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# Zonal instability and wave trapping

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**Abstract.** This paper presents a model for zonal flow generation based on a wave kinetic equation coupled to a poloidal momentum equation in a regime where wave trapping matters. Several models of the wave collision operator have been tested: Krook, diffusion and diffusion plus an instability growth rate. Conditions for zonal instability have been identified. It is found that a zonal instability is possible in all cases. However the force is a power law of the zonal velocity, so different from the quasi-linear case of random phases that produces a force that is linear in velocity. Also the zonal force may change sign, leading to flow radial profiles that are not sinusoidal.

## 1. Introduction

The generation and sustainment of zonal flows is commonly studied by solving a poloidal momentum equation coupled to a wave kinetic equation. In this description, zonal flows are driven by turbulence via the divergence of the Reynolds stress [1]. The Reynolds stress is related to a moment of the wave action density, itself solution of a wave kinetic equation. This set of coupled equations is often solved by using a random phase approximation, similar to the quasi-linear theory [2]. Coherent non linear solutions of the wave kinetic/momentum equations have been documented in the past [3, 4]. In particular, periodic radial structure of zonal flows have been identified [3]. Solutions have also been found where a coherent zonal mode propagates radially together with drift waves trapped in the zonal structure [5]. These solutions can be solitons, shocks or wave trains. Wave trapping appears to provide a mechanism for turbulence saturation. In this seminal work, the structure of the zonal potential that enters the wave kinetic equation was kind of ad-hoc, thus leaving some freedom in the nature of the solutions. It is shown here that in fact the solutions are strongly constrained by the non linear “collision” operator that enters the wave kinetic equation. This operator brings in the physics of wave-wave interactions, which therefore have an indirect impact on the zonal instability. It appears that most solutions are zonally unstable. However the zonal force, i.e. minus the divergence of the Reynolds stress, is a non linear function of the zonal flow velocity, as anticipated by Kaw and co-workers. The calculation is done here for conventional drift waves in slab geometry, as described in the Hasegawa-Mima model [7]. It can be nevertheless extended to turbulence in tokamaks by using a formalism recently developed by Gillot and co-workers [8]. It is also related

to a methodology developed in [9, 10] for zonal flows and Geodesic Acoustic Modes.

The paper is organised as follows. Section 2 presents the basic model of a zonal momentum equation coupled to a wave kinetic equation. Solutions are calculated in Section 3 for some specific examples of collision operator in the wave kinetic equation. For each case, the conditions for zonal instability are determined. A conclusion follows.

## 2. Wave kinetic model of zonal flows

### 2.1. Basic model

The wave kinetic equation reads for drift waves [6]

$$\frac{\partial N}{\partial t} - \{H, N\} = \mathcal{D}[N] \quad (1)$$

where  $N(\mathbf{x}, \mathbf{k}, t)$  is the drift wave action density,  $(\mathbf{x}, \mathbf{k})$  are the position and wave number of wave packets, and the bracket  $\{, \}$  is the usual Poisson bracket in  $(\mathbf{x}, \mathbf{k})$  coordinates. The wave Hamiltonian reads  $H = \omega_{\mathbf{k}} + k_y V(x, t)$ , where  $\omega_{\mathbf{k}} = k_y / (1 + k_x^2 + k_y^2)$  is the linear frequency of a drift wave,  $k_y$  its poloidal wave number, and  $V(x, t)$  the zonal flow velocity, oriented in the “poloidal” direction  $y$ . Lengths are normalised to the ion “sound” gyroradius  $\rho_s$ , while the time unit is the density gradient length divided by the sound speed. Finally the operator  $\mathcal{D}[N]$  that appears on the right hand side of Eq.(1) describes the physics of wave-wave coupling. It plays the same role as the collision operator in a conventional kinetic equation. Explicit expressions exist, but are intricate and difficult to handle [6]. We resort to approximate forms in the following. The large scale limit (in the “radial” direction  $x$ ) is considered, so that the corrections introduced by [11] to treat the zonal flow “ultraviolet” catastrophe are not included here.

The time evolution of the poloidal velocity of zonal flows is given by a momentum equation

$$\frac{\partial V}{\partial t} = \mathcal{F} - \nu V + \mu \frac{d^2 V}{dx^2} \quad (2)$$

where  $\mathcal{F}(x, t)$  is the force exerted by the drift wave turbulent background, i.e. minus the divergence of the Reynolds stress. The coefficients  $\nu$  and  $\mu$  correspond to damping terms, associated respectively with neoclassical friction forces and flow viscous damping [3]. The force  $\mathcal{F}$  is related to the wave action density via the equation [1]

$$\mathcal{F}(x, t) = \int \frac{d^2 \mathbf{k}}{4\pi^2} \frac{1}{(1 + k_{\perp}^2)^2} k_x k_y \frac{\partial N}{\partial x} \quad (3)$$

The set of equations Eqs.(1,2,3) describes the dynamics of drift waves coupled to zonal flows.

### 2.2. BGK solutions

Following the methodology proposed in [5], we look for Bernstein-Greene-Kruskal (BGK) solutions [12], i.e. solutions of the form  $N(x - ut, \mathbf{k})$  that are stationary in a frame that moves at speed  $u$ . The variable  $x' = x - ut$  is noted  $x$  to simplify the notations. The action density  $N(x, \mathbf{k})$  is a periodic in  $x$ . Both the velocity  $u$  and the period  $L$  are parameters that characterise respectively the speed and spatial periodicity of a zonal flow. Using a Taylor development in  $k_x$  near a reference wave number  $k_{x0}$ , typically the wave number at which the spectrum is maximum, the wave kinetic equation can be written

$$C_k k_y \{w, N\} = \mathcal{D}[N] \quad (4)$$

where the reduced Hamiltonian is  $w(x, \tilde{k}_x) = \tilde{k}_x^2/2 - v(x)$ ,  $C_k = -\partial_{k_x k_x}^2 \omega_{\mathbf{k}}/k_y$  the Hamiltonian curvature,  $v = V/C_k$  the normalised zonal flow velocity, and  $\tilde{k}_x = k_x - \bar{k}_x$  is a shifted wave number

$$\bar{k}_x = k_{x0} - \frac{1}{k_y C_k} \left( u - \frac{\partial \omega_{\mathbf{k}}}{\partial k_x} \Big|_{\mathbf{k}=\mathbf{k}_0} \right) \quad (5)$$

where  $\mathbf{k}_0 = (k_{x0}, k_y)$ . It is assumed that  $(\bar{k}_x, \tilde{k}_x) \ll k_y$ . Lines of constant  $w$  draw an island in the wave phase space  $(x, \tilde{k}_x)$  (see Fig.1). In absence of dissipation  $\mathcal{D}[N] = 0$ , the action density is a function of  $N_0(w, \sigma)$  only,  $\sigma$  being the sign of  $\tilde{k}_x$ . For finite, but weak, dissipation, it is therefore convenient to treat perturbatively the wave action density, i.e.  $N(w, x, \sigma) = N_0(w, \sigma) + \epsilon_D N_1(w, x, \sigma) + \dots$ , where  $\epsilon_D$  is an expansion parameter that depends on the details of the operator  $\mathcal{D}[N]$ . Eq.(4) becomes at first order in  $\epsilon_D$

$$-C_k k_y \tilde{k}_x \frac{\partial N_1}{\partial x} = \mathcal{D}[N_0] \quad (6)$$

The shape of  $N_0$  is provided by a solvability constraint that is derived from Eq.(6) after a division by  $\tilde{k}_x$  and integration in  $x$  over a zonal flow spatial period  $L$

$$\oint \frac{dx}{L} \frac{1}{\tilde{k}_x} \mathcal{D}[N_0] = 0 \quad (7)$$

where the integral is computed along a line at constant energy  $w$ . Solutions can be found, at least for some classes of operator  $\mathcal{D}$ . Note that once  $N_0$  is known, the zonal force  $\mathcal{F}$  can be calculated by combining Eq.(3) and Eq.(6)

$$\mathcal{F}(x) = \int_{-\infty}^{+\infty} \frac{dk_y}{4\pi^2} \frac{\Lambda_k}{C_k} I_k(x) \quad (8)$$

where  $\Lambda_k = 1/(1 + k_y^2)^2$  and  $I_k$  is an ‘‘intensity’’ given by

$$I_k(v) = - \int_{-v}^{+\infty} \frac{dw}{w+v} \left( \bar{k}_x \mathcal{D}_{odd}[N_0] + \sqrt{2(w+v)} \mathcal{D}_{even}[N_0] \right) \quad (9)$$

### 3. Solutions of the wake kinetic equation

As mentioned, an analytical solution of Eq.(6) is beyond reach for the complete non linear wave-wave ‘‘collision’’ operator. We build here approximate solutions for some typical collision operators with increasing complexity: Krook, diffusive without or with drive. Krook operators are widely used in the literature [2, 3], but cannot regularise fast spatial variations of the wave action, in contrast with a diffusion operator. Adding a growth rate accounts for a turbulence drive. In particular it acts as a source of trapped waves inside the phase-space island. In each case, the integral  $I_k$  is computed and the zonal instability characterised.

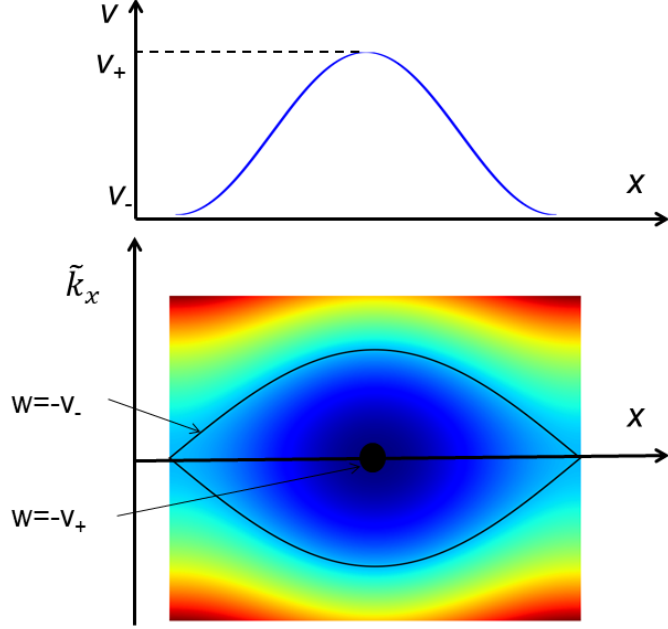
#### 3.1. Krook operator

Let us consider first the Krook operator

$$\mathcal{D}[N] = -\eta_k (N - N_{eq}) \quad (10)$$

The Krook operator Eq.(10) is alike a return friction force to an equilibrium solution that is linearised in  $k_x$  near the wave number  $(\bar{k}_x, k_y)$

$$N_{eq}(\mathbf{k}) = N_{eq}(\bar{\mathbf{k}}) + \frac{\partial N_{eq}}{\partial k_x} \Big|_{\mathbf{k}=\bar{\mathbf{k}}} \tilde{k}_x \quad (11)$$



**Figure 1.** Island shape for a sinusoidal zonal velocity  $v(x) = \cos(x)$ . The flow radial profile  $v(x)$  is shown on the top: its minima and maxima are  $v_-$  and  $v_+$ . The lower panel shows the contour lines of the effective Hamiltonian  $w(x, \tilde{k}_x) = \tilde{k}_x^2/2 - v(x)$ . The island separatrix is the curve  $w = -v_-$  and the island O point is prescribed by the condition  $w = -v_+$ .

The expansion parameter is  $\epsilon_D = \eta_k / (C_k k_y \delta_{isl})$ , where  $\delta_{isl}$  is the island width. In the trapped domain, the solvability constraint Eq.(7) on  $N_0$  reads

$$\oint \frac{dx}{L} \frac{\mathcal{D}[N_0]}{\tilde{k}_x} = 2 \int_{-x_0(w)}^{x_0(w)} \frac{dx}{L} \frac{1}{\sqrt{2(w+v(x))}} (\mathcal{D}[N_0](w, x, \sigma = +1) + \mathcal{D}[N_0](w, x, \sigma = -1)) = 0 \quad (12)$$

The function  $N_0$  is an even function of  $\sigma$  since both branches  $\sigma = \pm 1$  are connected within an island. The unperturbed solution  $N_{eq}$  Eq.(11) is an odd function of  $\tilde{k}_x$ , and therefore cancels out when averaging over both branches. As a result  $N_0(w) = 0$  for  $-v_+ < w < -v_-$ . In the passing domain, the solvability condition must be treated separately for  $\sigma = +1$  and  $\sigma = -1$ . One gets

$$N_0(w, \sigma) = N_{eq}(\bar{\mathbf{k}}) + \left. \frac{\partial N_{eq}}{\partial k_x} \right|_{\mathbf{k}=\bar{\mathbf{k}}} \sigma Q(w) \Theta(w + v_-) \quad (13)$$

where

$$Q(w) = \oint \frac{dx}{L} \sqrt{2(w+v)} \quad (14)$$

This solution does not match the inner solution  $N_0(w) = 0$  at  $w + v_- = 0$ . This is unavoidable since no regularising operator is present in this model, that would smooth out discontinuities. A proxy for  $N_0$  that is continuous at the separatrix and matches asymptotically the unperturbed wave action at  $w \rightarrow \infty$ , inspired from previous works on magnetic islands in tokamaks [13, 14],

is

$$N_0(w, \sigma) \simeq N_{eq}(\bar{\mathbf{k}}) + \left. \frac{\partial N_{eq}}{\partial k_x} \right|_{\mathbf{k}=\bar{\mathbf{k}}} \sigma \sqrt{2(w+v_-)} \Theta(w+v_-) \quad (15)$$

where  $\Theta$  is a Heaviside function. Referring to Eq.(9), it is important to identify the odd and even parts in  $\sigma$  (the sign of  $\tilde{k}_x$ ) of  $\mathcal{D}[N_0]$ . Moving to the variables  $(w, v(x), \sigma)$ , the Krook dissipation operator reads

$$\begin{aligned} \mathcal{D}[N_0] &= -\eta_k (N_0 - N_{eq}) \\ &\simeq \eta_k \left. \frac{\partial N_{eq}}{\partial k_x} \right|_{\mathbf{k}=\bar{\mathbf{k}}} \sigma \left( \sqrt{2(w+v)} \Theta(w+v) - \sqrt{2(w+v_-)} \Theta(w+v_-) \right) \end{aligned} \quad (16)$$

It is therefore an odd function of  $\sigma$ . The force is then equal to (see details in Appendix A.1)

$$\mathcal{F} = \sqrt{2}\pi \int_{-\infty}^{+\infty} \frac{dk_y}{4\pi^2} \frac{\Lambda_k}{C_k^{3/2}} \eta_k \left( -\bar{k}_x \left. \frac{\partial N_{eq}}{\partial k_x} \right|_{\mathbf{k}=\bar{\mathbf{k}}} \right) (V - V_-)^{1/2} \quad (17)$$

There is zonal flow instability when the force is positive, i.e. when

$$-\bar{k}_x \left. \frac{\partial N_{eq}}{\partial k_x} \right|_{\mathbf{k}=\bar{\mathbf{k}}} > 0 \quad (18)$$

This condition is similar to the result found when using a quasilinear approach [2]. Note however that the force is not a linear function of the zonal flow velocity, rather its square root.

### 3.2. Diffusion operator

Here we consider the diffusion collision operator

$$\mathcal{D}[N] = \frac{\partial}{\partial k_x} \left( D_k \frac{\partial N}{\partial k_x} \right) \quad (19)$$

In this operator  $k_x$  can be changed safely in  $\tilde{k}_x$  since they differ by a constant only. Also the unperturbed solution Eq.(11) satisfies  $\mathcal{D}[N_{eq}] = 0$  so that  $N$  can be replaced by  $N - N_{eq}$ . The expansion parameter is  $\epsilon_D = D_k / (C_k k_y \delta_{isl}^3)$ , where  $\delta_{isl}$  is the island width. The solvability constraint Eq.(7) leads to the solution

$$\frac{\partial N_0}{\partial w} = \frac{\Gamma_0}{Q(w)} \quad (20)$$

where the constant  $\Gamma_0$  is different in the trapped and passing domains. It must vanish in the trapped domain since  $Q(w = -v_+) = 0$ . Hence a non zero value of  $\Gamma_0$  would lead to a singular solution at the O point. The constant  $\Gamma_0$  is determined in the passing domain by using the boundary condition

$$\frac{\partial N}{\partial \tilde{k}_x} \rightarrow \left. \frac{\partial N_{eq}}{\partial k_x} \right|_{\mathbf{k}=\bar{\mathbf{k}}} \quad \text{when } \tilde{k}_x \rightarrow \pm\infty \quad (21)$$

It turns out that  $\Gamma_0 = \left. \frac{\partial N_{eq}}{\partial k_x} \right|_{\mathbf{k}=\bar{\mathbf{k}}}$  is just the unperturbed wave action density gradient. Hence the solution

$$\frac{\partial N_0}{\partial w} = \sigma \left. \frac{\partial N_{eq}}{\partial k_x} \right|_{\mathbf{k}=\bar{\mathbf{k}}} \frac{\Theta(w+v_-)}{Q(w)} \quad (22)$$

The operator Eq.(19) applied to  $N_0$  reads

$$\mathcal{D}[N_0] = \tilde{k}_x \frac{\partial}{\partial w} \left( D_k \tilde{k}_x \frac{\partial N_0}{\partial w} \right) \quad (23)$$

where  $\partial_w N_0$  is given by Eq.(22). It appears that  $\mathcal{D}[N_0]$  is an odd function of  $\sigma$ , which greatly simplifies the integral  $I_k$  given by Eq.(9). Also the contribution from trapped waves vanishes since the action density is flat inside the island. The solution Eq.(22) leads to a divergence of the integral  $I_k$ , and therefore of the zonal force  $\mathcal{F}$ . This is due to a singularity in  $\mathcal{D}[N_0]$  near the separatrix  $w + v_- = 0$ . The latter is caused by the discontinuity of the first derivative of  $N_0$ . This means that the initial kinetic equation should be solved non-perturbatively, a hard task in the general case. Some simplification occurs when adding a regularising term in  $x$ , i.e. a diffusion in space - that would correspond physically to turbulence diffusive spreading. That problem was solved in the past [15] to compute the temperature field near a magnetic island in presence of heat diffusion. Still the solution remains hardly tractable for the problem at hand. Hence we have to resort to some appropriate smoothing of the solution Eq.(20) near the separatrix. This regularisation comes from the diffusive boundary layer that built up in this region. The first step in this process is to notice that the wave action density  $N$  can be replaced by  $N - N_{eq}$  in the operator  $\mathcal{D}[N]$  since  $N = N_{eq}$  is a linear function in  $\tilde{k}_x$  and is therefore in the kernel of the diffusion operator, i.e.  $\mathcal{D}[N_{eq}] = 0$ . This grants the convergence of the integrals when  $\tilde{k}_x \rightarrow +\infty$ . The second step consists in casting the function  $N_0 - N_{eq}$  in an equivalent form

$$N_0 - N_{eq} = N_0(w) - N_{eq}(\bar{\mathbf{k}}) - \left. \frac{\partial N_{eq}}{\partial k_x} \right|_{\mathbf{k}=\bar{\mathbf{k}}} \tilde{k}_x = \left. \frac{\partial N_{eq}}{\partial k_x} \right|_{\mathbf{k}=\bar{\mathbf{k}}} \mathcal{G}(w, v) \quad (24)$$

where  $\sigma$  has been set to +1. As mentioned before, the dependence of the function  $\mathcal{G}(w, v)$  on  $x$  is in fact a dependence on  $v(x)$  because the radial wave number is itself a function of  $x$  via  $v(x)$ , i.e.  $\tilde{k}_x = \sqrt{2(w + v)}$ . The integral  $I_k$  given by Eq.(9) then reads

$$I_k(v) = -2\bar{k}_x \left. \frac{\partial N_{eq}}{\partial k_x} \right|_{\mathbf{k}=\bar{\mathbf{k}}} \int_{-v_-}^{+\infty} \frac{dw}{\sqrt{w + v}} \frac{\partial}{\partial w} \left( \sqrt{w + v} \frac{\partial \mathcal{G}}{\partial w} \right) \quad (25)$$

The function  $\mathcal{G}(w, v)$  converges to 0 when  $w \rightarrow +\infty$  since boundary conditions imposes  $\partial_w N_0 \rightarrow \partial_w N_{eq}$ . In principle its first derivative is discontinuous at the separatrix. A reasonable smoothing must be found, such that  $\mathcal{G}(w, v)$  is regular at the separatrix  $w = -v_-$ . This smoothing factor is  $v$  dependent, i.e. depends on the position along the separatrix. Let us assume first that a suitable regularisation has been found. It allows an integration by parts, which leads to the expression

$$I_k(v) = -2\bar{k}_x \left. \frac{\partial N_{eq}}{\partial k_x} \right|_{\mathbf{k}=\bar{\mathbf{k}}} L_k(v) \quad (26)$$

where

$$L_k(v) = \int_{-v_-}^{+\infty} \frac{dw}{2(w + v)} \frac{\partial \mathcal{G}}{\partial w} \quad (27)$$

Using again the fit  $Q(w) \simeq \sqrt{2(w + v_-)}$ , and in absence of regularisation, the expression of  $\frac{\partial \mathcal{G}}{\partial w}$  is

$$\frac{\partial \mathcal{G}}{\partial w}(w, v) = \frac{1}{\sqrt{2(w + v_-)}} - \frac{1}{\sqrt{2(w + v)}} \quad (28)$$

As expected, the result diverges when  $w + v_- \rightarrow 0$ . A “reasonable” regularisation is

$$\frac{\partial \mathcal{G}}{\partial w}(w, v) = \frac{v - v_-}{\sqrt{2(w + v)}} \frac{\sqrt{2(w + v_-)}}{[2(w + v_-) + \delta^2]^{3/2}} \quad (29)$$

where  $\delta$  is related to the size of the diffusive boundary layer, and should be smaller than the island width  $\delta_{isl}$ . The total force reads finally

$$\mathcal{F}(x) = 2 \int_{-\infty}^{+\infty} \frac{dk_y}{4\pi^2} \frac{\Lambda_k}{C_k} D_k \left( -\bar{k}_x \left. \frac{\partial N_{eq}}{\partial k_x} \right|_{\mathbf{k}=\bar{\mathbf{k}}} \right) L_k(v) \quad (30)$$

where the integral  $L_k(v)$  can be calculated numerically. An analytical fit valid when  $v - v_- \ll \delta^2$  can be estimated as (see Appendix A.2)

$$L_k(v) = -\frac{1}{2\delta^3} (v - v_-) \left[ \ln \left( \frac{v - v_-}{8\delta^2} \right) + o \left( \frac{v - v_-}{\delta^2} \right) \right] \quad (31)$$

It appears that a zonal instability develops when the condition Eq.(18) is fulfilled. Indeed the function  $L_k$  is positive whenever  $0 < v - v_- < \delta^2$ . Again, the dependence of the force on the zonal flow velocity, given by Eq.(31) is non linear.

### 3.3. Diffusion with drive

So far, all the cases analysed were characterised by flat profiles of the wave action density within the island. This is no longer true in presence of a drive. This point is illustrated by computing the contribution of trapped waves to the zonal force. Before entering the details of the calculation, it is useful to draw some general results. The function  $N_0(w)$  is even in  $\tilde{k}_x$  within the island since both branches  $\sigma = \pm 1$  are connected on a curve  $w = cte$ . Let us write the operator as a derivative of a flux  $\Gamma$

$$\mathcal{D}[N] = \frac{\partial \Gamma}{\partial k_x} = \tilde{k}_x \frac{\partial \Gamma(v, w, \sigma)}{\partial w} \quad (32)$$

The flux  $\Gamma(v, \tilde{k}_x)$  can be split in odd and even parts in  $\tilde{k}_x$ . Also  $\Gamma$  can be chosen null at  $\tilde{k}_x = 0$  since a constant can always be removed safely. An equivalent expression of the integral  $I_k$  Eq.(9), restricted to the island contribution  $I_{k,t}(v)$ , reads

$$I_{k,t}(v) = -2\tilde{k}_x \int_{-v}^{-v_-} \frac{dw}{\sqrt{2(w+v)}} \frac{\partial \Gamma_{even}}{\partial w} - 2\Gamma_{odd} \left( v, \sqrt{2(v-v_-)} \right) \quad (33)$$

We now address the example of a diffusion operator plus drive

$$\mathcal{D}[N] = \frac{\partial}{\partial k_x} \left( D_k \frac{\partial N}{\partial k_x} \right) + 2\gamma_{\mathbf{k}} N \quad (34)$$

where  $\gamma_{\mathbf{k}}$  is the drift wave growth rate. The regularising diffusive term is large near the separatrix, and within the island, compared to the drive. Hence the term  $\gamma_{\mathbf{k}} N$  can be replaced by its unperturbed value  $\gamma_{\mathbf{k}} N_{eq}$ . In other words the drive acts as a source. The latter is replaced by the divergence in  $\tilde{k}_x$  of a flux  $\Gamma_\gamma$  such that

$$\Gamma_\gamma(\tilde{k}_x) = 2 \int_0^{\tilde{k}_x} d\tilde{k}'_x \gamma_{\tilde{k}_x + \tilde{k}'_x, k_y} N_{eq}(\tilde{k}'_x) \quad (35)$$

This flux can be written as a function  $(w, v, \sigma)$  by using the relation  $\tilde{k}_x = \sigma \sqrt{2(w+v)}$ . The total flux then reads

$$\Gamma(w, v, \sigma) = D_k \tilde{k}_x(w, v, \sigma) \frac{dN_0}{dw} + \Gamma_\gamma(w, v, \sigma) \quad (36)$$

The solvability constraint imposes

$$\frac{dN_0}{dw} = -\frac{1}{D_k} \frac{1}{Q(w)} (\langle \Gamma_\gamma \rangle(w) - \langle \Gamma_\gamma \rangle(-v_+)) \quad (37)$$

where  $\langle \Gamma_\gamma \rangle(w)$  is the average of  $\Gamma_\gamma$  over a surface  $w = cte$ , defined as

$$\langle \Gamma \rangle = \oint \frac{dx}{L} \Gamma(x, w, \sigma) \quad (38)$$



where the property  $Q(-v_+) = 0$  has been used. Note that only the odd part of  $\Gamma_\gamma$  contributes to its average. Hence the flux reads

$$\Gamma(w, v, \sigma) = -\frac{1}{Q(w)} (\langle \Gamma_\gamma \rangle (w) - \langle \Gamma_\gamma \rangle (-v_+)) \sigma \sqrt{2(w+v)} + \Gamma_\gamma(w, v, \sigma) \quad (39)$$

It is convenient to introduce the normalised flux  $\hat{\Gamma}_\gamma = \frac{\Gamma_\gamma}{2N_{eq,k}\gamma_k}$ , where  $N_{eq,k}$  and  $\gamma_k$  are the wave action density  $N_{eq}$  and growth rate  $\gamma_k$  calculated at the surface  $k_x = \bar{k}_x$ . The flux  $\hat{\Gamma}_\gamma$  is then expanded near the resonant surface  $\tilde{k}_x = 0$

$$\hat{\Gamma}_\gamma(\tilde{k}_x) = \tilde{k}_x + \beta_k \tilde{k}_x^2 + \lambda_k \tilde{k}_x^3 + o(\tilde{k}_x^4) \quad (40)$$

Each monomial  $\tilde{k}_x^j$  ( $j = 1, 2, 3, \dots$ ) brings a contribution  $\hat{I}_{k,t}^{(j)}$  to the intensity (with  $\hat{I} = \frac{I}{2N_{eq,k}\gamma_k}$ ). The linear term in  $\tilde{k}_x$  does not contribute to the flux because of a cancellation between the first and last terms of Eq.(39) and the property  $\langle \hat{\Gamma}_\gamma \rangle (-v_+) = 0$ , so that  $\hat{\Gamma}^{(1)} = 0$ , and therefore  $I_{k,t}^{(1)} = 0$ . This is expected since the part of the flux linear in  $\tilde{k}_x$  is produced by the unperturbed drive  $2\gamma_{eq}N_{eq}$ , which is already incorporated in the calculation of the unperturbed action density. The quadratic term  $\beta_k \tilde{k}_x^2$  produces an average flux that vanishes  $\langle \hat{\Gamma}_\gamma^{(2)} \rangle = 0$ , because of its even parity. Hence its contribution to the total flux reduces to  $\hat{\Gamma}^{(2)}(w, v, \sigma) = 2\beta_k(w+v)$ , an even function of  $\sigma$ . It contributes to the intensity Eq.(33) via  $\Gamma_{even}$

$$\hat{I}_{k,t}^{(2)}(v) = -4N_{eq,k}\bar{k}_x [2(v-v_-)]^{1/2} \quad (41)$$

Finally the cubic term  $\lambda_k \tilde{k}_x^3$  produces a flux

$$\hat{\Gamma}^{(3)}(w, v) = -\lambda_k \frac{R(w)}{Q(w)} \sigma [2(w+v)]^{1/2} + \lambda_k \sigma [2(w+v)]^{3/2} \quad (42)$$

where

$$R(w) = \oint \frac{dx}{L} \tilde{k}_x^3 = 2 \int_{-x_0(w)}^{x_0(w)} \frac{dx}{L} [2(w+v)]^{3/2} \quad (43)$$

The flux  $\hat{\Gamma}^{(3)}$ , odd in  $\sigma$ , contributes to the intensity Eq.(33) via the odd flux  $\Gamma_{odd}$

$$\hat{I}_{k,t}^{(3)}(v) = 2\lambda_k \frac{R(-v_-)}{Q(-v_-)} [2(v-v_-)]^{1/2} - 2\lambda_k [2(v-v_-)]^{3/2} \quad (44)$$

Adding even and odd contributions, one finds

$$\hat{I}_{k,t}(v) = -2\Xi_k [2(v-v_-)]^{1/2} - 2\lambda_k [2(v-v_-)]^{3/2} \quad (45)$$

where

$$\Xi_k = 2\beta_k \bar{k}_x - \lambda_k \frac{R(-v_-)}{Q(-v_-)} \quad (46)$$

The corresponding action density gradient within the island is

$$\frac{dN_0}{dw} = -2N_{eq,k}\gamma_k \frac{\lambda_k R(w)}{D_k Q(w)} \quad (47)$$

The total zonal force is

$$\begin{aligned} \mathcal{F} &= -4\sqrt{2} \int_{-\infty}^{+\infty} \frac{dk_y}{4\pi^2} \frac{\Xi_k \Lambda_k}{C_k^{3/2}} N_{eq,k} \gamma_k (V - V_-)^{1/2} \\ &\quad - 8\sqrt{2} \int_{-\infty}^{+\infty} \frac{dk_y}{4\pi^2} \frac{\Lambda_k}{C_k^{5/2}} \lambda_k N_{eq,k} \gamma_k (V - V_-)^{3/2} \end{aligned} \quad (48)$$

Let us stress that this solution does not incorporate the boundary layer contribution Eq.(30), reason why it goes to 0 in absence of drive  $\gamma_k = 0$ . The underlying assumption is that the contribution from trapped waves prevail over the contribution from the thin diffusive boundary layer whenever the growth rate is large enough and/or in the special case  $\bar{k}_x = 0$ . Two limits are of interest. The coefficient of  $(V - V_-)^{1/2}$  is positive whenever  $\Xi_k < 0$  over a significant range of the fluctuation spectrum. A zonal flow instability then occurs, with a force proportional to the square root of the velocity, similar to the Krook case. In the opposite limit where  $\Xi_k > 0$ , but also  $\lambda_k < 0$  over a large part of the spectrum, the coefficient of  $(V - V_-)^{1/2}$  is negative, while the pre-factor of  $(V - V_-)^{3/2}$  is positive. In this case a zonal instability occurs, but for a range of velocities strictly above the velocity minimum. This situation occurs typically near maxima of the growth rate in  $k_x$ , where  $\beta_k \sim 0$ ,  $\lambda_k < 0$ . Hence it appears that zonal instability is again possible, but depends sensitively on the turbulence spectra and sign of the Taylor development of the growth rate times the wave action density near  $k_x = \bar{k}_x$ . Also flow minima and maxima should not behave in the same way since the O point is locally unstable (flow maximum) while the drive is negative near the separatrix (flow minimum).

#### 4. Conclusion

In conclusion, non linear solutions have been found for the wave kinetic equation coupled to the momentum equation of zonal flows in situations where wave trapping plays an important role. The nature of the solutions depends sensitively on the collision operator that represents the non linear wave-wave interaction. Three model operators have been studied: Krook, diffusion and diffusion plus drive. It appears that in all cases a zonal instability is possible. However the zonal force is not linear with the zonal flow velocity, in contrast with the quasilinear result [2]. The dependence is usually an algebraic dependence on the difference between the velocity and its minimum value. The diffusion plus drive operator is interesting, since solutions behave differently from those found with other model operators. First the condition for zonal instability is quite different from the quasilinear constraint. Second the zonal force may actually be negative near the flow minima, hence leading to situations where the flow radial profile is not sinusoidal. The optimum situation is a growth rate that is maximum at vanishing radial wave number, but with a maximum negative curvature with wave number. It must be noted that wave packets propagate equally in both directions in this simple model. Admitting that wave packets are proxy for avalanches, this feature is to be contrasted with gyrokinetic simulations where avalanches have a preferential direction [16, 17]. This model can be reconciled with the results of simulations by adding a mean shear flow. The resulting shift in the radial wave number confers a sign to the drift wave group velocity, in agreement with previous works [16, 17].

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#### Appendix

## Appendix A. Calculation of some useful integrals

### Appendix A.1. Zonal force for a Krook operator

We start from the force Eq.(8), which requires the value of the integral  $I_k$  as defined in Eq.(9).

Using the expression of  $\mathcal{D}[N_0]$  given by Eq.(16), one finds

$$\mathcal{F} = \int_{-\infty}^{+\infty} \frac{dk_y}{4\pi^2} \frac{\Lambda_k}{C_k} \eta_k \left( -\bar{k}_x \left. \frac{\partial N_{eq}}{\partial k_x} \right|_{\mathbf{k}=\bar{\mathbf{k}}} \right) \bar{I}(v) \quad (\text{A.1})$$

where

$$\begin{aligned} \bar{I}(v) &= \sqrt{2} \int_{-v}^{+\infty} \frac{dw}{\sqrt{w+v}} \\ &- \sqrt{2} \int_{-v_-}^{+\infty} \frac{dw}{w+v} \sqrt{w+v_-} \end{aligned}$$

This integral is better calculated as

$$\bar{I}(v) = \lim_{L \rightarrow \infty} (\bar{I}_1(L, v) - \bar{I}_2(L, v)) \quad (\text{A.2})$$

where

$$\begin{aligned} \bar{I}_1(L, v) &= \sqrt{2} \int_{-v}^{+L^2} \frac{dw}{\sqrt{w+v}} \\ \bar{I}_2(L, v) &= \sqrt{2} \int_{-v_-}^{+L^2} \frac{dw}{w+v} \sqrt{w+v_-} \end{aligned}$$

One finds readily  $\bar{I}_1(L, v) = 2\sqrt{2}L$ . The second integral is calculated by using the change of variables  $u^2 = 2(v - v_-)$  and  $z^2 = 2(w + v_-)$ , which yields

$$\bar{I}_2(L, v) = 2 \int_0^{\sqrt{2}L} \frac{dz z^2}{z^2 + u^2} = 2\sqrt{2}L - \pi u \quad (\text{A.3})$$

Combining the two expressions yield  $\bar{I}(L, v) = \pi u = \pi \sqrt{2(v - v_-)} = \pi \sqrt{2 \frac{V - V_-}{C_k}}$ .

### Appendix A.2. Calculations of the force for a diffusion operator

Combining Eqs.(27,29), one finds

$$L_k(v) = (v - v_-) \int_{-v_-}^{+\infty} \frac{dw}{[2(w+v)]^{3/2}} \frac{\sqrt{2(w+v_-)}}{[2(w+v_-) + \delta^2]^{3/2}} \quad (\text{A.4})$$

Using the same change of variables  $u^2 = 2(v - v_-)$  and  $z^2 = 2(w + v_-)$ , one finds  $L_k(v) = \frac{1}{2} u^2 \bar{L}(u)$

$$\bar{L}(u) = \int_0^{+\infty} \frac{dz z^2}{(z^2 + u^2)^{3/2}} \frac{1}{(z^2 + \delta^2)^{3/2}} \quad (\text{A.5})$$

In the case  $u \ll \delta$  an approximate estimate can be done by splitting the integral in two pieces  $\bar{L} = \bar{L}_1 + \bar{L}_2$ , where

$$\begin{aligned} \bar{L}_1 &= \frac{1}{\delta^3} \int_0^\ell \frac{dz z^2}{(z^2 + u^2)^{3/2}} \\ \bar{L}_2 &= \int_\ell^{+\infty} \frac{dz}{z} \frac{1}{(z^2 + \delta^2)^{3/2}} \end{aligned}$$

where  $u \ll \ell \ll \delta$ . Following Gradshteyn and [18], one has the following properties

$$\int \frac{dx x^2}{(x^2 + u^2)^{3/2}} = -\frac{x}{\sqrt{x^2 + u^2}} + \ln \left( x + \sqrt{x^2 + u^2} \right)$$

$$\int \frac{dx}{x(x^2 + \delta^2)^{3/2}} = \frac{1}{\delta^2} \frac{1}{\sqrt{x^2 + \delta^2}} + \frac{1}{2\delta^3} \ln \left( \frac{\sqrt{x^2 + \delta^2} - \delta}{\sqrt{x^2 + \delta^2} + \delta} \right)$$

Expanding  $L_1$  for large values of  $\ell/u$  and  $L_2$  for small values of  $\ell/\delta$ , the following series are found

$$\delta^3 L_1 = -\ell + \ln \ell + \ln 2 - \ln u + o\left(\frac{u^2}{\ell^2}\right)$$

$$\delta^3 L_2 = \ell - \ln \ell + \ln 2 + \ln \delta + o\left(\frac{\ell^2}{\delta^2}\right)$$

When summing  $L_1$  and  $L_2$ , divergent terms in  $\ell$  and  $\ln \ell$  cancel each other to yield

$$L_k(u) = -\frac{u^2}{\delta^3} \left[ \ln \left( \frac{u}{4\delta} \right) + o\left(\frac{u^2}{\delta^2}\right) \right] \quad (\text{A.6})$$

or equivalently

$$L_k(v) = -\frac{1}{2\delta^3} (v - v_-) \left[ \ln \left( \frac{v - v_-}{8\delta^2} \right) + o\left(\frac{v - v_-}{\delta^2}\right) \right] \quad (\text{A.7})$$

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