conditions

Boundary Hessian Conditions (BHC) are conditions on the minimizers of a polynomial f on a basic semialgebraic set S introduced by Marshall in [Mar06] and [Mar09]. These conditions are particular cases of the so called local-global principle, which allows one to prove global properties of polynomials (e.g. $f \in Q$) analyzing local properties (e.g. checking the BHC at the minimizers of f on $\mathcal{S}(Q)$). We refer to [Sch05a], [Sch06] and [Mar08, ch. 9] for more details. We introduce BHC conditions following [Sch09].

Definition 4.5 (Boundary Hessian Conditions)**.** Let $V \subset \mathbb{R}^n$ be a variety, and let Q be a finitely generated Archimedean quadratic module in $\mathbb{R}[V] \cong \frac{\mathbb{R}[\mathbf{X}]}{\mathcal{I}(V)}$ (or equivalently $Q + \mathcal{I}(V)$ is Archimedean in $\mathbb{R}[\mathbf{X}]$). Let $S = \mathcal{S}(Q) \cap V$ and $f \in \mathrm{Pos}(S)$. We say that the *Boundary Hessian Conditions* holds at $x \in \mathcal{V}(f) \cap S$ if there exists $t_1, \dots, t_m \in Q$ such that:

- $t_1, \dots, t_m$ are part of a regular system of parameters for V at x;
- $\nabla f(x) = a_1 \nabla t_1(x) + \cdots + a_m \nabla f(x)$, where a_i are strictly positive real numbers;
- the Hessian of f restricted to $\mathcal{V}(t_1, \dots, t_m) \cap V$ is positive definite at x.

When BHC holds, the minimizers are non-singular, isolated points and thus finite. It is proved in [Mar09] that if BHC holds at every minimizer of f on $\mathcal{S}(\mathbf{g})$ then $f \in \mathcal{Q}(\mathbf{g})$, which implies that the SoS relaxation $f^*_{\mathrm{SoS},d}$ is exact. [Nie14] proved that the BHC at every minimizer of f, which hold generically, implies the SoS finite convergence property.

In this section, we prove that, if the BHC hold, then the MoM relaxation is exact. We need some preliminary lemmas.

Lemma 4.6. *Let $p, g \in \mathbb{R}[\mathbf{X}]$, $k \geq \deg p + \deg g$ and $d \geq 2k + \deg g$. If $\sigma = \sigma^{[d]} \in \mathcal{L}_d(g)$ then: $\left\langle \sigma^{[d]} \middle| p^2 g \right\rangle = 0$ implies $pg \in \ker H^k_\sigma$.*

Proof. Let $h \in \mathbb{R}[\mathbf{X}]_k$ and $\sigma = \sigma^{[d]} \in \mathcal{L}_d(\mathbf{g})$. Since $\sigma^{[d]}$ is positive on $\mathcal{Q}_d(g)$ and $\left\langle \sigma^{[d]} \middle| p^2 g \right\rangle = 0$, then $\forall t \in \mathbb{R}$:

$$0 \leq \left\langle \sigma^{[d]} \middle| (p + th)^2 g \right\rangle = t^2 \left\langle \sigma^{[d]} \middle| h^2 g \right\rangle + 2t \left\langle \sigma^{[2d]} \middle| phg \right\rangle.$$

As a function of t the last expression is non-negative, and equal to 0 for $t = 0$. Then $t = 0$ must be a double root, and thus $\left\langle \sigma^{[d]} \middle| phg \right\rangle = \left\langle (fg \star \sigma)^{[k]} \middle| h \right\rangle = 0$ for all $h \in \mathbb{R}[\mathbf{X}]_k$. But this means $pg \in H^k_\sigma$. □

Lemma 4.7. *Let $f \in \mathcal{Q}_l(\mathbf{g})$. Then for k and $d \geq 2k$ big enough, if $\sigma \in \mathcal{L}_d(\mathbf{g})$ then: $\langle \sigma | f \rangle = 0$ if and only if $f \in \ker H^k_\sigma$.*

Proof. The *if* part is obvious.

For the *only if* part, we set $g_0 = 1$ for notation convenience. Since $f \in \mathcal{Q}_l(\mathbf{g})$, then $f = \sum_i s_i g_i$, with $s_i = \sum_j p^2_{i,j}$ and $\deg s_i g_i \leq l$. Let $d \geq \max_{i,j}\{2\deg(p_{i,j}) + \deg(g_i)\}$ and $\sigma \in \mathcal{L}_d(\mathbf{g})$. By hypothesis:

$$0 = \left\langle \sigma^{[d]} \middle| f \right\rangle = \sum_{i,j} \left\langle \sigma^{[d]} \middle| p^2_{i,j} g_i \right\rangle,$$

which implies $\left\langle \sigma^{[d]} \middle| p^2_{i,j} g_i \right\rangle = 0$ for all i and j. Let $k_{i,j}$ and $d_{i,j}$ be given from Lemma 4.6 (applied to $p_{i,j}$ and g_i). Let $k' \geq \max_{i,j}\{k_{i,j}\}$. Then $p_{i,j} g_i \in \ker H^{k'}_\sigma$ for all i and j which implies that $p^2_{i,j} g_i \in$

$\ker H_\sigma^{k'-\deg p_{i,j}}$. Letting $k = \min_{i,j}\{k' - \deg p_{i,j}\}$, we finally get $p_{i,j}^2 g_i \in \ker H_\sigma^k$ for all i and j, and $f = \sum_{i,j} p_{i,j}^2 g_i \in \ker H_\sigma^k$. □

Theorem 4.8. *Let $f \in \mathbb{R}[\mathbf{X}]$, $Q = \mathcal{Q}(\mathbf{g})$ be an Archimedean finitely generated quadratic module and assume that the BHC hold at every minimizer of f on $S = \mathcal{S}(\mathbf{g})$. Then the moment relaxation $(\mathcal{L}_{2d}(\mathbf{g}))_{d\in\mathbb{N}}$ is exact. For $t \in \mathbb{N}$ and $d, e \geq \frac{t}{2}$ big enough:*

$$\mathcal{L}_{2d}^{\min}(\mathbf{g})^{[t]} = \mathcal{L}_{2e}(\mathbf{g}, \pm(f - f^*))^{[t]} = \operatorname{conv}(\mathbf{e}_{\xi_1}, \ldots, \mathbf{e}_{\xi_r})^{[t]}.$$

where $\{\xi_1, \ldots, \xi_r\}$ is the finite set of minimizers of f on S. Moreover, if $d \geq t \geq \rho(\xi_1, \ldots, \xi_r)$ and $\sigma^* \in \mathcal{L}_{2d}^{\min}(\mathbf{g})$ is generic, then $(\ker H_{\sigma^*}^t) = \mathcal{I}(\xi_1, \ldots, \xi_r)$ is the vanishing ideal of the minimizers of f on S.

Proof. We can assume without loss of generality that $f^* = 0$. For d, e big enough, if $\sigma \in \mathcal{L}_{2d}^{\min}(\mathbf{g})$ then $f \in \ker H_\sigma^e$ by Lemma 4.7. This implies that $\mathcal{L}_{2d}^{\min}(\mathbf{g})^{[2e]} \subset \mathcal{L}_{2e}(\mathbf{g}, \pm f)$. Since the BHC hold, we know that $\dim \frac{\mathbb{R}[\mathbf{X}]}{\operatorname{supp}(Q+(f))} = 0$ (see the proof of [Mar06, th. 2.3]). By Theorem 3.23 applied to $\mathcal{L}_{2e}(\mathbf{g}, \pm f)$, we have $\mathcal{L}_{2e}(\mathbf{g}, \pm f)^{[t]} = \operatorname{conv}(\mathbf{e}_{\xi_1}, \ldots, \mathbf{e}_{\xi_r})^{[t]}$ for $t \in \mathbb{N}$ and e big enough. Since $\operatorname{conv}(\mathbf{e}_{\xi_1}, \ldots, \mathbf{e}_{\xi_r})^{[t]} \subset \mathcal{L}_{2d}^{\min}(\mathbf{g})^{[t]}$ by definition, we proved the first part: up to restriction, functional minimizers are coming from convex sums of evaluations at the minimizers of f.

The proof of the second part is equal to that of Theorem 4.1. □

Remark. In Theorem 4.8 we use BHC to prove the following:

- $f - f^* \in Q$ (i.e. SoS exactness);
- $\dim \frac{\mathbb{R}[\mathbf{X}]}{\operatorname{supp}(Q+(f-f^*))} = 0$.

If the previous two conditions holds, the conclusions of Theorem 4.8 remain valid.

We show now that moment exactness holds *generically*. For polynomials $f \in \mathbb{R}[\mathbf{X}]_d$ and $g_1 \in \mathbb{R}[\mathbf{X}]_{d_1}, \ldots, g_s \in \mathbb{R}[\mathbf{X}]_{d_s}$, we say that a property holds generically (or that the property holds for generic $f, g_1, \ldots, g_s$) if there exists finitely many nonzero polynomials $\phi_1, \ldots, \phi_l$ in the coefficients of polynomials in $\mathbb{R}[\mathbf{X}]_d$ and $\mathbb{R}[\mathbf{X}]_{d_1}, \ldots, \mathbb{R}[\mathbf{X}]_{d_s}$ such that, when $\phi_1(f, \mathbf{g}) \neq 0, \ldots, \phi_l(f, \mathbf{g}) \neq 0$, the property holds.

Corollary 4.9. *For $f \in \mathbb{R}[\mathbf{X}]_d$ and $g_1 \in \mathbb{R}[\mathbf{X}]_{d_1}, \ldots, g_s \in \mathbb{R}[\mathbf{X}]_{d_s}$ generic, the moment relaxation $(\mathcal{L}_{2d}(\mathbf{g}))_{d\in\mathbb{N}}$ is exact.*

Proof. By [Nie14, th. 1.2] BHC hold generically. We apply Theorem 4.8 to conclude. □

4.3 Finite minimizers

In this section, we consider Polynomial Optimization Problems for which the non-empty set of minimizers is finite and we propose a strategy to recover them.

If the set of minimizers is non-empty and finite, and we know the minimum f^* of f on $S = \mathcal{S}(\mathbf{g})$, by adding the equation $f - f^*$ to the definition of the truncated quadratic module, we obtain a quadratic module $Q' = \mathcal{Q}(\mathbf{g}, \pm(f - f^*))$, which defines the finite set $\mathcal{S}(Q')$ of minimizers of f on S. We can then apply the results of Section 4.1 to the relaxation $(\mathcal{L}_{2d}(\mathbf{g}, \pm(f - f^*)))_d$ or $(\mathcal{L}_{2d}(\Pi\mathbf{g}, \pm(f - f^*)))_d$.

Corollary 4.10. *Let $f \in \mathbb{R}[\mathbf{X}]$, $Q = \mathcal{Q}(\mathbf{g})$ be a finitely generated quadratic module. Assume that the minimizers of f on $S = \mathcal{S}(\mathbf{g})$ are finite: $\{x \in S \mid f(x) = f^*\} = \{\xi_1, \ldots, \xi_r\}$. Then for any $t \in \mathbb{N}$ and $d \geq \frac{t}{2}$ big enough:*

$$\mathcal{L}_{2d}(\Pi\mathbf{g}, \pm(f - f^*))^{[t]} = \operatorname{conv}(\mathbf{e}_{\xi_1}, \ldots, \mathbf{e}_{\xi_r})^{[t]}.$$

Moreover, if $d \geq t \geq \rho = \rho(\xi_1, \ldots, \xi_r)$ and $\sigma \in \mathcal{L}_{2d}^{\min}(\mathbf{g}, \pm(f - f^*))$ is generic, then $(\ker H_\sigma^t) = \mathcal{I}(\xi_1, \ldots, \xi_r)$ is the vanishing ideal of the minimizers of f on S.

In practice, the minimum f^* is usually not known. Since the computation of moment minimizers $\sigma^* \in \mathcal{L}_d^{\min}(\mathbf{g}, \pm(f - f^*))$ is based on numerical Semi-Definite Program (SDP) solvers, we can replace f^* by an approximate value, taking for instance $f^*_{\mathrm{MoM},d} = \inf\left\{ \langle \sigma | f \rangle \in \mathbb{R} \mid \sigma \in \mathcal{L}_{2d}^{(1)}(\mathbf{g}) \right\} \leq f^*$ for $d \in \mathbb{N}$. Notice that if $v < f^*$ then $\mathcal{L}_{2d}^{(1)}(\mathbf{g}, \pm(f - v))$ is empty since $\mathcal{S}(\Pi\mathbf{g}, \pm(f - v))$ is empty. If v is not close to f^*, the SDP solvers can detect the feasibility/infeasibility of the relaxation, that is if $\mathcal{L}_d(\Pi\mathbf{g}, \pm(f - v))$ is empty or not.

Notice also that by [Wö98] (or [Mar08, th. 6.1.1]) $\mathcal{O}(\mathbf{g})$ is Archimedean if the semialgebraic set is finite. If also $\mathcal{Q}(\mathbf{g})$ is Archimedean, since the SDP solvers perform approximate numerical computations, and since $\lim_{d\to\infty} \mathrm{d_H}(\mathcal{L}_d^{(1)}(\mathbf{g})^{[k]}, \mathcal{L}_d^{(1)}(\Pi\mathbf{g})^{[k]}) = 0$ for k big enough by Corollary 3.20, we can also replace the relaxation associated to the preordering by the relaxation associated to the quadratic module. This leads to the following algorithm for the computation of finite minimizers.

Algorithm 1: Finite Minimizers

input : $d \in \mathbb{N}$, $f, g_1, \ldots g_r \in \mathbb{R}[\mathbf{X}]_d$ such that f has a finite set of minimizers on $S = \mathcal{S}(\mathbf{g})$.
output: The minimizers $\{\xi_1, \ldots, \xi_r\}$ of f on S and $f^* = \inf_{x \in S} f(x)$.
$k = \lceil \frac{d}{2} \rceil$
repeat
  Compute $f^*_{\mathrm{MoM},k} = \inf\left\{ \langle \sigma | f \rangle \in \mathbb{R} \mid \sigma \in \mathcal{L}_{2k}^{(1)}(\mathbf{g}) \right\}$.
  Compute a generic element $\sigma^* \in \mathcal{L}_{2k}^{(1)}(\mathbf{g}, \pm(f - f^*_{\mathrm{MoM},k}))$
  Extract of the minimizers $\xi_1, \ldots, \xi_r$ from $H^t_{\sigma^*}$ for $t \leq k$ big enough.
  $k = k + 1$
until *minimizer extraction success*
return *the minimizers* $\{\xi_1, \ldots, \xi_r\}$ *and* $f^* = \langle \sigma^* | f \rangle$

Each loop of this algorithm requires two calls to SDP solvers. The first one is to compute $f^*_{\mathrm{MoM},k}$ on the convex set $\mathcal{L}_{2k}^{(1)}(\mathbf{g})$. The second one is to compute an interior or generic point σ^* of $\mathcal{L}_{2k}^{(1)}(\mathbf{g}, \pm(f - f^*_{\mathrm{MoM},k}))$, using an interior point SDP solver.

The extraction of minimizers from the Hankel matrix $H^t_{\sigma^*}$ is based on the algorithm of polynomial-exponential decomposition of series described in [Mou18]. It involves numerical linear algebra functions such as SVD, eigenvalue and eigenvector computation. It provides an approximation of the linear functional σ^* as a weighted sum of evaluations $\sigma^* \approx \sum_{i=1}^r \omega_i \mathbf{e}_{\xi_i}$. We consider that the minimizer extraction succeeds when such an approximation of σ^* is obtained within a given threshold.

If the set of minimizers is finite and the moment relaxation $(\mathcal{L}_{2d}(\mathbf{g}))_{d\in\mathbb{N}}$ is exact for f then this algorithm terminates. When this is not the case, it shall also terminates using approximate computation. Indeed, increasing the degree k, we obtain better approximations of f^* and of a generic element of $\mathcal{L}_{2d}(\Pi\mathbf{g}, \pm(f - f^*))^{[2t]} = \mathrm{conv}(\mathbf{e}_{\xi_1}, \ldots, \mathbf{e}_{\xi_r})^{[2t]}$. When a sufficiently good approximation of a generic element of $\mathrm{conv}(\mathbf{e}_{\xi_1}, \ldots, \mathbf{e}_{\xi_r})^{[2t]}$ is obtained, the minimizer extraction succeeds and Algorithm 1 outputs an approximation of the minimizers $\{\xi_1, \ldots, \xi_r\}$ and the minimum f^*. We will illustrate it in Example 4.14.

4.4 Gradient, KKT and Polar ideals

Another approach which has been investigated to make the relaxations exact, is to add equality constraints satisfied by the minimizers (and independent of the minimum f^*) to a Polynomial Optimization Program.

For global optimization we can consider the gradient equations (see [NDS06]): obviously $\nabla f(x^*) = \mathbf{0}$ for all the minimizers x^* of f on $S = \mathbb{R}^n$. For constrained optimization we can consider Karush–Kuhn–Tucker (KKT) constraints, adding new variables (see [DNP07]) or projecting them to the variables $\mathbf{X}$ (Jacobian equations, see [Nie13a]). We shortly describe them.

Let $g_1,\dots,g_r,h_1,\dots,h_s \in \mathbb{R}[\mathbf{X}]$ defining $S = \mathcal{S}(\mathbf{g},\pm\mathbf{h})$, and let $f \in \mathbb{R}[\mathbf{X}]$ be the objective function. Let $\mathbf{\Lambda} = (\Lambda_1,\dots,\Lambda_r)$ and $\mathbf{\Gamma} = (\Gamma_1,\dots,\Gamma_s)$ be variables representing the *Lagrange multipliers* associated with $\mathbf{g}$ and $\mathbf{h}$. The *KKT constraints* associated to the optimization problem $\min f(x)\colon x \in \mathcal{S}(\mathbf{g},\pm\mathbf{h})$ are:

$$\begin{cases} \dfrac{\partial f}{\partial X_i} - \displaystyle\sum_{k=1}^{r} \Lambda_k^2 \frac{\partial g_k}{\partial X_i} - \sum_{j=1}^{s} \Gamma_j \frac{\partial h_j}{\partial X_i} = 0 \quad \forall i \\ \Lambda_k g_k = 0, \quad h_j = 0, \quad g_k \geq 0 \qquad\qquad \forall j,k, \end{cases} \tag{6}$$

where the polynomials belong to $\mathbb{R}[\mathbf{X},\mathbf{\Gamma},\mathbf{\Lambda}]$. These are sufficient but not necessary conditions for $x^* \in S$ being a minimizer.

For $x \in S$, we say that g_i is an *active constraint at* x if $g_i(x) = 0$. Let $x^* \in S$ and $g_{i_1},\dots g_{i_k}$ be the active constraints at x^*. The KKT constraints are necessary if Linear Independence Constraint Qualification (LICQ) holds, that is, if $\nabla h_1(x^*),\dots,\nabla h_s(x^*),\nabla g_{i_1}(x^*),\dots,\nabla g_{i_k}(x^*)$ are linearly independent at the minimizer $x^* \in S$ (see [NW06, th. 12.1]). We cannot avoid the LICQ hypothesis: for example if $f = X_1 \in \mathbb{R}[X_1]$ and $g_1 = X_1^3 \in \mathbb{R}[X_1]$, then $x^* = 0$ is a minimizer, but the KKT equations are not satisfied at $x^* = 0$.

To avoid this problem we define the *polar ideal*. Observe from eq. (6) that, if KKT constraints are satisfied at x and

- if g_i is not an active constraint at x, then $\Lambda_i = 0$;
- if $g_{i_1},\dots g_{i_k}$ are the active constraints at x, then the gradients $\nabla f(x),\nabla h_1(x),\dots,\nabla h_r(x),\nabla g_{i_1}(x)$, $\dots,\nabla g_{i_k}(x)$ are linearly dependent.

Definition 4.11. For $f,g_1,\dots,g_r,h_1,\dots,h_s \in \mathbb{R}[\mathbf{X}]$ as before, the *polar ideal* is defined as follows:

$$J := (\mathbf{h}) + \prod_{\{a_1,\dots,a_k\}\subset\{1,\dots r\}} \Big((g_{a_1},\dots,g_{a_k}) + \Big(\operatorname{rank}\operatorname{Jac}(f,\mathbf{h},g_{a_1},\dots,g_{a_k})\Big) < s+k+1\Big).$$

where $\Big(\Big(\operatorname{rank}\operatorname{Jac}(f,\mathbf{h},g_{a_1},\dots,g_{a_k})\Big) < l\Big)$ is the ideal generated by the $l \times l$ minors of the Jacobian matrix $\operatorname{Jac}(f,\mathbf{h},g_{a_1},\dots,g_{a_k})$. The generators of J besides $\mathbf{h}$ are the product of active constraints and the generators of rank ideals.

In this definition, we could replace the product of ideals by their intersection and the $l \times l$ minors of the Jacobian matrices by polynomials defining the same varieties.

We prove that every minimizer belongs to $\mathcal{V}_{\mathbb{R}}(J)$.

Lemma 4.12. *Let x^* be a minimizer of f on $S = \mathcal{S}(\mathbf{g},\pm\mathbf{h})$. Then $x^* \in \mathcal{V}_{\mathbb{R}}(J)$.*

Proof. Since $x^* \in S$, then $x^* \in \mathcal{V}_{\mathbb{R}}(\mathbf{h})$.

If the LICQ hold at x^*, then x^* is a KKT point (see [NW06, th. 12.1]) and $\nabla f(x) = \sum_j \gamma_j \nabla \mathbf{h}_j(x) + \sum_j \lambda_j^2 \nabla \mathbf{g}_j(x)$ for some γ_j and λ_i in $\mathbb{R}$. As $\lambda_k = 0$ if g_k is not an active constraint, we have that

$$\nabla f(x^*),\nabla h_1(x^*),\dots,\nabla h_r(x^*),\nabla g_{i_1}({}^*x),\dots,\nabla g_{i_k}(x^*)$$

are linearly dependent, where $g_{i_1},\dots g_{i_k}$ are the active constraints at x^*. Thus $x^* \in \mathcal{V}_{\mathbb{R}}(g_{a_1},\dots,g_{a_k})$ and $\Big(\operatorname{rank}\operatorname{Jac}(f,h_1,\dots,h_s,g_{a_1},\dots,g_{a_k})(x^*)\Big) < s+k+1$. This implies $x^* \in \mathcal{V}_{\mathbb{R}}(J)$.

If the LICQ do not hold at x^* and $g_{i_1},\dots,g_{i_k}$ are the active constraints, then the gradients $\nabla h_1(x^*),\dots,\nabla h_s(x^*)$ and $\nabla g_{i_1}(x^*),\dots,\nabla g_{i_k}(x^*)$ are linearly dependent. This implies that $\nabla f(x^*),\nabla h_1(x^*)$, $\dots,\nabla h_s(x^*)$ and $\nabla g_{i_1}(x^*),\dots,\nabla g_{i_k}(x^*)$ are also linearly dependent, and we conclude as in the previous case. □

Theorem 4.13. *Let $Q = \mathcal{Q}(\mathbf{g},\pm\mathbf{h})$ and $J = (\mathbf{h}')$ be the polar ideal, where $\mathbf{h}' \subset \mathbb{R}[\mathbf{X}]$ is a finite set of generators. If $\mathcal{V}_{\mathbb{R}}(J)$ is finite then the moment relaxation $(\mathcal{L}_{2d}(\mathbf{g},\pm\mathbf{h}'))_{d\in\mathbb{N}}$ is exact.*

Proof. Minimizers belongs to $\mathcal{V}_{\mathbb{R}}(J)$ by Lemma 4.12. Then MoM exactness follows from Corollary 4.3. □

The assumption in [NDS06], [DNP07] and [Nie13a] for finite convergence and SoS exactness are smoothness conditions or radicality assumptions on the associated complex variety. Our condition for MoM exactness is of a different nature, since it is on the finiteness of the real polar variety (see Example 4.17).

Notice that by taking equations $\mathbf{h}'$ such that $(\mathbf{h}') = \sqrt[\mathbb{R}]{J}$ instead of generators of J, we have the same MoM relaxation (by Lemma 3.11 and following remark). Then the SoS exactness property under the (real) radicality assumption implies SoS exactness for the extended relaxation $(\mathcal{Q}_{2d}(\mathbf{g}, \mathbf{h}'))_{d\in\mathbb{N}}$.

4.5 Examples

We give some examples where we compute the minimum and the minimizers for some POP, which MoM relaxation is exact. Computations were performed with the Julia package `MomentTools.jl`[2] using the SDP solver `Mosek`, based on an interior point method.

Example 4.14 (Motzkin polynomial)**.** We find the global minimizers of the bivariate Motzkin polynomial $f = x^4y^2 + x^2y^4 - 3x^2y^2 + 1$. This is an example of a (globally) positive polynomial which is not sum of squares (and then the SoS relaxation cannot be exact). Its minimum is $f^* = 0$ and the four minimizers are $(\pm 1, \pm 1) \in \mathbb{R}^2$ (see [Rez96]).

```
v0, M = minimize(f, [], [], X, 4, Mosek.Optimizer)
```

Here $f^*_{\mathrm{MoM},4} \approx v_0 = -1.23437 \cdot 10^{-10}$, but we cannot recover the minimizers: exactness does not hold. We add $f - f^*_{\mathrm{MoM},4} = 0$ to find them, i.e. use $\mathcal{L}_d(\pm(f - f^*_{\mathrm{MoM},4}))$.

```
v1, M = minimize(f, [f-v0], [], X, 4, Mosek.Optimizer)
```

Here the new optimum if $v_1 \approx 1.84908 \cdot 10^{-10}$. In this case the approximation of the minimum is of the same order as before, but we can recover the minimizers by Corollary 4.10:

```
w, Xi = get_measure(M)
```

We obtain the following approximation of the 4 minimizers:

$$\xi_1 = (1.0000009448913, 1.00000094519956) \qquad \xi_2 = (1.00000094499890, -1.00000094499890)$$
$$\xi_3 = (-1.0000009448913, 1.00000094499890) \qquad \xi_4 = (-1.000000945184, -1.00000094519956).$$

Example 4.15 (Robinson form)**.** We find the minimizers of the Robinson form $f = x^6 + y^6 + z^6 + 3x^2y^2z^2 - x^4(y^2+z^2) - y^4(x^2+z^2) - z^4(x^2+y^2)$ on the unit sphere $h = x^2 + y^2 + z^2 - 1$. The Robinson polynomial has minimum $f^* = 0$ (globally and on the unit sphere), and the minimizers on $\mathcal{V}_{\mathbb{R}}(h)$ are:

$$\frac{\sqrt{3}}{3}(\pm 1, \pm 1, \pm 1), \frac{\sqrt{2}}{2}(0, \pm 1, \pm 1), \frac{\sqrt{2}}{2}(\pm 1, 0, \pm 1), \frac{\sqrt{2}}{2}(\pm 1, \pm 1, 0).$$

BHC are satisfied at every minimizer (see [Nie14, ex. 3.2]) and we can recover the minimizers by Theorem 4.8.

```
v, M = minimize(f, [h], [], X, 5, Mosek.Optimizer)
w, Xi = get_measure(M)
```

$\xi_{\mathbf{x}}$:	0.577351068999	$8.8124779306401\,10^{-12}$	0.707107158043	0.707107157553
$\xi_{\mathbf{y}}$:	0.577351069076	0.707107158048	$1.2717294461251\,10^{-13}$	0.707107157555
$\xi_{\mathbf{z}}$:	0.577351066102	0.707107158048	0.707107158042	$2.4787712013401\,10^{-9}$

Here $f^*_{\mathrm{MoM},5} \approx v = -1.27211 \cdot 10^{-7}$ and the minimizers with positive coordinates are (all the twenty minimizers are found):

Example 4.16 (Gradient ideal)**.** We compute the minimizers of Example 2.11.Let $f = (X^4Y^2 + X^2Y^4 + Z^6 - 2X^2Y^2Z^2) + X^8 + Y^8 + Z^8 \in \mathbb{R}[X,Y,Z]$. We want to minimize f over the gradient variety $\mathcal{V}_{\mathbb{R}}\left(\frac{\partial f}{\partial X}, \frac{\partial f}{\partial Y}, \frac{\partial f}{\partial Z}\right)$.

```
v, M = minimize(f, differentiate(f,X), [], X, 4, Mosek.Optimizer)
w, Xi = get_measure(M, 2.e-2)
```

The approximation of the minimum $f^* = 0$ is $v = -1.6279 \cdot 10^{-9}$, and the decomposition with a threshold of $2 \cdot 10^{-2}$ gives the following numerical approximation of the minimizer (the origin):

$$\xi = (2.976731510689691\,10^{-17}; -9.515032317137384\,10^{-19}; 3.763401209219283\,10^{-18}).$$

Example 4.17 (Singular minimizer)**.** We minimize $f = x$ on the compact semialgebraic set $S = \mathcal{S}(x^3 - y^2, 1 - x^2 - y^2)$. The only minimizer is the origin, which is a singular point of the boundary of S. Thus BHC do not hold. The regularity conditions for the Jacobian and KKT constraints are not satisfied, but the real polar variety is finite. Adding the polar constraints, we have an exact MoM relaxation. We can recover an approximation of the minimizer from the MoM relaxation of order 5:

```
v, M = polar_minimize(f, [], [x^3-y^2,1-x^2-y^2], X, 5, Mosek.Optimizer)
w, Xi = get_measure(M, 2.e-3)
```

The approximation of the minimum $f^* = 0$ is $v = -0.0045$, and the decomposition of the moment sequence with a threshold of $2 \cdot 10^{-3}$ gives the following approximation of the minimizer (the origin):

$$\xi = (-0.004514367348787526, 2.1341684460860045\,10^{-21}).$$

The error of approximation on the minimizer is of the same order than the error on the minimum f^*.

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