Topologically integrable derivations and additive group actions on affine ind-schemes

Roberto Diaz, Adrien Dubouloz, Alvaro Liendo

TOPOLOGICALLY INTEGRABLE DERIVATIONS AND ADDITIVE GROUP ACTIONS ON AFFINE IND-SCHEMES

ROBERTO DÍAZ, ADRIEN DUBOULOZ, AND ALVARO LIENDO

ABSTRACT. Affine ind-varieties are infinite dimensional generalizations of algebraic varieties which appear naturally in many different contexts, in particular in the study of automorphism groups of affine spaces. In this article we introduce and develop the basic algebraic theory of topologically integrable derivations of complete topological rings. We establish a bijective algebro-geometric correspondence between additive group actions on affine ind-varieties and topologically integrable derivations of their coordinate pro-rings which extends the classical fruitful correspondence between additive group actions on affine varieties and locally nilpotent derivations of their coordinate rings.

INTRODUCTION

Motivated by the study of algebro-geometric properties of some "infinite dimensional" groups which appear naturally in algebraic geometry, such as for instance the group of algebraic automorphisms of the affine n-space $\mathbb{A}^n_k$ over a field k, $n \geq 2$, Shafarevich [20, 21] introduced and developed some notions of infinite-dimensional affine algebraic variety and infinite-dimensional affine algebraic group. In Shafarevich's sense, an affine ind-variety over an algebraically closed field k is a topological space X which is homeomorphic to the colimit $\varinjlim_{n\in\mathbb{N}} X_n$ of a countable inductive system of closed embeddings $X_0 \hookrightarrow X_1 \hookrightarrow X_2 \hookrightarrow \cdots$ of ordinary affine algebraic k-varieties, endowed with the final topology. One declares that a morphism between two such ind-varieties $\varinjlim_{n\in\mathbb{N}} X_n$ and $\varinjlim_{n\in\mathbb{N}} Y_n$ consists of a collection of compatible morphisms of ordinary affine algebraic varieties between the corresponding inductive systems, and a group object in the so-defined category is then called an affine ind-group. Since Shafarevich pioneering work, this notion has been developed further by many authors [16, 15, 22, 17, 9, 6] driven mainly by its numerous applications to the study of algebraic automorphism groups of affine varieties.

A different approach to affine ind-varieties, closer to the Grothendieck theory of ind-representable functors and formal schemes [10, 1], was proposed by Kambayashi [12, 13, 14] in the form of a category of locally ringed spaces anti-equivalent to the category whose objects are linearly topologized complete k-algebras $\mathcal{A}$ which admit fundamental systems of open neighborhoods of 0 consisting of a countable families of ideals $(\mathfrak{a}_n)_{n\in\mathbb{N}}$, with the property that all the quotients $A_n = \mathcal{A}/\mathfrak{a}_n$ are integral finitely generated k-algebras. The underlying topological space of an affine ind-k-variety in Kambayashi's sense is defined as the set $\mathrm{Spf}(\mathcal{A})$ of open prime ideals of $\mathcal{A}$, endowed with the subspace topology inherited from the Zariski topology on the usual prime spectrum $\mathrm{Spec}(\mathcal{A})$. Morphisms between such ind-k-varieties are in turn simply determined by continuous homomorphisms between the corresponding topological algebras, see Section 2.

It is known that these two notions of ind-k-varieties are not equivalent, even already at the topological level (see [22] for an in-depth comparison). Despite its natural definition and its algebraic flavor which allows to easily extend it to more general complete topological algebras, leading to a natural theory of

2010 *Mathematics Subject Classification.* 13J10 13N15 14R20 14L30 .

Key words and phrases. affine ind-schemes, additive group actions, complete topological rings, restricted power series, restricted exponential homomorphisms, topologically integrable derivations.

The first author was partially supported by CONICYT-PFCHA/Doctorado Nacional/2016-folio 21161165 and by the grant 346300 for IMPAN from the Simons Foundation and the matching 2015-2019 Polish MNiSW fund. The second was partially supported by the French ANR projects FIBALGA ANR-18-CE40-0003-01 and ISITE-BFC ANR-15-IDEX-OOO8. The third author was partially supported by Fondecyt project 1200502.

affine ind-schemes, so far, the applications of Kambayashi's notion of affine ind-varieties have not been researched as much as those of Shafarevich's version.

The main goal of this paper is to develop the basic tools to extend the existing rich algebro-geometric theory of additive group actions on affine varieties and schemes to Kambayashi's affine ind-varieties and ind-schemes. To explain our results and put them into context, we restrict ourselves in this introduction to affine schemes and ind-schemes defined over an algebraically closed field k of characteristic zero. Every algebraic action of the additive group $\mathbb{G}_{a,k} = \mathrm{Spec}(k[T])$ on an affine k-scheme is uniquely determined by its comorphism $\mu\colon A \to A \otimes_k k[T] = A[T]$. The fact that μ is the comorphism of a $\mathbb{G}_{a,k}$-action implies that the map which associates to every $f \in A$ the element $\frac{d}{dT}(\mu(f))|_{T=0}$ is a k-derivation ∂ of A, which corresponds geometrically to the velocity vector field along the orbits of the action on X. Conversely, an algebraic vector field ∂ on X determines an algebraic action of $\mathbb{G}_{a,k}$ on X if and only if its formal flow is algebraic, that is, if and only if the formal exponential homomorphism

$$\exp(T\partial)\colon A \to A[[T]], \qquad f \mapsto \sum_n \frac{\partial^n(f)}{n!}T^n$$

factors through the polynomial ring $A[T] \subset A[[T]]$. Clearly, a k-derivation ∂ of A satisfies this polynomial integrability property if and only if for every $f \in A$, there exists $n \in \mathbb{N}$ such that $\partial^n(f) = 0$. Derivations with this property are called *locally nilpotent*, and we obtain the well-known correspondence between $\mathbb{G}_{a,k}$-actions on an affine k-variety $X = \mathrm{Spec}(A)$ and locally nilpotent k-derivations of A.

Let now $\mathcal{A}$ be linearly topologized complete k-algebra which admits a fundamental system of open neighborhoods of 0 consisting of a countable family $(\mathfrak{a}_n)_{n\in\mathbb{N}}$ of ideals of $\mathcal{A}$. We call a continuous k-derivation ∂ of $\mathcal{A}$ *topologically integrable* if the sequence of k-linear endomorphisms $(\partial^n)_{i\in\mathbb{N}}$ of $\mathcal{A}$ converges continuously to the zero homomorphism, that is, if for every $f \in \mathcal{A}$ and every $i \in \mathbb{N}$, there exist an indices $n_0, j \in \mathbb{N}$ such that $\partial^n(f + \mathfrak{a}_j) \subset \mathfrak{a}_i$ for every integer $n \geq n_0$ (see Definition 1.8). Note that in the case where the topology on $\mathcal{A}$ is the discrete one, a k-derivation of $\mathcal{A}$ is topologically integrable precisely when it is locally nilpotent. Our main result is the following extension of the classical correspondence for affine k-varieties to the case of affine ind-k-schemes (see Theorem 3.6 for the general version which applies to arbitrary relative affine ind-schemes over a base affine ind-scheme).

Theorem. *Let $\mathfrak{X} = \mathrm{Spf}(\mathcal{A})$ be the affine ind-k-scheme associated to a linearly topologized complete k-algebra $\mathcal{A}$ which admits a fundamental system of open neighborhoods of 0 consisting of a countable family of ideals. Then there exists a one-to-one correspondence between $\mathbb{G}_{a,k}$-actions on $\mathfrak{X}$ and topologically integrable k-derivations of $\mathcal{A}$.*

This correspondence is made explicit as follows. The topological integrability of a continuous k-derivation ∂ of $\mathcal{A}$ is equivalent to the property that its associated formal exponential homomomorphism $\exp(T\partial)$ factors through a continuous homomorphism with values in the subring $\mathcal{A}\{T\} \subset \mathcal{A}[[T]]$ of *restricted power series*, consisting of formal power series $\sum_{i\in\mathbb{N}} a_i T^i$ whose coefficients a_i tend to 0 for the topology on $\mathcal{A}$ when n tends to infinity. The topological ring $\mathcal{A}\{T\}$ is isomorphic to the completed tensor product $\mathcal{A}\widehat{\otimes}_k k[T]$ with respect to the given topology on $\mathcal{A}$ and the discrete topology on $k[T]$. The resulting continuous homomorphism

$$\exp(T\partial)\colon \mathcal{A} \to \mathcal{A}\{T\} \cong \mathcal{A}\widehat{\otimes}_k k[T]$$

determines through Kambayashi's definition a morphism of affine ind-k-schemes

$$\mathbb{G}_{a,k} \times_k \mathrm{Spf}(\mathcal{A}) \cong \mathrm{Spf}(\mathcal{A}\{T\}) \to \mathrm{Spf}(\mathcal{A})$$

which satisfies the axioms of an action of the additive group $\mathbb{G}_{a,k}$ on the affine ind-k-scheme $\mathrm{Spf}(\mathcal{A})$. We show conversely that every continuous homomorphism $\mathrm{e}\colon \mathcal{A} \to \mathcal{A}\{T\}$ which satisfies the axioms of a comorphism of a $\mathbb{G}_{a,k}$-action on an affine ind-k-scheme $\mathrm{Spf}(\mathcal{A})$ is the restricted exponential homomorphism $\exp(T\partial)$ associated to a topologically integrable k-derivation ∂ of $\mathcal{A}$ (see Theorem 2.26.)

One of the cornerstones of the algebraic theory of locally nilpotent derivations is the existence for every nonzero such derivation ∂ of a k-algebra A of a so-called local slice, that is, an element $s \in A$ such that $\partial(s) \in \ker(\partial) \setminus \{0\}$. Not every nonzero topologically integrable derivation k-derivation ∂ of a

linearly topologized complete k-algebra $\mathcal{A}$ admits a local slice (see Example 2.13 for a counterexample). On the other hand, we establish that the theory of topologically integrable derivations with local slices closely resembles the usual finite-dimensional case: after an appropriate localization, these derivations admit a Dixmier-Reynolds operator (see Definition 2.15) which provides a retraction of $\mathcal{A}$ onto their kernels. In particular, we have the following result (see Proposition 2.16 and Corollary 2.18 for the general case).

Theorem. *Let $\mathcal{A}$ be linearly topologized complete k-algebra and let $\partial\colon \mathcal{A} \to \mathcal{A}$ be a topologically integrable derivation admitting a slice s such that $\partial(s) = 1$. Then $\mathcal{A} \cong (\ker\partial)\{s\}$ and $\exp(T\partial)$ coincides with the homomorphism of topological $(\ker\partial)$-algebras*

$$(\ker\partial)\{s\} \to (\ker\partial)\{s\}\{T\} \cong (\ker\partial)\{s,T\}, \quad s \mapsto s+T.$$

The paper is organized as follows. In Section 1 we collect and review essential definitions and results on the classes of topological groups, rings and modules which are relevant in the context of Kambayashi's definition of affine ind-schemes. In Section 2, we develop the basic algebraic theory of restricted exponential homomorphisms and their correspondence with topologically integrable iterated higher derivations. Section 3 is devoted to the geometric side of the picture: there, for the convenience of the reader, we first review the main steps of the construction of the affine ind-scheme associated to a linearly topologized complete ring, and then, we illustrate the resulting anti-equivalence between restricted exponential homomorphisms and additive group actions on affine ind-schemes.

CONTENTS

1. PRELIMINARIES ON TOPOLOGICAL GROUPS AND RINGS

In this section we recall and gather general results on topological groups, rings and modules. Standard references for theses topics are for instance Bourbaki [3, Chapter III], [4, Chapter III] and Northcott [19].

Recall that a topological abelian group is an abelian group G endowed with a topology for which the map $G \times G \to G$, $(x,y) \mapsto x-y$ is continuous. The topology on G is called linear if G has a fundamental system of open neighborhoods of its neutral element 0 consisting of open subgroups. In what follows, we only consider topological abelian groups G endowed with a linear topology which satisfy the following additional condition:

($\star$) *There exists a fundamental system of open neighborhoods of the neutral element 0 consisting of a countable family of open subgroups.*

A topological group satisfying this property is in particular a first-countable topological space. For simplicity, we refer them simply to as topological groups and we refer those fundamental systems of open neighborhoods of the neutral element simply to as fundamental systems of open subgroups of G. Given such a fundamental system $(H_n)_{n\in\mathbb{N}}$ parametrized by the set $\mathbb{N}$ of non-negative integers, we henceforth also always assume in addition that $H_0 = G$ and that $H_m \subseteq H_n$ whenever $m \geq n$.

A continuous homomorphism $f\colon G \to G'$ between topological groups is refered to as a *homomorphism of topological groups*. Note that such a homomorphism is automatically uniformly continuous in the sense of [3, II.2.1].

1.1. **Separated completions of topological groups.** A topological group G is separated as a topological space if and only if the intersection of all open subgroups of G consists of the neutral element 0 only, hence, since every open subgroup of a topological group is also closed [3, III.2.1 Corollary to Proposition 4], if and only if $\{0\}$ is a closed subset of G.

Given a topological group G, the collection of topological groups G/H, where H ranges through the set Γ of open subgroups of G, together with the canonical surjective homomorphisms $p_{H',H}\colon G/H' \to G/H$ whenever $H' \subseteq H$ form an inverse system of topological groups when each G/H is endowed with the quotient topology, which is the discrete one as H is open. Note that with respect to these topologies, the canonical homomorphisms $p_H\colon G \to G/H$, $H \in \Gamma$, are homomorphisms of topological groups. The limit $\widehat{G} = \varprojlim_{H\in\Gamma} G/H$ of this system endowed with the inverse limit topology is a linearly topologized abelian group. We denote by $\widehat{p}_H\colon \widehat{G} \to G/H$, $H \in \Gamma$, the associated canonical continuous homomorphisms and by $c\colon G \to \widehat{G}$ the continuous homomorphism induced by the homomorphisms $p_H\colon G \to G/H$, $H \in \Gamma$.

Proposition 1.1. *Let G be a topological group and let $(H_n)_{n\in\mathbb{N}}$ be a fundamental system open subgroups of G. Then the following hold:*

1) The group $\widehat{G}$ is a separated topological group canonically isomorphic to the group $\varprojlim_{n\in\mathbb{N}} G/H_n$ endowed with the inverse limit topology,

2) The canonical projections $\widehat{p}_H\colon \widehat{G} \to G/H$ are surjective homomorphisms of topological groups,

3) The canonical map $c\colon G \to \widehat{G}$ is a homomorphism of topological groups whose image is a dense subgroup of $\widehat{G}$ and whose kernel is equal to the closure of $\{0\}$ in G. Furthermore, the induced morphism of topological groups $c\colon G \to c(G)$ is open.

Proof. Since all the G/H are endowed with the discrete topology, $\{0\}$ is closed in G/H and so, $\{0\}$ is closed in $\widehat{G}$ by definition of the inverse limit topology. This yields that $\widehat{G}$ is separated. Since $(H_n)_{n\in\mathbb{N}}$ is a cofinal subset of Γ, the canonical homomorphism $\widehat{G} \to \varprojlim_{n\in\mathbb{N}} G/H_n$ is an isomorphism of topological groups. A countable fundamental system of open subgroups of $\widehat{G}$ is given by the kernels of the projections $\widehat{p}_{H_n}$, $n \in \mathbb{N}$. This shows that $\widehat{G}$ is a topological group in our sense. Since each $p_{H',H}\colon G/H' \to G/H$, $H, H' \in \Gamma$, is surjective and $(H_n)_{n\in\mathbb{N}}$ is a countable cofinal subset of Γ, by Mittag-Leffler theorem [3, II.3.5 Corollary I], the canonical homomorphisms $\widehat{p}_H\colon \widehat{G} \to G/H$ are all surjective. Assertion 3) is [3, III.7.3 Proposition 2]. □

Definition 1.2. The topological group $\widehat{G}$ is called the *separated completion* of the topological group G. We say that a topological group is *complete* if the canonical homomorphism $c\colon G \to \widehat{G}$ is an isomorphism of topological groups.

The separated separated completion $c\colon G \to \widehat{G}$ is characterized by the following universal property [3, III.3.4 Proposotion 8]: *For every homomorphism of topological groups $f\colon G \to G''$ where G'' is complete, there exists a unique homomorphism of topological groups $\widehat{f}\colon \widehat{G} \to G''$ such that $f = \widehat{f} \circ c$.*

Remark 1.3. A separated topological group G is metrizable. Indeed, given a countable fundamental system of open subgroups $(H_n)_{n\in\mathbb{N}}$, a metric d inducing the topology on G is for instance defined by

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ \frac{1}{2^n} & \text{if } x - y \in H_n \setminus H_{n+1}. \end{cases}$$

For such a group, the notion of completeness of Definition 1.2 is equivalent to the fact that the metric space (G,d) is a complete in the usual sense, see also Remark 1.7 below.

Proposition 1.4. *Let G and G' be topological groups with respective separated completions $c\colon G \to \widehat{G}$ and $c'\colon G' \to \widehat{G'}$. Then for every homomorphism of topological groups $h\colon G \to G'$ there exists a unique homomorphism of topological groups $\widehat{h}\colon \widehat{G} \to \widehat{G'}$ such that $c' \circ h = \widehat{h} \circ c$.*

Conversely, every homomorphism of topological groups $\widehat{h}\colon \widehat{G} \to \widehat{G'}$ is uniquely determined by its restriction $\widehat{h} \circ c\colon G \to \widehat{G'}$ to G.

Proof. The first assertion is an immediate consequence of the universal property of separated completion. The second assertion follows from the fact that the image of the separated completion homomorphism $c\colon G \to \widehat{G}$ is dense. □

Lemma 1.5. *Let $(G_n)_{n\in\mathbb{N}}$ be an inverse system of complete topological groups with surjective transition homomorphisms $p_{m,n}\colon G_m \to G_n$ for every $m \geq n \geq 0$. Then the limit $\mathcal{G} = \varprojlim_{n\in\mathbb{N}} G_n$ endowed with the inverse limit topology is a complete topological group and each canonical projection $\widehat{p}_n\colon \mathcal{G} \to G_n$ is a surjective homomorphism of topological groups.*

Proof. The fact that $\mathcal{G}$ endowed with the inverse limit topology is a linearly topologized abelian group and the fact that the canonical projections $\widehat{p}_n\colon \mathcal{G} \to G_n$ are continous homomorphisms are clear. The surjectivity of $\widehat{p}_n$ follows again from Mittag-Leffler theorem [3, II.3.5 Corollary I]. A countable fundamental system of open subgroups of $\mathcal{G}$ is given for instance by the collection of inverse images of such fundamental systems of each G_n by the homomorphisms $\widehat{p}_n$. Finally, since each G_n is complete, it follows from [3, II.3.5 Corollary to Proposition 10] that $\mathcal{G}$ is complete. □

1.2. **Convergence and summability in topological groups.**

Definition 1.6. Let G be a topological group and let $(x_i)_{i\in I}$ be a family of elements of G parametrized by a countable infinite index set I. For every finite subset $J \subset I$, set $s_J = s_J((x_i)_{i\in I}) = \sum_{j\in J} x_j \in G$.

a) The family $(x_i)_{i\in I}$ is said to be *Cauchy* if for every open subgroup H of G there exists a finite subset $J(H)$ of I such that $x_i - x_j \in H$ for all $i, j \in I \setminus J(H)$.

b) The family $(x_i)_{i\in I}$ is said to *converge* to an element $x \in G$ if for every open subgroup H of G there exists a finite subset $J(H)$ such that $x_i - x \in H$ for all $i \in I \setminus J(n)$.

c) The family $(x_i)_{i\in I}$ is said to be *summable* of sum $s \in G$ if for every open subgroup H of G there exists a finite subset $J(H) \subset I$ such that $s_J - s \in H$ for every every finite subset $J \supset J(H)$ of I.

If G is separated then an element $x \in G$ to which a family $(x_i)_{i\in I}$ converges is unique if it exists, we call it the limit of the family $(x_i)_{i\in I}$. We say that a family $(x_i)_{i\in I}$ is convergent if it converges to an element $x \in G$. Similarly, an element $s \in G$ such that $(x_i)_{i\in I}$ is summable of sum $s \in G$ is unique if it exists. We call it the sum of the family $(x_i)_{i\in I}$ and we write $s = \sum_{i\in I} x_i$.

Proposition 1.7. *(*[4, III.2.6 Proposition 5] *For a separated topological group G, the following conditions are equivalent:*

a) G is a complete topological group,

b) Every Cauchy family $(x_i)_{i\in I}$ of elements of G is convergent in G,

c) Every family $(x_i)_{i\in I}$ of element of G which converges to 0 is summable in G.

Definition 1.8. Let G and G' be topological groups, let $f_n\colon G \to G'$, $n \in \mathbb{N}$, be a sequence of homomomorphisms of groups and let $f\colon G \to G'$ be a homomorphism of groups.

a) The sequence $(f_n)_{n\in\mathbb{N}}$ is said to converge *pointwise* to f if for every $g \in G$ and every open subgroup H' of G', there exists an index n_0 such that $f_n(g) - f(g) \in H'$ for every integer $n \geq n_0$,

b) The sequence $(f_n)_{n\in\mathbb{N}}$ is said to converge *continuously* to f if every f_n, $n \in \mathbb{N}$, is continuous and for every $g \in G$ and every open subgroup H' of G', there exists an open subgroup H of G and an index n_0 such that $f_n(g + x) - f(g + x) \in H'$ for every element $x \in H$ and every integer $n \geq n_0$.

Clearly, a sequence $(f_n)_{n\in\mathbb{N}}$ which converges continuously to a homomorphism f converges pointwise to this homomorphism.

Lemma 1.9. *Let G be a topological group, let G' be separated topological group and let $f_n\colon G \to G'$, $n \in \mathbb{N}$, be a sequence of homomorphisms of topological groups. Then the following properties are equivalent:*

a) The sequence $(f_n)_{n\in\mathbb{N}}$ converges continuously to a homomorphism $f\colon G \to G'$,

b) There exists a homomorphism of topological groups $f\colon G \to G'$ such that the sequence $f_n - f$ converges continuously to the zero homomorphism,

c) The sequence $(f_n)_{n\in\mathbb{N}}$ is pointwise convergent to a homomorphism of topological groups $f\colon G \to G'$, and for every open subgroup H' of G', there exists an open subgroup H of G and an integer $n_0 \geq 0$ such that $(f_n - f)(H) \subset H'$ for every $n \geq n_0$.

In particular, if a sequence $(f_n)_{n\in\mathbb{N}}$ of homomorphisms of topological groups converges continuously to a homomorphism $f\colon G \to G'$, then f is continuous.

Proof. Denote by $(h_n)_{n\in\mathbb{N}}$ the sequence of group homomorphisms defined by $h_n = f_n - f$ for every $n \in \mathbb{N}$. The implication b) $\Rightarrow$ a) is clear. Conversely, assume that the sequence $(f_n)_{n\in\mathbb{N}}$ converges continuously to a homomorphism $f\colon G \to G'$. Applying the definition of continuous convergence to the point 0 of G, it follows that for every open subgroup H' of G', there exists an open subgroup H_1 of G such that $(f_n - f)(H_1) \subset H'$ for all sufficiently large n. On the other hand, since f_n is continuous for every n, there exists an open subgroup $H_2(n)$ of G such that $f_n(H_2(n)) \subset H'$. Choosing n sufficiently large, we have $-f(x) = (f_n(x)-f(x))-f_n(x) \in H'$ for every $x \in H = H_1 \cap H_2(n)$. Thus $f(H) \subset H'$ which shows that f is continuous at 0, hence continuous since it is a homomorphism of groups. Then $(h_n)_{n\in\mathbb{N}}$ is a sequence of homomorphisms of topological groups which converges continuously to the zero homomorphism. Thus, a) implies b).

Now assume that for some homomorphism of topological groups f the sequence $h_n = f_n - f$, $n \in \mathbb{N}$, converges continuously to the zero homomorphism. Applying the definition of continuous convergence to the element $0 \in G$, we conclude that there exist an open subgroup H of G and an integer n_0 such that $h_n(H) \subset H'$ for every $n \geq n_0$. So b) implies c). Conversely, assume that c) holds, let H' be an open subgroup of G' and let g be an element of G. Since by hypothesis the sequence $(f_n(g))_{n\in\mathbb{N}}$ converges to an element g' of G, there exists an integer $n_1 \geq 0$ such that $f_n(g) - g' \in H'$ for every $n \geq n_1$. It follows that $f(g) - g' = f(g) - f_n(g) + f_n(g) - g'$ belongs to H', hence that $g' = f(g)$ since G' is separated. This implies in turn that the sequence $(h_n(g))_{n\in\mathbb{N}}$ converges to 0 in G', so that there exists an integer $n_2 \geq 0$ such that $h_n(g) \in H'$ for every $n \geq n_2$. Since on the other hand there exists by hypothesis an open subgroup H of G and an integer $n_3 \geq 0$ such that $h_n(H) \subset H'$ for every $n \geq n_3$, we conclude that $h_n(g + H) \subset H'$ for every $n \geq \max(n_2, n_3)$. So the sequence $(h_n)_{n\in\mathbb{N}}$ converges continuously to 0, which shows that c) implies b). □

Lemma 1.10. *Let G be a topological group and let G' be a separated topological group with respective separated completions $c\colon G \to \widehat{G}$ and $c'\colon G' \to \widehat{G'}$. Let $f_n\colon G \to G'$, $n \in \mathbb{N}$, be a sequence of homomorphisms of topological groups, let $\widetilde{f}_n = c' \circ f_n\colon G \to \widehat{G'}$, $n \in \mathbb{N}$, and let $\widehat{f}_n\colon \widehat{G} \to \widehat{G'}$, $n \in \mathbb{N}$, be the sequence of homomorphisms of topological groups deduced from the sequence $(\widetilde{f}_n)_{n\in\mathbb{N}}$ by the universal property of the separated completion.*

If the sequence $(f_n)_{n\in\mathbb{N}}$ converges continuously to a homomorphism $f\colon G \to G'$ then the sequences $(\widetilde{f}_n)_{n\in\mathbb{N}}$ and $(\widehat{f}_n)_{n\in\mathbb{N}}$ converges continuously respectively to the homomorphism of topological groups $\widetilde{f} = c' \circ f$ and to the homomorphism of topological groups $\widehat{f}\colon \widehat{G} \to \widehat{G'}$ deduced from $\widetilde{f}$ by the universal property of the separated completion.

Proof. Note that $\widetilde{f_n}$ and $\widehat{f_n}$ are homomorphisms of topological groups for every $n \in \mathbb{N}$. By Lemma 1.9, f is a homomorphism of topological groups so that $\widetilde{f}$ and $\widehat{f}$ are homomorphisms of topological groups as well. Let $g \in G$ and let $\widehat{H'}$ be an open subgroup of $\widehat{G'}$. Then $H' = {c'}^{-1}(\widehat{H'})$ is an open subgroup of G'. Since $(f_n)_{n\in\mathbb{N}}$ converges continuously to f, there exists an open subgroup H of G and an index n_0 such that $f_n(g+x) - f(g+x) \in H'$ for every element $x \in H$ and every integer $n > n_0$. Since $c'(H') \subset \widehat{H'}$, this implies that $\widetilde{f}_n(g+x) - \widetilde{f}(g+x) \in \widehat{H'}$, which shows that $(\widetilde{f}_n)_{n\in\mathbb{N}}$ converges continuously to $\widetilde{f}$. Replacing G' by $\widehat{G'}$, the sequence f_n by the sequence $\widetilde{f}_n$ and the homomorphism f by $\widetilde{f}$ we can now assume that G' is complete. By Lemma 1.9, it remains to show that the sequence of group homomorphisms $(\widehat{h}_n)_{n\in\mathbb{N}}$ defined by $\widehat{h}_n = \widehat{f}_n - \widehat{f}$ converges continuously to the zero homomorphism on $\widehat{G}$. By definition of $(\widehat{h}_n)_{n\in\mathbb{N}}$, continuous convergence holds in restriction to the subgroup $G_0 = c(G)$ of $\widehat{G}$. Since $\widehat{h}_n$ is uniformly continuous and G_0 is dense in $\widehat{G}$, it follows that $(\widehat{h}_n)_{n\in\mathbb{N}}$ converges pointwise to the zero homomorphism on $\widehat{G}$. Let H' be an open subgroup of G'. Then there exist an integer n_0 and open subgroup H of $\widehat{G}$ such that $\widehat{h}_n(z) \in H'$ for every $z \in H_0 = G_0 \cap H$ and every $n \geq n_0$. Since H_0 is dense in H and H is a first-countable topological space, for every $x \in H$, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of elements of H_0 which converges to x. Setting $y_n = x - x_n$, the sequence $(\widehat{h}_i(y_j))_{(i,j)\in\mathbb{N}^2}$ converges to 0 in $\widehat{G'}$. This implies in particular that there exists a strictly increasing map $\varphi\colon \mathbb{N} \to \mathbb{N}$ and an integer $n_1 \geq 0$ such that $\widehat{h}_n(y_{\varphi(n)}) \in H'$ for every $n \geq n_1$. It follows that for every $n \geq \max(n_0, n_1)$, $\widehat{h}_n(x) = \widehat{h}_n(x_{\varphi(n)}) + \widehat{h}_n(y_{\varphi(n)})$ belong to H', which shows, by Lemma 1.9 c), that the sequence $(\widehat{h}_n)_{n\in\mathbb{N}}$ converges continuously to the zero homomorphism on G. □

Corollary 1.11. *Let G be a topological group and let G' be separated topological group with separated completion $c'\colon G' \to \widehat{G'}$. Let $h_n\colon G \to G'$, $n \in \mathbb{N}$, be a sequence of homomorphisms of topological groups which converges continuously to the zero homomorphism. Then the sequence of homomorphisms $s_N = \sum_{n=0}^{N} c' \circ h_n$, $N \in \mathbb{N}$, converges continuously to the homomorphism*

$$s = \sum_{n\in\mathbb{N}} c' \circ h_n\colon G \to \widehat{G'}, \quad g \mapsto \sum_{n\in\mathbb{N}} c'(h_n(g)).$$

Proof. Let $\widetilde{h}_n = c' \circ h_n\colon G \to \widehat{G'}$. First note that since for every $g \in G$ the sequence $(h_n(g))_{n\in\mathbb{N}}$ converges to 0 in G', it follows from Proposition 1.7 that the family $(\widetilde{h}_n(g))_{n\in\mathbb{N}}$ of elements of $\widehat{G'}$ is summable, so that the map s is indeed well defined. Since every $\widetilde{h}_n$ is a homomorphism of groups, for every $g_1, g_2 \in G$ and every integer $N \in \mathbb{N}$, we have $s_N(g_1 + g_2) = s_N(g_1) + s_N(g_2)$. Since $\widehat{G'}$ is separated, this implies that $s(g_1 + g_2) = s(g_1) + s(g_2)$, showing that $s\colon G \to \widehat{G'}$ is a homomorphism. Let $\widehat{H'}$ be an open subgroup of $\widehat{G'}$. Since the sequence $(\widetilde{h}_n)_{n\in\mathbb{N}}$ converges continuously to the zero homomorphism, Lemma 1.9 implies that there exists an integer $n_0 \geq 0$ and an open subgroup H_1 of G such that $\widetilde{h}_n(H_1) \subset \widehat{H'}$ for every $n \geq n_0$. Since for every $n \in \mathbb{N}$, $\widetilde{h}_n$ is continuous, hence in particular continuous at 0, there exists an open subgroup H_2 of G such that $\widetilde{h}_n(H_2) \subset \widehat{H'}$ for every $n = 0, \ldots, n_0$. Putting $H = H_1 \cap H_2$, we have $\widetilde{h}_n(H) \subset \widehat{H'}$ for every $n \in \mathbb{N}$, which implies in turn that $s_N(H) \subset \widehat{H'}$ for every $N \in \mathbb{N}$. Since $\widehat{H'}$ is an open subgroup of $\widehat{G}$, it also closed. It follows that for every $g \in H$, the limit $s(g)$ of the sequence $(s_n(g))_{n\in\mathbb{N}}$ belong to $\widehat{H'}$, so that $s(H) \subset \widehat{H'}$. This shows that s is continuous at 0, hence continuous since it is a homomorphism. □

1.3. **Recollection on topological rings and modules.** Recall that a commutative topological ring $\mathcal{A}$ is a topological abelian group endowed with a ring structure for which the multiplication $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is continuous. A module M over a topological ring $\mathcal{A}$ is a called a topological $\mathcal{A}$-module if it is a topological abelian group and the scalar multiplication $\mathcal{A} \times M \to M$ is continuous, where $\mathcal{A} \times M$ is endowed with product topology. In the sequel, unless otherwise specified, the term *topological ring* (resp. *topological module*) will refer to a commutative topological ring $\mathcal{A}$ (resp. topological module M over some topological ring $\mathcal{A}$) endowed with a linear topology for which there exists a fundamental system of neighbourhoods of 0 consisting of a countable family $(\mathfrak{a}_n)_{n\in\mathbb{N}}$ of ideals of $\mathcal{A}$ (resp. endowed with a linear

topology with a fundamental system of neighbourhoods of 0 consisting of a countable family of open submodules $(M_n)_{n\in\mathbb{N}}$). We also always assume that $\mathfrak{a}_0 = \mathcal{A}$ and that $\mathfrak{a}_m \subseteq \mathfrak{a}_n$ whenever $m \geq m$ and similarly that $M_0 = M$ and $M_m \subseteq M_n$ for whenever $m \geq n$. A continuous homomorphism $f\colon \mathcal{A} \to \mathcal{A}'$ between topological rings is refered to as a *homomorphism of topological rings*. We denote by $\mathcal{C}\mathrm{Hom}(\mathcal{A},\mathcal{B})$ the subgroup of the abelian group $\mathrm{Hom}(\mathcal{A},\mathcal{B})$ consisting of continuous homomorphisms. Similarly, a continuous homomorphism of topological modules $f\colon M \to N$ over a topological ring $\mathcal{A}$ is refered to as a *homomorphism of topological $\mathcal{A}$-modules* and we denote by $\mathcal{C}\mathrm{Hom}_{\mathcal{A}}(M,N)$ the $\mathcal{A}$-module of such continuous homomorphisms.

Given a topological ring $\mathcal{A}$ (resp. a topological module M over a topological ring $\mathcal{A}$) the separated completion $\widehat{\mathcal{A}}$ of $\mathcal{A}$ (resp. $\widehat{M}$ of M) as a topological group carries the structure of a topological ring (resp.of a topological $\mathcal{A}$-module) and the canonical homomorphism of topological groups $c\colon \mathcal{A} \to \widehat{\mathcal{A}}$ (resp. $c\colon M \to \widehat{M}$) is a homomorphism of topological rings (resp. of topological $\mathcal{A}$-modules). We say that $\mathcal{A}$ (resp. M) is a complete topological ring (resp. a complete topological $\mathcal{A}$-module) if $c\colon \mathcal{A} \to \widehat{\mathcal{A}}$ (resp. $c\colon M \to \widehat{M}$) is an isomorphism.

For every complete topological ring $\mathcal{B}$ the composition with $c\colon \mathcal{A} \to \widehat{\mathcal{A}}$ induces an isomorphism

$$c^*\colon \mathcal{C}\mathrm{Hom}(\widehat{\mathcal{A}},\mathcal{B}) \to \mathcal{C}\mathrm{Hom}(\mathcal{A},\mathcal{B}), \quad \widehat{f} \mapsto \widehat{f}\circ c.$$

Let $\mathcal{A}$ be a topological ring and let $\mathcal{B}$ be a complete topological ring, with fundamental systems $(\mathfrak{a}_n)_{n\in\mathbb{N}}$ and $(\mathfrak{b}_n)_{n\in\mathbb{N}}$ of open neighborhoods of 0, respectively. Set $A_n = \mathcal{A}/\mathfrak{a}_n$ and $B_m = \mathcal{B}/\mathfrak{b}_m$ so that we have $\widehat{\mathcal{A}} \cong \varprojlim_{n\in\mathbb{N}} A_n$ and $\mathcal{B} \cong \varprojlim_{m\in\mathbb{N}} B_m$. Every homomorphism of topological rings $f\colon \widehat{\mathcal{A}} \to \mathcal{B}$ is equivalently described by an inverse system of continuous homomorphisms $f_m\colon \widehat{\mathcal{A}} \to B_m$. The kernel of each such f_m being an open ideal of $\widehat{\mathcal{A}}$, it contains some open ideal $\mathfrak{a}_n$ and so, f_m factors through a homomorphism $f_{n,m}\colon A_n \to B_m$. Summing up, we have:

$$\begin{array}{rcl} \mathcal{C}\mathrm{Hom}(\widehat{\mathcal{A}},\mathcal{B}) = \mathcal{C}\mathrm{Hom}(\varprojlim_{n\in\mathbb{N}} A_n, \varprojlim_{m\in\mathbb{N}} B_m) & \cong & \varprojlim_{m\in\mathbb{N}}(\mathcal{C}\mathrm{Hom}(\varprojlim_{n\in\mathbb{N}} A_n, B_m)) \\ & \cong & \varprojlim_{m\in\mathbb{N}}(\varinjlim_{n\in\mathbb{N}} \mathrm{Hom}(A_n, B_m)). \end{array}$$

1.3.1. *Completed tensor product.* We recall basic properties of completed tensor products of topological modules, see [4, III] and [10, 0.7.7].

Definition 1.12. ([4, III Exercise 28]) Let M and N be topological modules over a topological ring $\mathcal{A}$. The *completed tensor product* $M\widehat{\otimes}_{\mathcal{A}}N$ of M and N over $\mathcal{A}$ is the separated completion $\widehat{M \otimes_{\mathcal{A}} N}$ of the tensor product $M \otimes_{\mathcal{A}} N$ with respect to the linear topology generated by open neighborhoods of 0 of the form $U \otimes N + M \otimes V$, where U and V run respectively through the set of open $\mathcal{A}$-submodules of M and N.

We denote by $\tau\colon M \times N \to M\widehat{\otimes}_{\mathcal{A}}N$ the composition of the canonical homomorphism of topological $\mathcal{A}$-modules $M \times N \to M \otimes_{\mathcal{A}} N$, where $M \times N$ is endowed with the product topology, with the separated completion homomorphism $c\colon M \otimes_{\mathcal{A}} N \to M\widehat{\otimes}_{\mathcal{A}}N$

It follows from the universal properties of the tensor product and of the separated completion that the canonical homomorphism of topological $\mathcal{A}$-modules $\tau\colon M \times N \to M\widehat{\otimes}_{\mathcal{A}}N$ satisfies the following universal property: *For every continuous $\mathcal{A}$-bilinear homomorphism $\Phi\colon M \times N \to E$ into a complete topological $\mathcal{A}$-module E, there exists a unique homomorphism of topological $\mathcal{A}$-modules $\widehat{\varphi}\colon M\widehat{\otimes}_{\mathcal{A}}N \to E$ such that $\Phi = \widehat{\varphi}\circ\tau$.* As for the usual tensor product, this universal property implies the following associativity result whose proof is a direct adaptation of that of [5, II.3.8, Proposition 8]:

Lemma 1.13. *Let $\mathcal{A}$ be a topological ring, let M and $\mathcal{B}$ be respectively a topological $\mathcal{A}$-module and a topological $\mathcal{A}$-algebra and let N and P be topological $\mathcal{B}$-modules. Then there is a canonical isomorphism of complete topological $\mathcal{B}$-modules*

$$(M\widehat{\otimes}_{\mathcal{A}}N)\widehat{\otimes}_{\mathcal{B}}P \cong M\widehat{\otimes}_{\mathcal{A}}(N\widehat{\otimes}_{\mathcal{B}}P)$$

where $M\widehat{\otimes}_{\mathcal{A}}N$ is viewed as topological $\mathcal{B}$-module via the $\mathcal{B}$-module structure of N.

In the case where $M = \mathcal{B}_1$ and $N = \mathcal{B}_2$ are topological $\mathcal{A}$-algebras, the completed tensor product $\mathcal{B}_1\widehat{\otimes}_{\mathcal{A}}\mathcal{B}_2$ is a complete topological $\mathcal{A}$-algebra and the composition $\sigma_1\colon \mathcal{B}_1 \to \mathcal{B}_1\widehat{\otimes}_{\mathcal{A}}\mathcal{B}_2$ (resp. $\sigma_2\colon \mathcal{B}_2 \to \mathcal{B}_1\widehat{\otimes}_{\mathcal{A}}\mathcal{B}_2$) of $\mathrm{id}_{\mathcal{B}_1} \otimes 1\colon \mathcal{B}_1 \to \mathcal{B}_1 \otimes_{\mathcal{A}} \mathcal{B}_2$ (resp. $1 \otimes \mathrm{id}_{\mathcal{B}_2}\colon \mathcal{B}_2 \to \mathcal{B}_1 \otimes_{\mathcal{A}} \mathcal{B}_2$) with the separated completion homomorphism $\mathcal{B}_1 \otimes_{\mathcal{A}} \mathcal{B}_2 \to \mathcal{B}_1\widehat{\otimes}_{\mathcal{A}}\mathcal{B}_2$ is a homomorphism of topological $\mathcal{A}$-algebras. The $\mathcal{A}$-algebra $\mathcal{B}_1\widehat{\otimes}_{\mathcal{A}}\mathcal{B}_2$ satisfies the following universal property: *For every complete topological $\mathcal{A}$-algebra $\mathcal{C}$ and every pair of homomorphisms of topological $\mathcal{A}$-algebras $f_i\colon \mathcal{B}_i \to \mathcal{C}$ there exists a unique homomorphism of topological $\mathcal{A}$-algebras $f\colon \mathcal{B}_1\widehat{\otimes}_{\mathcal{A}}\mathcal{B}_2 \to \mathcal{C}$ such that $f_i = f \circ \sigma_i$, $i = 1, 2$.*

In general, given a finitely generated algebra R over the field k, the covariant functor $R \otimes_k -$ which associates to a k-algebra A the k-algebra $R \otimes_k A$ is not representable in the category of k-algebras. The following example shows in contrast that the natural extension of $R\widehat{\otimes}_k-$ of $R \otimes_k -$ to the category of complete topological k-algebras is representable.

Example 1.14. Let R be a finitely generated algebra over a field k. Then the covariant functor

$$R\widehat{\otimes}_k-\colon (\mathrm{CTop}_{/}k) \to (\mathrm{Sets})$$

associating to a complete topological k-algebra $\mathcal{A}$ the completed tensor product $R\widehat{\otimes}_k\mathcal{A}$ is representable.

Proof. Since R is finitely generated, it is a countable k-vector space. We can thus write $R = \varinjlim_{n\in\mathbb{N}} V_n = \bigcup_{n\in\mathbb{N}} V_n$ where the V_n are an increasing sequence of finite dimensional k-vector spaces $V_0 \subset \cdots \subset V_n \subset V_{n+1} \subset \cdots$ which form an exhaustion of R. For every $n \leq n'$, the inclusion $V_n \subset V_{n'}$ induces a dual surjection $V_{n'}^\vee \to V_n^\vee$ between the duals of $V_{n'}$ and V_n respectively, hence a surjective k-algebra homomorphism $\mathrm{Sym}^\cdot(V_{n'}^\vee) \to \mathrm{Sym}^\cdot(V_n^\vee)$ between the symmetric k-algebras of $V_{n'}^\vee$ and $V_n^\vee$, respectively. Viewing each $R_n = \mathrm{Sym}^\cdot(V_n^\vee)$ as endowed with the discrete topology, the ring $\mathcal{R} = \varprojlim_{n\in\mathbb{N}} R_n$ endowed with the initial topology is a complete topological k-algebra whose isomorphism type is independent on the choice of a particular exhaustion $\{V_n\}_{n\in\mathbb{N}}$ of R by finite dimensional k-vector subspaces.

Now let $\mathcal{A} = \varprojlim_{m\in\mathbb{N}} A_m$ be a complete topological k-algebra. Since tensor product commutes with colimits and the k-vector spaces V_n are finite dimensional, we have natural isomorphisms

$$\begin{aligned} R\widehat{\otimes}_k\mathcal{A} = \varprojlim_{m\in\mathbb{N}}(R \otimes_k A_m) &= \varprojlim_{m\in\mathbb{N}}((\varinjlim_{n\in\mathbb{N}} V_n) \otimes_k A_m) \\ &\cong \varprojlim_{m\in\mathbb{N}}(\varinjlim_{n\in\mathbb{N}}(V_n \otimes_k A_m)) \\ &\cong \varprojlim_{m\in\mathbb{N}}(\varinjlim_{n\in\mathbb{N}}(\mathrm{Hom}_{k-\mathrm{mod}}(V_n^\vee, A_m)). \end{aligned}$$

The universal property of symmetric algebras provides in turn natural isomorphisms

$$\mathrm{Hom}_{k-\mathrm{mod}}(V_n^\vee, A_m) \cong \mathrm{Hom}_{k-\mathrm{alg}}(\mathrm{Sym}^\cdot(V_n^\vee), A_m) = \mathrm{Hom}_{k-\mathrm{alg}}(R_n, A_m).$$

Summing up, we obtain for every $\mathcal{A}$ a natural isomorphism

$$\Phi_{\mathcal{A}} : \mathcal{C}\mathrm{Hom}_k(\mathcal{R}, \mathcal{A}) = \varprojlim_{m\in\mathbb{N}}(\varinjlim_{n\in\mathbb{N}}(\mathrm{Hom}_{k-\mathrm{alg}}(R_n, A_m)) \xrightarrow{\cong} R\widehat{\otimes}_k\mathcal{A}.$$

These isomorphisms are easily seen from the construction to be functorial in $\mathcal{A}$, defining an isomorphism of covariant functors $\Phi\colon \mathcal{C}\mathrm{Hom}(-, \mathcal{R}) \to R\widehat{\otimes}_k-$ which shows that $\mathcal{R}$ represents the functor $R\widehat{\otimes}_k-$.

The universal element $u = \Phi_{\mathcal{R}}(\mathrm{id}_{\mathcal{R}}) \in R\widehat{\otimes}_k\mathcal{R}$ can be described as follows. For every $n \in \mathbb{N}$, let $u_n \in R \otimes_k R_n = R \otimes_k \mathrm{Sym}^\cdot(V_n^\vee)$ be the image by the natural homomorphism

$$V_n \otimes_k V_n^\vee \to R \otimes_k \mathrm{Sym}^\cdot(V_n^\vee)$$

of the element corresponding to id_{V_n} via the isomorphism $\mathrm{Hom}_k(V_n, V_n) \cong V_n \otimes_k V_n^\vee$. The collection of elements $u_n \in R \otimes_k R_n$ is an inverse system with the respect to the projection homomorphisms $R \otimes_k R_{n'} \to R \otimes_k R_n$, $n \leq n'$ and we have $u = \varprojlim_{n\in\mathbb{N}} u_n \in \varprojlim_{n\in\mathbb{N}} R \otimes_k R_n = R\widehat{\otimes}_k\mathcal{R}$. □

1.3.2. *Separated completed localization.* In what follows by a *multiplicatively closed subset* of a ring A, we mean a subset S of A containing 1 and stable under multiplication. We now recall basic results on separated completed localizations of topological rings and modules, see [10, 0.7.6].

Definition 1.15. Let $\mathcal{A}$ be a be a topological ring and let $S \subset \mathcal{A}$ be a multiplicatively closed subset of $\mathcal{A}$. The *separated completed localization* $\widehat{S^{-1}\mathcal{A}}$ of $\mathcal{A}$ with respect to S is the separated completion of the usual localization $S^{-1}\mathcal{A}$ endowed with the topology co-induced by the localization homomorphism $j\colon \mathcal{A} \to S^{-1}\mathcal{A}$. The composition

$$\widetilde{j} = c \circ j\colon \mathcal{A} \xrightarrow{j} S^{-1}\mathcal{A} \xrightarrow{c} \widehat{S^{-1}\mathcal{A}}$$

of the usual localization homomorphism with the separated completion homomorphism is a homomorphism topological ring which we call *separated completed localization homomorphism* of $\mathcal{A}$ with respect to S.

Notation 1.16. Given a topological ring $\mathcal{A}$ and element $f \in \mathcal{A}$ (resp. a prime ideal $\mathfrak{p}$ of $\mathcal{A}$), we denote by $\widehat{\mathcal{A}_f}$ (resp. $\widehat{\mathcal{A}_\mathfrak{p}}$) the separated completed localization of $\mathcal{A}$ with respect to the multiplicatively closed subset $S = \{f^n\}_{n\geq 0}$ (resp. $S = \mathcal{A} \setminus \mathfrak{p}$).

The separated completed localization enjoys the following universal property:

Proposition 1.17. *With the notation above, let $\mathcal{B}$ be a complete topological ring and let $\varphi\colon \mathcal{A} \to \mathcal{B}$ be a homomorphism of topological rings such that $\varphi(S) \subset \mathcal{B}^*$. Then there exists a unique homomorphism of topological rings $\widehat{S^{-1}\varphi}\colon \widehat{S^{-1}\mathcal{A}} \to \mathcal{B}$ such that $\varphi = \widehat{S^{-1}\varphi} \circ \widetilde{j}$.*

Proof. By the universal property of the usual localization $j\colon \mathcal{A} \to S^{-1}\mathcal{A}$, the exists a unique homomorphism $S^{-1}\varphi\colon S^{-1}\mathcal{A} \to \mathcal{B}$ such that $\varphi = S^{-1}\varphi \circ j$. The homomorphism $S^{-1}\varphi$ is continuous for the topology on $S^{-1}\mathcal{A}$ co-induced by that on $\mathcal{A}$, and since $\mathcal{B}$ is complete, it follows that there exists a unique homomorphism of topological rings $\widehat{S^{-1}\varphi}\colon \widehat{S^{-1}\mathcal{A}} \to \mathcal{B}$ such that $S^{-1}\varphi = \widehat{S^{-1}\varphi} \circ c$. We then have $\varphi = S^{-1}\varphi \circ j = \widehat{S^{-1}\varphi} \circ c \circ j = \widehat{S^{-1}\varphi} \circ \widetilde{j}$. □

Lemma 1.18. *Let $\mathcal{A}$ be a topological ring with separated completion $c\colon \mathcal{A} \to \widehat{\mathcal{A}}$, let $S \subset \mathcal{A}$ be a multiplicatively closed subset and let $\widehat{S} \subset \widehat{\mathcal{A}}$ be the closure of $c(S)$ in $\widehat{\mathcal{A}}$. Then there exists a canonical isomorphism $\widehat{S^{-1}\mathcal{A}} \cong \widehat{\widehat{S}^{-1}\widehat{\mathcal{A}}}$ of complete topological rings.*

Proof. Let $(\mathfrak{a}_n)_{n\in\mathbb{N}}$ be a fundamental system of open ideal in $\mathcal{A}$, let $\mathrm{p}_n\colon \mathcal{A} \to A_n = \mathcal{A}/\mathfrak{a}_n$, $n \in \mathbb{N}$, be the quotient homomorphisms and let $S_n = \mathrm{p}_n(S) \subset A_n$. Then $\widehat{S} = \varprojlim_{n\in\mathbb{N}} S_n \subset \varprojlim_{n\in\mathbb{N}} A_n = \widehat{\mathcal{A}}$ so that, by definition,

$$\widehat{S^{-1}\mathcal{A}} \cong \varprojlim_{n\in\mathbb{N}} S_n^{-1}A_n \cong \widehat{\widehat{S}^{-1}\widehat{\mathcal{A}}}.$$

□

Corollary 1.19. *Let $\mathcal{A}$ be a topological ring, let $c\colon \mathcal{A} \to \widehat{\mathcal{A}}$ be its separated completion and let $S \subset \mathcal{A}$ be a multiplicatively closed subset. Then $\widehat{S^{-1}\mathcal{A}}$ is the zero ring if an only if 0 belongs to the closure $\widehat{S}$ of $c(S)$ in $\widehat{\mathcal{A}}$.*

Proof. In view of Lemma 1.18, we are reduced to the case where $\mathcal{A}$ is complete and S is closed in $\mathcal{A}$. Now if $0 \in S$ then $S^{-1}\mathcal{A}$ is the zero ring, and so $\widehat{S^{-1}\mathcal{A}}$ is the zero ring as well. Conversely, using the notation of the proof of Lemma 1.18, if $\widehat{S^{-1}\mathcal{A}} = \varprojlim_{n\in\mathbb{N}} S_n^{-1}A_n$ is the zero ring, then $S_n^{-1}A_n$ is the zero ring for every $n \in \mathbb{N}$, which implies that $0 \in S_n$ for every $n \in \mathbb{N}$. It follows that 0 belongs $\varprojlim_{n\in\mathbb{N}} S_n = S$ as S is closed. □

Example 1.20. Let $\mathcal{A} = \mathbb{C}[u]$ endowed with the u-adic topology, with fundamental system of neighbourhoods of 0 given by the ideals $\mathfrak{a}_n = u^n\mathbb{C}[u]$, $n \geq 0$, and let $S = \{u^m\}_{m\geq 0}$. We have then $\widehat{\mathcal{A}} \cong \mathbb{C}[[u]]$ endowed with the u-adic topology and $\widehat{S} = c(S) = \{u^m\}_{m\geq 0}$. Since $A_n = \mathbb{C}[u]/(u^n) = \mathbb{C}[[u]]/(u^n)$ the images $S_n = \pi_n(S) = \pi_n(\widehat{S})$, $n \geq 1$, all contain the element 0. Thus $S_n^{-1}A_n = \{0\}$ for every $n \geq 0$ from which it follows that $\widehat{S^{-1}\mathcal{A}}$ is the zero ring. On the other hand, we have $S^{-1}\mathcal{A} = \mathbb{C}[u^{\pm 1}]$ and $\widehat{S}^{-1}\widehat{\mathcal{A}} = \mathbb{C}[[u^{\pm 1}]] = \mathbb{C}((u))$ and the images of the ideal $\mathfrak{a}_n$ and $c(\mathfrak{a}_n)$ by the respective localization

homomorphisms are all equal to the unit ideals in $\mathbb{C}[u^{\pm 1}]$ and $\mathbb{C}((u))$ respectively. The induced topologies on $S^{-1}\mathcal{A}$ and $\widehat{S}^{-1}\widehat{\mathcal{A}}$ are thus the trivial ones, which implies that the separated completions of these rings are both isomorphic to the zero ring.

Lemma 1.21. *Let $i\colon \mathcal{A} \to \mathcal{B}$ be an injective closed homomorphism of complete topological rings and let S be a multiplicatively closed subset of $\mathcal{A}$. Then $\widehat{S^{-1}i}\colon \widehat{S^{-1}A} \to \widehat{i(S)^{-1}\mathcal{B}}$ is an injective homomorphism of topological rings.*

Proof. Since $\mathcal{A}$ (resp. $\mathcal{B}$) is complete, the kernel of the separated completed localization homomorphism $\mathcal{A} \to \widehat{S^{-1}\mathcal{A}}$ (resp. $\mathcal{B} \to \widehat{i(S)^{-1}\mathcal{B}}$) consists of elements of $\mathcal{A}$ (resp. $\mathcal{B}$) which are anihilated by the multiplication by an element of the closure of S in $\mathcal{A}$ (resp. of the closure of $i(S)$ in $\mathcal{B}$). On the other hand, since i is a closed homomorphism of topological rings, $i(\mathcal{A})$ is complete subspace of $\mathcal{B}$, hence a closed subspace, so that the closures of $i(S)$ in $i(\mathcal{A})$ and $\mathcal{B}$ coincide. □

Remark 1.22. Note that the conclusion of Lemma 1.21 does not hold if $i\colon \mathcal{A} \to \mathcal{B}$ is not a closed homomorphism. For instance, the inclusion $\mathbb{C}[u] \to \mathbb{C}[[u]]$ where $\mathbb{C}[u]$ and $\mathbb{C}[[u]]$ are endowed respectively with the discrete topology and the u-adic topology is continuous but not closed. The separated completed localizations of these topological rings with respect to the multiplicatively closed subset $S = \{u^m\}_{m\geq 0}$ are respectively isomorphic to $\mathbb{C}[u, u^{-1}]$ endowed with the discrete topology and to the zero ring, so that $\widehat{S^{-1}i}$ is not injective in this case.

Definition 1.23. Let $\mathcal{A}$ be a be a topological ring, let M be a topological $\mathcal{A}$-module and let $S \subset \mathcal{A}$ multiplicatively closed subset of $\mathcal{A}$. The *separated completed localization* $\widehat{S^{-1}M}$ of M with respect to S is the separated completion of the $S^{-1}\mathcal{A}$-module $S^{-1}M = S^{-1}\mathcal{A} \otimes_{\mathcal{A}} M$ with respect to the topology co-induced by the localization homomorphism $j_M\colon M \to S^{-1}M$

The natural structure of topological $S^{-1}\mathcal{A}$-module on $S^{-1}M$ induces a structure of complete topological $\widehat{S^{-1}\mathcal{A}}$-module on $\widehat{S^{-1}M}$. The composition

$$\widetilde{j}_M = c \circ j_M\colon M \xrightarrow{j} S^{-1}M \xrightarrow{c} \widehat{S^{-1}M}$$

of the usual localization homomorphism with the separated completion homomorphism is a homomorphism of topological modules which we call the *separated completed localization homomorphism* of M with respect to S.

For every homomorphism of topolgical $\mathcal{A}$-modules $f\colon M \to N$, we denote by $\widehat{S^{-1}f}\colon \widehat{S^{-1}M} \to \widehat{S^{-1}N}$ the homomorphism of topological $\widehat{S^{-1}\mathcal{A}}$-modules induced by the universal properties of usual localization and separated completion.

1.4. **Restricted power series.** We recall properties of restricted power series rings with coefficients in a topological ring following [4, III.4.2], see also [10, 0.7.5].

Definition 1.24. Let $\mathcal{A}$ be a topological ring with separated completion $c\colon \mathcal{A} \to \widehat{\mathcal{A}}$ and let $T_1, \dots, T_r$ be a collection of indeterminates.

The ring of *restricted power series* with coefficients in $\widehat{\mathcal{A}}$ is the separated completion $\widehat{\mathcal{A}}\{T_1, \dots, T_r\}$ of the polynomial ring $\mathcal{A}[T_1, \dots, T_r]$ endowed with the topology generated by the ideals $\mathfrak{a}[T_1, \dots, T_r]$, where $\mathfrak{a}$ runs throught the set of open ideals of $\mathcal{A}$.

We denote by $i_0\colon \widehat{\mathcal{A}} \to \widehat{\mathcal{A}}\{T_1, \dots, T_r\}$ the homomorphism of topological rings deduced by the universal property of $c\colon \mathcal{A} \to \widehat{\mathcal{A}}$ from the composition of the inclusion $\mathcal{A} \hookrightarrow \mathcal{A}[T_1, \dots, T_r]$ as the subring of constant polynomials with the separated completion homomorphism $\mathcal{A}[T_1, \dots, T_r] \to \widehat{\mathcal{A}}\{T_1, \dots, T_r\}$. The elements in the image of i_0 are said to be *constant restricted power series*.

Letting $(\mathfrak{a}_n)_{n\in\mathbb{N}}$ be a fundamental system of open ideals of $\mathcal{A}$, it follows from the definition that

$$\widehat{\mathcal{A}}\{T_1, \dots, T_r\} \cong \varprojlim_{n\in\mathbb{N}} (\mathcal{A}/\mathfrak{a}_n)[T_1, \dots, T_r].$$

Identifying a polynomial in $\widehat{\mathcal{A}}[T_1,\dots,T_r]$ with the family $(a_I)_{I\in\mathbb{N}^r}$ of its coefficients, we see that the elements of $\widehat{\mathcal{A}}\{T_1,\dots,T_r\}$ are represented by families $(a_I)_{I\in\mathbb{N}^r}$ of elements of $\widehat{\mathcal{A}}$ which converge to 0 in the sense of Definition 1.6, namely, a family of elements $(a_I)_{I\in\mathbb{N}^r}$ of $\widehat{\mathcal{A}}$ represents a restricted power series if and only if for every open ideal $\mathfrak{a}$ of $\widehat{\mathcal{A}}$, all but finitely many of the a_I belong to $\mathfrak{a}$.

Following [4, III.4.2], we henceforth identify $\widehat{\mathcal{A}}\{T_1,\dots,T_r\}$ with the $\widehat{\mathcal{A}}$-subalgebra of the algebra of formal power series with coefficients in $\widehat{\mathcal{A}}$ consisting of formal power series

$$\sum_{I=(i_1,\dots,i_r)\in\mathbb{N}^r} a_I T_1^{i_1}\cdots T_r^{i_r}$$

such that the family $(a_I)_{I\in\mathbb{N}^r}$ converges to 0. Note that such a family is summable in $\widehat{\mathcal{A}}$ by Proposition 1.7, so that with our identification, the family $(a_I T_1^{i_1}\cdots T_r^{i_r})_{I\in\mathbb{N}^r}$ of elements of $\widehat{\mathcal{A}}\{T_1,\dots,T_r\}$ is summable, with sum $\sum_{I\in\mathbb{N}^r} a_I T_1^{i_1}\cdots T_r^{i_r}$. Recall [3, 5.3 Proposition 2 and Theorem 2] that for every partition $(J_\lambda)_{\lambda\in L}$ of $\mathbb{N}^r$, the subfamily $(a_I)_{I\in J_\lambda}$ is summable, say of sum $s_\lambda\in\widehat{\mathcal{A}}$, and the family $(s_\lambda)_{\lambda\in L}$ is summable, with the same sum as the family $(a_I)_{I\in\mathbb{N}^r}$, so that, with our identification, we have

$$\sum_{I\in\mathbb{N}^r} a_I T_1^{i_1}\cdots T_r^{i_r} = \sum_{\lambda\in L}\sum_{I\in J_\lambda} a_I T_1^{i_1}\cdots T_r^{i_r}.$$

Proposition 1.25. *The ring $\widehat{\mathcal{A}}\{T_1,\dots,T_r\}$ satisfies the following universal property: for every continuous ring homomorphism $f\colon\mathcal{A}\to\mathcal{B}$ to a complete topological ring $\mathcal{B}$ and every choice of r elements $b_1,\dots,b_r$ of $\mathcal{B}$, there exists a unique continuous ring homomorphism $\overline{f}\colon\widehat{\mathcal{A}}\{T_1,\dots,T_r\}\to\mathcal{B}$ such that $\overline{f}|_{c(\mathcal{A})}=\widehat{f}$ and such that $\overline{f}(T_i)=b_i$ for every $i=1,\dots,r$.*

Proof. This follows from [4, III.4.2 Proposition 4] and the universal property of the separated completion homomorphism $c\colon\mathcal{A}\to\widehat{\mathcal{A}}$. □

Corollary 1.26. *Let $\mathcal{A}$ be a complete topological ring and let $\mathcal{B}$ be a complete topological $\mathcal{A}$-algebra. Then for every $r\geq 1$, there are canonical isomorphisms*

$$\mathcal{C}\mathrm{Hom}_{\mathcal{A}-\mathrm{alg}}(\mathcal{A}\{T_1,\dots,T_r\},\mathcal{B})\cong\mathcal{C}\mathrm{Hom}_{\mathcal{A}-\mathrm{mod}}(\mathcal{A}^{\oplus r},\mathcal{B})\cong\mathcal{B}^{\oplus r}.$$

Notation 1.27. Given a complete topological ring $\mathcal{A}$ and a subset $J\subset\{1,\dots,r\}$, we denote by

$$\pi_{(1,J)}\colon\mathcal{A}\{T_i\}_{i\in\{1,\dots,r\}}\to\mathcal{A}\{T_i\}_{i\notin J}$$

the unique homomorphism of topological $\mathcal{A}$-algebras defined by $\pi_{(1,J)}(T_i)=1$ if $i\in J$ and $\pi_{(1,J)}(T_i)=T_i$ otherwise. For $J=\{1,\dots,r\}$, we denote the corresponding homomorphism $\mathcal{A}\{T_1,\dots,T_r\}\to\mathcal{A}$ simply by $\pi_{(1,\dots,1)}$.

For every collection $a_1,\dots,a_r$ of elements of $\mathcal{A}$, we denote by $\lambda(a_1,\dots,a_r)$ the unique endomorphisms of topological $\mathcal{A}$-algebras of $\mathcal{A}\{T_1,\dots,T_r\}$ defined by $T_i\mapsto a_iT_i$, $i=1,\dots,r$.

Finally, we denote by $\Delta\colon\mathcal{A}\{T_1,\dots,T_r\}\to\mathcal{A}\{T\}$ the unique homomorphism of topological $\mathcal{A}$-algebras that maps T_i to T for every $i=1,\dots,r$.

It follows from the definition of the completed tensor product that we have canonical isomorphisms

$$\widehat{\mathcal{A}}\{T_1,\dots,T_r\}\cong\mathcal{A}\widehat{\otimes}_{\mathbb{Z}}\mathbb{Z}[T_1,\dots,T_r]\cong\widehat{\mathcal{A}}\widehat{\otimes}_{\mathbb{Z}}\mathbb{Z}[T_1,\dots,T_r]$$

where $\mathbb{Z}[T_1,\dots,T_r]$ is endowed with the discrete topology. The following lemma is then a straightforward consequence of Lemma 1.13.

Lemma 1.28. *For every complete topological ring $\mathcal{A}$ and every set of variables $T_1,\dots,T_s,T_{s+1},\dots,T_r$, there exist canonical isomorphisms of complete topological $\mathcal{A}$-algebras*

$$\mathcal{A}\{T_1,\dots,T_s,T_{s+1},\dots T_r\}\cong\mathcal{A}\{T_1,\dots,T_r\}\widehat{\otimes}_{\mathcal{A}}\mathcal{A}\{T_{s+1},\dots,T_r\}\cong\mathcal{A}\{T_1,\dots,T_s\}\{T_{s+1},\dots T_r\}.$$

Lemma 1.29. *Let $\mathcal{B}$ be the limit of a countable inverse system $(\mathcal{B}_n)_{n\in\mathbb{N}}$ of complete topological rings with surjective continuous transition homomorphisms $p_{m,n}\colon \mathcal{B}_m \to \mathcal{B}_n$ for every $m \geq n \geq 0$ and let $T_1, \ldots, T_r$ be indeterminates. Then the canonical homomorphism of complete topological rings*

$$\mathcal{B}\{T_1,\ldots,T_r\} \cong (\varprojlim_{n\in\mathbb{N}} \mathcal{B}_n)\widehat{\otimes}_{\mathbb{Z}}\mathbb{Z}[T_1,\ldots,T_r] \to \varprojlim_{n\in\mathbb{N}}(\mathcal{B}_n\widehat{\otimes}_{\mathbb{Z}}\mathbb{Z}[T_1,\ldots T_r]) \cong \varprojlim_{n\in\mathbb{N}}(\mathcal{B}_n\{T_1,\ldots,T_r\})$$

is an isomorphism.

Proof. Let $p_n\colon \mathcal{B} \to \mathcal{B}_n$, $n \in \mathbb{N}$ be the canonical projection homomorphisms. By definition, elements of $\mathcal{B}\{T_1,\ldots,T_r\}$ are represented by families $(b_I)_{I\in\mathbb{N}^r}$ of elements of $\mathcal{B}$ which converge to 0 in $\mathcal{B}$. Since the projection homomorphisms p_n are surjective, it follows from the definition of the topology on $\mathcal{B}$ that these families are in one-to-one correspondence with collections of families $(b_{n,I})_{I\in\mathbb{N}^r}$ of elements of $\mathcal{B}_n$, $n \in \mathbb{N}$, such that $(b_{n,I})_{I\in\mathbb{N}^r}$ converges to 0 in $\mathcal{B}_n$ for every $n \in \mathbb{N}$ and such that $b_{n,I} = p_{m,n}(b_{m,I})$ for every $m \geq n \geq 0$ and every $I \in \mathbb{N}^r$. □

Lemma 1.30. *Let $\mathcal{A}$ be a topological ring, let $\mathcal{B}$ be a separated topological ring with separated completion $c\colon \mathcal{B} \to \widehat{\mathcal{B}}$ and let $h_n\colon \mathcal{A} \to \mathcal{B}$ be a sequence of homomorphisms of groups which converges pointwise to the zero homomorphism. Then the map*

$$s\colon \mathcal{A} \to \widehat{\mathcal{B}}\{T\}, \quad a \mapsto \sum_{n\in\mathbb{N}} c(h_n(a))T^n$$

is a well-defined homomorphism of groups and the following assertions are equivalent:

a) The homomorphism $s\colon \mathcal{A} \to \widehat{\mathcal{B}}\{T\}$ is continuous,

b) Every h_n, $n \in \mathbb{N}$, is continuous and the sequence $(h_n)_{n\in\mathbb{N}}$ converges continuously to the zero homomorphism.

Proof. Let $u_n\colon \mathcal{A} \to \widehat{\mathcal{B}}\{T\}$, $n \in \mathbb{N}$, be the sequence of homomorphisms of groups defined by $u_n(a) = c(h_n(a))T^n$. Since the sequence $(h_n)_{n\in\mathbb{N}}$ converges pointwise to the zero homomorphism, it follows from the definition of the topology on $\widehat{\mathcal{B}}\{T\}$ that the sequence $(u_n)_{n\in\mathbb{N}}$ converges pointwise to the zero homomorphism. Arguing as in the proof of Corollary 1.11, we conclude that the map s is a well-defined homomorphism of groups, and that the sequence of homomorphisms $(s_N)_{N\in\mathbb{N}}$ defined by $s_N = \sum_{n=0}^{N} u_n$ converges pointwise to s.

If each h_n, $n \in \mathbb{N}$, is continuous and the sequence $(h_n)_{n\in\mathbb{N}}$ converges continuously to the zero homomorphism, then s is a homomorphism of topological groups by Corollary 1.11. Conversely, assume that s is continuous. By definition of the topology on

$$\widehat{\mathcal{B}}\{T\} \subset \widehat{\mathcal{B}}[[T]] = \prod_{n\in\mathbb{N}} \widehat{\mathcal{B}},$$

every projection $p_n\colon \widehat{\mathcal{B}}\{T\} \to \widehat{\mathcal{B}}$, $(b_n)_{n\in\mathbb{N}} \mapsto b_n$, $n \in \mathbb{N}$, is a homomorphism of topological groups. It follows that $p_n \circ s = c \circ h_n$ is continuous for every $n \in \mathbb{N}$, which implies in turn since $c\colon \mathcal{B} \to \widehat{\mathcal{B}}$ is open onto its image that h_n is continuous. Furthermore, for every open ideal $\widehat{\mathfrak{b}}$ of $\widehat{\mathcal{B}}$, there exists an index $n \geq 0$ such that $c(h_n(a)) \in \widehat{\mathfrak{b}}$ for every $a \in \mathcal{A}$ and every $n \geq n_0$. This implies that the sequence $(c \circ h_n)_{n\in\mathbb{N}}$ converges continuously to the zero homomorphism, hence that the sequence $(h_n)_{n\in\mathbb{N}}$ converges to the zero homomorphism. □

Lemma 1.31. *Let $\mathcal{A}$ be topological ring with separated completion $c\colon \mathcal{A} \to \widehat{\mathcal{A}}$ and let $S \subset \mathcal{A}$ be a multiplicatively closed subset. Let $T_1, \ldots, T_r$ be a set of indeterminates and let $\widehat{S}_0 \subset \widehat{\mathcal{A}}\{T_1,\ldots,T_r\}$ be the image of S by the composition of c with the inclusion $i_0\colon \widehat{\mathcal{A}} \hookrightarrow \widehat{\mathcal{A}}\{T_1,\ldots,T_r\}$. Then there exists a canonical isomorphism of complete topological $\widehat{S^{-1}\mathcal{A}}$-algebras*

$$\widehat{S^{-1}\mathcal{A}}\{T_1,\ldots,T_r\} \cong (\widehat{S}_0^{-1}(\widehat{\mathcal{A}\{T_1,\ldots,T_r\}})),$$

where $(\widehat{S_0}^{-1}(\widehat{\mathcal{A}\{T_1,\dots,T_r\}}))$ *is viewed as an* $\widehat{S^{-1}\mathcal{A}}$*-algebra via the unique homomorphism of topological rings deduced from the homomorphism of topological rings*

$$\mathcal{A} \overset{i_0\circ c}{\to} \widehat{\mathcal{A}}\{T_1,\dots,T_r\} \to (\widehat{S_0}^{-1}(\widehat{\mathcal{A}\{T_1,\dots,T_r\}}))$$

by the universal property of separated completed localization.

Proof. Indeed, it follows from the definition of the separated completed localization and the definition of the restricted power series rings that these topological rings are both isomorphic, as topological $\widehat{S^{-1}\mathcal{A}}$-algebras, to the separated completion of the ring $S^{-1}\mathcal{A}[T_1,\dots,T_r] = S^{-1}\mathcal{A}\otimes_{\mathbb{Z}}\mathbb{Z}[T_1,\dots,T_r]$ with respect to the topology generated by the open ideals $S^{-1}\mathfrak{a}[T_1,\dots,T_r]$, where $\mathfrak{a}$ ranges through the set of open ideals of $\mathcal{A}$. □

Notation 1.32. Given a multiplicatively closed subset S of a topological ring $\mathcal{A}$, we denote by

$$\widetilde{j}_{T_1,\dots,T_r}\colon \widehat{\mathcal{A}}\{T_1,\dots,T_r\} \to \widehat{S^{-1}\mathcal{A}}\{T_1,\dots,T_r\}$$

the homomorphism of topological $\widehat{\mathcal{A}}$-algebras such $\widetilde{j}_{T_1,\dots,T_r}(T_i) = T_i$ for every $i = 1,\dots,r$.

2. Restricted exponential homomorphisms and topologically integrable derivations

In this section, we develop the basic algebraic theory of restricted exponential homomorphisms, which are the counterpart for topological rings of the co-action homomorphisms $e\colon B \to B\otimes_{\mathbb{Z}}\mathbb{Z}[T]$ of the Hopf algebra $\mathbb{Z}[T]$ of the additive group scheme $\mathbb{G}_{a,\mathbb{Z}}$ in the category of rings. We establish a one-to-one correspondence between restricted exponential homomorphisms and suitable systems of continuous iterated higher derivations which extends the classical correspondence between algebraic exponential homomorphisms $e\colon B \to B\otimes_{\mathbb{Z}}\mathbb{Z}[T]$ and locally finite iterative higher derivations of the ring B [18, 2, 8].

2.1. **Restricted exponential homomorphisms.** Recall that the ring $\mathbb{Z}[T]$, where T is an indeterminate, carries the structure of a cocommutative Hopf algebra whose comuliplication, coinverse and counit are given respectively by the following $\mathbb{Z}$-algebra homomorphisms:

$$\begin{array}{ll} m\colon \mathbb{Z}[T] \to \mathbb{Z}[T]\otimes_{\mathbb{Z}}\mathbb{Z}[T] \cong \mathbb{Z}[T,T'], & T \mapsto T+T' \\ \iota\colon \mathbb{Z}[T] \to \mathbb{Z}[T], & T \mapsto -T \\ \epsilon\colon \mathbb{Z}[T] \to \mathbb{Z}, & T \mapsto 0. \end{array}$$

Given any complete topological ring $\mathcal{A}$, the complete topological ring $\mathcal{A}\{T\} = \mathcal{A}\widehat{\otimes}\mathbb{Z}[T]$ inherits the structure of a cocommutative topological Hopf $\mathcal{A}$-algebra with comultiplication $\mathrm{id}_{\mathcal{A}}\widehat{\otimes}m$, coinverse $\mathrm{id}_{\mathcal{A}}\widehat{\otimes}\iota$ and counit $\mathrm{id}_{\mathcal{A}}\widehat{\otimes}\epsilon$.

Definition 2.1. Let $\mathcal{A}$ be a complete topological ring and let $\mathcal{B}$ be a complete topological $\mathcal{A}$-algebra. A *restricted exponential $\mathcal{A}$-homomorphism* is a homomorphism of topological $\mathcal{A}$-algebras

$$\mathrm{e}\colon \mathcal{B} \to \mathcal{B}\{T\} = \mathcal{B}\widehat{\otimes}_{\mathbb{Z}}\mathbb{Z}[T]$$

which defines a coaction of the Hopf $\mathcal{A}$-algebra $\mathcal{A}\{T\}$ on $\mathcal{B}$. This means equivalently that the following diagrams of $\mathcal{A}$-algebra homomorphisms are commutative:

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{\ \mathrm{e}\ } & \mathcal{B}\{T\} \\ {\scriptstyle \mathrm{e}}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_{\mathcal{B}}\widehat{\otimes}m} \\ \mathcal{B}\{T\} & \xrightarrow{\ \mathrm{e}\widehat{\otimes}\mathrm{id}_{\mathbb{Z}[T]}\ } & \mathcal{B}\{T'\}\{T\} = \mathcal{B}\{T',T\} \end{array} \qquad \begin{array}{ccc} \mathcal{B} & \xrightarrow{\ \mathrm{e}\ } & \mathcal{B}\{T\} \\ & {\scriptstyle \mathrm{id}_{\mathcal{B}}}\searrow & \downarrow{\scriptstyle q=\mathrm{id}_{\mathcal{B}}\widehat{\otimes}\epsilon} \\ & & \mathcal{B} = \mathcal{B}\{T\}/T\mathcal{B}\{T\}. \end{array}$$

Let $p_i\colon \mathcal{B}[[T]] = \prod_{i\in\mathbb{N}} \mathcal{B} \to \mathcal{B}$ denote the i-th projection. Then by definition of the topology on $\mathcal{B}\{T\}$, the composition $\mathrm{e}_i = p_i \circ \mathrm{e}\colon \mathcal{B} \to \mathcal{B}$ is a homomorphism of topological $\mathcal{A}$-modules for every $i \in \mathbb{N}$ for which we can write

$$\mathrm{e} = \sum_{i\in\mathbb{N}} \mathrm{e}_i T^i.$$

The commutativity of the right hand side diagram of Definition 2.1 means that $\mathrm{e}_0 = \mathrm{id}_{\mathcal{B}}$. On the other hand, with the identifications made, the homomorphisms $\mathrm{e}\widehat{\otimes}\mathrm{id}_{\mathbb{Z}[T]}$ and $\mathrm{id}_{\mathcal{B}}\widehat{\otimes} m$ are given by

$$\mathrm{e}\widehat{\otimes}\mathrm{id}_{\mathbb{Z}[T]}\colon \mathcal{B}\{T\} \to \mathcal{B}\{T', T\},\ \sum_{i\in\mathbb{N}} b_i T^i \mapsto \sum_{i\in\mathbb{N}} \mathrm{e}(b_i)T^i = \sum_{(i,j)\in\mathbb{N}^2} \mathrm{e}_j(b_i)T'^j T^i$$

$$\mathrm{id}_{\mathcal{B}}\widehat{\otimes} m\colon \mathcal{B}\{T\} \to \mathcal{B}\{T', T\},\ \sum_{i\in\mathbb{N}} b_i T^i \mapsto \sum_{i\in\mathbb{N}} b_i (T'+T)^i.$$

The commutativity of the left hand side diagram in Definition 2.1 says that in $\mathcal{B}\{T, T'\}$, we have

$$\sum_{(i,j)\in\mathbb{N}^2} (\mathrm{e}_j \circ \mathrm{e}_i)T'^j T^i = \sum_{\ell\in\mathbb{N}} \mathrm{e}_\ell (T'+T)^\ell. \tag{2.1}$$

In the next subsections, we first establish general properties of restricted exponential homomorphisms e. Then we discuss in more detail the properties of their associated collections $(\mathrm{e}_i)_{i\in\mathbb{N}}$ of homomorphisms of topological modules. In what follows, unless otherwise specified all topological modules and rings and are assumed to be modules and algebras over a fixed topological ring $\mathcal{A}$, and homomorphisms between these are assumed to be homomorphisms of $\mathcal{A}$-modules and $\mathcal{A}$-algebras respectively.

2.2. **Basic properties of restricted exponential automorphisms.**

2.2.1. *Rings of invariants and associated restricted exponential homomorphisms.*

Definition 2.2. Let $\mathcal{B}$ be a complete topological ring and let $\mathrm{e}\colon \mathcal{B} \to \mathcal{B}\{T\}$ be a restricted exponential homomorphism. We say that en element $b \in \mathcal{B}$ is e-*invariant* if $\mathrm{e}(b) = i_0(b)$. We denote by

$$\mathcal{B}^{\mathrm{e}} = \mathrm{Ker}(\mathrm{e} - i_0) \subseteq \mathcal{B}$$

the subset of all e-invariant elements of $\mathcal{B}$, endowed with the induced topology.

Proposition 2.3. *Let $\mathcal{B}$ be a complete topological ring and let $\mathrm{e} = \sum_{i\in\mathbb{N}} \mathrm{e}_i T^i\colon \mathcal{B} \to \mathcal{B}\{T\}$ be a restricted exponential homomorphism. Then the following hold:*

a) The set $\mathcal{B}^{\mathrm{e}}$ is a complete topological subring of $\mathcal{B}$.

b) For every $i \geq 1$, the homomorphism $\mathrm{e}_i\colon \mathcal{B} \to \mathcal{B}$ is a homomorphism of topological $\mathcal{B}^{\mathrm{e}}$-modules.

c) If $\mathcal{B}$ admits a fundamental system $(\mathfrak{b}_n)_{n\in\mathbb{N}}$ of open ideals consisting of prime ideals of $\mathcal{B}$ then $\mathcal{B}^{\mathrm{e}}$ is factorially closed in $\mathcal{B}$. In particular, every invertible element of $\mathcal{B}$ is contained in $\mathcal{B}^{\mathrm{e}}$.

Proof. The fact that $\mathcal{B}^{\mathrm{e}}$ is a subgring of $\mathcal{B}$ is clear. Since $\mathcal{B}\{T\}$ is complete, hence separated, $\{0\}$ is a closed subset of $\mathcal{B}\{T\}$. Since $\mathrm{e} - i_0\colon \mathcal{B} \to \mathcal{B}\{T\}$ is a homomorphism of topological groups, $\mathcal{B}^{\mathrm{e}}$ is a closed subgroup of $\mathcal{B}$, hence a complete topological group since $\mathcal{B}$ is complete. Assertion b) is clear from the definition of $\mathcal{B}^{\mathrm{e}}$ and the homomorphisms e_i.

Now let $(\mathfrak{b}_n)_{n\in\mathbb{N}}$ be fundamental system of open ideal of $\mathcal{B}$ consisting of prime ideals of $\mathcal{B}$ and let

$$\pi_n\colon \mathcal{B}\{T\} = \varprojlim_{n\in\mathbb{N}} (\mathcal{B}/\mathfrak{b}_n)[T] \to (\mathcal{B}/\mathfrak{b}_n)[T], \quad n \in \mathbb{N},$$

be the canonical projections. Write $\mathrm{e} = \sum_{i\in\mathbb{N}} \mathrm{e}_i T^i$, where $\mathrm{e}_0 = \mathrm{id}_{\mathcal{B}}$, and $b, b' \in \mathcal{B}$. Assume that $\mathrm{e}(bb') = \mathrm{e}(b)\mathrm{e}(b') = bb'$ then for every $n \geq 0$, we have

$$\pi_n(\mathrm{e}(b)\mathrm{e}(b')) = (\sum_{i\in\mathbb{N}} \pi_n(\mathrm{e}_i(b))T^i)(\sum_{i\in\mathbb{N}} \pi_n(\mathrm{e}_i(b'))T^i) = \pi_n(bb') = \pi_n(b)\pi_n(b')$$

in the integral domain $(\mathcal{B}/\mathfrak{b}_n)[T]$. It follows that $\pi_n(\mathrm{e}_i(b)) = \pi_n(\mathrm{e}_i(b')) = 0$ for every $i \geq 1$. This implies that for every $i \geq 1$, $\mathrm{e}_i(b)$ and $\mathrm{e}_i(b)'$ belong to $\bigcap_{n\geq 1} \mathfrak{b}_n = \{0\}$ as $\mathcal{B}$ is separated. Thus $\mathrm{e}(b) = b$ and $\mathrm{e}(b') = b'$. Finally, if $b \in \mathcal{B}$ is invertible, then $bb^{-1} = 1 \in \mathcal{B}^{\mathrm{e}}$ and so, b and b^{-1} belong to $\mathcal{B}^{\mathrm{e}}$. $\square$

Example 2.4. If the topology on $\mathcal{B}$ is the discrete one, the existence of a fundamental system of open prime ideals of $\mathcal{B}$ is equivalent to the property that $\mathcal{B}$ is an integral domain. This is no longer true in general, and the conclusion of assertion b) in Proposition 2.3 does not hold under the weaker assumption that $\mathcal{B}$ is integral. Indeed, let $\mathcal{B} = \varprojlim_{n\in\mathbb{N}} \mathbb{C}[u]/(u^n) \cong \mathbb{C}[[u]]$ be the completion of $\mathbb{C}[u]$ for the u-adic topology. The homomorphism of $\mathbb{C}$-algebras $\mathbb{C}[u] \to \mathbb{C}[[u]]\{T\}$ defined by

$$u \mapsto u\sum_{i\in\mathbb{N}}(uT)^i = \sum_{i\in\mathbb{N}} u^{i+1}T^i$$

induces a uniquely determined continuous homomorphism $\mathrm{e}\colon \mathcal{B} \to \mathcal{B}\{T\}$ which satisfies the axioms of a restricted exponential $\mathbb{C}$-homomorphism. The ring of invariants $\mathcal{B}^e$ is equal to the subring of constant formal power series. In particular, the invertible element $1-u$ of $\mathcal{B}$ does not belong to $\mathcal{B}^{\mathrm{e}}$, so that $\mathcal{B}^{\mathrm{e}}$ is not factorially closed in $\mathcal{B}$.

Proposition 2.5. *Let $\mathcal{B}$ be a complete topological ring and let $\mathrm{e}\colon \mathcal{B} \to \mathcal{B}\{T\}$ be a restricted exponential homomorphism. Then for every element $a \in \mathcal{B}^{\mathrm{e}}$, the homomorphism*

$$\mathrm{e}_{\lambda(a)} := \lambda(a)\circ \mathrm{e}\colon \mathcal{B} \xrightarrow{\mathrm{e}} \mathcal{B}\{T\} \xrightarrow{T\mapsto aT} \mathcal{B}\{T\}$$

is a restricted exponential homomorphism.

Proof. Since the homomorphisms e and $\lambda(a)$ are continuous, so is $\mathrm{e}_{\lambda(a)}$. The commutativity of the right hand side diagram in Definition 2.1 for $\mathrm{e}_{\lambda(a)}$ is clear. Writing $\mathrm{e} = \sum_{i\in\mathbb{N}} \mathrm{e}_i T^i$, we have

$$\mathrm{e}_{\lambda(a)} = \sum_{i\in\mathbb{N}} \mathrm{e}_i(aT)^i = \sum_{i\in\mathbb{N}} (a^i\mathrm{e}_i)T^i,$$

where $a^i\mathrm{e}_i$ is the homomorphism defined by $b \mapsto a^i\mathrm{e}_i(b)$ for every $b \in \mathcal{B}$. Since $a \in \mathcal{B}^{\mathrm{e}}$, $a^i \in \mathcal{B}^{\mathrm{e}}$ for every $i \geq 0$. Since by Proposition 2.3 b) each e_i, $i \geq 1$, is a homomorphism of topological $\mathcal{B}^{\mathrm{e}}$-module, applying (2.1), we obtain

$$\begin{array}{rcl}(\mathrm{e}_{\lambda(a)}\widehat{\otimes}\mathrm{id}_{\mathbb{Z}[T]})\circ \mathrm{e}_{\lambda(a)} &=& \sum_{i\in\mathbb{N}}(\mathrm{e}_{\lambda(a)}\circ(a^i\mathrm{e}_i))T^i \\ &=& \sum_{(i,j)\in\mathbb{N}^2}(\mathrm{e}_j\circ(a^ie_i))(aT')^jT^i \\ &=& \sum_{(i,j)\in\mathbb{N}^2}(\mathrm{e}_j\circ \mathrm{e}_i)(aT')^j(aT)^i \\ &=& \sum_{\ell\in\mathbb{N}} \mathrm{e}_\ell(aT'+aT)^\ell = (\mathrm{id}_{\mathcal{B}}\widehat{\otimes}m)\circ \mathrm{e}_{\lambda(a)}.\end{array}$$

This shows that the commutativity of left hand side diagram in Definition 2.1 is satisfied for $\mathrm{e}_{\lambda(a)}$. □

Proposition 2.6. *Let $\mathcal{B}$ be a complete topological ring and let $\mathrm{e}\colon \mathcal{B} \to \mathcal{B}\{T\}$ be a restricted exponential homomorphism. Let $S \subset \mathcal{B}^{\mathrm{e}}$ be a multiplicatively closed subset, let $\widetilde{j}\colon \mathcal{B} \to \widehat{S^{-1}\mathcal{B}}$ be the separated completed localization homomorphism and let $\widetilde{j}_T\colon \mathcal{B}\{T\} \to \widehat{S^{-1}\mathcal{B}}\{T\}$ be the induced homomorphism.*

Then there exists a unique restricted exponential homomorphism $\widehat{S^{-1}\mathrm{e}}\colon \widehat{S^{-1}\mathcal{B}} \to \widehat{S^{-1}\mathcal{B}}\{T\}$ such that $\widetilde{j}_T\circ \mathrm{e} = \widehat{S^{-1}\mathrm{e}}\circ\widetilde{j}$

Proof. We identify $S \subset \mathcal{B}$ with $i_0(S) \subset \mathcal{B}\{T\}$ and $S^{-1}\widehat{(\mathcal{B}\{T\})}$ with $\widehat{S^{-1}\mathcal{B}}\{T\}$ by the canonical isomorphism of Lemma 1.31. Since $S \subset \mathcal{B}^{\mathrm{e}}$,we have $\mathrm{e}(S) = S \subset \mathcal{B}\{T\}$ so that by the universal property of separated completed localization, there exists a unique homomorphism of topological rings

$$\widehat{S^{-1}e}\colon \widehat{S^{-1}\mathcal{B}} \to \widehat{S^{-1}\mathcal{B}}\{T\}$$

such that $\widetilde{j}_T\circ \mathrm{e} = \widehat{S^{-1}\mathrm{e}}\circ\widetilde{j}$. Write $\mathrm{e} = \sum_{i\in\mathbb{N}} \mathrm{e}_iT^i$. Since $S \subset \mathcal{B}^{\mathrm{e}}$ and since e_i is a homomorphism of topological $\mathcal{B}^{\mathrm{e}}$-module for every $i \geq 0$ by Proposition 2.3 b), it follows that each e_i, $i \in \mathbb{N}$, induces a uniquely determined homomorphism of topological $S^{-1}\mathcal{B}^{\mathrm{e}}$-modules $S^{-1}\mathrm{e}_i\colon S^{-1}\mathcal{B} \to S^{-1}\mathcal{B}$, hence by the universal property of separated completed localization, a homomorphism $\widehat{S^{-1}\mathrm{e}_i}\colon \widehat{S^{-1}\mathcal{B}} \to \widehat{S^{-1}\mathcal{B}}$ of topological $\widehat{S^{-1}\mathcal{B}^{\mathrm{e}}}$-modules. By construction, we then have

$$\widehat{S^{-1}\mathrm{e}} = \sum_{i\in\mathbb{N}} \widehat{S^{-1}\mathrm{e}_i}T^i.$$

To show that $\widehat{S^{-1}\mathrm{e}}$ is a restricted exponential homomorphism, it is enough to check the commutativity of the two diagrams of Definition 2.1 in restriction to the dense image of $S^{-1}\mathcal{B}$ in $\widehat{S^{-1}\mathcal{B}}$ by the separated completion morphism $c\colon S^{-1}\mathcal{B} \to \widehat{S^{-1}\mathcal{B}}$. Let $x = s^{-1}b \in S^{-1}\mathcal{B}$, where $b \in \mathcal{B}$. Then by definition of $\widehat{S^{-1}\mathrm{e}}$, we have

$$\widehat{S^{-1}\mathrm{e}}(c(x)) = \sum_{i\in\mathbb{N}} \widehat{S^{-1}\mathrm{e}_i}(c(x))T^i = c_T\big(\sum_{i\in\mathbb{N}}(S^{-1}\mathrm{e}_i)(x)T^i\big) = c_T\big(\sum_{i\in\mathbb{N}} s^{-1}\mathrm{e}_i(b)T^i\big),$$

where $c_T\colon S^{-1}\mathcal{B}\{T\} \to \widehat{S^{-1}\mathcal{B}}\{T\}$ denotes the separated completion homomorphism. This immediately implies the commutativity of the right hand side diagram of Definition 2.1. On the other hand, letting $c_{T,T'}\colon S^{-1}\mathcal{B}\{T,T'\} \to \widehat{S^{-1}\mathcal{B}}\{T,T'\}$ be the separated completion homomorphism, we have

$$\begin{array}{rcl}
(\widehat{S^{-1}e}\widehat{\otimes}\mathrm{id}_{\mathbb{Z}[T]})(\widehat{S^{-1}e}(c(x)) & = & \sum_{i\in\mathbb{N}} \widehat{S^{-1}e}(c(s^{-1}e_i(b)))T^i \\
& = & \sum_{(i,j)\in\mathbb{N}} \widehat{S^{-1}e_j}(c(s^{-1}e_i(b)))T'^jT^i \\
& = & c_{T,T'}(\sum_{(i,j)\in\mathbb{N}^2} S^{-1}e_j((s^{-1}e_i(b)))T'^jT^i) \\
& = & c_{T,T'}(\sum_{(i,j)\in\mathbb{N}^2} s^{-1}e_j(e_i(b))T'^jT^i) \\
& = & c_{T,T'}(\sum_{\ell\in\mathbb{N}} s^{-1}e_\ell(b)(T'+T)^\ell) \\
& = & c_{T,T'}(\sum_{\ell\in\mathbb{N}} S^{-1}e_\ell(x)(T'+T)^\ell) \\
& = & (\sum_{\ell\in\mathbb{N}} \widehat{S^{-1}e_\ell}(c(x))(T'+T)^\ell) = (\mathrm{id}_{\mathcal{B}}\widehat{\otimes}m)(\widehat{S^{-1}e}(c(x)),
\end{array}$$

which shows the commutativity of the left hand side diagram. □

Proposition 2.7. *Let $\pi\colon \mathcal{B} \to \mathcal{C}$ be a surjective open homomorphism of complete topological rings. Let $I = \mathrm{Ker}(\pi)$ and let $\mathrm{e}\colon \mathcal{B} \to \mathcal{B}\{T\}$ be a restricted exponential homomorphism. Assume that $\mathrm{e}(I) \subset i_0(I)\mathcal{B}\{T\}$. Then there exists a unique restricted exponential homomorphism $\overline{\mathrm{e}}\colon \mathcal{C} \to \mathcal{C}\{T\}$ such that $\pi_T \circ \mathrm{e} = \overline{\mathrm{e}} \circ \pi$, where $\pi_T\colon \mathcal{B}\{T\} \to \mathcal{C}\{T\}$ is the unique homomorphism of topological $\mathcal{B}$-algebras which maps T to T.*

Proof. The assumptions imply that e induces a unique homomorphism of rings

$$\widetilde{\mathrm{e}}\colon \mathcal{C} = \mathcal{B}/I \to \mathcal{B}\{T\}/i_0(I)\mathcal{B}\{T\} \cong \mathcal{B}/I \otimes \mathcal{B}\{T\} \cong \mathcal{C} \otimes \mathcal{B}\{T\}$$

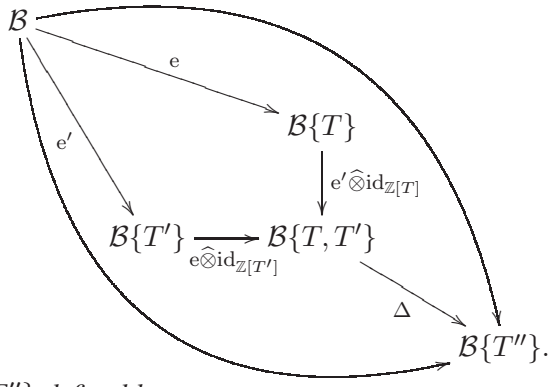
such that $(\pi \otimes \mathrm{id}_{\mathcal{B}\{T\}}) \circ \mathrm{e} = \widetilde{\mathrm{e}} \circ \pi$. Since π is open, $\widetilde{\mathrm{e}}$ is continuous when we endow the ring $\mathcal{C} \otimes \mathcal{B}\{T\}$ with the linear topology generated by open ideals of the form $U_{\mathcal{C}} \otimes \mathcal{B}\{T\} + \mathcal{C} \otimes V_{\mathcal{B}\{T\}}$ where $U_{\mathcal{C}}$ and $V_{\mathcal{B}\{T\}}$ run through the sets of open ideals of $\mathcal{C}$ and $\mathcal{B}\{T\}$ respectively. Note also that $\pi \otimes \mathrm{id}_{\mathcal{B}\{T\}}$ is an open homomorphism of topological rings. The composition

$$\overline{\mathrm{e}}\colon \mathcal{C} \to \mathcal{C}\widehat{\otimes}\mathcal{B}\{T\} \cong \mathcal{C}\{T\}$$

of $\widetilde{\mathrm{e}}$ with the separated completion homomorphism $c\colon \mathcal{C} \otimes \mathcal{B}\{T\} \to \mathcal{C}\widehat{\otimes}\mathcal{B}\{T\}$ is then a homomorphism of topological rings. The commutativity of the two diagrams in Definition 2.1 is straightforward to check. □

2.2.2. *Operations on restricted exponential homomorphism.*
Recall Notation 1.27 that for a complete topological ring $\mathcal{B}$, $\Delta\colon \mathcal{B}\{T,T'\} \to \mathcal{B}\{T''\}$ denotes the unique continuous $\mathcal{B}$-algebra homomorphism which maps T and T' to T''.

Proposition 2.8. *Let $\mathcal{B}$ be a complete topological ring and let $\mathrm{e}\colon \mathcal{B} \to \mathcal{B}\{T\}$ and $\mathrm{e}'\colon \mathcal{B} \to \mathcal{B}\{T'\}$ be restricted exponential homomorphisms such that the following diagram commutes*



Then the map $\mathrm{e}''\colon \mathcal{B} \to \mathcal{B}\{T''\}$ *defined by*

$$\mathrm{e}'' = \Delta \circ (\mathrm{e}\widehat{\otimes}\mathrm{id}_{\mathbb{Z}[T']}) \circ \mathrm{e}' = \Delta \circ (\mathrm{e}'\widehat{\otimes}\mathrm{id}_{\mathbb{Z}[T]}) \circ \mathrm{e}$$

is a restricted exponential homomorphism.

Proof. Being the composition of homomorphisms of topological rings, e'' is a homomorphism of topological rings. Denoting T'' by T_0, we have, by definition of e'',

$$\mathrm{e}'' = \sum_{n\in\mathbb{N}} \mathrm{e}''_n T_0^n = \sum_{(i,j)\in\mathbb{N}^2} (\mathrm{e}'_j \circ \mathrm{e}_j) T_0^{i+j} = \sum_{(i,j)\in\mathbb{N}^2} (\mathrm{e}_j \circ \mathrm{e}'_i) T_0^{i+j}. \tag{2.2}$$

This implies in particular that $\mathrm{e}''_0 = \mathrm{e}'_0 \circ \mathrm{e}_0 = \mathrm{id}_{\mathcal{B}}$, hence that the commutativity of the right hand side diagram of Definition 2.1 holds for e''. Combining Equation (2.2) above with Equation (2.1) for e and e', we obtain on the other hand that

$$\begin{aligned}
(\mathrm{id}_{\mathcal{B}}\widehat{\otimes}m) \circ \mathrm{e}'' &= \textstyle\sum_{(i,j)\in\mathbb{N}^2} (\mathrm{e}'_j \circ \mathrm{e}_i)(T_0 + T'_0)^{i+j} \\
&= \textstyle\sum_{(i,k,\ell)\in\mathbb{N}^3} (\mathrm{e}'_\ell \circ \mathrm{e}'_k \circ \mathrm{e}_i) {T'_0}^\ell T_0^k (T_0 + T'_0)^i \\
&= \textstyle\sum_{(i,k,\ell)\in\mathbb{N}^3} (\mathrm{e}_i \circ \mathrm{e}'_\ell \circ \mathrm{e}'_k)(T_0 + T'_0)^i T'_0 T_0^k \\
&= \textstyle\sum_{(m,n,k,\ell)\in\mathbb{N}^4} (\mathrm{e}_n \circ \mathrm{e}_m \circ \mathrm{e}'_\ell \circ \mathrm{e}'_k) {T'_0}^n T_0^m {T'_0}^\ell T_0^k \\
&= \textstyle\sum_{(m,n,k,\ell)\in\mathbb{N}^4} (\mathrm{e}_n \circ \mathrm{e}'_\ell \circ \mathrm{e}_m \circ \mathrm{e}'_k) {T'_0}^{n+\ell} T_0^{k+m} \\
&= \textstyle\sum_{(i,n,\ell)\in\mathbb{N}^3} (\mathrm{e}_n \circ \mathrm{e}'_\ell \circ \mathrm{e}''_i) {T'_0}^{n+\ell} T_0^i \\
&= \textstyle\sum_{(i,j)\in\mathbb{N}^2} (\mathrm{e}''_j \circ \mathrm{e}''_i) {T'_0}^j T_0^i \\
&= (\mathrm{e}''\widehat{\otimes}\mathrm{id}_{\mathbb{Z}[T_0]}) \circ \mathrm{e}'',
\end{aligned}$$

which shows that e'' is a restricted exponential homomorphism. □

Proposition 2.9. *Let* $(\mathcal{B}_n)_{n\in\mathbb{N}}$ *be a countable inverse system of complete topological rings with continuous and surjective transition homomorphisms* $p_{m,n}\colon \mathcal{B}_m \to \mathcal{B}_n$ *for every* $m \geq n \geq 0$*. Let* $\mathcal{B}$ *be its limit and let* $p_n\colon \mathcal{B} \to \mathcal{B}_n$*,* $n \in \mathbb{N}$*, be the canonical continuous projections. Let* $\mathrm{e}_n\colon \mathcal{B}_n \to \mathcal{B}_n\{T\}$*,* $n \in \mathbb{N}$*, be a collection of restricted exponential homomorphisms such that*

$$\mathrm{e}_n \circ p_{m,n} = (p_{n,m}\widehat{\otimes}\mathrm{id}_{\mathbb{Z}[T]}) \circ \mathrm{e}_n \qquad \forall m \geq n \geq 0.$$

Then there exists a unique restricted exponential homomorphism $\underline{\mathrm{e}} = \varprojlim_{n\in\mathbb{N}} \mathrm{e}_n\colon \mathcal{B} \to \mathcal{B}\{T\}$ *such that* $\mathrm{e}_n \circ p_n = (p_n\widehat{\otimes}\mathrm{id}_{\mathbb{Z}[T]}) \circ \underline{\mathrm{e}}$ *for every* $n \in \mathbb{N}$*.*

Proof. The hypothesis combined with Lemma 1.28 implies the existence of a unique homomorphism of topological rings

$$\underline{\mathrm{e}} = \varprojlim_{n\in\mathbb{N}} \mathrm{e}_n\colon \mathcal{B} = \varprojlim_{n\in\mathbb{N}} \mathcal{B}_n \to \varprojlim_{n\in\mathbb{N}} \mathcal{B}_n\{T\} \cong \mathcal{B}\{T\}$$

such that $\mathrm{e}_n \circ p_n = (p_n\widehat{\otimes}\mathrm{id}_{\mathbb{Z}[T]}) \circ \underline{\mathrm{e}}$ for every $n \in \mathbb{N}$. The equality $\mathrm{e}_{n,0} = \mathrm{id}_{\mathcal{B}}$ for every $n \in \mathbb{N}$ implies that $\underline{\mathrm{e}}_0 = \mathrm{id}_{\mathcal{B}}$. Similarly, since $(\mathrm{id}_{\mathcal{B}}\widehat{\otimes}m) \circ \mathrm{e}_n = (\mathrm{e}_n\widehat{\otimes}\mathrm{id}_{\mathbb{Z}[T]}) \circ \mathrm{e}_n$ for every $n \in \mathbb{N}$, it follows that $(\mathrm{id}_{\mathcal{B}}\widehat{\otimes}m) \circ \underline{\mathrm{e}} = (\underline{\mathrm{e}}\widehat{\otimes}\mathrm{id}_{\mathbb{Z}[T]}) \circ \underline{\mathrm{e}}$, showing that $\underline{\mathrm{e}}$ is a restricted exponential homomorphism. □

2.2.3. *Restricted exponential homomorphisms and automorphisms.*

Proposition 2.10. *Let $\mathcal{B}$ be a complete topological ring and let $\mathrm{e}\colon \mathcal{B} \to \mathcal{B}\{T\}$ be a restricted exponential homomorphism. Then for every continuous ring automorphism α of $\mathcal{B}$, the composition*

$$ {}^{\alpha}\mathrm{e} := (\alpha\widehat{\otimes}\mathrm{id}_{\mathbb{Z}[T]}) \circ \mathrm{e} \circ \alpha^{-1}\colon \mathcal{B} \to \mathcal{B}\{T\} $$

is a restricted exponential homomorphism.

Proof. It is clear that ${}^{\alpha}\mathrm{e}$ is homomorphism of topological rings. By definition, we have

$$ {}^{\alpha}\mathrm{e} = \sum_{i\in\mathbb{N}} {}^{\alpha}\mathrm{e}_i T^i = \sum_{i\in\mathbb{N}} (\alpha \circ \mathrm{e}_i \circ \alpha^{-1}) T^i. $$

Since $\mathrm{e}_0 = \mathrm{id}_{\mathcal{B}}$, we have ${}^{\alpha}\mathrm{e}_0 = \mathrm{id}_{\mathcal{B}}$ showing that commutativity of the right hand side diagram of Definition 2.1 holds for ${}^{\alpha}\mathrm{e}(b)$. On the other hand, applying Equation (2.1), we have

$$ \begin{array}{rcl} (\mathrm{id}_{\mathcal{B}}\widehat{\otimes}m) \circ {}^{\alpha}\mathrm{e} & = & \sum_{\ell\in\mathbb{N}} (\alpha \circ \mathrm{e}_\ell \circ \alpha^{-1})(T+T')^\ell \\ & = & \sum_{(i,j)\in\mathbb{N}^2} (\alpha \circ \mathrm{e}_i \circ \mathrm{e}_j \circ \alpha^{-1}) T'^j T^i \\ & = & \sum_{(i,j)\in\mathbb{N}^2} \big((\alpha \circ \mathrm{e}_i \circ \alpha^{-1}) \circ (\alpha \circ \mathrm{e}_j \circ \alpha^{-1})\big) T'^j T^i \\ & = & ({}^{\alpha}\mathrm{e}\widehat{\otimes}\mathrm{id}_{\mathbb{Z}[T]}) \circ {}^{\alpha}\mathrm{e}, \end{array} $$

which shows that ${}^{\alpha}\mathrm{e}$ is a restricted exponential homomorphism. □

Proposition 2.11. *Let $\mathcal{B}$ be a complete topological ring and let $\mathrm{e}\colon \mathcal{B} \to \mathcal{B}\{T\}$ be a restricted exponential homomorphism. Then the compositions*

$$ \varphi = \pi_{(1)} \circ e\colon \mathcal{B} \xrightarrow{\mathrm{e}} \mathcal{B}\{T\} \xrightarrow{T\mapsto 1} \mathcal{B} \quad \textit{and} \quad \psi = \pi_{(-1)} \circ e\colon \mathcal{B} \xrightarrow{\mathrm{e}} \mathcal{B}\{T\} \xrightarrow{T\mapsto -1} \mathcal{B} $$

are continuous ring automorphisms of $\mathcal{B}$ inverse to each other.

Proof. The homomorphisms φ and ψ are clearly continuous. Note that by definition, $\psi = \pi_{(1)} \circ \mathrm{e}_{\lambda(-1)}$. Furthermore, letting $\pi_{(1,1)} = \pi_{(1)} \circ \Delta\colon \mathcal{B}\{T,T'\} \to \mathcal{B}$ be the unique continuous $\mathcal{B}$-algebra homomorphism that maps T and T' to 1, we have

$$ \psi \circ \varphi = \pi_{(1,1)} \circ (\mathrm{e}_{\lambda(-1)}\widehat{\otimes}\mathrm{id}_{\mathbb{Z}[T]}) \circ e \quad \text{and} \quad \varphi \circ \psi = \pi_{(1,1)} \circ (\mathrm{e}\widehat{\otimes}\mathrm{id}_{\mathbb{Z}[T']}) \circ \mathrm{e}_{\lambda(-1)}. $$

Let $f\colon \mathcal{B} \to \mathcal{B}\{T,T'\}$ be the composition of e with the unique continuous $\mathcal{B}$-algebra homomorphism $\mathcal{B}\{T\} \to \mathcal{B}\{T,T'\}$ that maps T to $T - T'$. Since e is a restricted exponential homomorphism, Equation (2.1) implies that

$$ (\mathrm{e}_{\lambda(-1)}\widehat{\otimes}\mathrm{id}_{\mathbb{Z}[T]}) \circ e = f = (\mathrm{e}\widehat{\otimes}\mathrm{id}_{\mathbb{Z}[T']}) \circ \mathrm{e}_{\lambda(-1)}(b). $$

Since $\pi_{(1,1)} \circ f = \mathrm{id}_{\mathcal{B}}$, the assertion follows. □

2.3. **Sliced restricted exponential homomorphisms.**

Definition 2.12. Let $\mathcal{B}$ be a complete topological ring and let $\mathrm{e}\colon \mathcal{B} \to \mathcal{B}\{T\}$ be a restricted exponential homomorphism. A *local slice* for e is element $s \in \mathcal{B}$ such that $\mathrm{e}(s) \in \mathcal{B}\{T\}$ is a polynomial of degree 1.

Example 2.13. Let $\mathcal{B} = \mathbb{C}[u]$ endowed with the discrete topology, so that $\mathcal{B}\{T\} = \mathbb{C}[u][T]$. Then $\mathrm{e}\colon \mathbb{C}[u] \to \mathbb{C}[u][T]$ defined by $P(u) \mapsto P(u+T)$ is a restricted exponential homomorphism which has the element $s = u$ as a slice. On the other hand, letting $\mathcal{B} = \mathbb{C}[[u]]$ endowed with the u-adic topology, the restricted exponential homomorphism

$$ \mathrm{e}\colon \mathbb{C}[[u]] \to \mathbb{C}[[u]]\{T\}, \quad u \mapsto u\sum_{i\in\mathbb{N}} (uT)^i = \sum_{i\in\mathbb{N}} u^{i+1}T^i $$

of Example 2.4 does not admit a local slice.

Let $\mathrm{e} = \sum_{i\in\mathbb{N}} \mathrm{e}_i T^i \colon \mathcal{B} \to \mathcal{B}\{T\}$ be a restricted exponential homomorphism and let s be a local slice. Then, by definition, we have $\mathrm{e}(s) = \mathrm{e}_0(s) + \mathrm{e}_1(s)T = s + \mathrm{e}_1(s)T$. Applying $\mathrm{e}\widehat{\otimes}\mathrm{id}_{\mathbb{Z}[T]}$ to this last equation we obtain

$$s+\mathrm{e}_1(s)(T+T') = ((\mathrm{id}_{\mathcal{B}}\widehat{\otimes} m)\circ \mathrm{e})(s) = ((\mathrm{e}\widehat{\otimes}\mathrm{id}_{\mathbb{Z}[T]})\circ\mathrm{e})(s) = \mathrm{e}(s)+\mathrm{e}(\mathrm{e}_1(s))T = s+\mathrm{e}_1(s)T'+\mathrm{e}(\mathrm{e}_1(s))T$$

in $\mathcal{B}\{T,T'\}$. It follows that $s_1 := \mathrm{e}_1(s)$ belong to $\mathcal{B}^{\mathrm{e}}$.

Let

$$\widetilde{j}_{s_1} \colon \mathcal{B} \to \widehat{\mathcal{B}_{s_1}} = \widehat{S^{-1}\mathcal{B}} \quad \text{and} \quad \bar{j}_{s_1} \colon \mathcal{B}^{\mathrm{e}} \to \widehat{\mathcal{B}^{\mathrm{e}}_{s_1}} = \widehat{S^{-1}\mathcal{B}^{\mathrm{e}}}$$

be the separated completed localization homomorphisms of $\mathcal{B}$ and $\mathcal{B}^{\mathrm{e}}$ with respect to the multiplicative subset $S = \{s_1^n\}_{n\geq 0}$ of $\mathcal{B}^{\mathrm{e}} \subseteq \mathcal{B}$. Since $\mathcal{B}^{\mathrm{e}}$ is closed in $\mathcal{B}$ by Proposition 2.3, the homomorphism of topological rings $\widehat{i}_{s_1} \colon \widehat{\mathcal{B}^{\mathrm{e}}_{s_1}} \to \widehat{\mathcal{B}_{s_1}}$ induced by the inclusion $i \colon \mathcal{B}^{\mathrm{e}} \hookrightarrow \mathcal{B}$ is injective by Lemma 1.21. We henceforth consider $\widehat{\mathcal{B}_{s_1}}$ as a $\widehat{\mathcal{B}^{\mathrm{e}}_{s_1}}$-algebra via this homomorphism. By Proposition 2.6, there exists a unique restricted exponential homomorphism $\widehat{\mathrm{e}_{s_1}} := \widehat{S^{-1}e} \colon \widehat{\mathcal{B}_{s_1}} \to \widehat{\mathcal{B}_{s_1}}\{T\}$ such that the following diagram commutes

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{\mathrm{e}} & \mathcal{B}\widehat{\otimes}_{\mathbb{Z}}\mathbb{Z}[T] \\ \downarrow{\scriptstyle \widetilde{j}_{s_1}} & & \downarrow{\scriptstyle \widetilde{j}_{s_1}\widehat{\otimes}\mathrm{id}} \\ \widehat{\mathcal{B}_{s_1}} & \xrightarrow{\widehat{\mathrm{e}_{s_1}}} & \widehat{\mathcal{B}_{s_1}}\widehat{\otimes}_{\mathbb{Z}}\mathbb{Z}[T]. \end{array}$$

Let σ be the image of the element $s_1^{-1}s \in \mathcal{B}_{s_1}$ in $\widehat{\mathcal{B}_{s_1}}$ by the separated completion homomorphism.

Lemma 2.14. *The image $\sigma \in \widehat{\mathcal{B}_{s_1}}$ of the element $s_1^{-1}s \in \mathcal{B}_{s_1}$ by the separated completion homomorphism is a regular element of $\widehat{\mathcal{B}_{s_1}}$. Furthermore, if $\widehat{\mathcal{B}_{s_1}}$ is not the zero ring, then $\widehat{\mathrm{e}_{s_1}}(\sigma) = \sigma + T$.*

Proof. If $\widehat{\mathcal{B}_{s_1}}$ is the zero ring, there is nothing to prove. We can thus assume that $\widehat{\mathcal{B}_{s_1}} \neq \{0\}$. By Corollary 1.19, 0 does not belong to the closure of S in $\mathcal{B}$, so that the image of s_1 in $\widehat{\mathcal{B}_{s_1}}$ is a nonzero invertible element. By definition of $\widehat{\mathrm{e}_{s_1}}$, we have $\widehat{\mathrm{e}_{s_1}}(\sigma) = \sigma + T$, so that σ is a local slice for $\widehat{\mathrm{e}_{s_1}}$. Now let $b \in \widehat{\mathcal{B}_{s_1}}$ be an element such that $\sigma b = 0$. Then we have

$$0 = \widehat{\mathrm{e}_{s_1}}(\sigma b) = \widehat{\mathrm{e}_{s_1}}(\sigma)\widehat{\mathrm{e}_{s_1}}(b) = (\sigma + T)\sum_{i\in\mathbb{N}} \widehat{\mathrm{e}_{s_1}}_i(b)T^i = \sigma b + \sum_{i\geq 1}(\sigma\widehat{\mathrm{e}_{s_1}}_i(b) + \widehat{\mathrm{e}_{s_1}}_{i-1}(b))T^i,$$

from which we infer by induction that

$$b = \widehat{\mathrm{e}_{s_1}}_0(b) = (-1)^i\widehat{\mathrm{e}_{s_1}}_i(b)\sigma^i \text{ for every } i \in \mathbb{N}.$$

Since $\widehat{\mathrm{e}_{s_1}}(b) \in \widehat{\mathcal{B}_{s_1}}\{T\}$, the sequence $(\widehat{\mathrm{e}_{s_1}}_i(b))_{i\in\mathbb{N}}$ converges to 0 in $\widehat{\mathcal{B}_{s_1}}$ by definition of the topology on $\widehat{\mathcal{B}_{s_1}}\{T\}$. This implies in turn that the sequence$((-1)^i\widehat{\mathrm{e}_{s_1}}_i(b)\sigma^i)_{i\in\mathbb{N}}$ also converges to 0, hence that $b = 0$. So σ is a regular element of $\widehat{\mathcal{B}_{s_1}}$. □

We let

$$\upsilon_{-\sigma} \colon \widehat{\mathcal{B}_{s_1}}\{T\} \to \widehat{\mathcal{B}_{s_1}} \quad \text{and} \quad \upsilon_{\sigma} \colon \widehat{\mathcal{B}_{s_1}}\{T\} \to \widehat{\mathcal{B}_{s_1}}$$

be the unique homomorphisms of topological $\widehat{\mathcal{B}_{s_1}}$-algebras mapping T to $-\sigma$ and σ respectively.

Definition 2.15. With the notation above, we call the topological ring homomorphisms

$$R_s = \upsilon_{-\sigma}\circ\widehat{\mathrm{e}_{s_1}} \colon \widehat{\mathcal{B}_{s_1}} \to \widehat{\mathcal{B}_{s_1}} \quad \text{and} \quad \theta_s = \upsilon_{\sigma}\circ(\widehat{i_{s_1}}\widehat{\otimes}\mathrm{id}) \colon \widehat{\mathcal{B}^{\mathrm{e}}_{s_1}}\{T\} \to \widehat{\mathcal{B}_{s_1}}$$

the *Dixmier-Reynolds homomomorphism* and the *cylinder homomorphism* associated to the local slice s.

Proposition 2.16. *Let $\mathcal{B}$ be a complete topological ring, let $\mathrm{e} \colon \mathcal{B} \to \mathcal{B}\{T\}$ be a restricted exponential homomorphism and let s be a local slice for e. Then the following hold:*

a) The homomorphism $R_s \colon \widehat{\mathcal{B}_{s_1}} \to \widehat{\mathcal{B}_{s_1}}$ is a homomorphism of topological $\widehat{\mathcal{B}^{\mathrm{e}}_{s_1}}$-algebras with image equal to $\widehat{\mathcal{B}^{\mathrm{e}}_{s_1}}$ and such that $R_s(R_s(b)b') = R_s(b)R_s(b')$ for every $b, b' \in \widehat{\mathcal{B}_{s_1}}$.

b) The homomorphism $\theta_s\colon \widehat{\mathcal{B}^{\mathrm{e}}_{s_1}}\{T\} \to \widehat{\mathcal{B}_{s_1}}$ is an isomorphism of topological $\widehat{\mathcal{B}^{\mathrm{e}}_{s_1}}$-algebras.

Proof. Since by Proposition 2.3 $\mathcal{B}^{\mathrm{e}}$ is closed in $\mathcal{B}$, it follows that 0 belongs to closure of $S = \{s_1^n\}_{n\geq 0}$ in $\mathcal{B}^{\mathrm{e}}$ if and only if it belongs to the closure $\overline{S}$ of S in $\mathcal{B}$. This implies that $\widehat{\mathcal{B}^{\mathrm{e}}_{s_1}}$ and $\widehat{\mathcal{B}_{s_1}}$ are equal to the zero ring if and only if $0 \in \overline{S}$. If $0 \in \overline{S}$ then assertions a) and b) hold trivially. We thus assume from now on that $0 \notin \overline{S}$ so that the images of s_1 in $\widehat{\mathcal{B}^{\mathrm{e}}_{s_1}}$ and $\widehat{\mathcal{B}_{s_1}}$ are non-zero invertible elements. By Lemma 2.14, σ is a local slice for $\widehat{\mathrm{e}_{s_1}}$. Replacing $\mathcal{B}$ by $\widehat{\mathcal{B}_{s_1}}$, e by $\widehat{\mathrm{e}_{s_1}}$ and s by σ, we may thus assume without loss of generality from the very beginning that $\mathrm{e}(s) = s + T$.

To prove assertion a), we first observe that since $\mathcal{B}^{\mathrm{e}} = \mathrm{Ker}(\mathrm{e} - i_0)$, where $i_0\colon \mathcal{B} \to \mathcal{B}\{T\}$ is the inclusion of $\mathcal{B}$ as the subring of constant restricted power series, we have $R_s|_{\mathcal{B}^{\mathrm{e}}} = \mathrm{id}_{\mathcal{B}^{\mathrm{e}}}$ so that R_s is indeed a $\mathcal{B}^{\mathrm{e}}$-algebra homomorphism. Now given an element $b \in \mathcal{B}$, we have $R_s(b) = \sum_{i\in\mathbb{N}} \mathrm{e}_i(b)(-s)^i$ and then

$$\begin{array}{rcl}(\mathrm{e}\circ R_s)(b) & = & \mathrm{e}(\sum_{i\in\mathbb{N}} \mathrm{e}_i(b)(-s)^i) \\ & = & \sum_{i\in\mathbb{N}} \mathrm{e}(\mathrm{e}_i(b))(-\mathrm{e}(s))^i \\ & = & \sum_{i\in\mathbb{N}} \mathrm{e}(\mathrm{e}_i(b))(-s-T)^i \\ & = & \sum_{(i,j)\in\mathbb{N}} (\mathrm{e}_i\circ \mathrm{e}_j)(b)T^j(s-T)^i \\ & = & \sum_{\ell\in\mathbb{N}} \mathrm{e}_\ell(b)(T-s-T)^\ell = R_s(b).\end{array}$$

This show that $\mathrm{e}\circ R_s = R_s$ hence that the image of R_s is contained in $\mathcal{B}^{\mathrm{e}} \subset \mathcal{B}$. Finally, since $R_s|_{\mathcal{B}^{\mathrm{e}}} = \mathrm{id}_{\mathcal{B}^{\mathrm{e}}}$, we have for every $b, b' \in \mathcal{B}$, $R_s(R_s(b)b') = R_s(R_s(b))R_s(b') = R_s(b)R_s(b')$. This shows a).

To prove assertion b), we observe that the composition

$$\big((\mathrm{id}_{\mathbb{Z}[T']}\widehat{\otimes}\upsilon_s)\circ(\mathrm{id}_{\mathbb{Z}[T]}\widehat{\otimes}\upsilon_{-s})\big)\circ(\mathrm{id}\widehat{\otimes}m)\circ\mathrm{e}\colon \mathcal{B}\to\mathcal{B}\{T\}\to\mathcal{B}\{T,T'\}\to\mathcal{B}$$

is equal to the identity. Writing for $(\mathrm{id}_{\mathbb{Z}[T']}\widehat{\otimes}\upsilon_s)\circ(\mathrm{id}_{\mathbb{Z}[T]}\widehat{\otimes}\upsilon_{-s}) = \upsilon_s\widehat{\otimes}\upsilon_{-s}$ for simplicity, it follows that for every element $b \in \mathcal{B}$, we have

$$\begin{array}{rcl}b = ((\upsilon_s\widehat{\otimes}\upsilon_{-s})\circ(\mathrm{id}\widehat{\otimes}m)\circ\mathrm{e})(b) & = & ((\upsilon_s\widehat{\otimes}\upsilon_{-s})\circ(\mathrm{e}\widehat{\otimes}\mathrm{id})\circ\mathrm{e})(b) \\ & = & (\upsilon_s\widehat{\otimes}\upsilon_{-s})(\sum_{(i,j)\in\mathbb{N}^2}(\mathrm{e}_j\circ\mathrm{e}_i)(b)T'^jT^i) \\ & = & \sum_{i\in\mathbb{N}}(\upsilon_{-s}\circ\mathrm{e})(\mathrm{e}_i(b))s^i \\ & = & \sum_{i\in\mathbb{N}} R_s(e_i(b))s^i\end{array}$$

Since $R_s(b_i) \in \mathcal{B}^{\mathrm{e}}$ for every $i \in \mathbb{N}$, this implies that θ is surjective. Suppose that θ is not injective and let $a = \sum_{i\in\mathbb{N}} a_iT^i \in \mathcal{B}^{\mathrm{e}}\{T\}$ be a nonzero element of minimal order $\mathrm{ord}_0(a) = \min\{i,\ a_i \neq 0\}$ such that $\theta(a) = \sum_{i\in\mathbb{N}} a_is^i = 0$. Since by Lemma 2.14 s is a regular element of $\mathcal{B}$, the minimality of $\mathrm{ord}_0(a)$ implies that that $a_0 \neq 0$. On the other hand, since $R_s(s) = 0$, we have

$$a_0 = \sum_{i\in\mathbb{N}} a_iR_s(s)^i = R_s(\theta(a)) = 0,$$

a contradiction. This shows that θ is injective, hence an isomorphism. □

Remark 2.17. For a restricted exponential homomorphism $\mathrm{e}\colon \mathcal{B} \to \mathcal{B}\{T\}$ with a local slice s such that $\mathrm{e}(s) = s + s_1T$, assertion a) in Proposition 2.16 says that the homomorphism of topological rings

$$R_s = \upsilon_{-s_1^{-1}s}\circ\widehat{\mathrm{e}_{s_1}}\colon \widehat{\mathcal{B}_{s_1}} \to \widehat{\mathcal{B}_{s_1}}$$

of Definition 2.15 is an idempotent endomorphism that satisfies the property of a Reynolds operator with values in the subalgebra of $\widehat{\mathrm{e}_{s_1}}$-invariant elements of $\widehat{\mathcal{B}_{s_1}}$. The composition $\pi_s\colon \mathcal{B} \to \widehat{\mathcal{B}_{s_1}}$ of the separated completed localization homomorphism $\mathcal{B} \to \widehat{\mathcal{B}_{s_1}}$ with R_s is the analog in the context of restricted exponential homomorphisms of the Dixmier map introduced in [8, 1.1.9].

Corollary 2.18. *Let $\mathcal{B}$ be a complete topological ring and let $\mathrm{e}\colon \mathcal{B} \to \mathcal{B}\{T\}$ be a restricted exponential homomorphism. Assume that e has a local slice $s \in \mathcal{B}$ such that $\mathrm{e}(s) = s + T$. Then $\mathcal{B} \cong \mathcal{B}^{\mathrm{e}}\{s\}$ and e coincides with the homomorphism of topological $\mathcal{B}^{\mathrm{e}}$-algebras*

$$\mathcal{B}^{\mathrm{e}}\{s\} \to \mathcal{B}^{\mathrm{e}}\{s\}\{T\} \cong \mathcal{B}^{\mathrm{e}}\{s,T\}, \quad s \mapsto s+T.$$

2.4. **Topologically integrable iterated higher derivations.** In the subsection, we extend to arbitrary complete topological rings the correspondence between locally finite iterative higher derivations and exponential homomorphisms which classically holds for discretely topologized rings.

2.4.1. *Continuous iterated higher derivations.*

Definition 2.19. Let $\mathcal{A}$ be a topological ring, let $\mathcal{B}$ be a topological $\mathcal{A}$-algebra and M be a topological $\mathcal{B}$-module. A continuous $\mathcal{A}$-derivation of $\mathcal{B}$ into M is a homomorphism of topological $\mathcal{A}$-modules $\partial\colon \mathcal{B} \to M$ which satisfies the Leibniz rule $\partial(bb') = b \cdot \partial(b') + \partial(b) \cdot b'$ for all $b, b' \in B$.

Lemma 2.20. *With the notation of Definition 2.19, let $c_{\mathcal{B}}\colon \mathcal{B} \to \widehat{\mathcal{B}}$ and $c_M\colon M \to \widehat{M}$ be the separated completions of $\mathcal{B}$ and M respectively. Then for every continuous $\mathcal{A}$-derivation $\partial\colon \mathcal{B} \to M$ there exists a unique continuous $\mathcal{A}$-derivation $\widehat{\partial}\colon \widehat{\mathcal{B}} \to \widehat{M}$ such that $\partial \circ c_M = \widehat{\partial} \circ c_B$.*

Proof. The existence of a unique homomorphism of topological $\mathcal{A}$-modules $\widehat{\partial}\colon \widehat{\mathcal{B}} \to \widehat{M}$ such that $c_M \circ \partial = \widehat{\partial} \circ c_{\mathcal{B}}$ follows from Lemma 1.4. By construction, $\widehat{\partial}$ is the homomorphism obtained from $\widetilde{\partial} = c_M \circ \partial$ by the universal property of the separated completion homomorphism $c_{\mathcal{B}}$. Since $\widetilde{\partial}$ satisfies the Leibniz rule, it follows that $\widehat{\partial}$ satisfies the Leibniz rule in restriction to the image of $c_{\mathcal{B}}$, hence on $\widehat{\mathcal{B}}$ because $\widehat{\partial}$ is uniformly continuous and $c_{\mathcal{B}}(\mathcal{B})$ is dense in $\mathcal{B}$. □

Definition 2.21. Let $\mathcal{A}$ be a topological ring and let $\mathcal{B}$ be a topological $\mathcal{A}$-algebra. A continuous *iterated higher $\mathcal{A}$-derivation* of $\mathcal{B}$ is a collection $D = \left\{D^{(i)}\right\}_{i\geq 0}$ of homomorphisms of topological $\mathcal{A}$-modules $D^{(i)}\colon \mathcal{B} \to \mathcal{B}$ which satisfy the following properties:

(1) The homomorphism $D^{(0)}$ is the identity homomorphism of $\mathcal{B}$,
(2) For every $i \geq 0$, the *Leibniz rule* $D^{(i)}(bb') = \sum_{j=0}^{i} D^{(j)}(b)D^{(i-j)}(b')$ holds for every pair of elements $b, b' \in \mathcal{B}$,
(3) For every $i, j \geq 0$, $D^{(i)} \circ D^{(j)} = \binom{i+j}{i} D^{(i+j)}$.

Note that the first two properties imply in particular that $\partial = D^{(1)}\colon \mathcal{B} \to \mathcal{B}$ is a continuous $\mathcal{A}$-derivation of $\mathcal{B}$ into itself. If $\mathcal{A}$ contains the field $\mathbb{Q}$ then the third property implies that $D^{(i)} = \frac{1}{i!}\partial^i$ for every $i \geq 0$, where ∂^i denotes the i-th iterate of ∂. In this case, a continuous iterated higher $\mathcal{A}$-derivation is then uniquely determined by a continuous $\mathcal{A}$-derivation ∂ of $\mathcal{B}$ into itself. The notion of higher derivation was first introduced by Hasse and Schmidt in [11].

Definition 2.22. Let $\mathcal{A}$ be a topological ring and let $\mathcal{B}$ be a topological $\mathcal{A}$-algebra. A *topologically integrable iterated higher $\mathcal{A}$-derivation* of $\mathcal{B}$ (an $\mathcal{A}$-TIIHD for short) is a continuous iterated higher $\mathcal{A}$-derivation $D = \left\{D^{(i)}\right\}_{i\geq 0}$ such that the sequence of homomorphims of topological $\mathcal{A}$-modules $(D^{(i)})_{n\in\mathbb{N}}$ converges continuously to the zero endomorphism of $\mathcal{B}$.

When $\mathcal{A}$ contains the field $\mathbb{Q}$, we say that a continuous $\mathcal{A}$-derivation ∂ is *topologically integrable* if its associated continuous iterated higher $\mathcal{A}$-derivation $D = \left\{\frac{1}{i!}\partial^i\right\}_{i\geq 0}$ is topologically integrable, equivalently, if the sequence of homomorphims of topological $\mathcal{A}$-modules $(\partial^i)_{i\in\mathbb{N}}$ converges continuously to the zero endomorphism of $\mathcal{B}$.

Remark 2.23. When $\mathcal{A}$ and $\mathcal{B}$ are topological rings endowed with the discrete topology, the condition that the sequence $(D^{(i)})_{n\in\mathbb{N}}$ is continuously convergent to the zero map means equivalently that for every element $b \in B$ there exists an integer i_0 such that $D^{(i)}(b) = 0$ for every $i \geq i_0$. We thus recover in this case the classical notions of locally finite iterated higher $\mathcal{A}$-derivation of $\mathcal{B}$ ([18], [2]) and locally nilpotent $\mathcal{A}$-derivation of $\mathcal{B}$ ([8]).

Lemma 2.24. *With the notation of Definition 2.21, let $c\colon \mathcal{B} \to \widehat{\mathcal{B}}$ be the separated completion of $\mathcal{B}$. Then for every iterated higher $\mathcal{A}$-derivation $D = \left\{D^{(i)}\right\}_{i\geq 0}$ of $\mathcal{B}$, there exists a unique iterated higher $\mathcal{A}$-derivation $\widehat{D} = \left\{\widehat{D}^{(i)}\right\}_{i\geq 0}$ of $\widehat{\mathcal{B}}$ such that $\widehat{D}^{(i)} \circ c = c \circ D^{(i)}$ for every $i \geq 0$.*

Furthermore, if D is topologically integrable, then so is $\widehat{D}$.

Proof. The existence of a unique collection of homomorphism of topological $\mathcal{A}$-modules $\widehat{D}^{(i)}$ such that $\widehat{D}^{(i)} \circ c = c \circ D^{(i)}$ for every $i \geq 0$ follows from the universal property of the separated completion. The fact that $\widehat{D} = \left\{\widehat{D}^{(i)}\right\}_{i\geq 0}$ satisfies the properties of an iterated higher $\mathcal{A}$-derivation follows from the same argument as in the proof of Lemma 2.20. The second assertion follows from Lemma 1.10. □

Example 2.25. Let $\mathcal{A}$ be a complete topological and let $\mathcal{A}[T]$ be endowed with the topology generated by the ideals $\mathfrak{a}[T]$, where $\mathfrak{a}$ runs throught the set of open ideals of $\mathcal{A}$. Let $\Delta\colon \mathcal{A}[T] \to \mathcal{A}[T,S]$ be the homomorphism of topological $\mathcal{A}$-algebras $T \mapsto T+S$ and let for every $i \geq 0$,

$$D^{(i)} = (\frac{1}{i!}\frac{\partial^i}{\partial S^i}|_{S=0}) \circ \Delta\colon \mathcal{A}[T] \to \mathcal{A}[T]$$

be the homomorphism of $\mathcal{A}$-modules which associates to a polynomial $P(T) \in \mathcal{A}[T]$ the i-th coefficient of the Taylor expansion at 0 of $P(T+S)$ with respect to the variable S. In particular, $D^{(1)}$ is simply the $\mathcal{A}$-derivation $\frac{\partial}{\partial T}$ of $\mathcal{A}[T]$.

It is straightforward to check that $D = \{D^{(i)}\}_{i\geq 0}$ is a continuous locally finite iterated higher $\mathcal{A}$-derivation of $\mathcal{A}[T]$. In particular, the sequence $(D^{(i)})_{i\in\mathbb{N}}$ is pointwise convergent to the zero homomorphism. Since on the other hand $D^{(i)}(\mathfrak{a}\mathcal{A}[T]) \subseteq \mathfrak{a}\mathcal{A}[T]$ for every $i \geq 0$ and every open ideal $\mathfrak{a}$ of $\mathcal{A}$, it follows from Lemma 1.9 that the sequence $(D^{(i)})_{i\in\mathbb{N}}$ converges continuously to the zero map (see Exemple 1.9). The collection $D = \{D^{(i)}\}_{i\geq 0}$ is thus a topologically integrable iterated higher $\mathcal{A}$-derivation of $\mathcal{A}[T]$. By Lemma 2.24, its canonical extension $\widehat{D} = \{\widehat{D}^{(i)}\}_{i\geq 0}$ to the separated completion $\mathcal{A}\{T\}$ of $\mathcal{A}[T]$ is a topologically integrable iterated higher $\mathcal{A}$-derivation of $\mathcal{A}\{T\}$.

2.4.2. *The correspondence between restricted exponential homomorphisms and topologically integrable iterated higher derivations.*

Theorem 2.26. *Let $\mathcal{A}$ be complete topological ring let $\mathcal{B}$ be a complete topological $\mathcal{A}$-algebra. Then there exists a one-to-one correspondence between restricted exponential $\mathcal{A}$-homomorphisms $\mathcal{B} \to \mathcal{B}\{T\}$ and topologically integrable iterated higher $\mathcal{A}$-derivations of $\mathcal{B}$.*

The correspondence is defined as follows:

1) Given a restricted exponential homomorphism $\mathrm{e} = \sum_{i\in\mathbb{N}} \mathrm{e}_i T^i\colon \mathcal{B} \to \mathcal{B}\{T\}$, it follows from the identities (2.1) expressing the commutativity of the two diagrams in Definition 2.1 that the collection of homomorphisms of topological $\mathcal{A}$-modules $D^{(i)} = \mathrm{e}_i$, $i \in \mathbb{N}$, is a continuous iterated higher $\mathcal{A}$-derivation of $\mathcal{B}$. Since e is continuous, it follows in turn from Lemma 1.30 that the so-defined sequence $(D^{(i)})_{i\in\mathbb{N}}$ converges continuously to the zero homomorphism. This shows that $D = \left\{D^{(i)}\right\}_{i\in\mathbb{N}}$ is a topologically integrable iterated higher $\mathcal{A}$-derivation of $\mathcal{B}$.

2) Conversely, given a topologically integrable iterated higher $\mathcal{A}$-derivation $D = \left\{D^{(i)}\right\}_{i\in\mathbb{N}}$ of $\mathcal{B}$, it follows from Lemma 1.30 again that the $\mathcal{A}$-module homomorphism

$$e = \exp(TD) := \sum_{i\in\mathbb{N}} D^{(i)}T^i\colon \mathcal{B} \to \mathcal{B}\{T\}$$

is well-defined and continuous. The properties of an iterated higher $\mathcal{A}$-derivation listed in Definition 2.21 guarantee precisely that e satisfies the axioms of a restricted exponential homomorphism.

In the case where the base topological ring $\mathcal{A}$ contains the field $\mathbb{Q}$, the fact that a continuous iterated higher $\mathcal{A}$-derivation $D = \left\{D^{(i)}\right\}_{i\in\mathbb{N}}$, is uniquely determined by the continuous $\mathcal{A}$-derivation $\partial = D^{(1)}$ of $\mathcal{B}$ implies in turn that a restricted exponential $\mathcal{A}$-homomorphism $\mathrm{e} = \sum_{i\in\mathbb{N}} \mathrm{e}_i T^i\colon \mathcal{B} \to \mathcal{B}\{T\}$ is uniquely determined by the topologically integrable $\mathcal{A}$-derivation

$$\partial = \mathrm{e}_1 = \tfrac{\partial}{\partial T}|_{T=0} \circ \mathrm{e}.$$

The following example illustrates the importance of continuous convergence in the correspondence between topologically integrable $\mathcal{A}$-derivation and restricted exponential homomorphisms.

Example 2.27. Let $\mathcal{A}$ be complete topological ring containing $\mathbb{Q}$, with fundamental system of open ideals $(\mathfrak{a}_n)_{n\in\mathbb{N}}$, and let $\mathcal{A}[(X_i)_{i\in\mathbb{N}}] = \mathcal{A}\otimes_{\mathbb{Z}}\mathbb{Z}[(X_i)_{i\in\mathbb{N}}]$ be the polynomial ring in countably many variables X_i, $i\in\mathbb{N}$. Let $\mathcal{B}$ be the separated completion of $\mathcal{A}[(X_i)_{i\in\mathbb{N}}]$ with respect to the topology induced by the fundamental system of open ideals

$$\overline{\mathfrak{a}}_n = \mathfrak{a}_n\otimes_{\mathbb{Z}}\mathbb{Z}[(X_i)_{i\in\mathbb{N}}] + \mathcal{A}\otimes_{\mathbb{Z}}(X_i)_{i\geq n},\quad n\in\mathbb{N}.$$

Note that since $\mathcal{A}$ is separated and $\bigcap_{i\in\mathbb{N}}(X_i)_{i\geq n} = \{0\}$, $\mathcal{A}[(X_i)_{i\in\mathbb{N}}]$ is a separated topological ring so that the separated completion homomorphism $c\colon \mathcal{A}[(X_i)_{i\in\mathbb{N}}]\to\mathcal{B}$ is injective.

Let ∂ be the $\mathcal{A}$-derivation of $\mathcal{A}[(X_i)_{i\in\mathbb{N}}]$ defined by

$$\partial(X_0) = X_1,\quad \partial(X_{2i-1}) = X_{2i+1},\text{ and }\quad \partial(X_{2i}) = X_{2i-2}\quad \forall i\geq 1.$$

It is easily seen that ∂ is continuous and that the sequence of homomorphisms $(\partial^i)_{i\in\mathbb{N}}$ converges pointwise to the zero homomorphism. However, it does not converge continuously to the zero homomorphism. Indeed, since $\partial^\ell(X_{2\ell}) = X_0$, it follows that for any given $i\in\mathbb{N}$ there cannot exist any integer n_0 such that $\partial^n(\overline{\mathfrak{a}}_i)\subseteq\overline{\mathfrak{a}}_1$ for every $n\geq n_0$. Since $c\colon\mathcal{A}[(X_i)_{i\in\mathbb{N}}]\to\mathcal{B}$ is injective, this implies in turn that the associated $\mathcal{A}$-derivation $\widehat{\partial}$ of $\mathcal{B}$ is not topologically integrable. The associated $\mathcal{A}$-algebra homomorphism $\exp(T\widehat{\partial})\colon\mathcal{B}\to\mathcal{B}\{T\}$ is well-defined but not continuous, hence is not a restricted exponential homomorphism.

We now briefly translate some of the main basic properties of restricted exponential homomorphisms established in subsection 2.2 in the language of topologically integrable derivations. We first observe that the ring of invariants $\mathcal{B}^{\mathrm{e}}$ of the restricted exponential homomorphism associated to topologically integrable $\mathcal{A}$-derivation ∂ of $\mathcal{B}$ is equal to the kernel of ∂.

Proposition 2.28. *Let $\mathcal{A}$ be a topological ring containing $\mathbb{Q}$, let $\mathcal{B}$ be a complete topological $\mathcal{A}$-algebra and let ∂ be a topologically integrable $\mathcal{A}$-derivation of $\mathcal{B}$. Then the following hold :*

a) For every element $f\in\operatorname{Ker}(\partial)$, the $\mathcal{A}$-derivation $f\partial$ of $\mathcal{B}$ is topologically integrable.

b) For every multiplicatively closed subset S of $\operatorname{Ker}(\partial)$, the $\mathcal{A}$-derivation $\widehat{S^{-1}\partial}$ of $\widehat{S^{-1}\mathcal{B}}$ is topologically integrable.

c) For every surjective open homomorphism of complete topological rings $\pi\colon\mathcal{B}\to\mathcal{C}$ such that $\partial(\operatorname{Ker}\pi)\subset\operatorname{Ker}\pi$, the induced $\mathcal{A}$-derivation $\overline{\partial}$ of $\mathcal{C}\cong\mathcal{B}/\operatorname{Ker}\pi$ is topologically integrable.

d) For every topologically integrable $\mathcal{A}$-derivation ∂' of $\mathcal{B}$ such $\partial\circ\partial' = \partial'\circ\partial$, the $\mathcal{A}$-derivation $\partial'' = \partial+\partial'$ of $\mathcal{B}$ is topologically integrable.

Proof. The assertions follow respectively from Propositions 2.5, 2.6, 2.7 and 2.8 □

Similarly, the following proposition is a consequence of Proposition 1.29:

Proposition 2.29. *Let $\mathcal{A}$ be a topological ring containing $\mathbb{Q}$ and let $(\mathcal{B}_n)_{n\in\mathbb{N}}$ be a countable inverse system of complete topological $\mathcal{A}$-algebras with continuous and surjective transition homomorphisms $p_{m,n}\colon\mathcal{B}_m\to\mathcal{B}_n$ for every $m\geq n\geq 0$. Let $\mathcal{B}$ be its limit and let $p_n\colon\mathcal{B}\to\mathcal{B}_n$, $n\in\mathbb{N}$, be the canonical continuous projections. Let $\partial_n\colon\mathcal{B}_n\to\mathcal{B}_n$, $n\in\mathbb{N}$, be a sequence of topologically integrable $\mathcal{A}$-derivations such that $\partial_n\circ p_{m,n} = p_{m,n}\circ\partial_m$ for all $m\geq n\geq 0$. Then there exists a unique topologically integrable $\mathcal{A}$-derivation $\partial = \varprojlim_{n\in\mathbb{N}}\partial_n$ of $\mathcal{B}$ such that $\partial_n\circ p_n = p_n\circ\partial$ for every $n\in\mathbb{N}$.*

Proposition 2.16 can be translated as follows:

Proposition 2.30. *Let $\mathcal{A}$ be a topological ring containing $\mathbb{Q}$, let $\mathcal{B}$ be a complete topological $\mathcal{A}$-algebra and let ∂ be a topologically integrable $\mathcal{A}$-derivation of $\mathcal{B}$. If $\operatorname{Ker}\partial^2\setminus\operatorname{Ker}\partial$ is not empty then for every $s\in\operatorname{Ker}\partial^2\setminus\operatorname{Ker}\partial$ with $\partial(s) = s_1\in\operatorname{Ker}\partial$, there exists an isomorphism of topological $\widehat{(\operatorname{Ker}\partial)_{s_1}}$-algebras*

$$\widehat{\mathcal{B}_{s_1}}\xrightarrow{\cong}\widehat{(\operatorname{Ker}\partial)_{s_1}}\{S\}$$

which maps the induced topologically integrable $\widehat{(\operatorname{Ker}\partial)_{s_1}}$-derivation $\widehat{\partial_{s_1}}$ of $\widehat{\mathcal{B}_{s_1}}$ onto the topologically integrable $\widehat{(\operatorname{Ker}\partial)_{s_1}}$-derivation $\frac{\partial}{\partial S}$ of $\widehat{(\operatorname{Ker}\partial)_{s_1}}\{S\}$.

In contrast with the case of usual locally nilpotent derivations of discrete topological rings containing $\mathbb{Q}$, there exist topologically integrable derivations ∂ of complete topological rings containing $\mathbb{Q}$ for which $\operatorname{Ker}\partial^n = \operatorname{Ker}\partial$ for every $n \geq 1$ so that in particular $\operatorname{Ker}\partial^2 \setminus \operatorname{Ker}\partial = \emptyset$. This is the case for instance for the topologically integrable derivation $u^2\frac{\partial}{\partial u}$ of $\mathbb{C}[[u]]$ corresponding to the restricted exponential homomorphism of Example 2.13. The following example provides another illustration of this phenomenon.

Example 2.31. As in Example 2.27, let $\mathcal{A}$ be a complete topological ring containing $\mathbb{Q}$ with fundamental system of open ideals $(\mathfrak{a}_n)_{n\in\mathbb{N}}$ and let $\mathcal{B}$ be the separated completion of the polynomial ring $\mathcal{A}[(X_i)_{i\in\mathbb{N}}] = \mathcal{A}\otimes_{\mathbb{Z}}\mathbb{Z}[(X_i)_{i\in\mathbb{N}}]$ in countably many variables with respect to the topology induced by the fundamental system of open ideals

$$\overline{\mathfrak{a}}_n = \mathfrak{a}_n \otimes_{\mathbb{Z}} \mathbb{Z}[(X_i)_{i\in\mathbb{N}}] + \mathcal{A}\otimes_{\mathbb{Z}}(X_i)_{i\geq n}, \quad n\in\mathbb{N}.$$

Then the $\mathcal{A}$-derivation $\widehat{\partial}_+$ of $\mathcal{B}$ induced by the $\mathcal{A}$-derivation of $\mathcal{A}[(X_i)_{i\in\mathbb{N}}]$ defined by

$$\partial_+(X_i) = (i+1)X_{i+1}, \quad \forall i \geq 0,$$

is topologically integrable with $\operatorname{Ker}\widehat{\partial}_+^n = \mathcal{A}$ for all $n \geq 1$.

Proof. For every $n \geq 1$, let $\widehat{\mathcal{B}}_n = \mathcal{A}\{X_0,\ldots,X_{n-1}\}$ be the separated completion of

$$\mathcal{A}[X_0,\ldots,X_{n-1}] = \mathcal{A}\widehat{\otimes}_{\mathbb{Z}}\mathbb{Z}[X_0,\ldots,X_{n-1}]$$

with respect to the topology generated by the ideals $\mathfrak{a}_i\mathcal{A}[X_0,\ldots,X_{n-1}]$, $i\in\mathbb{N}$. The topological rings $\widehat{\mathcal{B}}_0 = \mathcal{A}$ and $\widehat{\mathcal{B}}_n$, $n\geq 1$, form an inverse system of complete topological $\mathcal{A}$-algebras for the collection of continuous and surjective transition homomorphisms $p_{m,n}\colon \widehat{\mathcal{B}}_m \to \widehat{\mathcal{B}}_n$ with kernels $(X_n,\ldots,X_{m-1})\widehat{\mathcal{B}}_m$, $m\geq n\geq 0$, whose limit is canonically isomorphic to $\mathcal{B}$. We denote by

$$p_n\colon \mathcal{B} = \varprojlim_{n\in\mathbb{N}} \widehat{\mathcal{B}}_n \to \widehat{\mathcal{B}}_n, \quad n\in\mathbb{N}$$

the canonical continuous surjective homomorphisms. For every $n\geq 1$, the $\mathcal{A}$-derivation $\widehat{\delta}_n\colon \widehat{\mathcal{B}}_n \to \widehat{\mathcal{B}}_{n+1}$ induced by the $\mathcal{A}$-derivation

$$\delta_n = \sum_{i=0}^{n-1}(i+1)X_{i+1}\frac{\partial}{\partial X_i}\colon \mathcal{A}[X_0,\ldots,X_{n-1}] \to \mathcal{A}[X_0,\ldots,X_n]$$

is continuous and the composition of $\widehat{\delta}_n$ with the projection $p_{n+1,n}\colon \widehat{\mathcal{B}}_{n+1} \to \widehat{\mathcal{B}}_n$ is a topologically integrable $\mathcal{A}$-derivation $\widehat{\partial}_{+,n}$ of $\widehat{\mathcal{B}}_n$. Since by construction $p_n\circ\widehat{\partial}_+ = \widehat{\partial}_{+,n}\circ p_n$ for every $n\in\mathbb{N}$, it follows from Proposition 2.29 that $\widehat{\partial}_+$ is topologically integrable.

Using the canonical isomorphisms $\mathcal{A}\{X_0,\ldots,X_{n-1}\} \cong \mathcal{A}\{X_0,\ldots,X_{n-2}\}\{X_{n-1}\}$ of Lemma 1.28, it is straightforward to check by induction on n that for every $n\geq 1$, the kernel of the $\mathcal{A}$-derivation $\widehat{\delta}_n$ is equal to $\mathcal{A}$. Since $p_{n+1}\circ\partial_+ = \widehat{\delta}_n\circ p_n$ for every $n\in\mathbb{N}$, this implies that $\operatorname{Ker}\widehat{\partial}_+ = \mathcal{A}$. Finally, if $\operatorname{Ker}\widehat{\partial}_+^n \setminus \operatorname{Ker}\widehat{\partial}_+ \neq \emptyset$ for some $n\geq 2$, then there would exist an element $s\in\operatorname{Ker}\widehat{\partial}_+^2 \setminus \operatorname{Ker}\widehat{\partial}_+$. Letting $s_1 = \partial_+(s)$, it would follows from Proposition 2.30 that $\widehat{\mathcal{B}_{s_1}} \cong \widehat{\mathcal{A}_{s_1}}\{S\}$, which is impossible. Thus $\operatorname{Ker}\widehat{\partial}_+^n = \mathcal{A}$ for every $n\geq 1$, which completes the proof. □

3. Geometric interpretation: additive group actions on affine ind-schemes

In this section, we recall a construction due to Kambayashi which associates to every complete topological ring $\mathcal{A}$ a locally topologically ringed space $(\operatorname{Spf}(\mathcal{A}), \mathcal{O}_{\operatorname{Spf}(\mathcal{A})})$ called the affine ind-scheme of $\mathcal{A}$. We then establish that restricted exponential homomorphisms correspond through this construction to actions of the additive group ind-scheme on affine ind-schemes.

As in the previous sections, we use the term topological ring to refer to a linearly topologized ring $\mathcal{A}$ which admits a fundamental system of open neighborhoods of 0 consisting of a countable family $(\mathfrak{a}_n)_{n\in\mathbb{N}}$ of ideals of $\mathcal{A}$.

3.1. **Recollection on affine ind-schemes.** We review the basic steps of the construction of the affine ind-scheme associated to a complete topological ring $\mathcal{A}$ following Kambayashi [12, 13].

Definition 3.1. A *topologically ringed space* is a ringed space $(\mathfrak{X}, \mathcal{O}_\mathfrak{X})$ such that for every open subset V of $\mathfrak{X}$, $\mathcal{O}_\mathfrak{X}(V)$ is a topological ring and such that for every pair of open subsets $V' \subseteq V$ of $\mathfrak{X}$, the restriction homomorphism $\mathcal{O}_\mathfrak{X}(V) \to \mathcal{O}_\mathfrak{X}(V')$ is a continuous homomorphism of topological rings.

A *morphism of topologically ringed spaces* from $(\mathfrak{X}, \mathcal{O}_\mathfrak{X})$ to $(\mathfrak{Y}, \mathcal{O}_\mathfrak{Y})$ is a pair $(f, f^\sharp)$ consisting of a continuous map $f\colon \mathfrak{X} \to \mathfrak{Y}$ and a homomorphism $f^\sharp\colon \mathcal{O}_\mathfrak{Y} \to f_*\mathcal{O}_\mathfrak{X}$ of sheaves of rings on $\mathfrak{Y}$ such that for every open subset V of $\mathfrak{Y}$ the homomorphism $f^\sharp(V)\colon \mathcal{O}_\mathfrak{Y}(V) \to f_*\mathcal{O}_\mathfrak{X}(V) = \mathcal{O}_\mathfrak{X}(f^{-1}(V))$ is a continuous homomorphism of topological rings.

Let $\mathcal{A}$ be a complete topological ring with a fundamental system $(\mathfrak{a}_n)_{n\in\mathbb{N}}$ of open neighborhoods of 0. For every $m \geq n \geq 0$, let $\pi_n\colon \mathcal{A} \to A_n = \mathcal{A}/\mathfrak{a}_n$ and $\pi_{m,n}\colon A_m \to A_n$ be the quotient morphisms and let $(j_n, j_n^\sharp)\colon X_n = (\mathrm{Spec}(A_n), \mathcal{O}_{\mathrm{Spec}(A_n)}) \to X = (\mathrm{Spec}(\mathcal{A}), \mathcal{O}_{\mathrm{Spec}(\mathcal{A})})$ and

$$(j_{m,n}, j_{m,n}^\sharp)\colon X_n = (\mathrm{Spec}(A_n), \mathcal{O}_{\mathrm{Spec}(A_n)}) \to X_m = (\mathrm{Spec}(A_m), \mathcal{O}_{\mathrm{Spec}(A_m)})$$

be the corresponding closed immersion of schemes.

We let $\mathrm{Spf}(\mathcal{A})$ be the subset of $\mathrm{Spec}(\mathcal{A})$ consisting of open prime ideals of $\mathcal{A}$, endowed with the subspace topology induced by the usual Zariski topology on $\mathrm{Spec}(\mathcal{A})$. For every $n \geq 0$, the structure sheaf of X_n induces a sheaf of rings $\mathcal{O}_{\mathrm{Spf}(\mathcal{A}),n} = (j_{n,*}\mathcal{O}_{\mathrm{Spec}(A_n)})|_{\mathrm{Spf}(\mathcal{A})}$ on $\mathrm{Spf}(\mathcal{A})$. Since $j_n = j_m \circ j_{m,n}$ for every $m \geq n$, the collection of homomorphisms

$$j_{m,n}^\sharp\colon \mathcal{O}_{\mathrm{Spec}(A_m)} \to (j_{m,n})_*\mathcal{O}_{\mathrm{Spec}(A_n)}$$

induces an inverse system of homomorphisms of sheaves of rings $\varphi_{m,n}\colon \mathcal{O}_{\mathrm{Spf}(\mathcal{A}),m} \to \mathcal{O}_{\mathrm{Spf}(\mathcal{A}),n}$. Considering each of the sheaves $\mathcal{O}_{\mathrm{Spf}(\mathcal{A}),n}$ as sheaves of discrete topological rings on $\mathrm{Spf}(\mathcal{A})$, we let $\mathcal{O}_{\mathrm{Spf}(\mathcal{A})}$ be the limit of this inverse system in the category of sheaves of topological rings. So, for every open subset $V = U \cap \mathrm{Spf}(\mathcal{A})$ of $\mathrm{Spf}(\mathcal{A})$, where U is a Zariski open subset of $\mathrm{Spec}(\mathcal{A})$, we have

$$\mathcal{O}_{\mathrm{Spf}(\mathcal{A})}(V) = (\varprojlim_{n\in\mathbb{N}} \mathcal{O}_{\mathrm{Spf}(\mathcal{A}),n})(V) = \varprojlim_{n\in\mathbb{N}} \mathcal{O}_{\mathrm{Spec}(A_n)}(j_n^{-1}(U))$$

endowed with the initial topology.

Definition 3.2. The *affine ind-scheme of a complete topological ring* $\mathcal{A}$ is the topologically ringed space $(\mathrm{Spf}(\mathcal{A}), \mathcal{O}_{\mathrm{Spf}(\mathcal{A})})$.

Let $\mathcal{A}$ and $\mathcal{B}$ be complete topological rings with respective fundamental systems $(\mathfrak{a}_n)_{n\in\mathbb{N}}$ and $(\mathfrak{b}_n)_{n\in\mathbb{N}}$ of open neighborhoods of 0, and let $\varphi\colon \mathcal{A} \to \mathcal{B}$ be a homomorphism of topological rings. The continuity of φ implies that the image of the restriction to $\mathrm{Spf}(\mathcal{B}) \subseteq \mathrm{Spec}(B)$ of the continuous map $\mathrm{Spec}(\mathcal{B}) \to \mathrm{Spec}(\mathcal{A})$ is contained in the subset $\mathrm{Spf}(\mathcal{A}) \subseteq \mathrm{Spec}(\mathcal{A})$. Furthermore, for every ideal $\mathfrak{b}_n \subseteq \mathcal{B}$, there exists an index $n' = n'(n)$ such that $\mathfrak{a}_{n'} \subseteq \varphi^{-1}(\mathfrak{b}_n)$. It follows that φ induces a morphism of schemes $\alpha_{n,n'}\colon \mathrm{Spec}(\mathcal{B}/\mathfrak{b}_n) \to \mathrm{Spec}(\mathcal{A}/\mathfrak{a}_{n'})$ and that, letting $\alpha\colon \mathrm{Spf}(\mathcal{B}) \to \mathrm{Spf}(\mathcal{A})$ be the continuous map induced by φ, the following diagram of continuous maps of topological spaces commutes

$$\begin{array}{ccc} \mathrm{Spf}(\mathcal{B}) & \xrightarrow{\alpha} & \mathrm{Spf}(\mathcal{A}) \\ {\scriptstyle j_{\mathcal{B},n}}\uparrow & & \uparrow{\scriptstyle j_{\mathcal{A},n'}} \\ \mathrm{Spec}(\mathcal{B}/\mathfrak{b}_n) & \xrightarrow{\alpha_{n,n'}} & \mathrm{Spec}(\mathcal{A}/\mathfrak{a}'_n). \end{array}$$

Pushing forward the homomorphism $\alpha_{n,n'}^\sharp\colon \mathcal{O}_{\mathrm{Spec}(\mathcal{A}/\mathfrak{a}_{n'})} \to (\alpha_{n,n'})_*\mathcal{O}_{\mathrm{Spec}(\mathcal{B}/\mathfrak{b}_n)}$ by $j_{\mathcal{A},n'}$ gives a homomorphism of sheaves of rings

$$\mathcal{O}_{\mathrm{Spf}(\mathcal{A}),n'} = (j_{\mathcal{A},n'})_*\mathcal{O}_{\mathrm{Spec}(\mathcal{A}/\mathfrak{a}_{n'})} \to \alpha_*(j_{\mathcal{B},n})_*\mathcal{O}_{\mathrm{Spec}(\mathcal{B}/\mathfrak{b}_n)} = \alpha_*\mathcal{O}_{\mathrm{Spf}(\mathcal{B}),n}$$

on $\mathrm{Spf}(\mathcal{A})$. The homomorphism of sheaves of topological rings

$$\mathcal{O}_{\mathrm{Spf}(\mathcal{A})} = \varprojlim_{n'\in\mathbb{N}} \mathcal{O}_{\mathrm{Spf}(\mathcal{A}),n'} \to \mathcal{O}_{\mathrm{Spf}(\mathcal{A}),n'} \to \alpha_*\mathcal{O}_{\mathrm{Spf}(\mathcal{B}),n}$$

is continuous and independent on the choice of an index n' such that $\mathfrak{a}_{n'} \subseteq \varphi^{-1}(\mathfrak{b}_n)$. This yields in turn a canonical continuous homomorphism of sheaves of topological rings

$$\alpha^\sharp \colon \mathcal{O}_{\mathrm{Spf}(\mathcal{A})} \to \varprojlim_{n\in\mathbb{N}} \alpha_* \mathcal{O}_{\mathrm{Spf}(\mathcal{B}),n} = \alpha_* \mathcal{O}_{\mathrm{Spf}(\mathcal{B})}.$$

The morphism topologically ringed spaces

$$(\alpha, \alpha^\sharp) \colon (\mathrm{Spf}(\mathcal{B}), \mathcal{O}_{\mathrm{Spf}(\mathcal{B})}) \to (\mathrm{Spf}(\mathcal{A}), \mathcal{O}_{\mathrm{Spf}(\mathcal{A})})$$

is called the *morphism of affine ind-schemes associated to* φ. We henceforth denote it simply by $\mathrm{Spf}(\varphi)$.

Remark 3.3. Let $\mathcal{A}$ be a complete topological ring with a fundamental system $(\mathfrak{a}_n)_{n\in\mathbb{N}}$ of open neighborhoods of 0. Since a prime ideal $\mathfrak{p} \in \mathrm{Spec}(\mathcal{A})$ is open if and only it is equal to $\pi_n^{-1}(\pi_n(\mathfrak{p}))$ for some $n \geq 0$, the set $\mathrm{Spf}(\mathcal{A}) \subset \mathrm{Spec}(\mathcal{A})$ is equal to the union of the images of the closed immersions $j_n \colon \mathrm{Spec}(A_n) \hookrightarrow \mathrm{Spec}(\mathcal{A})$, $n \geq 0$. Furthermore the induced canonical map $\varinjlim_{n\in\mathbb{N}} \mathrm{Spec}(A_n) \to \mathrm{Spf}(\mathcal{A})$ is bijective and continuous with respect to the final topology on $\varinjlim_{n\in\mathbb{N}} \mathrm{Spec}(A_n)$ and the Zariski topology on $\mathrm{Spf}(\mathcal{A})$. Note however that this canonical map is in general not a homeomorphism, i.e. the final topology on $\varinjlim_{n\in\mathbb{N}} \mathrm{Spec}(A_n)$ is strictly finer than the Zariski topology, see [22].

For every $f \in \mathcal{A}$, we let

$$\mathfrak{D}(f) = \mathrm{Spf}(\mathcal{A}) \cap D(f) = \{\mathfrak{p} \in \mathrm{Spf}(\mathcal{A}) \mid f \notin \mathfrak{p}\}$$

where $D(f)$ is the usual principal open subset $\{\mathfrak{p} \in \mathrm{Spec}(A) \mid f \notin \mathfrak{p}\}$ of $\mathrm{Spec}(\mathcal{A})$. These open subsets $\mathfrak{D}(f)$ form a basis of the Zariski topology on $\mathrm{Spf}(\mathcal{A})$. Let $\widehat{\mathcal{A}_f} = \varprojlim_{n\in\mathbb{N}} (A_n)_{\pi_n(f)}$ be the separated completed localization of $\mathcal{A}$ with respect to the multiplicatively closed $\{f^n\}_{n\in\mathbb{N}}$ (see Definition 1.15 and Notation 1.16). Then the collection of canonical projections

$$\widehat{\mathcal{A}_f} \to (A_n)_{\pi_n(f)} = \mathcal{O}_{\mathrm{Spec}(A_n)}(D(\pi_n(f))) = \mathcal{O}_{\mathrm{Spf}(\mathcal{A}),n}(\mathfrak{D}(f)), \quad n \geq 0$$

induces a canonical isomorphism of topological rings $\widehat{\mathcal{A}_f} \to \varprojlim_{n\in\mathbb{N}} \mathcal{O}_{\mathrm{Spf}(\mathcal{A}),n}(\mathfrak{D}(f)) = \mathcal{O}_{\mathrm{Spf}(\mathcal{A})}(\mathfrak{D}(f))$. Since the canonical homomorphism $\mathcal{A} \to \widehat{\mathcal{A}_1} = \varprojlim_{n\in\mathbb{N}} A_n$ is an isomorphism because $\mathcal{A}$ is complete, we have in particular $\mathcal{A} \cong \mathcal{O}_{\mathrm{Spf}(\mathcal{A})}(\mathrm{Spf}(\mathcal{A}))$ and the restriction homomorphism

$$\mathcal{A} \cong \mathcal{O}_{\mathrm{Spf}(\mathcal{A})}(\mathrm{Spf}(\mathcal{A})) \to \mathcal{O}_{\mathrm{Spf}(\mathcal{A})}(\mathfrak{D}(f)) = \widehat{\mathcal{A}_f}$$

coincides with the separated completed localization homomorphism $\tilde{j}_f \colon \mathcal{A} \to \widehat{\mathcal{A}_f}$. The morphism of affine ind-schemes $\mathrm{Spf}(\tilde{j}_f) = (\alpha, \alpha^\sharp) \colon \mathrm{Spf}(\widehat{\mathcal{A}_f}) \to \mathrm{Spf}(\mathcal{A})$ is an open immersion with image equal to $\mathfrak{D}(f)$.

Given a point $\mathfrak{p} \in \mathrm{Spf}(\mathcal{A})$, the stalk $\mathcal{O}_{\mathrm{Spf}(\mathcal{A}),\mathfrak{p}}$ of $\mathcal{O}_{\mathrm{Spf}(\mathcal{A})}$ at $\mathfrak{p}$ can be described as follows. Let S be the multiplicatively closed subset of $\mathcal{A}$ consisting of elements f such that $\mathfrak{p} \in \mathfrak{D}(f)$. Then S is a directed set under the relation $f \leq g$ if and only if $\mathfrak{D}(g) \subseteq \mathfrak{D}(f)$, equivalently, if and only if the separated completed localization homomorphism $\tilde{j}_g \colon \mathcal{A} \to \widehat{\mathcal{A}_g}$ factors through $\tilde{j}_f \colon \widehat{\mathcal{A}} \to \widehat{\mathcal{A}_f}$. Since by definition of the Zariski topology on $\mathrm{Spf}(\mathcal{A})$, the open subsets $\mathfrak{D}(f)$, $f \in S$, form a cofinal subset of the set of all open neighborhoods of $\mathfrak{p}$ in $\mathrm{Spf}(\mathcal{A})$, we have a canonical isomorphism of rings

$$\mathcal{O}_{\mathrm{Spf}(\mathcal{A}),\mathfrak{p}} = \varinjlim_{U\ni\mathfrak{p}} \mathcal{O}_{\mathrm{Spf}(\mathcal{A})}(U) \cong \varinjlim_{f\in S} \mathcal{O}_{\mathrm{Spf}(\mathcal{A})}(\mathfrak{D}(f)) \cong \varinjlim_{f\in S} \widehat{\mathcal{A}_f}.$$

By [13, Theorem 2.2.3.], the ring $\mathcal{O}_{\mathrm{Spf}(\mathcal{A}),\mathfrak{p}}$ is local and the canonical homomorphism $h \colon \mathcal{O}_{\mathrm{Spf}(\mathcal{A}),\mathfrak{p}} \to \widehat{\mathcal{A}_\mathfrak{p}}$ induced by the collection of compatible canonical homomorphisms $\widehat{\mathcal{A}_f} \to \widehat{\mathcal{A}_\mathfrak{p}}$ has dense image in the local topological ring $\widehat{\mathcal{A}_\mathfrak{p}}$. Moreover, the image in $\mathcal{O}_{\mathrm{Spf}(\mathcal{A}),\mathfrak{p}}$ by the canonical homomorphism $\widehat{\mathcal{A}_f} \to \mathcal{O}_{\mathrm{Spf}(\mathcal{A}),\mathfrak{p}}$ of an element $a_f = (a_{f,n}) \in \widehat{\mathcal{A}_f} = \varprojlim_{n\in\mathbb{N}} (A_n)_{\pi_n(f)}$ belongs to the kernel of h if and

only for every $n \geq 0$ then exists $g = g(n, f) \geq f$ such that $a_{f,n}$ belongs to the kernel of the restriction homomorphism

$$(A_n)_{\pi_n(f)} = \mathcal{O}_{\mathrm{Spec}(A_n)}(D(\pi_n(f))) \to \mathcal{O}_{\mathrm{Spec}(A_n)}(D(\pi_n(g))) = (A_n)_{\pi_n(g)}$$

corresponding to the inclusion of principal open subsets $D(\pi_n(g)) \subset D(\pi_n(f))$ of $\mathrm{Spec}(A_n)$.

It follows in particular that the affine ind-scheme $\mathrm{Spf}(\mathcal{A}) = (\mathrm{Spf}(\mathcal{A}), \mathcal{O}_{\mathrm{Spf}(\mathcal{A})})$ of a complete topological ring $\mathcal{A}$ is a locally topologically ringed space. One can also check that the morphism of topologically ringed spaces $\mathrm{Spf}(\varphi)\colon \mathrm{Spf}(\mathcal{B}) \to \mathrm{Spf}(\mathcal{A})$ associated to a continuous homomorphism of complete topological rings $\varphi\colon \mathcal{A} \to \mathcal{B}$ is a morphism of locally topologically ringed spaces.

Definition 3.4. An *affine ind-scheme* is a locally topologically ringed space $(\mathfrak{X}, \mathcal{O}_\mathfrak{X})$ isomorphic to the affine ind-scheme $(\mathrm{Spf}(\mathcal{A}), \mathcal{O}_{\mathrm{Spf}(\mathcal{A})})$ of a complete topological ring $\mathcal{A}$.

A *morphism between affine ind-schemes* $(\mathfrak{Y}, \mathcal{O}_\mathfrak{Y})$ and $(\mathfrak{X}, \mathcal{O}_\mathfrak{X})$ is a morphism of locally topologically ringed spaces

$$(f, f^\sharp)\colon (\mathfrak{Y}, \mathcal{O}_\mathfrak{Y}) \cong (\mathrm{Spf}(\mathcal{B}), \mathcal{O}_{\mathrm{Spf}(\mathcal{B})}) \to (\mathrm{Spf}(\mathcal{A}), \mathcal{O}_{\mathrm{Spf}(\mathcal{A})}) \cong (\mathfrak{X}, \mathcal{O}_\mathfrak{X})$$

which is induced by a continuous homomorphism of complete topological rings $\varphi\colon \mathcal{A} \to \mathcal{B}$.

Given an affine ind-scheme $\mathfrak{S} = \mathrm{Spf}(\mathcal{A})$, an affine ind-$\mathfrak{S}$-scheme is an affine ind-scheme $\mathfrak{X} \cong \mathrm{Spf}(\mathcal{B})$ with a morphism of affine ind-schemes $f\colon \mathfrak{X} \to \mathfrak{S}$. A morphism between affine ind-$\mathfrak{S}$-schemes $g\colon \mathfrak{Y} \to \mathfrak{S}$ and $f\colon \mathfrak{X} \to \mathfrak{S}$ is a morphism of affine-ind-scheme $h\colon \mathfrak{Y} \to \mathfrak{X}$ such that $g = f \circ h$. The category $(\mathrm{AffInd}_{/\mathfrak{S}})$ of affine ind-$\mathfrak{S}$-schemes is by construction anti-equivalent to the category of complete topological $\mathcal{A}$-algebras $\varphi\colon \mathcal{A} \to \mathcal{B}$. Given two such affine ind-$\mathfrak{S}$-schemes $\mathfrak{X} \cong \mathrm{Spf}(\mathcal{B})$ and $\mathfrak{X}' \cong \mathrm{Spf}(\mathcal{B}')$ corresponding to complete topological $\mathcal{A}$-algebras $\varphi\colon \mathcal{A} \to \mathcal{B}$ and $\varphi'\colon \mathcal{A} \to \mathcal{B}'$ respectively, it follows from the universal property of the completed tensor product (see subsection 1.3.1) that the affine ind-$\mathfrak{S}$-scheme $\mathrm{Spf}(\mathcal{B}\widehat{\otimes}_\mathcal{A}\mathcal{B}')$ together with the projections morphisms $p_1\colon \mathrm{Spf}(\mathcal{B}\widehat{\otimes}_\mathcal{A}\mathcal{B}') \to \mathrm{Spf}(\mathcal{B})$ and $p_2\colon \mathrm{Spf}(\mathcal{B}\widehat{\otimes}_\mathcal{A}\mathcal{B}') \to \mathrm{Spf}(\mathcal{B}')$ induced respectively by the canonical homomorphisms $\sigma_1\colon \mathcal{B} \to \mathcal{B}\widehat{\otimes}_\mathcal{A}\mathcal{B}'$ and $\sigma_2\colon \mathcal{B} \to \mathcal{B}\widehat{\otimes}_\mathcal{A}\mathcal{B}'$ is the fibered product of $\mathfrak{X}$ and $\mathfrak{X}'$ in the category of affine ind-$\mathfrak{S}$-schemes. We denote it by $\mathfrak{X}\widehat{\times}_\mathfrak{S}\mathfrak{X}'$.

Example 3.5. Let R be a finitely generated algebra over a field k and let $X = \mathrm{Spec}(R)$ be the associated affine k-scheme of finite type. Then the functor

$$\widehat{F} = \mathrm{Mor}(X, \mathbb{A}^1_k) : (\mathrm{AffIndSch}/_k)^\circ \to (\mathrm{Sets})$$

which associates to every affine ind-k-scheme $\mathfrak{S}$ the set of morphisms of affine ind-$\mathfrak{S}$-schemes from $X\widehat{\times}_k\mathfrak{S}$ to $\mathbb{A}^1_k\widehat{\times}_k\mathfrak{S}$ is representable.

Proof. Indeed, given an affine ind-k-scheme $\mathfrak{S} = \mathrm{Spf}(\mathcal{A})$, we have

$$\widehat{F}(\mathfrak{S}) = \mathrm{Hom}_\mathfrak{S}(X\widehat{\times}_k\mathfrak{S}, \mathbb{A}_k\widehat{\times}_k\mathfrak{S}) \cong \mathcal{C}\mathrm{Hom}_{\mathcal{A}-\mathrm{alg}}(\mathcal{A}\{T\}, R\widehat{\otimes}_k\mathcal{A}) \cong R\widehat{\otimes}_k\mathcal{A},$$

where the last isomorphism follows from Corollary 1.26. So $\widehat{F}$ coincides via the anti-equivalence between the category of affine ind-k-schemes and the category of complete topological k-algebras to the covariant functor $R\widehat{\otimes}_k -$. By Example 1.14 and its proof, the latter is represented by the complete topological k-algebra $\mathcal{R} = \varprojlim_{n\in\mathbb{N}} \mathrm{Sym}^\cdot(V_n^\vee)$, where $\{V_n\}_{n\in\mathbb{N}}$ is any exhaustion of R by finite dimensional k-vector subspaces. It follows that $\widehat{F}$ is represented by the affine ind-k-scheme $\mathfrak{X} = \mathrm{Spf}(\mathcal{R})$. The universal element $u \in R\widehat{\otimes}_k\mathcal{R}$ defined in the proof of Example 1.14 corresponds in turn to a morphism of affine ind-k-schemes $\upsilon\colon X\widehat{\times}_k\mathfrak{X} \to \mathbb{A}^1_k$.

By construction, the set $\mathfrak{X}(k) = \mathrm{Mor}_k(\mathrm{Spec}(k), \mathfrak{X})$ of k-rational points of $\mathfrak{X}$ is equal to the union of the sets of k-rational points of the schemes $\mathrm{Spec}(\mathrm{Sym}^\cdot(V_n^\vee))$, hence to $\bigcup_{n\in\mathbb{N}} V_n = R$. The map $\upsilon(k)\colon (X\widehat{\times}_k\mathfrak{X})(k) = X(k) \times \mathfrak{X}(k) \to \mathbb{A}^1_k(k) = k$ is the universal evaluation map which associates to pair (x, f) consisting of a k-rational point x of X and a k-rational point $f \in R$ of $\mathfrak{X}$ the element $\upsilon(x, f) = f(x)$ of k. □

3.2. **Additive group ind-scheme actions.** We now give the geometric interpretation of restricted exponential homomorphisms as comorphisms of actions of the additive group ind-scheme on affine ind-schemes.

Let $\mathcal{A}$ be a complete topological ring and let $\mathfrak{S} = \mathrm{Spf}(\mathcal{A})$ be its associated affine ind-scheme. Let $\mathcal{A}\{T\} \cong \mathcal{A}\widehat{\otimes}_{\mathbb{Z}}\mathbb{Z}[T]$ be the ring of restricted power series in one variable over $\mathcal{A}$, let $i_0\colon \mathcal{A} \to \mathcal{A}\{T\}$ be the canonical inclusion homomorphism of $\mathcal{A}$ as the subring of constants restricted power series (see subsection 1.4). Recall subsection 2.1 that $(\mathcal{A}\{T\}, m, \iota, \epsilon)$, where $m\colon \mathcal{A}\{T\} \to \mathcal{A}\{T, T'\}$ is the unique continuous homomorphism of topological $\mathcal{A}$-algebras that maps T to $T + T'$, $\iota\colon \mathcal{A}\{T\} \to \mathcal{A}\{T\}$ is the unique continuous homomorphism of topological $\mathcal{A}$-algebras that maps T to $-T$ and $\epsilon\colon \mathcal{A}\{T\} \to \mathcal{A}$ is the unique continuous homomorphism of topological $\mathcal{A}$-algebras that maps T to $0 \in \mathcal{A}$ is a cocommutative topological Hopf $\mathcal{A}$-algebra.

The affine ind-$\mathfrak{S}$-scheme $\mathrm{Spf}(i_0)\colon \mathrm{Spf}(\mathcal{A}\{T\}) \to \mathfrak{S}$ is then an abelian group object in the category of affine ind-$\mathfrak{S}$-schemes, with respective group law and neutral section given by the morphisms

$$\mathrm{Spf}(m)\colon \mathrm{Spf}(\mathcal{A}\{T\})\widehat{\times}_{\mathfrak{S}}\mathrm{Spf}(\mathcal{A}\{T\}) \cong \mathrm{Spf}(\mathcal{A}\{T, T'\}) \to \mathrm{Spf}(\mathcal{A}\{T\})$$

and $\mathrm{Spf}(\epsilon)\colon \mathfrak{S} = \mathrm{Spf}(\mathcal{A}) \to \mathrm{Spf}(\mathcal{A}\{T\})$, which is isomorphic to the affine ind-$\mathfrak{S}$-group scheme $\mathbb{G}_{a,\mathbb{Z}}\widehat{\times}_{\mathbb{Z}}\mathfrak{S}$, where $\mathbb{G}_{a,\mathbb{Z}} = \mathrm{Spec}(\mathbb{Z}[T])$ is the usual additive group scheme. We henceforth denote this affine ind-$\mathfrak{S}$-group scheme by $\mathbb{G}_{a,\mathfrak{S}}$ and call it the *additive group ind-scheme over* $\mathfrak{S}$.

Given any complete topological $\mathcal{A}$-algebra $\varphi\colon \mathcal{A} \to \mathcal{B}$, it follows from Proposition 1.25 that the map $\mathcal{B} \to \mathrm{Hom}_{\mathcal{B}\text{-alg}}(\mathcal{A}\{T\}, \mathcal{B})$, $b \mapsto \overline{\varphi}_b$, where $\overline{\varphi}_b$ is the unique continuous $\mathcal{A}$-algebra homomorphism $\bar{\varphi}_b\colon \mathcal{A}\{T\} \to \mathcal{B}$ such $\bar{\varphi}_b(T) = b$, is an isomorphism of topological abelian groups. This implies in turn that the affine ind-$\mathfrak{S}$-group scheme $\mathbb{G}_{a,\mathfrak{S}}$ represents the covariant functor

$$\begin{array}{lcl} \Gamma\colon (\mathrm{AffInd}_{/\mathfrak{S}})^{\mathrm{opp}} & \to & (\mathrm{TopAbGrps}) \\ (\mathrm{Spf}(\mathcal{B}), \mathcal{O}_{\mathrm{Spf}(\mathcal{B})}) & \mapsto & \Gamma(\mathrm{Spf}(\mathcal{B}), \mathcal{O}_{\mathrm{Spf}(\mathcal{B})}) = \mathcal{O}_{\mathrm{Spf}(\mathcal{B})}(\mathrm{Spf}(\mathcal{B})) = \mathcal{B} \end{array}$$

from the opposite category of affine ind-$\mathfrak{S}$-schemes to the category of topological abelian groups.

Now let $\varphi\colon \mathcal{A} \to \mathcal{B}$ be a complete topological $\mathcal{A}$-algebra and let $\mathfrak{X} = \mathrm{Spf}(\mathcal{B})$ be the affine ind-$\mathfrak{S}$-scheme of $\mathcal{B}$. Since $\mathcal{B}$ is complete we have a canonical isomorphism $\mathcal{B}\{T\} \cong \mathcal{B}\widehat{\otimes}_{\mathcal{A}}\mathcal{A}\{T\}$. Let $e\colon \mathcal{B} \to \mathcal{B}\{T\} \cong \mathcal{B}\widehat{\otimes}_{\mathcal{A}}\mathcal{A}\{T\}$ be a restricted exponential $\mathcal{A}$-homomorphism as in Definition 2.1. Then $\mathrm{Spf}(\mathcal{B}\{T\}) \cong \mathbb{G}_{a,\mathfrak{S}}\widehat{\times}_{\mathfrak{S}}\mathfrak{X}$ and the morphism of ind-$\mathfrak{S}$-schemes $\mathrm{Spf}(e)\colon \mathbb{G}_{a,\mathfrak{S}}\widehat{\times}_{\mathfrak{S}}\mathfrak{X} \to \mathfrak{X}$ satisfies the axioms of an action of the ind-$\mathfrak{S}$-group scheme $\mathbb{G}_{a,\mathfrak{S}}$ on $\mathfrak{X}$, namely, the commutativity of the following two diagrams

$$\begin{array}{ccc} \mathbb{G}_{a,\mathfrak{S}}\widehat{\times}_{\mathfrak{S}}\mathbb{G}_{a,\mathfrak{S}}\widehat{\times}_{\mathfrak{S}}\mathfrak{X} & \xrightarrow{id\times\mu} & \mathbb{G}_{a,\mathfrak{S}}\widehat{\times}_{\mathfrak{S}}\mathfrak{X} \\ \downarrow{\scriptstyle m\times id} & & \downarrow{\scriptstyle \mu} \\ \mathbb{G}_{a,\mathfrak{S}}\widehat{\times}_{\mathfrak{S}}\mathfrak{X} & \xrightarrow{\mu} & \mathfrak{X} \end{array} \qquad \begin{array}{ccc} \mathfrak{S}\widehat{\times}_{\mathfrak{S}}\mathfrak{X} & \xrightarrow{\mathrm{Spf}(\epsilon)\times id} & \mathbb{G}_{a,\mathfrak{S}}\widehat{\times}_{\mathfrak{S}}\mathfrak{X} \\ & \searrow^{\cong} & \downarrow{\scriptstyle \mu} \\ & & \mathfrak{X} \end{array}$$

Conversely, given an action $\mu\colon \mathbb{G}_{a,\mathfrak{S}}\widehat{\times}_{\mathfrak{S}}\mathfrak{X} \to \mathfrak{X}$ of $\mathbb{G}_{a,\mathfrak{S}}$ on $\mathfrak{X} = \mathrm{Spf}(\mathcal{B})$, the homomorphism of complete topological $\mathcal{A}$-algebras

$$e = \mu^\sharp(\mathrm{Spf}(\mathcal{B}))\colon \mathcal{B} = \mathcal{O}_{\mathrm{Spf}(\mathcal{B})}(\mathrm{Spf}(\mathcal{B})) \to \mu_*\mathcal{O}_{\mathbb{G}_{a,\mathfrak{S}}\widehat{\times}_{\mathfrak{S}}\mathfrak{X}}(\mathrm{Spf}(\mathcal{B})) \cong \mu_*\mathcal{O}_{\mathrm{Spf}(\mathcal{B}\{T\})}(\mathrm{Spf}(\mathcal{B})) \cong \mathcal{B}\{T\}$$

satisfies the axioms of a restricted exponential $\mathcal{A}$-homomorphism. In other words, via the anti-equivalence between the category of affine ind-$\mathfrak{S}$schemes $f\colon \mathfrak{X} \to \mathfrak{S}$ and the category of complete topological $\mathcal{A}$-algebras $\varphi\colon \mathcal{A} \to \mathcal{B}$, restricted exponential $\mathcal{A}$-homomorphisms $e\colon \mathcal{B} \to \mathcal{B}\{T\}$ correspond to $\mathbb{G}_{a,\mathfrak{S}}$-actions on the affine ind-$\mathfrak{S}$-scheme $\mathrm{Spf}(\mathcal{B})$.

Combined with Theorem 2.26, this yields the following extension of the classical correspondence between $\mathbb{G}_{a,S}$-actions on an affine scheme $X = \mathrm{Spec}(B)$ over an affine scheme $S = \mathrm{Spec}(A)$ and locally finite higher iterative A-derivations (A-LFHID) of B:

Theorem 3.6. *Let $\mathcal{A}$ be a complete topological ring and let $\varphi\colon \mathcal{A} \to \mathcal{B}$ be a complete topological $\mathcal{A}$-algebra. Let $\mathfrak{S} = \mathrm{Spf}(\mathcal{A})$ and let $f\colon \mathfrak{X} = \mathrm{Spf}(\mathcal{B}) \to \mathfrak{S}$ be the corresponding affine ind-scheme and affine ind-$\mathfrak{S}$-scheme respectively.*

Then actions $\mathbb{G}_{a,\mathfrak{S}}\widehat{\times}_{\mathfrak{S}}\mathfrak{X} \to \mathfrak{X}$ of the additive group ind-scheme $\mathbb{G}_{a,\mathfrak{S}}$ on $\mathfrak{X}$ are in one-to-one correspondence with topologically integrable iterated higher $\mathcal{A}$-derivations $D = \{D^{(i)}\}_{i\geq 0}$ of $\mathcal{B}$.

We now consider examples of affine ind-schemes with actions of the additive group ind-scheme. A first natural example is given by the affine ind-scheme $\mathrm{Mor}(X, \mathbb{A}^1_k)$ associated to an affine k-scheme of finite type X endowed with a non-trivial $\mathbb{G}_{a,k}$-action.

Example 3.7. Let $X = \mathrm{Spec}(R)$ be an affine scheme of finite type over a field k of characteristic zero endowed with a non-trivial $\mathbb{G}_{a,k}$-action $\mu : \mathbb{G}_{a,k} \times_k X \to X$. Let $\mathfrak{X} = \mathrm{Mor}(X, \mathbb{A}^1_k)$ be the ind-scheme of Example 3.5 and let $\widehat{\mu} : \mathbb{G}_{a,k}\widehat{\times}_k\mathfrak{X} \to \mathfrak{X}$ be the morphism of functors given by the composition at the source by the k-automorphisms of X associated to the $\mathbb{G}_{a,k}$-action μ. Then $\widehat{\mu}$ is a morphism of affine ind-schemes which defines a $\mathbb{G}_{a,k}$-action on $\mathfrak{X}$.

Proof. The assertion is an immediate consequence of Yoneda embedding lemma. Nevertheless, let us give a constructive argument following the lines of that of Example 3.5. Let δ be the non-zero locally nilpotent k-derivation corresponding the $\mathbb{G}_{a,k}$-action μ. Then R admits an exhaustion by a countable family $\mathcal{W} = \{W_n\}_{n\in\mathbb{N}}$ of finite dimensional δ-stable k-vector subspaces. Indeed, given any exhaustion of R by a countable family $\mathcal{V} = \{V_n\}_{n\in\mathbb{N}}$ of finite dimensional k-vector subspaces, the fact that δ is locally nilpotent implies that for every $n \in \mathbb{N}$, the k-vector subspace W_n generated by the elements $\delta^m(f)$, $m \geq 0$, where the elements f run through a k-basis of V_n is finite dimensional and δ-stable. Furthermore, since $V_n \subseteq V_m$ for every $m \geq n$ and $V_n \subseteq W_n$, we have $V_n \subseteq W_n \subseteq W_m$ so that the W_m form an increasing exhaustion of R by δ-stable finite dimensional k-vector subspaces.

Let $\mathcal{W} = \{W_m\}$ be a δ-stable exhaustion of R as above. Then, for every $m \in \mathbb{N}$, the restriction of δ to W_m is a nilpotent linear endomorphism δ_m of W_m. The dual endomorphism $\delta_m^\vee$ of $W_m^\vee$ defines a unique k-derivation ∂_m of the symmetric algebra $\mathrm{Sym}^\cdot(W_m^\vee)$ of $W_m^\vee$, which is locally nilpotent. The collection of so-defined locally nilpotent k-derivations ∂_m of the k-algebras $\mathrm{Sym}^\cdot(W_m^\vee)$ form an inverse system with respect to the surjective projection homomorphisms $p_{m,n}\colon \mathrm{Sym}^\cdot(W_m^\vee) \to \mathrm{Sym}^\cdot(W_n^\vee)$ associated to the inclusions $W_n \subseteq W_m$, $m \geq n$. By Proposition 2.30, there exists a unique topologically integrable k-derivation ∂ of the topological k-algebra $\mathcal{R} = \varprojlim_{n\in\mathbb{N}} \mathrm{Sym}^\cdot(W_n^\vee)$ such that for every $n \in \mathbb{N}$, we have $\partial_n \circ p_n = p_n \circ \partial$, where $p_n\colon \mathcal{R} \to \mathrm{Sym}^\cdot(W_n^\vee)$ is the canonical continuous projection.

By Example 3.5, the affine ind-scheme $\mathfrak{X} = \mathrm{Spf}(\mathcal{R})$ of $\mathcal{R}$ represents the functor $\mathrm{Mor}(X, \mathbb{A}^1_k)$. The $\mathbb{G}_{a,k}$-action $\widehat{\mu} : \mathbb{G}_{a,k}\widehat{\times}_k\mathfrak{X} \to \mathfrak{X}$ is then that associated to the topologically integrable k-derivations ∂ of $\mathcal{R}$. Note that the k-derivation $\Delta = -\delta\widehat{\otimes}_k\mathrm{id}_{\mathcal{R}} + \mathrm{id}_R\widehat{\otimes}_k\partial$ of $R\widehat{\otimes}_k\mathcal{R}$ is also topologically integrable. It defines a $\mathbb{G}_{a,k}$-action on the affine ind-k-scheme $X\widehat{\times}_k\mathfrak{X} = \mathrm{Spf}(R\widehat{\otimes}_k\mathcal{R})$ for which, by construction, the universal evaluation morphism $\upsilon\colon X\widehat{\times}_k\mathfrak{X} \to \mathbb{A}^1_k$ of Example 3.5 is $\mathbb{G}_{a,k}$-invariant. □

Remark 3.8. With the notation of Examples 3.5 and 3.7, the restriction to the set $\mathfrak{X}(k) = R$ of k-rational points of $\mathfrak{X}$ of the $\mathbb{G}_{a,k}$-action $\widehat{\mu} : \mathbb{G}_{a,k}\widehat{\times}_k\mathfrak{X} \to \mathfrak{X}$ coincides with the contragredient representation of $(k, +)$ on R defined for every f in R by

$$t \cdot f = \exp((-t)\delta)(f) = \sum_{n\geq 0}(-1)^n\frac{t^n}{n!}\delta^n(f).$$

In particular, for every k-rational point x of X, we have $(t \cdot f)(x) = f((-t) \cdot x)$ and hence

$$\upsilon(t \cdot (x, f)) = (t \cdot f)(t \cdot x) = f(x) = \upsilon(x, f).$$

Example 3.9. As a concrete illustration of Example 3.7, consider the locally nilpotent k-derivation $\delta = \partial/\partial x$ of $R = k[x]$ corresponding to the action of $\mathbb{G}_{a,k}$ on $\mathbb{A}^1_k$ by translations and the exhaustion of R by the δ-stable subspaces

$$W_n = k[x]_{\leq n} = k\langle x^0, \ldots, x^n\rangle, \quad n \in \mathbb{N},$$

consisting of polynomials of degree less than or equal to n. For every $n \in \mathbb{N}$, the algebra $\mathrm{Sym}^{\cdot}(W_n^\vee)$ is isomorphic to the polynomial ring $k[X_0, \ldots, X_n]$, where $(X_0, \ldots, X_n, \ldots)$ is the family of elements of the dual $R^\vee$ of R as a k-vector space defined by $X_i(x_j) = \delta_{i,j}$ for every $i, j \in \mathbb{N}$. The complete topological k-algebra $\mathcal{R} = \varprojlim_{n\in\mathbb{N}} \mathrm{Sym}^{\cdot}(W_n^\vee)$ is isomorphic to the separated completion of the polynomial ring $k[(X_i)_{i\in\mathbb{N}}]$ with respect to the topology induced by the fundamental system of open ideals $\mathfrak{a}_n = (X_i)_{i\geq n} k[(X_i)_{i\in\mathbb{N}}]$. The universal element $u \in R\widehat{\otimes}_k\mathcal{R}$ of the proof of Example 1.14 can be represented by the formal power series $\sum_{n\in\mathbb{N}} x^n X_n \in k[x][[(X_i)_{i\in\mathbb{N}}]]$.

On the other hand, the k-derivation ∂_n of $\mathrm{Sym}^{\cdot}(W_n^\vee)$ is given by $\partial_n(X_i) = (i+1)X_{i+1}$, if $i \leq n-1$ and $\partial_n(X_n) = 0$. The corresponding topologically integrable k-derivation $\partial = \varprojlim_{n\in\mathbb{N}} \partial_n$ of $\mathcal{R}$ induced by the inverse system of locally nilpotent k-derivations ∂_n, $n \in \mathbb{N}$, coincides with the topologically integrable k-derivation with trivial kernel k of Example 2.31. Note that in contrast, for the topologically integrable k-derivation $\Delta = -\delta\widehat{\otimes}_k \mathrm{id}_{\mathcal{A}} + \mathrm{id}_R\widehat{\otimes}_k\partial$ of $R\hat{\otimes}_k\mathcal{R}$, we have

$$\Delta(u) = \sum_{n\in\mathbb{N}} \Delta(x^n X_n) = \sum_{n\in\mathbb{N}} (-nx^{n-1}X_n + (n+1)x^n X_{n+1}) = 0.$$

For an affine scheme X of finite over a field k, there are many natural affine ind-k-schemes that can be constructed from the ind-k-scheme $\mathfrak{X} = \mathrm{Mor}(X, \mathbb{A}^1_k)$ (see e.g. [9]). These include for instance the ind-k-scheme $\mathrm{Mor}_k(X, Y)$ where Y is any affine k-scheme and the ind-k-scheme $\mathrm{Aut}_k(X)$ of k-automorphisms of X. Since every non-trivial $\mathbb{G}_{a,k}$-action on X gives rise to a non-trivial $\mathbb{G}_{a,k}$-action on these affine ind-k-schemes, this provides a large supply of natural affine ind-k-schemes with interesting natural $\mathbb{G}_{a,k}$-actions.

Another family of examples is given by the following ind-scheme counterpart of Danielewski hypersurfaces.

Example 3.10. Let $\mathcal{A}$ be an integral complete topological algebra over a field k of characteristic zero. Let $\mathcal{A}\{y, z\}$ be the restricted power series ring in two variables over $\mathcal{A}$. Let $x \in \mathcal{A}$ be a non-zero element and let $P(y) \in \mathcal{A}\{y\}$ be a non-zero restricted power series. Then there exists a unique continuous $\mathcal{A}$-derivation ∂ of $\mathcal{A}\{y, z\}$ such that, $\partial(y) = x$ and $\partial(z) = P'(y)$, where $P'(y) \in \mathcal{A}\{y\}$ denote the derivative of the restricted power series $P(y)$. The so defined $\mathcal{A}$-derivation ∂ is topologically integrable. Indeed, let $(\mathfrak{a}_n)_{n\in\mathbb{N}}$ be a fundamental system of open ideals of $\mathcal{A}$ so that we have by definition $\mathcal{A}\{y, z\} = \varprojlim_{n\in\mathbb{N}} A_n[y, z]$, where $A_n = \mathcal{A}/\mathfrak{a}_n$. Let $x_n \in A_n$ and $P_n(y) \in A_n[y]$ denote the respective residue classes of $x \in \mathcal{A}$ and $P(y) \in \mathcal{A}\{y\}$, $n \geq 0$. Then we have $\partial = \varprojlim_{n\in\mathbb{N}} \partial_n$, where ∂_n is the triangular, hence locally nilpotent, A_n-derivation of $A_n[y, z]$ defined by $\partial_n(y) = x_n$ and $\partial_n(z) = P_n'(y)$. Thus ∂ is topologically integrable by Proposition 2.29. Note that the element $xz - P(y)$ belong to the kernel $\mathrm{Ker}\partial$ of ∂, in particular, in contrast with the example considered in Example 2.31, $\mathcal{A}$ is a proper sub-algebra of $\mathrm{Ker}\partial$.

Now assume in addition that x and $P(y)$ are chosen so that the principal ideal $I = (xz - P(y))$ of $\mathcal{A}\{y, z\}$ is prime and closed. Then $\mathcal{B} = \mathcal{A}\{y, z\}/I$ is a topological $\mathcal{A}$-algebra when endowed with the quotient topology, and the quotient homomorphism $q\colon \mathcal{A}\{y, z\} \to \mathcal{B}$ is continuous. Since $\partial(I) \subset I$, ∂ induces a topologically integrable $\mathcal{A}$-derivation $\overline{\partial}$ of $\mathcal{B}$. Letting $\mathfrak{X} = \mathrm{Spf}(\mathcal{A})$, the homomorphism $\mathcal{A}\{y, z\} \to \mathcal{B}$ corresponds to a closed embedding of affine ind-$\mathfrak{X}$-schemes

$$\mathfrak{Y} = \mathrm{Spf}(\mathcal{B}) \hookrightarrow \mathbb{A}^2_{\mathfrak{X}} = \mathrm{Spf}(\mathcal{A}\{y, z\})$$

which is equivariant for the $\mathbb{G}_{a,\mathfrak{X}}$-actions on $\mathfrak{Y}$ and $\mathbb{A}^2_{\mathfrak{X}}$ associated to $\overline{\partial}$ and ∂ respectively.

For a concrete illustration, consider the completion $\mathcal{A}$ of the polynomial ring $k[(X_i)_{i\in\mathbb{N}}]$ in countably many variables with respect to the topology generated by the ideals $\mathfrak{a}_n = (X_i - 1)_{i>n}$. Let $x \in \mathcal{A}$ be the element represented by the Cauchy sequence $x_n = \prod_{i=0}^n X_i \in k[X_0, \ldots, X_n]$. Choosing for the restricted power series $P(y)$ a non-constant polynomial $P(y) \in k[y] \subset \mathcal{A}\{y\}$, we obtain an associated affine ind-$\mathfrak{X}$-scheme $\mathfrak{Y} \subset \mathbb{A}^2_{\mathfrak{X}}$ which is a colimit of so-called *Danielewski varieties* in $\mathbb{A}^{n+3}_k$ defined by equations of the form $\prod_{i=0}^n X_i z = P(y)$ (see e.g. [7]). One can also choose non-polynomial restricted power-series $P(y) \in \mathcal{A}\{y\}$, for instance, the one $P(y) = \sum_{i=0}^\infty (X_i - 1)^i y^{i+1}$.

REFERENCES

[1] M. Artin, A. Grothendieck and J.-L. Verdier, *Théorie des topos et cohomologie étale des schémas*, *Tome 1: Théorie des topos*, Séminaire de Géométrie Algébrique du Bois-Marie 1963-1964 (SGA 4). Lecture Notes in Mathematics 269, Springer 1972.

[2] A. Crachiola, L. Makar-Limanov, *On the rigidity of small domains*, J. Algebra, 284 (2005), pp. 1-12.

[3] N. Bourbaki, *Elements of mathematics. General topology*, Chapters 1-4. Translated from the French. Hermann, Paris; Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont. 1966

[4] N. Bourbaki, *Elements of mathematics. Commutative algebra*, Chapters 1-7. Translated from the French. Translated from the French. Hermann, Paris; Addison-Wesley Publishing Co., Reading, Mass., 1972.

[5] N. Bourbaki, *Elements of mathematics. Algebra I*, Chapters 1-3. Translated from the French. Hermann, Paris; Addison-Wesley Publishing Co., Reading Mass., 1974.

[6] R. Díaz, and A. Liendo, *On toric ind-variety and pro-affine semigroups*, J. Algebra 569 (2021) 442–465.

[7] A. Dubouloz, *Additive group actions on Danielewski varieties and the Cancellation Problem*, Math. Z. 255 (2007), no. 1, 77-93.

[8] G. Freudenburg, *Algebraic theory of locally nilpotent derivations.* Encyclopaedia of Mathematical Sciences 136, Invariant Theory and Algebraic Transformation Groups 7, Springer, 2017.

[9] J.-P. Furter and H. Kraft *On the geometry of the automorphism groups of affine varieties*, arXiv:1809.04175 [math.AG] (2018).

[10] A. Grothendieck and J. Dieudonné, Jean *Éléments de géométrie algébrique: I. Le langage des schémas*, Grundlehren der Mathematischen Wissenschaften (in French). 166 (2nd ed.). Berlin; New York: Springer-Verlag. ISBN 978-3-540-05113-8.

[11] H. Hasse, F.K. Schmidt,*Noch eine Bergründung der Theorie der höheren Differential quotienten in einemalgebraischen Functionenkörper einer Unbestimmten,*J. Reine Angew. Math. 177 (1937) 215–237

[12] T. Kambayashi, *Pro-affine algebras, ind-affine groups and the Jacobian Problem*, Journal of Algebra, 185 (1996), 481-501.

[13] T. Kambayashi, *Some basic results on pro-afine algebras and ind-afine schemes*, Osaka J. Math. 40 (2003), no. 3, 621-638.

[14] T. Kambayashi, *Inverse limits of polynomial rings,* Osaka J. Math., 41 (2004), 617-624.

[15] T. Kambayashi and M. Miyanishi, *On two recent views of the Jacobian conjecture*, Affine algebraic geometry, 113-138, Contemp. Math., 369, Amer. Math. Soc., Providence, RI, 2005.

[16] S. Kumar, *Kac-Moody groups, their ag varieties and representation theory, Progress in Mathematics*, vol. 204, Birkhäuser Boston Inc., Boston, MA, 2002.

[17] H. Kraft and A. Regeta, *Automorphisms of the Lie algebra of vector fields on affine n-space*, J. Eur. Math. Soc. (JEMS) 19 (2017), no. 5, 1577-1588.

[18] M. Miyanishi,*A remark on an iterative infinite higher derivation*, J. Math. Kyoto Univ. 8 (1968) 411-415.

[19] D. G Northcott, *Lessons on rings, modules and multiplicities*, Cambridge University Press Bentley House, 1968.

[20] I. R. Shafarevich, *On some infinite-dimensional groups*, Rend. Mat. Appl. (5) 25 (1-2) (1966) 208-212.

[21] I. R. Shafarevich, *On some infinite-dimensional groups. II*, Math. USSR Izvestija 18 (1982), 185-194.

[22] I. Stampfli, *On the topologies on ind-varieties and related irreducibility questions*, J. Algebra 372 (2012), 531-541.

INSTITUTO DE MATEMÁTICA Y FÍSICA, UNIVERSIDAD DE TALCA, CASILLA 721, TALCA, CHILE.
Email address: robediaz@utalca.cl

IMB UMR5584, CNRS, UNIV. BOURGOGNE FRANCHE-COMTÉ, EIPHI GRADUATE SCHOOL ANR-17-EURE-0002, F-21000 DIJON, FRANCE.
Email address: adrien.dubouloz@u-bourgogne.fr

INSTITUTO DE MATEMÁTICA Y FÍSICA, UNIVERSIDAD DE TALCA, CASILLA 721, TALCA, CHILE.
Email address: aliendo@inst-mat.utalca.cl