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# Carleman estimate and null controllability of a fourth order parabolic equation in dimension $N \geq 2$

*S. Guerrero\**, *K. Kassab†*

## Abstract

In this paper, we consider a fourth order parabolic equation in a bounded smooth domain  $\Omega$  with homogeneous Dirichlet boundary conditions on the solution and the laplacian of the solution. The main result we establish is a Carleman inequality with observation in an arbitrarily small open set  $\omega$  included in  $\Omega$ , which leads to a null controllability result at any time  $T > 0$  for the associated control system with a control function acting through  $\omega$ .

## Résumé

Dans ce papier, on considère une équation parabolique de quatrième ordre dans un domaine borné régulier  $\Omega$  avec des conditions aux limites de type Dirichlet homogènes sur la solution et le laplacien de la solution. Le résultat principal concerne une inégalité de Carleman avec une observation dans un petit ouvert  $\omega$  inclus dans  $\Omega$ , ce qui conduit à la contrôlabilité à zéro pour tout  $T > 0$  du système de contrôle associé avec un contrôle agissant sur  $\omega$ .

MSC : 35K35, 93B05, 93B07.

**Keywords:** Fourth order parabolic equation, global Carleman estimate, controllability, observability.

## 1 Introduction

In the present paper, we consider  $\Omega \subset \mathbb{R}^N$  with ( $N \geq 2$ ) a bounded connected open set whose boundary  $\partial\Omega$  is regular enough. Let  $\omega \subset \Omega$  be a (small) nonempty open subset and let  $T > 0$ . We will use the notation  $Q = (0, T) \times \Omega$  and  $\Sigma = (0, T) \times \partial\Omega$  and we will denote by  $\vec{n}(x)$  the outward unit normal vector to  $\Omega$  at the point  $x \in \partial\Omega$ . On the other hand, we will denote by  $C_0$  a generic positive constant which may depend on  $\Omega$  and  $\omega$  but not on  $T$ .

Let us introduce the following control system :

$$\begin{cases} \partial_t y + \Delta^2 y = \chi_\omega v & \text{in } Q, \\ y = \Delta y = 0 & \text{on } \Sigma, \\ y(0, \cdot) = y_0(\cdot) & \text{in } \Omega, \end{cases} \quad (1)$$

where  $y_0 \in L^2(\Omega)$  is the initial condition and  $v \in L^2(Q)$  is the control function. Let us notice that  $y$  may represent a scaled film height and the term  $\Delta^2 y$  represents the capillarity-driven surface diffusion. Our objective is to establish the null controllability for the above system by means of a Carleman inequality for the associated adjoint problem.

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Let us present some interesting results about the existence and uniqueness of solutions for systems which are related to the system under view.

In [22], the authors studied the epitaxial growth of nanoscale thin films, which is modeled by the following system :

$$\begin{cases} \partial_t u + \Delta^2 u - \nabla \cdot (f(\nabla u)) = g & \text{in } \tilde{Q} , \\ \frac{\partial u}{\partial \tilde{n}} = \frac{\partial \Delta u}{\partial \tilde{n}} = 0 & \text{on } \tilde{\Sigma} , \\ u(0, \cdot) = u_0(\cdot) & \text{in } \tilde{\Omega} , \end{cases}$$

where  $\tilde{\Omega} = (0, L)^2$ ,  $\tilde{Q} = (0, T) \times \tilde{\Omega}$ ,  $\tilde{\Sigma} = (0, T) \times \partial \tilde{\Omega}$ ,  $u_0 \in L^2(\tilde{\Omega})$ ,  $f \in C^1(\mathbb{R}^N; \mathbb{R}^N)$  and  $g \in L^2((0, T) \times \tilde{\Omega})$ . The authors showed the existence, uniqueness and regularity of solutions in an appropriate functional space.

As long as degenerate fourth order parabolic equations are concerned, some results on the existence, (non) uniqueness, positivity and asymptotic behavior of weak solutions are presented in [9]. The system under view is

$$\begin{cases} \partial_t u + \nabla \cdot (m(u) \nabla \Delta u) = 0 & \text{in } Q , \\ \frac{\partial u}{\partial \tilde{n}} = \frac{\partial \Delta u}{\partial \tilde{n}} = 0 & \text{on } \Sigma , \\ u(0, \cdot) = u_0(\cdot) & \text{in } \Omega , \end{cases}$$

where  $m$  is a specific function and  $u_0$  is an initial data. Many related results have been proved concerning this kind of systems : see, for instance, [2], [17], [13], [1], [24] and [3]. Moreover, in [26] the authors present a unifying method to prove the existence and uniqueness of weak solutions for such initial-boundary value problems.

Furthermore, in [18] the authors prove the existence of a weak solution of the following system :

$$\begin{cases} \partial_t u + \nabla \cdot (|\nabla \Delta u|^{p(x)-2} \nabla \Delta u) = f(x, u) & \text{in } Q , \\ u = \Delta u = 0 & \text{on } \Sigma , \\ u(0, \cdot) = u_0(\cdot) & \text{in } \Omega , \end{cases}$$

where  $p$  and  $f$  are specific functions and  $u_0$  is an initial data. The previous model may describe some properties of medical magnetic resonance images in space and time. In the particular case where the nonlinear source is given by  $f(x, u) = u(x, t) - a(x)$ , the functions  $u(x, t)$  and  $a(x)$  represent, respectively, a digital image and its observation. Furthermore, for  $p \equiv 2$  the problem becomes the classical Cahn-Hilliard problem, which has been extendedly studied (for more details see [25]). Moreover, in [28] the authors worked on a similar system for which they proved the existence of a weak solution.

Now, let us start talking about the null controllability.

**Definition 1.1.** *System (1) is null controllable at any  $T > 0$  if for every initial data  $y_0 \in L^2(\Omega)$ , there exists a control  $v \in L^2((0, T) \times \omega)$  such that the solution  $y_v \in L^2(0, T; H^2(\Omega)) \cap C^0([0, T]; L^2(\Omega))$  of (1) satisfies*

$$y_v(T, \cdot) = 0 \text{ in } \Omega.$$

Let us introduce the non-homogeneous adjoint system associated to (1) :

$$\begin{cases} -\partial_t \varphi + \Delta^2 \varphi = f & \text{in } Q , \\ \varphi = \Delta \varphi = 0 & \text{on } \Sigma , \\ \varphi(T, \cdot) = \varphi_0(\cdot) & \text{in } \Omega , \end{cases} \quad (2)$$

where  $\varphi_0 \in L^2(\Omega)$  and  $f \in L^2(Q)$ . It is very well-known by now that the null controllability (and continuous dependence of  $\|v\|_{L^2((0, T) \times \omega)}$  with respect to  $\|y_0\|_{L^2(\Omega)}$ ) is equivalent to the observability inequality :

$$\exists C > 0 : \int_{\Omega} |\varphi(0, x)|^2 dx \leq C \iint_{(0, T) \times \omega} |\varphi|^2 dx dt, \forall \varphi_0 \in L^2(\Omega),$$

where  $\varphi$  is the solution of (2) with  $f \equiv 0$ . This estimate can be obtained from a so-called Carleman inequality and the dissipation of energy :

$$-\frac{d}{dt} \int_{\Omega} |\varphi(t, x)|^2 dx \leq 0, \quad \forall t \in (0, T).$$

Let us now present some interesting results concerning the null controllability of fourth order parabolic equations. There is an extended literature in dimension 1. The first Carleman inequality for a fourth order parabolic equation was proved in [7], where the local exact controllability to the trajectories of the Kuramoto-Sivashinsky equation with boundary controls was established. Then, the author in [29] proved a Carleman inequality with interior observation and deduced a null controllability result for the semilinear system

$$\begin{cases} \partial_t u + \partial_{xxxx} u + f(u) = \chi_{\omega} h & \text{in } (0, T) \times (0, 1) , \\ u(\cdot, 0) = u(\cdot, 1) = 0 & \text{in } (0, T) , \\ \partial_x u(\cdot, 0) = \partial_x u(\cdot, 1) = 0 & \text{in } (0, T) , \\ u(0, \cdot) = u_0(\cdot) & \text{in } (0, 1) , \end{cases}$$

where  $u_0 \in L^2(0, 1)$  and  $f$  is a globally Lipschitz continuous function. Using the Carleman inequality proved in [29], the existence of insensitizing controls for a fourth order parabolic equation is established in [16]. The local controllability to the trajectories of the stabilized Kuramoto-Sivashinsky system with an interior control acting on the heat equation is proved in [5]. In the recent work [6] the authors analyze the behavior of the cost of the null controllability of a fourth order parabolic equation through boundary controls with vanishing diffusion coefficient and a transport term. All the previous works used a Carleman inequality, nevertheless the first null controllability result concerning a fourth order parabolic equation was established in [19] without proving a Carleman inequality (see also [8] for a semilinear equation).

In higher dimensions, there has been limited publications on the controllability of fourth order parabolic equations. Among them, the approximate controllability and non-approximate controllability of higher order parabolic equations were studied in [11]. In addition, in [27] the author proved the null controllability of system (1) by using the ideas of [23]. Consequently, as far as we now, a Carleman inequality for a fourth order parabolic equation was an open problem whenever  $N \geq 2$ .

In order to state our Carleman inequality, we will need some weight functions :

$$\begin{aligned} \alpha(x, t) &= \frac{e^{4\lambda\|\eta\|_{\infty}} - e^{\lambda(2\|\eta\|_{\infty} + \eta(x))}}{t^{1/2}(T-t)^{1/2}}, \quad \xi(x, t) = \frac{e^{\lambda(2\|\eta\|_{\infty} + \eta(x))}}{t^{1/2}(T-t)^{1/2}}, \\ \xi_*(x, t) &= \xi|_{\partial\Omega}(x, t) = \frac{e^{2\lambda\|\eta\|_{\infty}}}{t^{1/2}(T-t)^{1/2}}, \quad \alpha_*(x, t) = \alpha|_{\partial\Omega}(x, t) = \frac{e^{4\lambda\|\eta\|_{\infty}} - e^{2\lambda\|\eta\|_{\infty}}}{t^{1/2}(T-t)^{1/2}}, \end{aligned}$$

where  $\eta$  satisfies:

$$\eta \in C^4(\bar{\Omega}), \quad \eta|_{\partial\Omega} = 0, \quad |\nabla\eta| \geq C_0 > 0 \text{ in } \Omega \setminus \bar{\omega}' ,$$

with  $\omega' \subset \omega$  an open set. For the existence of  $\eta$ , see [15].

The main objective of this paper is to prove the following theorem :

**Theorem 1.2.** *There exists a positive constant  $\tilde{C}_0 = \tilde{C}_0(\Omega, \omega)$  such that*

$$\begin{aligned} & \iint_Q e^{-2s\alpha} \left( s^6 \lambda^8 \xi^6 |\varphi|^2 + s^4 \lambda^6 \xi^4 |\nabla\varphi|^2 + s^3 \lambda^4 \xi^3 |\Delta\varphi|^2 \right. \\ & \quad \left. + s^2 \lambda^4 e^{-2s\alpha} \xi^2 |\nabla^2\varphi|^2 + s \lambda^2 \xi |\nabla\Delta\varphi|^2 + s^{-1} \xi^{-1} (|\partial_t\varphi|^2 + |\Delta^2\varphi|^2) \right) dxdt \\ & \leq \tilde{C}_0 \left( s^7 \lambda^8 \iint_{w \times (0, T)} e^{-2s\alpha} \xi^7 |\varphi|^2 dxdt + \iint_Q e^{-2s\alpha} |f|^2 dxdt \right) \end{aligned} \quad (3)$$

for any  $\lambda \geq \tilde{C}_0$  and any  $s \geq \tilde{C}_0(T^{1/2} + T)$ .

**Remark 1.3.** Let us denote

$$\nabla^3 \varphi = \left( \frac{\partial^3 \varphi}{\partial x_i \partial x_j \partial x_k} \right)_{i,j,k=1}^N \quad \text{and} \quad \nabla^4 \varphi = \left( \frac{\partial^4 \varphi}{\partial x_i \partial x_j \partial x_k \partial x_l} \right)_{i,j,k,l=1}^N.$$

Then, we can add the terms

$$\lambda^2 \iint_Q e^{-2s\alpha} |\nabla^3 \varphi|^2 dxdt \quad \text{and} \quad s^{-2} \iint_Q e^{-2s\alpha} \xi^{-2} |\nabla^4 \varphi|^2 dxdt$$

to the left-hand side of (3).

*Proof.* Since  $\varphi = 0$  on  $\Sigma$ , in order to add

$$\lambda^2 \iint_Q e^{-2s\alpha} |\nabla^3 \varphi|^2 dxdt \tag{4}$$

to the left-hand side of (3), it suffices to estimate

$$\|\Delta(\lambda e^{-s\alpha} \varphi)\|_{L^2(0,T;H^1(\Omega))}^2$$

by the terms in the left-hand side of (3). Then, we estimate

$$\|\Delta(s^{-1} e^{-s\alpha} \xi^{-1} \varphi)\|_{L^2(0,T;H^2(\Omega))}^2$$

in terms of (4) and the terms in the left-hand side of (3). This implies that

$$s^{-2} \iint_Q e^{-2s\alpha} \xi^{-2} |\nabla^4 \varphi|^2 dxdt$$

can be added to the left-hand side of (3). □

**Lemma 1.4.** Let  $a_0, a_1 \in L^\infty(Q; \mathbb{R})$ ,  $B_0, B_1 \in L^\infty(Q; \mathbb{R}^N)$ ,  $D \in L^\infty(Q; \mathbb{R}^{N^2})$  and  $E \in L^\infty(Q; \mathbb{R}^{N^3})$ . We define the following differential operator

$$P_3(\varphi) := a_0 \varphi + B_0 \cdot \nabla \varphi + D : \nabla^2 \varphi + a_1 \Delta \varphi + B_1 \cdot \nabla \Delta \varphi + \sum_{i,j,k=1}^N E_{ijk} \frac{\partial^3 \varphi}{\partial x_i \partial x_j \partial x_k}.$$

Then, the solution of the following system

$$\begin{cases} -\partial_t \varphi + \Delta^2 \varphi + P_3(\varphi) = f & \text{in } Q, \\ \varphi = \Delta \varphi = 0 & \text{on } \Sigma, \\ \varphi(T, \cdot) = \varphi_0(\cdot) & \text{in } \Omega, \end{cases} \tag{5}$$

satisfies the Carleman inequality (3) whenever  $\lambda \geq C_0(1 + \|E\|_\infty)$  and  $s \geq C_0(T^{1/2} + T(1 + \|a_0\|_\infty^{1/3} + \|B_0\|_\infty^{1/2} + \|a_1\|_\infty^{2/3} + \|D\|_\infty + \|B_1\|_\infty^2))$ .

*Proof.* Let us denote  $\tilde{f} = f - P_3(\varphi)$ . Let us notice that

$$\begin{aligned} \iint_Q e^{-2s\alpha} |\tilde{f}|^2 dxdt &\leq \iint_Q e^{-2s\alpha} |f|^2 dxdt + \iint_Q e^{-2s\alpha} |a_0 \varphi|^2 dxdt + \iint_Q e^{-2s\alpha} |B_0 \cdot \nabla \varphi|^2 dxdt \\ &\quad + \iint_Q e^{-2s\alpha} |D : \nabla^2 \varphi|^2 dxdt + \iint_Q e^{-2s\alpha} |a_1 \Delta \varphi|^2 dxdt \\ &\quad + \iint_Q e^{-2s\alpha} |B_1 \cdot \nabla \Delta \varphi|^2 dxdt + \iint_Q e^{-2s\alpha} \left| \sum_{i,j,k=1}^N E_{ijk} \frac{\partial^3 \varphi}{\partial x_i \partial x_j \partial x_k} \right|^2 dxdt. \end{aligned} \tag{6}$$

Now, we can absorb all the terms in the right-hand side of (6) ( except for the first one), with the help of inequality (3). We are going to give the details for the last two terms. The first one gives

$$\begin{aligned} \iint_Q e^{-2s\alpha} |B_1 \cdot \nabla \Delta \varphi|^2 dx dt &\leq \|B_1\|_\infty^2 \iint_Q e^{-2s\alpha} |\nabla \Delta \varphi|^2 dx dt \\ &\leq s \iint_Q e^{-2s\alpha} \xi |\nabla \Delta \varphi|^2 dx dt \end{aligned}$$

for  $s \geq T \|B_1\|_\infty^2$ . This term is absorbed by the fifth term in the left-hand side of (3) by taking  $\lambda \geq C_0$ . On the other hand, we have

$$\iint_Q e^{-2s\alpha} \left| \sum_{i,j,k=1}^N E_{ijk} \frac{\partial^3 \varphi}{\partial x_i \partial x_j \partial x_k} \right|^2 dx dt \leq \|E\|_\infty^2 \iint_Q e^{-2s\alpha} \left| \frac{\partial^3 \varphi}{\partial x_i \partial x_j \partial x_k} \right|^2 dx dt,$$

which is absorbed by (4) by taking  $\lambda \geq C_0 \|E\|_\infty$ . The same can be done for the other terms in (6).  $\square$

One of the main consequences of the Carleman inequality (3) is the null controllability of system (1) :

**Theorem 1.5.** *System (1) is null controllable at any time  $T > 0$ . Moreover, there exists a positive constant  $\bar{C}_0(\Omega, \omega)$  such that  $\|v\|_{L^2((0,T) \times \omega)} \leq \bar{C}_0 e^{\bar{C}_0/T^{1/2}} \|y_0\|_{L^2(\Omega)}$ .*

**Lemma 1.6.** *Let  $a_0, a_1 \in L^\infty(Q; \mathbb{R})$ ,  $B_0 \in L^\infty(Q; \mathbb{R}^N)$ ,  $D \in L^\infty(Q; \mathbb{R}^{N^2})$ ,  $B_1 \in C^1(\bar{Q}; \mathbb{R}^N)$  and  $E \in C^1(\bar{Q}; \mathbb{R}^{N^3})$ . We define the following differential operator*

$$P_3^*(y) := a_0 y - \nabla \cdot (B_0 y) + \sum_{ij=1}^N \partial_{ij} (D_{ij} y) + \Delta (a_1 y) - \Delta \nabla \cdot (B_1 y) - \sum_{ijk=1}^N \partial_{ijk} (E_{ijk} y).$$

Then the following system is null controllable at any time  $T > 0$

$$\begin{cases} \partial_t y + \Delta^2 y + P_3^*(y) = \chi_\omega v & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ \Delta y - \left( \sum_{ijk=1}^N E_{ijk} n_i n_j n_k + B_1 \cdot \vec{n} \right) \frac{\partial y}{\partial \vec{n}} = 0 & \text{on } \Sigma, \\ y(0, \cdot) = y_0(\cdot) & \text{in } \Omega. \end{cases} \quad (7)$$

*Proof.* Let us notice that by multiplying (7)<sub>1</sub> by  $\varphi$  solution of (5) with  $f \equiv 0$  and integrating by parts we find

$$\int_\Omega y(T) \varphi_0 dx - \int_\Omega y_0 \varphi(T) dx = \iint_Q \varphi v \chi_\omega.$$

By using Lemma 1.4, one can prove the null controllability of (7) is a classical way.  $\square$

The rest of the paper is devoted to the proof of the Carleman inequality (3).

## 2 Proof of the Carleman inequality

In this section we are going to prove Theorem 1.2. Before we start the proof we will introduce some remarks and lemmas.

We start with some essential properties on the weight functions :

**Remark 2.1.** *We have*

$$\begin{aligned} \nabla \xi &= \lambda \xi \nabla \eta \text{ in } Q, \quad \xi^{-1} \leq \frac{T}{2} \text{ in } Q, \quad \nabla \eta = \frac{\partial \eta}{\partial \vec{n}} \vec{n} \text{ on } \Sigma, \\ |\partial_t \alpha| + |\partial_t \xi| &\leq \frac{T}{2} \xi^3 \text{ in } Q. \end{aligned} \quad (8)$$

Now, we present a result for the Cauchy problem with right-hand sides in  $L^2(Q)$  and initial conditions in  $H^2(\Omega) \cap H_0^1(\Omega)$  :

**Lemma 2.2.** *Assume  $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$  and  $f \in L^2(Q)$  and let  $u$  be the solution of*

$$\begin{cases} -\partial_t u + \Delta^2 u = f & \text{in } Q, \\ u = \Delta u = 0 & \text{on } \Sigma, \\ u(T, \cdot) = u_0(\cdot) & \text{in } \Omega. \end{cases} \quad (9)$$

Then  $u \in L^2(0, T; H^4(\Omega)) \cap H^1(0, T; L^2(\Omega))$  and there exists  $C > 0$  (independent of  $u$ ) such that

$$\|u\|_{L^2(0, T; H^4(\Omega)) \cap H^1(0, T; L^2(\Omega))} \leq C(\|f\|_{L^2(Q)} + \|u_0\|_{H^2(\Omega)}).$$

Next, we use the previous result to estimate some weighted regular norms of  $\varphi$  (solution of (2)) in terms of the right-hand side  $f$  and the  $L^2(Q)$  weighted norm of  $\varphi$  :

**Lemma 2.3.** *There exists a positive constant  $C_0$  such that for all  $f \in L^2(Q)$  and all  $\varphi_0 \in L^2(\Omega)$  the associated solution to (2) satisfies*

$$\begin{aligned} \int_0^T e^{-2s\alpha_*} \|\varphi\|_{H^4(\Omega)}^2 dt + s^{\frac{9}{4}} \lambda^3 \int_0^T e^{-2s\alpha_*} \xi_*^{\frac{9}{4}} \|\nabla^2 \varphi\|_{L^2(\partial\Omega)}^2 dt + s^{\frac{3}{4}} \lambda \int_0^T e^{-2s\alpha_*} \xi_*^{\frac{3}{4}} \|\nabla^3 \varphi\|_{L^2(\partial\Omega)}^2 dt \\ \leq C_0 \left( s^6 \lambda^8 \iint_Q e^{-2s\alpha} \xi^6 |\varphi|^2 dx dt + \iint_Q e^{-2s\alpha} |f|^2 dx dt \right), \end{aligned} \quad (10)$$

for all  $\lambda \geq C_0$  and all  $s \geq C_0(T^{1/2} + T)$ .

*Proof.* Using interpolation results between the Hilbert spaces  $L^2(0, T; H^r(\Omega))$  ( $r \in [0, 4]$ ), (10) is a direct consequence of the classical trace inequality

$$\|\nabla^j \varphi\|_{L^2(\partial\Omega)}^2 \leq C \|\varphi\|_{H^j(\Omega)} \|\varphi\|_{H^{j+1}(\Omega)} \quad (j = 2, 3)$$

and the following estimate :

$$\int_0^T e^{-2s\alpha_*} \|\varphi\|_{H^4(\Omega)}^2 dt \leq C_0 \left( s^6 \iint_Q e^{-2s\alpha} \xi^6 |\varphi|^2 dx dt + \iint_Q e^{-2s\alpha} |f|^2 dx dt \right), \quad (11)$$

for all  $\lambda \geq C_0$  and all  $s \geq C_0(T + T^{1/2})$ .

To prove (11), we set  $\varphi_* = e^{-s\alpha_*} \varphi$ . This function satisfies

$$\begin{cases} \partial_t \varphi_* - \Delta^2 \varphi_* = f_* & \text{in } Q, \\ \varphi_* = \Delta \varphi_* = 0 & \text{on } \Sigma, \\ \varphi_*(T, \cdot) = 0 & \text{in } \Omega, \end{cases}$$

where  $f_* = e^{-s\alpha_*} f - \partial_t(e^{-s\alpha_*})\varphi$ . Using Lemma 2.2 and (8) we deduce that

$$\|\varphi_*\|_{L^2(0, T; H^4(\Omega)) \cap H^1(0, T; L^2(\Omega))}^2 \leq C_0 \left( s^6 \int_0^T e^{-2s\alpha} \xi_*^6 \|\varphi\|_{L^2(\Omega)}^2 dt + \iint_Q e^{-2s\alpha} |f|^2 dx dt \right),$$

for all  $s \geq C_0 T^{1/2}$ . In particular, we find (11).  $\square$

The next result is a Carleman inequality for the Laplace operator with homogeneous Dirichlet boundary conditions.

**Lemma 2.4.** *There exists a constant  $C_0(\Omega, \omega)$  such that, for any  $\lambda, \tau \geq C_0$ , the following inequality holds:*

$$\begin{aligned} & \tau^6 \lambda^8 \int_{\Omega} e^{6\lambda\eta(x)} e^{2\tau e^{\lambda\eta(x)}} |q|^2 dx dt + \tau^4 \lambda^6 \int_{\Omega} e^{4\lambda\eta(x)} e^{2\tau e^{\lambda\eta(x)}} |\nabla q|^2 dx dt \\ & \leq C_0 \left( \tau^6 \lambda^8 \int_{\omega'} e^{6\lambda\eta(x)} e^{2\tau e^{\lambda\eta(x)}} |q|^2 dx dt + \tau^3 \lambda^4 \int_{\Omega} e^{3\lambda\eta(x)} e^{2\tau e^{2\lambda\eta(x)}} |\Delta q|^2 dx dt \right), \end{aligned} \quad (12)$$

for all  $q \in C^2(\overline{Q})$  with  $q = 0$  on  $\Sigma$ .

The proof of (12) follows the steps of the proof of the Carleman inequality for the heat equation presented in [15] (Lemma 1.2).

**Remark 2.5.** *By taking  $\tau = s \frac{\exp(2\lambda\|\eta\|_{\infty})}{t^{1/2}(T-t)^{1/2}}$  in (12), multiplying (12) by  $\exp\left(-2s \frac{\exp(4\lambda\|\eta\|_{\infty})}{t^{1/2}(T-t)^{1/2}}\right)$  and integrating in  $(0, T)$ , we deduce*

$$\begin{aligned} & s^6 \lambda^8 \iint_Q e^{-2s\alpha} \xi^6 |q|^2 dx dt + s^4 \lambda^6 \iint_Q e^{-2s\alpha} \xi^4 |\nabla q|^2 dx dt \\ & \leq C_0 \left( s^6 \lambda^8 \iint_{\omega' \times (0, T)} e^{-2s\alpha} \xi^6 |q|^2 dx dt + s^3 \lambda^4 \iint_Q e^{-2s\alpha} \xi^3 |\Delta q|^2 dx dt \right), \end{aligned} \quad (13)$$

for  $\lambda \geq C_0$  and  $s \geq C_0 T$ .

Observe that (13) is a Carleman inequality for the Laplace operator with weight functions depending on  $x$  and  $t$ . However, the classical Carleman inequality for the heat equation (see [15], Lemma 1.2) is not true for these weight functions since the power of  $t(T-t)$  in the definitions of  $\xi$  and  $\alpha$  is lower than 1.

**Proof of Theorem 1.2.** We divide it in several steps.

*Step 1. A change of variable and first computations.*

Let us set

$$\psi(t, x) = e^{-s\alpha(t, x)} \varphi(t, x), \quad \forall (t, x) \in Q. \quad (14)$$

Then, using the boundary conditions satisfied by  $\varphi$  (see (2)), it is not difficult to see that

$$\Delta \psi = 2s\lambda\xi \frac{\partial \eta}{\partial \vec{n}} \frac{\partial \psi}{\partial \vec{n}} \text{ on } \Sigma, \quad (15)$$

and

$$\frac{\partial^2 \psi}{\partial \vec{n}^2} = 2s\lambda\xi \frac{\partial \eta}{\partial \vec{n}} \frac{\partial \psi}{\partial \vec{n}} + H \frac{\partial \psi}{\partial \vec{n}} \text{ on } \Sigma \text{ with } H = \operatorname{div}(\vec{n}). \quad (16)$$

By replacing  $\varphi$  in the equation  $-\partial_t \varphi + \Delta^2 \varphi = f$  (see (2)) by  $e^{s\alpha} \psi$ , we have

$$\begin{aligned} P(\psi) & \triangleq -s\alpha_t \psi - \partial_t \psi + e^{-s\alpha} (\Delta^2 e^{s\alpha} \psi + 4\nabla \Delta e^{s\alpha} \cdot \nabla \psi + 2\Delta e^{s\alpha} \Delta \psi \\ & + 4\nabla^2 e^{s\alpha} : \nabla^2 \psi + 4\nabla e^{s\alpha} \cdot \nabla \Delta \psi + e^{s\alpha} \Delta^2 \psi) = e^{-s\alpha} f \text{ in } Q, \end{aligned}$$

so that

$$\begin{aligned} P(\psi) & = -s\alpha_t \psi - \partial_t \psi + s^4 \lambda^4 \xi^4 |\nabla \eta|^4 \psi - 6s^3 \lambda^4 \xi^3 |\nabla \eta|^4 \psi + 7s^2 \lambda^4 \xi^2 |\nabla \eta|^4 \psi - s\lambda^4 \xi |\nabla \eta|^4 \psi \\ & - 4s^3 \lambda^3 \xi^3 (\nabla^2 \eta \nabla \eta \nabla \eta) \psi + 12s^2 \lambda^3 \xi^2 (\nabla^2 \eta \nabla \eta \nabla \eta) \psi - 4s\lambda^3 \xi (\nabla^2 \eta \nabla \eta \nabla \eta) \psi - 2s^3 \lambda^3 \xi^3 |\nabla \eta|^2 \Delta \eta \psi \\ & + 6s^2 \lambda^3 \xi^2 |\nabla \eta|^2 \Delta \eta \psi - 2s\lambda^3 \xi |\nabla \eta|^2 \Delta \eta \psi + 2s^2 \lambda^2 \xi^2 \operatorname{div}(\nabla^2 \eta \nabla \eta) \psi \\ & - 2s\lambda^2 \xi \operatorname{div}(\nabla^2 \eta \nabla \eta) \psi + 2s^2 \lambda^2 \xi^2 (\nabla \Delta \eta \cdot \nabla \eta) \psi - 2s\lambda^2 \xi (\nabla \Delta \eta \cdot \nabla \eta) \psi + s^2 \lambda^2 \xi^2 |\Delta \eta|^2 \psi - s\lambda^2 \xi |\Delta \eta|^2 \psi \end{aligned}$$



$$\begin{aligned}
& -s\lambda\xi\Delta^2\eta\psi - 4s^3\lambda^3\xi^3|\nabla\eta|^2(\nabla\eta\cdot\nabla\psi) + 12s^2\lambda^3\xi^2|\nabla\eta|^2(\nabla\eta\cdot\nabla\psi) + 8s^2\lambda^2\xi^2(\nabla^2\eta\nabla\eta\nabla\psi) - 4s\lambda^3\xi|\nabla\eta|^2(\nabla\eta\cdot\nabla\psi) \\
& - 8s\lambda^2\xi(\nabla^2\eta\nabla\eta\nabla\psi) - 4s\lambda^2\xi\Delta\eta(\nabla\eta\cdot\nabla\psi) + 4s^2\lambda^2\xi^2\Delta\eta(\nabla\eta\cdot\nabla\psi) - 4s\lambda\xi(\nabla\Delta\eta\cdot\nabla\psi) + 2s^2\lambda^2\xi^2|\nabla\eta|^2\Delta\psi \\
& - 2s\lambda^2\xi|\nabla\eta|^2\Delta\psi - 2s\lambda\xi\Delta\eta\Delta\psi + 4s^2\lambda^2\xi^2(\nabla^2\psi\nabla\eta\nabla\eta) - 4s\lambda^2\xi(\nabla^2\psi\nabla\eta\nabla\eta) \\
& - 4s\lambda\xi\nabla^2\eta:\nabla^2\psi - 4s\lambda\xi(\nabla\eta\nabla\Delta\psi) + \Delta^2\psi \text{ in } Q.
\end{aligned}$$

Let us consider the following functionals

$$\begin{aligned}
P_1(\psi) & \triangleq -4s^3\lambda^3\xi^3|\nabla\eta|^2\nabla\eta\cdot\nabla\psi - \partial_t\psi - 4s\lambda\xi(\nabla\eta\cdot\nabla\Delta\psi) - 6s^3\lambda^4\xi^3|\nabla\eta|^4\psi \\
& - 2s\lambda^2\xi|\nabla\eta|^2\Delta\psi - 12s^3\lambda^3\xi^3(\nabla^2\eta\nabla\eta\nabla\eta)\psi - 2s^3\lambda^3\xi^3|\nabla\eta|^2\Delta\eta\psi \\
& - 2s\lambda\xi\Delta\eta\Delta\psi + 4s\lambda\xi\nabla^2\psi:\nabla^2\eta - 4s\lambda^2\xi(\nabla^2\psi\nabla\eta\nabla\eta) \text{ in } Q,
\end{aligned} \tag{17}$$

$$\begin{aligned}
P_2(\psi) & \triangleq s^4\lambda^4\xi^4|\nabla\eta|^4\psi + \Delta^2\psi + 4s^2\lambda^2\xi^2(\nabla^2\psi\nabla\eta\nabla\eta) + 2s^2\lambda^2\xi^2|\nabla\eta|^2\Delta\psi + 12s^2\lambda^3\xi^2|\nabla\eta|^2(\nabla\eta\cdot\nabla\psi) \\
& + 8s^2\lambda^2\xi^2(\nabla^2\eta\nabla\eta\nabla\psi) + 4s^2\lambda^2\xi^2\Delta\eta(\nabla\psi\cdot\nabla\eta) \text{ in } Q
\end{aligned}$$

and

$$R(\psi) \triangleq (P - P_1 - P_2)(\psi) = e^{-s\alpha}f - (P_1 + P_2)(\psi) \text{ in } Q. \tag{18}$$

*Step 2. Computation of the product  $(P_1\psi, P_2\psi)_{L^2(Q)}$ .*

In this step, we will compute  $\iint_Q P_1(\psi)P_2(\psi)dxdt$  under the form  $\sum_{i=1}^{10}\sum_{j=1}^7 I_{ij}$  where  $I_{ij}$  is the scalar product in  $L^2(Q)$  of the  $i$ -th term of  $P_1(\psi)$  with the  $j$ -th term of  $P_2(\psi)$ .

In order to shorten the formulas used below, we define

$$\begin{aligned}
A & = A_1 + A_2 + A_3, \\
B & = B_1 + B_2 + B_3 + B_4,
\end{aligned}$$

where

$$\begin{aligned}
A_1 & \triangleq s^5\lambda^7 \iint_Q \xi^5(\lambda + s\xi)|\psi|^2 dxdt, & A_2 & \triangleq s^3\lambda^5 \iint_Q \xi^3(\lambda + s\xi)|\nabla\psi|^2 dxdt, \\
A_3 & \triangleq s\lambda^3 \iint_Q \xi(\lambda + s\xi)|\nabla^2\psi|^2 dxdt, & B_1 & \triangleq s^4\lambda^4 \iint_{\Sigma} \xi^4(\lambda + s\xi) \left(\frac{\partial\psi}{\partial\vec{n}}\right)^2 d\sigma dt, \\
B_2 & \triangleq \iint_{\Sigma} (\lambda + s\xi) \left(\frac{\partial\Delta\psi}{\partial\vec{n}}\right)^2 d\sigma dt, & B_3 & \triangleq s^2\lambda^3 \iint_{\Sigma} \xi_*^2 e^{-2s\alpha_*} |\nabla^2\varphi|^2 dxdt
\end{aligned} \tag{19}$$

and

$$B_4 \triangleq s^{1/2}\lambda \iint_{\Sigma} \xi_*^{1/2} e^{-2s\alpha_*} |\nabla^3\varphi|^2 dxdt. \tag{20}$$

For the calculation of the  $I'_{ij}$ 's we will be using integration by parts in space and time. First, we have

$$\begin{aligned}
I_{11} & = -4s^7\lambda^7 \iint_Q \xi^7|\nabla\eta|^6\nabla\eta\cdot\nabla\psi\psi dxdt \\
& = 14s^7\lambda^8 \iint_Q |\nabla\eta|^8\xi^7|\psi|^2 dxdt + 2s^7\lambda^7 \iint_Q \xi^7|\nabla\eta|^6\Delta\eta|\psi|^2 dxdt \\
& \quad + 12s^7\lambda^7 \iint_Q \xi^7|\nabla\eta|^4(\nabla^2\eta\nabla\eta\nabla\eta)|\psi|^2 dxdt
\end{aligned}$$

and

$$I_{21} = -s^4 \lambda^4 \iint_Q \xi^4 |\nabla \eta|^4 \psi \partial_t \psi \, dxdt \geq -C_0 A_1,$$

for  $s \geq C_0 T^{1/2}$ . In the last estimate we have used (8). Moreover,

$$\begin{aligned} I_{31} &= -4s^5 \lambda^5 \iint_Q \xi^5 |\nabla \eta|^4 (\nabla \eta \cdot \nabla \Delta \psi) \psi \, dxdt \geq -20s^5 \lambda^6 \iint_Q \xi^5 |\nabla \eta|^4 |\nabla \psi \cdot \nabla \eta|^2 \, dxdt \\ &\quad -16s^5 \lambda^5 \iint_Q \xi^5 |\nabla \eta|^2 (\nabla^2 \eta \nabla \eta \nabla \psi) (\nabla \eta \cdot \nabla \psi) \, dxdt - 4s^5 \lambda^5 \iint_Q \xi^5 |\nabla \eta|^4 (\nabla^2 \eta \nabla \psi \nabla \psi) \, dxdt \\ &\quad -10s^5 \lambda^6 \iint_Q \xi^5 |\nabla \eta|^6 |\nabla \psi|^2 \, dxdt - 8s^5 \lambda^5 \iint_Q \xi^5 |\nabla \eta|^2 (\nabla^2 \eta \nabla \eta \nabla \eta) |\nabla \psi|^2 \, dxdt \\ &\quad -2s^5 \lambda^5 \iint_Q \xi^5 |\nabla \eta|^4 \Delta \eta |\nabla \psi|^2 \, dxdt + 2s^5 \lambda^5 \iint_{\Sigma} \xi^5 \left( \frac{\partial \eta}{\partial \bar{n}} \right)^5 \left( \frac{\partial \psi}{\partial \bar{n}} \right)^2 \, d\sigma dt - C_0 A_1, \end{aligned}$$

for  $\lambda \geq C_0$ . On the other hand, we have

$$I_{41} = -6s^7 \lambda^8 \iint_Q \xi^7 |\nabla \eta|^8 |\psi|^2 \, dxdt$$

and

$$I_{51} = -2s^5 \lambda^6 \iint_Q \xi^5 |\nabla \eta|^4 \psi \Delta \psi \, dxdt \geq 2s^5 \lambda^6 \iint_Q \xi^5 |\nabla \eta|^6 |\nabla \psi|^2 \, dxdt - C_0 A_1,$$

for  $\lambda \geq C_0$ . Furthermore, we find

$$\begin{aligned} I_{61} &= -12s^7 \lambda^7 \iint_Q \xi^7 |\nabla \eta|^4 (\nabla^2 \eta \nabla \eta \nabla \eta) |\psi|^2 \, dxdt, \\ I_{71} &= -2s^7 \lambda^7 \iint_Q \xi^7 |\nabla \eta|^6 \Delta \eta |\psi|^2 \, dxdt \end{aligned}$$

and

$$I_{81} = -2s^5 \lambda^5 \iint_Q \xi^5 |\nabla \eta|^4 \Delta \eta \psi \Delta \psi \, dxdt \geq 2s^5 \lambda^5 \iint_Q \xi^5 |\nabla \eta|^4 \Delta \eta |\nabla \psi|^2 \, dxdt - C_0 A_1,$$

for  $\lambda \geq C_0$ . Next, we get

$$I_{91} = 4s^5 \lambda^5 \iint_Q \xi^5 |\nabla \eta|^4 \psi \nabla^2 \psi : \nabla^2 \eta \, dxdt \geq -4s^5 \lambda^5 \iint_Q \xi^5 |\nabla \eta|^4 (\nabla^2 \eta \nabla \psi \nabla \psi) \, dxdt - C_0 A_1,$$

for  $\lambda \geq C_0$ . Moreover, we have

$$I_{10,1} = -4s^5 \lambda^6 \iint_Q \xi^5 |\nabla \eta|^4 \psi (\nabla^2 \psi \nabla \eta \nabla \eta) \, dxdt \geq 4s^5 \lambda^6 \iint_Q \xi^5 |\nabla \eta|^4 |\nabla \eta \cdot \nabla \psi|^2 \, dxdt - C_0 A_1,$$

for  $\lambda \geq C_0$ . As long as  $I_{12}$  is concerned, we use the definition of  $B_1$  and  $B_3$  (see (19)) in order to estimate 3

boundary terms and we obtain

$$\begin{aligned}
I_{12} &\geq -4s^3\lambda^3 \iint_{\Sigma} \xi^3 \frac{\partial\psi}{\partial\bar{n}} \frac{\partial\Delta\psi}{\partial\bar{n}} \left(\frac{\partial\eta}{\partial\bar{n}}\right)^3 d\sigma dt + 4s^3\lambda^3 \iint_{\Sigma} \xi^3 \left(\frac{\partial\eta}{\partial\bar{n}}\right)^3 |\nabla^2\psi\bar{n}|^2 d\sigma dt \\
&\quad -2s^3\lambda^3 \iint_{\Sigma} \xi^3 \left(\frac{\partial\eta}{\partial\bar{n}}\right)^2 |\nabla^2\psi|^2 dx dt - 24s^3\lambda^4 \iint_Q \xi^3 |\nabla\eta|^2 |\nabla^2\psi\nabla\eta|^2 dx dt \\
&\quad -16s^3\lambda^3 \iint_Q \xi^3 [\nabla^2\psi(\nabla^2\psi\nabla\eta)(\nabla^2\eta\nabla\eta)] dx dt - 8s^3\lambda^3 \iint_Q \xi^3 |\nabla\eta|^2 \partial_{ik}\eta\partial_{ij}\psi\partial_{jk}\psi dx dt \\
&\quad +6s^3\lambda^4 \iint_Q \xi^3 |\nabla\eta|^4 |\nabla^2\psi|^2 dx dt + 4s^3\lambda^3 \iint_Q \xi^3 (\nabla^2\eta\nabla\eta\nabla\eta) |\nabla^2\psi|^2 dx dt \\
&\quad +2s^3\lambda^3 \iint_Q \xi^3 |\nabla\eta|^2 \Delta\eta |\nabla^2\psi|^2 dx dt - C_0 \left( A_2 + B_1 + B_3 \right),
\end{aligned}$$

for any  $\lambda \geq C_0$  and  $s \geq C_0(T^{1/2} + T)$ . Next, we find

$$\begin{aligned}
I_{22} &= - \iint_Q \Delta^2\psi\psi_t dx dt = \iint_Q \nabla\Delta\psi \cdot \nabla\psi_t dx dt = - \iint_Q \Delta\psi\partial_t\Delta\psi dx dt + \iint_{\Sigma} \Delta\psi \frac{\partial\psi_t}{\partial\bar{n}} d\sigma dt \\
&= -\frac{1}{2} \iint_Q \partial_t(\Delta\psi)^2 dx dt + \iint_{\Sigma} \Delta\psi \frac{\partial\psi_t}{\partial\bar{n}} d\sigma dt = \iint_{\Sigma} \Delta\psi \frac{\partial\psi_t}{\partial\bar{n}} d\sigma dt \geq -C_0B_1,
\end{aligned}$$

for  $s \geq C_0(T^{1/2} + T)$ . Here, we have used (8) and (15). Next, we get

$$\begin{aligned}
I_{32} &= -4s\lambda \iint_{\Sigma} \xi \frac{\partial\eta}{\partial\bar{n}} \left(\frac{\partial\Delta\psi}{\partial\bar{n}}\right)^2 d\sigma dt + 2s\lambda \iint_{\Sigma} \xi \frac{\partial\eta}{\partial\bar{n}} |\nabla\Delta\psi|^2 d\sigma dt + 4s\lambda^2 \iint_Q \xi |\nabla\Delta\psi \cdot \nabla\eta|^2 dx dt \\
&\quad +4s\lambda \iint_Q \xi (\nabla^2\eta\nabla\Delta\psi\nabla\Delta\psi) dx dt - 2s\lambda^2 \iint_Q \xi |\nabla\eta|^2 |\nabla\Delta\psi|^2 dx dt - 2s\lambda \iint_Q \xi \Delta\eta |\nabla\Delta\psi|^2 dx dt.
\end{aligned}$$

For the next 5 terms, we use (15) and we obtain, for any  $\lambda \geq C_0$  and any  $s \geq C_0T$

$$I_{42} = -6s^3\lambda^4 \iint_Q \xi^3 |\nabla\eta|^4 \psi \Delta^2\psi dx dt \geq -6s^3\lambda^4 \iint_Q \xi^3 |\nabla\eta|^4 |\Delta\psi|^2 dx dt - C_0 \left( A_2 + B_1 \right),$$

$$I_{52} = -2s\lambda^2 \iint_Q \xi |\nabla\eta|^2 \Delta\psi \Delta^2\psi dx dt \geq 2s\lambda^2 \iint_Q \xi |\nabla\eta|^2 |\nabla\Delta\psi|^2 dx dt - C_0 \left( A_3 + B_1 + B_2 \right),$$

$$I_{62} = -12s^3\lambda^3 \iint_Q \xi^3 (\nabla^2\eta\nabla\eta\nabla\eta) \psi \Delta^2\psi dx dt$$

$$\geq -12s^3\lambda^3 \iint_Q \xi^3 (\nabla^2\eta\nabla\eta\nabla\eta) |\Delta\psi|^2 dx dt - C_0 \left( A_2 + A_1 + B_1 \right),$$

$$I_{72} = -2s^3\lambda^3 \iint_Q \xi^3 |\nabla\eta|^2 \Delta\eta \psi \Delta^2\psi dx dt \geq -2s^3\lambda^3 \iint_Q \xi^3 \Delta\eta |\nabla\eta|^2 |\Delta\psi|^2 dx dt - C_0 \left( A_1 + A_2 + B_1 \right)$$

and

$$I_{82} = -2s\lambda \iint_Q \xi \Delta\eta \Delta\psi \Delta^2\psi dx dt \geq 2s\lambda \iint_Q \xi \Delta\eta |\nabla\Delta\psi|^2 dx dt - C_0 \left( A_3 + B_1 + B_2 \right).$$

For the next 2 terms, we use the definition of  $B_3$  and  $B_4$  (see (19)) and we obtain, for  $\lambda \geq C_0$  and  $s \geq C_0T$ ,

$$I_{92} = 4s\lambda \iint_Q \xi \nabla^2\psi : \nabla^2\eta \Delta^2\psi dx dt \geq -4s\lambda \iint_Q \xi (\nabla^2\eta\nabla\Delta\psi\nabla\Delta\psi) dx dt - C_0 \left( A_3 + B_3 + B_4 \right)$$

and

$$I_{10,2} = -4s\lambda^2 \iint_Q \xi(\nabla^2\psi\nabla\eta\nabla\eta)\Delta^2\psi dxdt \geq 4s\lambda^2 \iint_Q \xi|\nabla\Delta\psi \cdot \nabla\eta|^2 dxdt - C_0(A_3 + B_3 + B_4).$$

Furthermore,

$$\begin{aligned} I_{13} &\geq -8s^5\lambda^5 \iint_\Sigma \xi^5 \left(\frac{\partial\eta}{\partial\vec{n}}\right)^5 \left(\frac{\partial\psi}{\partial\vec{n}}\right)^2 d\sigma dt + 40s^5\lambda^6 \iint_Q \xi^5|\nabla\eta|^4|\nabla\psi \cdot \nabla\eta|^2 dxdt \\ &\quad + 16s^5\lambda^5 \iint_Q \xi^5(\nabla^2\eta\nabla\eta\nabla\eta)|\nabla\psi \cdot \nabla\eta|^2 dxdt + 16s^5\lambda^5 \iint_Q \xi^5|\nabla\eta|^2(\nabla^2\eta\nabla\eta\nabla\psi)(\nabla\psi \cdot \nabla\eta) dxdt \\ &\quad + 8s^5\lambda^5 \iint_Q \xi^5|\nabla\eta|^2\Delta\eta|\nabla\psi \cdot \nabla\eta|^2 dxdt \end{aligned}$$

and

$$\begin{aligned} I_{23} &\geq 8s^2\lambda^3 \iint_Q \xi^2|\nabla\eta|^2\partial_t\psi(\nabla\psi \cdot \nabla\eta) dxdt + 4s^2\lambda^2 \iint_Q \xi^2(\nabla^2\eta\nabla\eta\nabla\psi)\partial_t\psi dxdt, \\ &\quad + 4s^2\lambda^2 \iint_Q \xi^2\Delta\eta(\nabla\psi \cdot \nabla\eta)\partial_t\psi dxdt - C_0A_2. \end{aligned}$$

Then, by using (16) we find

$$\begin{aligned} I_{33} &\geq -64s^5\lambda^5 \iint_\Sigma \xi^5 \left(\frac{\partial\eta}{\partial\vec{n}}\right)^5 \left(\frac{\partial\psi}{\partial\vec{n}}\right)^2 d\sigma dt + 8s^3\lambda^3 \iint_\Sigma \xi^3 \left(\frac{\partial\eta}{\partial\vec{n}}\right)^3 |\nabla^2\psi \cdot \vec{n}|^2 d\sigma dt \\ &\quad + 48s^3\lambda^4 \iint_Q \xi^3|\nabla^2\psi\nabla\eta\nabla\eta|^2 dxdt + 16s^3\lambda^3 \iint_Q \xi^3(\nabla^2\psi\nabla\eta\nabla\eta)(\nabla^2\psi : \nabla^2\eta) dxdt \\ &\quad + 32s^3\lambda^3 \iint_Q \xi^3[\nabla^2\eta(\nabla^2\psi\nabla\eta)(\nabla^2\psi\nabla\eta)] dxdt - 16s^3\lambda^3 \iint_Q \xi^3[\nabla^2\psi(\nabla^2\eta\nabla\eta)(\nabla^2\psi\nabla\eta)] dxdt \\ &\quad - 8s^3\lambda^3 \iint_Q \xi^3\Delta\eta|\nabla^2\psi\nabla\eta|^2 dxdt - 24s^3\lambda^4 \iint_Q \xi^3|\nabla\eta|^2|\nabla^2\psi\nabla\eta|^2 dxdt - C_0B_1, \end{aligned}$$

for  $\lambda \geq C_0$ . Next,

$$I_{43} = -24s^5\lambda^6 \iint_Q \xi^5|\nabla\eta|^4\psi(\nabla^2\psi\nabla\eta\nabla\eta) dxdt \geq 24s^5\lambda^6 \iint_Q \xi^5|\nabla\eta|^4|\nabla\psi \cdot \nabla\eta|^2 - C_0A_1$$

and

$$I_{53} = -8s^3\lambda^4 \iint_Q \xi^3|\nabla\eta|^2\Delta\psi(\nabla^2\psi\nabla\eta\nabla\eta) dxdt \geq -8s^3\lambda^4 \iint_Q \xi^3|\nabla\eta|^2|\nabla^2\psi\nabla\eta|^2 dxdt - C_0(A_2 + B_1),$$

for  $\lambda \geq C_0$ . Here, we have used (16). Next, we have

$$I_{63} = 48s^5\lambda^5 \iint_Q \xi^5(\nabla^2\eta\nabla\eta\nabla\eta)\psi(\nabla^2\psi\nabla\eta\nabla\eta) dxdt \geq 48s^5\lambda^5 \iint_Q \xi^5(\nabla^2\eta\nabla\eta\nabla\eta)|\nabla\psi \cdot \nabla\eta|^2 dxdt - C_0A_1,$$

$$I_{73} = -8s^5\lambda^5 \iint_Q \xi^5|\nabla\eta|^2\Delta\eta\psi(\nabla^2\psi\nabla\eta\nabla\eta) dxdt \geq 8s^5\lambda^5 \iint_Q \xi^5|\nabla\eta|^2\Delta\eta|\nabla\psi \cdot \nabla\eta|^2 dxdt - C_0A_1$$

and

$$I_{83} = -8s^3\lambda^3 \iint_Q \xi^3\Delta\eta\Delta\psi(\nabla^2\psi\nabla\eta\nabla\eta) dxdt \geq -8s^3\lambda^3 \iint_Q \xi^3\Delta\eta|\nabla^2\psi\nabla\eta|^2 dxdt - C_0(A_2 + B_1),$$

for  $\lambda \geq C_0$ . Here, we have used again (16). From the definition of  $B_1$  and  $B_3$ , we deduce for  $\lambda \geq C_0$  and  $s \geq C_0(T^{1/2} + T)$

$$\begin{aligned} I_{93} &= 16s^3\lambda^3 \iint_Q \xi^3 (\nabla^2\psi : \nabla^2\eta)(\nabla^2\psi\nabla\eta\nabla\eta) dxdt \\ &\geq 16s^3\lambda^3 \iint_Q \xi^3 [\nabla^2\eta(\nabla^2\psi\nabla\eta)(\nabla^2\psi\nabla\eta)] dxdt - C_0(A_2 + B_1 + B_3). \end{aligned}$$

Then, we find

$$I_{10,3} = -16s^3\lambda^4 \iint_Q \xi^3 |\nabla^2\psi\nabla\eta\nabla\eta|^2 dxdt$$

and

$$\begin{aligned} I_{14} &= -4s^5\lambda^5 \iint_\Sigma \xi^5 \left(\frac{\partial\eta}{\partial\vec{n}}\right)^5 \left(\frac{\partial\psi}{\partial\vec{n}}\right)^2 d\sigma dt + 40s^5\lambda^6 \iint_Q \xi^5 |\nabla\eta|^4 |\nabla\psi \cdot \nabla\eta|^2 dxdt \\ &\quad + 32s^5\lambda^5 \iint_Q \xi^5 |\nabla\eta|^2 (\nabla^2\eta\nabla\eta\nabla\psi)(\nabla\psi \cdot \nabla\eta) dxdt + 8s^5\lambda^5 \iint_Q \xi^5 |\nabla\eta|^4 (\nabla^2\eta\nabla\psi\nabla\psi) dxdt \\ &\quad - 20s^5\lambda^6 \iint_Q \xi^5 |\nabla\eta|^6 |\nabla\psi|^2 dxdt - 16s^5\lambda^5 \iint_Q \xi^5 |\nabla\eta|^2 (\nabla^2\eta\nabla\eta\nabla\eta) |\nabla\psi|^2 dxdt \\ &\quad - 4s^5\lambda^5 \iint_Q \xi^5 |\nabla\eta|^4 \Delta\eta |\nabla\psi|^2 dxdt. \end{aligned}$$

Furthermore, we get

$$I_{24} \geq 4s^2\lambda^2 \iint_Q \partial_t\psi (\nabla\psi \cdot \nabla\eta) |\nabla\eta|^2 dxdt + 4s^2\lambda^2 \iint_Q \xi^2 (\nabla^2\eta\nabla\eta\nabla\eta) \partial_t\psi dxdt - C_0A_2,$$

for  $\lambda \geq C_0$ . Next, we have from (15)

$$\begin{aligned} I_{34} &= -16s^5\lambda^5 \iint_\Sigma \xi^3 \left(\frac{\partial\eta}{\partial\vec{n}}\right)^5 \left(\frac{\partial\psi}{\partial\vec{n}}\right)^2 d\sigma dt + 12s^3\lambda^4 \iint_Q \xi^3 |\nabla\eta|^4 |\Delta\psi|^2 dxdt \\ &\quad + 8s^3\lambda^3 \iint_Q \xi^3 (\nabla^2\eta\nabla\eta\nabla\eta) |\Delta\psi|^2 dxdt + 4s^3\lambda^3 \iint_Q \xi^3 |\nabla\eta|^2 \Delta\eta |\Delta\psi|^2 dxdt. \end{aligned}$$

Furthermore, we find

$$I_{44} = -12s^5\lambda^6 \iint_Q \xi^5 |\nabla\eta|^6 \psi \Delta\psi dxdt \geq 12s^5\lambda^6 \iint_Q \xi^5 |\nabla\eta|^6 |\nabla\psi|^2 dxdt - C_0A_1,$$

for  $\lambda \geq C_0$ . Next, we have

$$I_{54} = -4s^3\lambda^4 \iint_Q \xi^3 |\nabla\eta|^4 |\Delta\psi|^2 dxdt,$$

for  $\lambda \geq C_0$ . Moreover, we have

$$I_{64} = -24s^5\lambda^5 \iint_Q \xi^5 |\nabla\eta|^2 (\nabla^2\eta\nabla\eta\nabla\eta) \psi \Delta\psi dxdt \geq 24s^5\lambda^5 \iint_Q \xi^5 |\nabla\eta|^2 (\nabla^2\eta\nabla\eta\nabla\eta) |\nabla\psi|^2 dxdt - C_0A_1$$

and

$$I_{74} = -4s^5\lambda^5 \iint_Q \xi^5 |\nabla\eta|^4 \Delta\eta \psi \Delta\psi dxdt \geq 4s^5\lambda^5 \iint_Q \xi^5 |\nabla\eta|^4 \Delta\eta |\nabla\psi|^2 dxdt - C_0A_1,$$

for  $\lambda \geq C_0$ . Furthermore, we obtain

$$I_{84} = -4s^3\lambda^3 \iint_Q \xi^3 |\nabla\eta|^2 \Delta\eta |\Delta\psi|^2 dxdt$$

and

$$I_{94} = 8s^3\lambda^3 \iint_Q \xi^3 |\nabla\eta|^2 \partial_{ik}\eta \partial_{ij}\psi \partial_{jk}\psi dxdt - C_0(A_2 + B_1 + B_3),$$

for  $\lambda \geq C_0$ . Next, we have from (16)

$$I_{10,4} = -8s^3\lambda^4 \iint_Q \xi^3 |\nabla\eta|^2 \Delta\psi (\nabla^2\psi \nabla\eta \nabla\eta) dxdt \geq -8s^3\lambda^4 \iint_Q \xi^3 |\nabla\eta|^2 |\nabla^2\psi \nabla\eta|^2 dxdt - C_0(A_2 + B_1),$$

for  $\lambda \geq C_0$ . On the other hand, we have

$$I_{15} = -48s^5\lambda^6 \iint_Q \xi^5 |\nabla\eta|^4 |\nabla\eta \cdot \nabla\psi|^2 dxdt$$

and

$$I_{25} = -12s^2\lambda^3 \iint_Q \xi^2 |\nabla\eta|^2 (\nabla\eta \cdot \nabla\psi) \partial_t\psi dxdt.$$

By using (16), we can deduce

$$I_{35} = -48s^3\lambda^4 \iint_Q \xi^3 |\nabla\eta|^2 (\nabla\eta \cdot \nabla\psi) (\nabla\eta \cdot \nabla\Delta\psi) dxdt \geq 48s^3\lambda^4 \iint_Q \xi^3 |\nabla\eta|^2 |\nabla^2\psi \nabla\eta|^2 dxdt - C_0(A_2 + B_1),$$

for  $\lambda \geq C_0$ . Next, we have

$$\sum_{i=4}^9 I_{i5} \geq -C_0(A_1 + A_2 + B_1),$$

for  $\lambda \geq C_0$ . On the other hand, we can deduce

$$I_{10,5} = -48s^3\lambda^5 \iint_Q \xi^3 |\nabla\eta|^2 (\nabla\eta \cdot \nabla\psi) (\nabla^2\psi \nabla\eta \nabla\eta) dxdt \geq -C_0(A_2 + B_1),$$

for  $\lambda \geq C_0$ . Also, we have

$$I_{16} = -32s^5\lambda^5 \iint_Q \xi^5 |\nabla\eta|^2 (\nabla^2\eta \nabla\eta \nabla\psi) (\nabla\psi \cdot \nabla\eta) dxdt$$

and

$$I_{26} = 8s^2\lambda^2 \iint_Q \xi^2 (\nabla^2\eta \nabla\eta \nabla\psi) \partial_t\psi dxdt.$$

By using (16), we have

$$\begin{aligned} I_{36} &= -32s^3\lambda^3 \iint_Q \xi^3 (\nabla^2\eta \nabla\eta \nabla\psi) (\nabla\eta \cdot \nabla\Delta\psi) dxdt \\ &\geq 32s^3\lambda^3 \iint_Q \xi^3 [\nabla^2\psi (\nabla^2\eta \nabla\eta) (\nabla^2\psi \nabla\eta)] dxdt - C_0(A_2 + B_1), \end{aligned}$$

for  $\lambda \geq C_0$ . Furthermore, we have

$$\sum_{i=4}^{10} I_{i6} \geq -C_0 \left( A_1 + A_2 + B_1 \right),$$

for  $\lambda \geq C_0$ . Also, we have

$$I_{17} = -16s^5 \lambda^5 \iint_Q \xi^5 |\nabla \eta|^2 \Delta \eta |\nabla \psi \cdot \nabla \eta|^2 dx dt$$

and

$$I_{27} = -4s^2 \lambda^2 \iint_Q \xi^2 \Delta \eta \partial_t \psi (\nabla \eta \cdot \nabla \psi) dx dt,$$

for  $\lambda \geq C_0$ . By using (16), we deduce

$$I_{37} = -16s^3 \lambda^3 \iint_Q \xi^3 \Delta \eta (\nabla \eta \cdot \nabla \psi) (\nabla \eta \cdot \nabla \Delta \psi) dx dt \geq 16s^3 \lambda^3 \iint_Q \xi^3 \Delta \eta |\nabla^2 \psi \nabla \eta|^2 dx dt - C_0 \left( A_2 + B_1 \right),$$

for  $\lambda \geq C_0$ . Also, we have

$$\sum_{i=4}^{10} I_{i7} \geq -C_0 \left( A_1 + A_2 + B_2 \right).$$

*Step 3. Simplifications in the computation of  $(P_1 \psi, P_2 \psi)_{L^2(Q)}$  and first main estimate.*

In this step, we put together all the  $I'_{ij}$ s, we do some simplifications and we find an estimate for  $(P_1 \psi, P_2 \psi)_{L^2(Q)}$ . First, we have

$$(I_{11})_1 + I_{41} = 8s^7 \lambda^8 \iint_Q \xi^7 |\nabla \eta|^8 |\psi|^2 dx dt,$$

$$(I_{14})_5 + (I_{31})_4 + (I_{51})_1 + (I_{44})_1 = -16s^5 \lambda^6 \iint_Q \xi^5 |\nabla \eta|^6 |\nabla \psi|^2 dx dt,$$

$$(I_{12})_7 + (I_{34})_2 + I_{54} + (I_{42})_1 = 8s^3 \lambda^4 \iint_Q \xi^3 |\nabla \eta|^4 |\nabla^2 \psi|^2 dx dt,$$

$$(I_{31})_1 + (I_{10,1})_1 + (I_{13})_2 + (I_{43})_1 + (I_{14})_2 + I_{15} = 40s^5 \lambda^6 \iint_Q \xi^5 |\nabla \eta|^4 |\nabla \psi \cdot \nabla \eta|^2 dx dt,$$

$$(I_{33})_3 + I_{10,3} = 32s^3 \lambda^4 \iint_Q \xi^3 |\nabla^2 \psi \nabla \eta \nabla \eta|^2 dx dt.$$

By using (16), we find

$$\begin{aligned} (I_{12})_4 + (I_{33})_8 + (I_{53})_1 + (I_{10,4})_1 + (I_{35})_1 &= -16s^3 \lambda^4 \iint_Q \xi^3 |\nabla \eta|^2 |\nabla^2 \psi \nabla \eta|^2 dx dt \\ &\geq -4s \lambda^2 \iint_Q \xi^3 |\nabla \Delta \psi \cdot \nabla \eta|^2 dx dt \\ &\quad -16s^5 \lambda^6 \iint_Q \xi^5 |\nabla \eta|^4 |\nabla \psi \cdot \nabla \eta|^2 dx dt \\ &\quad -C_0 \left( A_2 + B_1 \right), \end{aligned}$$

for  $\lambda \geq C_0$ . Next, we have

$$\begin{aligned}
(I_{32})_3 + (I_{10,2})_1 &= 8s\lambda^2 \iint_Q \xi |\nabla \Delta \psi \cdot \nabla \eta|^2 dx dt, \\
(I_{13})_3 + (I_{63})_1 &= 64s^5 \lambda^5 \iint_Q \xi^5 (\nabla^2 \eta \nabla \eta \nabla \eta) |\nabla \psi \cdot \nabla \eta|^2 dx dt \\
&\geq -C_0 s^5 \lambda^5 \iint_{\omega' \times (0, T)} \xi^5 |\nabla \psi|^2 dx dt \\
&\quad - s^5 \lambda^6 \iint_Q \xi^5 |\nabla \eta|^4 |\nabla \psi \cdot \nabla \eta|^2 dx dt,
\end{aligned}$$

for  $\lambda \geq C_0$ . Moreover, we have

$$\begin{aligned}
(I_{31})_7 + (I_{13})_1 + (I_{33})_1 + (I_{14})_1 + (I_{34})_2 &= -90s^5 \lambda^5 \iint_{\Sigma} \xi^5 \left( \frac{\partial \eta}{\partial \bar{n}} \right)^5 \left( \frac{\partial \psi}{\partial \bar{n}} \right)^2 d\sigma dt, \\
(I_{12})_1 &= -4s^3 \lambda^3 \iint_{\Sigma} \xi^3 \left( \frac{\partial \eta}{\partial \bar{n}} \right)^3 \frac{\partial \Delta \psi}{\partial \bar{n}} \frac{\partial \psi}{\partial \bar{n}} d\sigma dt, \\
(I_{12})_2 + (I_{33})_2 &= 12s^3 \lambda^3 \iint_{\Sigma} \xi^3 \left( \frac{\partial \eta}{\partial \bar{n}} \right)^3 |\nabla^2 \psi \bar{n}|^2 d\sigma dt, \\
(I_{12})_3 &= -2s^3 \lambda^3 \iint_{\Sigma} \xi^3 \left( \frac{\partial \eta}{\partial \bar{n}} \right)^3 |\nabla^2 \psi|^2 d\sigma dt, \\
(I_{32})_1 &= -4s\lambda \iint_{\Sigma} \xi \frac{\partial \eta}{\partial \bar{n}} \left( \frac{\partial \Delta \psi}{\partial \bar{n}} \right)^2 d\sigma dt, \\
(I_{32})_2 &= 2s\lambda \iint_{\Sigma} \xi \frac{\partial \eta}{\partial \bar{n}} |\nabla \Delta \psi|^2 d\sigma dt.
\end{aligned}$$

Then, we also observe the simplifications

$$\begin{aligned}
(I_{11})_2 + I_{71} &= 0, & (I_{11})_3 + I_{61} &= 0, & (I_{31})_2 + (I_{13})_4 + (I_{14})_3 + I_{16} &= 0, \\
(I_{31})_3 + I_{91} + (I_{14})_4 &= 0, & (I_{31})_5 + (I_{14})_6 + (I_{64})_1 &= 0, & (I_{31})_6 + I_{81} &= 0, \\
(I_{12})_5 + (I_{33})_6 + (I_{36})_1 &= 0, & (I_{12})_6 + I_{94} &= 0, & (I_{12})_8 + (I_{62})_1 + (I_{34})_3 &\geq -C_0 \left( B_1 + A_2 \right).
\end{aligned}$$

For this last estimate, we have used the fact that

$$8s^3 \lambda^3 \iint_Q \xi^3 (\nabla^2 \eta \nabla \eta \nabla \eta) |\Delta \psi|^2 dx dt \geq -C_0 \left( A_2 + B_1 \right) + 8s^3 \lambda^3 \iint_Q \xi^3 (\nabla^2 \eta \nabla \eta \nabla \eta) |\nabla^2 \psi|^2 dx dt.$$

Here, we used (15) and took  $\lambda \geq C_0$  and  $s \geq C_0 T$ . Using this same argument, we find

$$(I_{12})_9 + I_{72} + (I_{34})_4 + I_{84} \geq -C_0 \left( B_1 + A_2 \right).$$

Moreover,

$$(I_{32})_4 + (I_{92})_1 = 0, \quad (I_{32})_5 + (I_{52})_1 = 0, \quad (I_{32})_6 + (I_{82})_1 = 0.$$

Next,

$$\begin{aligned}
(I_{13})_5 + (I_{73})_1 + I_{17} &= 0, & (I_{14})_7 + (I_{74})_1 &= 0, & (I_{23})_1 + (I_{24})_1 + I_{25} &= 0, \\
(I_{23})_2 + (I_{24})_2 + I_{26} &= 0, & (I_{23})_3 + I_{27} &= 0, & (I_{33})_7 + (I_{83})_1 + (I_{37})_1 &= 0.
\end{aligned}$$



Arguing as before, we find

$$\begin{aligned}
(I_{33})_4 &= 16s^3\lambda^3 \iint_Q \xi^3 (\nabla^2\psi \nabla\eta \nabla\eta) (\nabla^2\psi : \nabla^2\eta) dxdt \\
&\geq -C_0(B_1 + A_2) + 16s^3\lambda^3 \iint_Q \xi^3 [\nabla^2\eta (\nabla^2\psi \nabla\eta) (\nabla^2\psi \nabla\eta)] dxdt \\
&\geq -C_0(B_1 + A_2) - C_0s^3\lambda^3 \iint_Q \xi^3 |\nabla\Delta\psi \cdot \nabla\eta| |\nabla\psi \cdot \nabla\eta| dxdt,
\end{aligned}$$

so that

$$\begin{aligned}
(I_{33})_4 + (I_{33})_5 + (I_{93})_1 &\geq -C_0(B_1 + A_2) - C_0s^3\lambda^3 \iint_Q \xi^3 |\nabla\Delta\psi \cdot \nabla\eta| |\nabla\psi \cdot \nabla\eta| dxdt \\
&\geq -C_0(B_1 + A_2) - s\lambda^2 \iint_Q \xi |\nabla\Delta\psi \cdot \nabla\eta|^2 dxdt \\
&\quad - s^5\lambda^6 \iint_Q |\nabla\eta|^4 |\nabla\psi \cdot \nabla\eta|^2 dxdt - C_0s^5\lambda^5 \iint_{\omega' \times (0,T)} \xi^5 |\nabla\psi|^2 dxdt,
\end{aligned}$$

for  $\lambda \geq C_0$ . At the end, we can deduce that

$$\begin{aligned}
(P_1(\psi), P_2(\psi))_{L^2(Q)} &\geq 8s^7\lambda^8 \iint_Q \xi^6 |\nabla\eta|^8 |\psi|^2 dxdt - 16s^5\lambda^6 \iint_Q \xi^5 |\nabla\eta|^6 |\nabla\psi|^2 dxdt \\
&\quad + 8s^3\lambda^4 \iint_Q \xi^3 |\nabla\eta|^4 |\Delta\psi|^2 dxdt + 22s^5\lambda^6 \iint_Q \xi^5 |\nabla\eta|^4 |\nabla\eta \cdot \nabla\psi|^2 dxdt \\
&\quad + 32s^3\lambda^4 \iint_Q \xi^3 |\nabla^2\psi \nabla\eta \nabla\eta|^2 dxdt + 3s\lambda^2 \iint_Q \xi |\nabla\Delta\psi \cdot \nabla\eta|^2 dxdt \\
&\quad + \bar{B} - C_0(A + B + s^5\lambda^5 \iint_{\omega' \times (0,T)} \xi^5 |\nabla\psi|^2 dxdt),
\end{aligned} \tag{21}$$

where  $\bar{B}$  is given by

$$\begin{aligned}
\bar{B} &= -90s^5\lambda^5 \iint_{\Sigma} \xi^5 \left(\frac{\partial\eta}{\partial\bar{n}}\right)^5 \left(\frac{\partial\psi}{\partial\bar{n}}\right)^2 d\sigma dt - 4s^3\lambda^3 \iint_{\Sigma} \xi^3 \frac{\partial\psi}{\partial\bar{n}} \frac{\partial\Delta\psi}{\partial\bar{n}} \left(\frac{\partial\eta}{\partial\bar{n}}\right)^3 d\sigma dt \\
&\quad + 12s^3\lambda^3 \iint_{\Sigma} \xi^3 \left(\frac{\partial\eta}{\partial\bar{n}}\right)^3 |\nabla^2\psi \bar{n}|^2 d\sigma dt - 2s^3\lambda^3 \iint_{\Sigma} \xi^3 \left(\frac{\partial\eta}{\partial\bar{n}}\right)^2 |\nabla^2\psi|^2 d\sigma dt \\
&\quad - 4s\lambda \iint_{\Sigma} \xi \frac{\partial\eta}{\partial\bar{n}} \left(\frac{\partial\Delta\psi}{\partial\bar{n}}\right)^2 d\sigma dt + 2s\lambda \iint_{\Sigma} \xi \frac{\partial\eta}{\partial\bar{n}} |\nabla\Delta\psi|^2 d\sigma dt.
\end{aligned}$$

*Step 4. Estimate of the boundary terms.*

In this step, we are going to prove the following estimate for the boundary term  $\bar{B}$  :

$$\bar{B} \geq -44s^5\lambda^5 \iint_{\Sigma} \xi^5 \left(\frac{\partial\eta}{\partial\bar{n}}\right)^5 \left(\frac{\partial\psi}{\partial\bar{n}}\right)^2 d\sigma dt - s\lambda \iint_{\Sigma} \xi \frac{\partial\eta}{\partial\bar{n}} \left(\frac{\partial\Delta\psi}{\partial\bar{n}}\right)^2 d\sigma dt - C_0 \int_0^T e^{-2s\alpha_*} \|\varphi\|_{H^4(\Omega)}^2 dt, \tag{22}$$

for  $\lambda \geq C_0$  and  $s \geq C_0T$ .

Let's denote

$$\begin{aligned}
J &= 12s^3\lambda^3 \iint_{\Sigma} \xi^3 \left(\frac{\partial\eta}{\partial\bar{n}}\right)^3 |\nabla^2\psi \bar{n}|^2 d\sigma dt - 2s^3\lambda^3 \iint_{\Sigma} \xi^3 \left(\frac{\partial\eta}{\partial\bar{n}}\right)^2 |\nabla^2\psi|^2 d\sigma dt \\
&\quad - 4s\lambda \iint_{\Sigma} \xi \frac{\partial\eta}{\partial\bar{n}} \left(\frac{\partial\Delta\psi}{\partial\bar{n}}\right)^2 d\sigma dt + 2s\lambda \iint_{\Sigma} \xi \frac{\partial\eta}{\partial\bar{n}} |\nabla\Delta\psi|^2 d\sigma dt.
\end{aligned}$$

By using (15), (16) and the fact that any vector can be decomposed uniquely as a sum of two vectors, one tangent to the curve (called the tangential component of the vector) and another one perpendicular to the curve (called the normal component of the vector), we can deduce

$$\begin{aligned} J \geq & 40s^5\lambda^5 \iint_{\Sigma} \xi^5 \left( \frac{\partial\eta}{\partial\vec{n}} \right)^5 \left( \frac{\partial\psi}{\partial\vec{n}} \right)^2 d\sigma dt - 2s\lambda \iint_{\Sigma} \xi \frac{\partial\eta}{\partial\vec{n}} \left( \frac{\partial\Delta\psi}{\partial\vec{n}} \right)^2 d\sigma dt \\ & + 16s^3\lambda^3 \iint_{\Sigma} \xi^3 \left( \frac{\partial\eta}{\partial\vec{n}} \right)^3 \left( \frac{\partial^2\psi}{\partial\vec{\tau}\partial\vec{n}} \right)^2 d\sigma dt - C_0B_1. \end{aligned}$$

Let's estimate the last term. We use the interpolation inequality

$$\left\| \frac{\partial\psi}{\partial\vec{n}} \right\|_{H^1(\partial\Omega)} \leq C \left\| \frac{\partial\psi}{\partial\vec{n}} \right\|_{L^2(\partial\Omega)}^{3/5} \left\| \frac{\partial\psi}{\partial\vec{n}} \right\|_{H^{5/2}(\partial\Omega)}^{2/5}$$

and we deduce, from Young's inequality,

$$\begin{aligned} & \left| 16s^3\lambda^3 \iint_{\Sigma} \xi^3 \left( \frac{\partial\eta}{\partial\vec{n}} \right)^3 \left( \frac{\partial^2\psi}{\partial\vec{\tau}\partial\vec{n}} \right)^2 d\sigma dt \right| \leq C_0s^3\lambda^3 \int_0^T \xi^3 \left\| \frac{\partial\psi}{\partial\vec{n}} \right\|_{H^1(\partial\Omega)}^2 dt \\ & \leq -s^5\lambda^5 \iint_{\Sigma} \xi^5 \left( \frac{\partial\eta}{\partial\vec{n}} \right)^5 \left( \frac{\partial\psi}{\partial\vec{n}} \right)^2 d\sigma dt + C_0 \int_0^T \left\| \frac{\partial\psi}{\partial\vec{n}} \right\|_{H^{5/2}(\partial\Omega)}^2 dt \\ & \leq -s^5\lambda^5 \iint_{\Sigma} \xi^5 \left( \frac{\partial\eta}{\partial\vec{n}} \right)^5 \left( \frac{\partial\psi}{\partial\vec{n}} \right)^2 d\sigma dt + C_0 \int_0^T e^{-2s\alpha_*} \|\varphi\|_{H^4(\Omega)}^2 dt. \end{aligned}$$

For the last estimate, we have used the continuity of the trace operator.

We deduce

$$J \geq 42s^5\lambda^5 \iint_{\Sigma} \xi^5 \left( \frac{\partial\eta}{\partial\vec{n}} \right)^5 \left( \frac{\partial\psi}{\partial\vec{n}} \right)^2 d\sigma dt - 2s\lambda \iint_{\Sigma} \xi \frac{\partial\eta}{\partial\vec{n}} \left( \frac{\partial\Delta\psi}{\partial\vec{n}} \right)^2 d\sigma dt - C_0 \int_0^T e^{-2s\alpha_*} \|\varphi\|_{H^4(\Omega)}^2 dt.$$

Finally, from

$$-4s^3\lambda^3 \iint_{\Sigma} \xi^3 \frac{\partial\psi}{\partial\vec{n}} \frac{\partial\Delta\psi}{\partial\vec{n}} \left( \frac{\partial\eta}{\partial\vec{n}} \right)^3 d\sigma dt \geq 4s^5\lambda^5 \iint_{\Sigma} \xi^5 \left( \frac{\partial\eta}{\partial\vec{n}} \right)^5 \left| \frac{\partial\psi}{\partial\vec{n}} \right|^2 d\sigma dt + s\lambda \iint_{\Sigma} \xi \frac{\partial\eta}{\partial\vec{n}} \left| \frac{\partial\Delta\psi}{\partial\vec{n}} \right|^2 d\sigma dt,$$

we deduce the desired inequality (22).

Coming back to (21), we find

$$\begin{aligned} (P_1(\psi), P_2(\psi))_{L^2(Q)} \geq & 8s^7\lambda^8 \iint_Q \xi^6 |\nabla\eta|^8 |\psi|^2 dx dt - 16s^5\lambda^6 \iint_Q \xi^5 |\nabla\eta|^6 |\nabla\psi|^2 dx dt \\ & + 8s^3\lambda^4 \iint_Q \xi^3 |\nabla\eta|^4 |\Delta\psi|^2 dx dt + 22s^5\lambda^6 \iint_Q \xi^5 |\nabla\eta|^4 |\nabla\eta \cdot \nabla\psi|^2 dx dt \\ & + 3s\lambda^2 \iint_Q \xi |\nabla\eta \cdot \nabla\Delta\psi|^2 dx dt - s^5\lambda^5 \iint_{\Sigma} \xi^5 \left( \frac{\partial\eta}{\partial\vec{n}} \right)^5 \left| \frac{\partial\psi}{\partial\vec{n}} \right|^2 d\sigma dt \\ & - C_0 \left( A + B_3 + B_4 + \|e^{-s\alpha_*} \varphi\|_{L^2(0,T;H^4(\Omega))}^2 + s^5\lambda^5 \iint_{\omega' \times (0,T)} \xi^5 |\nabla\psi|^2 dx dt \right), \end{aligned} \tag{23}$$

for  $\lambda \geq C_0$  and  $s \geq C_0(T^{1/2} + T)$ .

*Step 5. Last arrangements and conclusion.*

Note that the first three terms in the right-hand side of (23) are positive up to a residual term:

$$\begin{aligned}
& 8s^7\lambda^8 \iint_Q \xi^6 |\nabla\eta|^8 |\psi|^2 dxdt - 16s^5\lambda^6 \iint_Q \xi^5 |\nabla\eta|^6 |\nabla\psi|^2 dxdt + 8s^3\lambda^4 \iint_Q \xi^3 |\nabla\eta|^4 |\Delta\psi|^2 dxdt \\
& \geq 8s^{-1} \iint_Q \xi^{-1} (s^4\lambda^4\xi^4 |\nabla\eta|^4 \psi + s^2\lambda^2\xi^2 |\nabla\eta|^2 \Delta\psi)^2 - Cs^5\lambda^8 \iint_Q \xi^5 |\psi|^2 dxdt,
\end{aligned} \tag{24}$$

for  $\lambda \geq C_0$ . Then, coming back to  $\varphi$ , we can prove that some term in  $|\Delta\varphi|^2$  is bounded by the first four terms in the right-hand side of (23). Indeed, from

$$\varphi = e^{s\alpha}\psi \Rightarrow \Delta\varphi = e^{s\alpha} (\Delta\psi + s^2\lambda^2\xi^2 |\nabla\eta|^2 \psi - 2s\lambda\xi \nabla\eta \cdot \nabla\psi - s\lambda^2\xi |\nabla\eta|^2 \psi - s\lambda\xi \Delta\eta\psi),$$

we deduce that

$$\begin{aligned}
s^3\lambda^4 \iint_Q \xi^3 e^{-2s\alpha} |\nabla\eta|^4 |\Delta\varphi|^2 dxdt &= s^{-1} \iint_Q \xi^{-1} (e^{-s\alpha} s^2\lambda^2\xi^2 |\nabla\eta|^2 \Delta\varphi)^2 dxdt \\
&= s^{-1} \iint_Q \xi^{-1} \left( s^2\lambda^2\xi^2 |\nabla\eta|^2 \Delta\psi + s^4\lambda^4\xi^4 |\nabla\eta|^4 \psi \right. \\
&\quad \left. - 2s^3\lambda^3\xi^3 |\nabla\eta|^2 \nabla\eta \cdot \nabla\psi - s^3\lambda^4\xi^3 |\nabla\eta|^4 \psi - s^3\lambda^3\xi^3 |\nabla\eta|^2 \Delta\eta\psi \right)^2 dxdt \\
&\leq 8s^{-1} \iint_Q \xi^{-1} (s^2\lambda^2\xi^2 |\nabla\eta|^2 \Delta\psi + s^4\lambda^4\xi^4 |\nabla\eta|^4 \psi)^2 dxdt \\
&\quad + 8s^5\lambda^6 \iint_Q \xi^5 |\nabla\eta|^4 |\nabla\eta \cdot \nabla\psi|^2 dxdt + C_0 s^5\lambda^8 \iint_Q \xi^5 |\psi|^2 dxdt.
\end{aligned}$$

Here we used  $(a+b+c)^2 \leq 8a^2 + 2b^2 + 3c^2$ . Combining this and (24) with (23), we obtain

$$\begin{aligned}
(P_1(\psi), P_2(\psi))_{L^2(Q)} &\geq s^3\lambda^4 \iint_Q \xi^3 e^{-2s\alpha} |\nabla\eta|^4 |\Delta\varphi|^2 dxdt + 14s^5\lambda^6 \iint_Q \xi^5 |\nabla\eta|^4 |\nabla\eta \cdot \nabla\psi|^2 dxdt \\
&\quad + 3s\lambda^2 \iint_Q \xi |\nabla\eta \cdot \nabla\Delta\psi|^2 dxdt - s^5\lambda^5 \iint_{\Sigma} \xi^5 \left( \frac{\partial\eta}{\partial\vec{n}} \right)^5 \left| \frac{\partial\psi}{\partial\vec{n}} \right|^2 d\sigma dt \\
&\quad - C_0 \left( A + B_3 + B_4 + \|e^{-s\alpha^*}\varphi\|_{L^2(0,T;H^4(\Omega))}^2 + s^5\lambda^5 \iint_{\omega' \times (0,T)} \xi^5 |\nabla\psi|^2 dxdt \right),
\end{aligned} \tag{25}$$

for  $\lambda \geq C_0$  and  $s \geq C_0(T + T^{1/2})$ . From the definition of  $P_1\psi$  (see (17)), we find

$$\begin{aligned}
s^{-1} \iint_Q \xi^{-1} |\partial_t\psi|^2 dxdt &\leq C_0 \left( \|P_1\psi\|_{L^2(Q)}^2 + s^5\lambda^6 \iint_Q \xi^5 |\nabla\eta|^4 |\nabla\eta \cdot \nabla\psi|^2 dxdt \right. \\
&\quad \left. + s\lambda^2 \iint_Q \xi |\nabla\eta \cdot \nabla\Delta\psi|^2 dxdt + A_1 + A_3 \right),
\end{aligned}$$

for  $s \geq C_0T$ . Using here the definition of  $\psi$  (see (14)) and the equation  $\Delta^2\varphi = \partial_t\varphi + f$ , we have

$$\begin{aligned}
s^{-1} \iint_Q \xi^{-1} e^{-2s\alpha} (|\partial_t\varphi|^2 + |\Delta^2\varphi|^2) dxdt &\leq C_0 \left( \|P_1\psi\|_{L^2(Q)}^2 + s^5\lambda^6 \iint_Q \xi^5 |\nabla\eta|^4 |\nabla\eta \cdot \nabla\psi|^2 dxdt \right. \\
&\quad \left. + \|e^{-s\alpha} f\|_{L^2(Q)}^2 + s\lambda^2 \iint_Q \xi |\nabla\eta \cdot \nabla\Delta\psi|^2 dxdt + A_1 + A_3 \right).
\end{aligned}$$

We use now the fact that

$$\|P_1\psi\|_{L^2(Q)}^2 + \|P_2\psi\|_{L^2(Q)}^2 + 2(P_1\psi, P_2\psi)_{L^2(Q)} = \|R\psi - e^{-s\alpha} f\|_{L^2(Q)}^2$$

(see (18)) and we find from (25)

$$\begin{aligned}
& s^3 \lambda^4 \iint_Q \xi^3 e^{-2s\alpha} |\Delta \varphi|^2 dxdt + s^{-1} \iint_Q \xi^{-1} e^{-2s\alpha} (|\partial_t \varphi|^2 + |\Delta^2 \varphi|^2) dxdt + s^5 \lambda^5 \iint_\Sigma \xi^5 e^{-2s\alpha} \left| \frac{\partial \varphi}{\partial \vec{n}} \right|^2 d\sigma dt \\
& \leq C_0 \left( A + B_3 + B_4 + \|e^{-s\alpha} f\|_{L^2(Q)}^2 + \|e^{-s\alpha_*} \varphi\|_{L^2(0,T;H^4(\Omega))}^2 \right. \\
& \quad \left. + s^3 \lambda^4 \iint_{\omega' \times (0,T)} \xi^3 (|\Delta \varphi|^2 + s^2 \lambda \xi^2 |\nabla \varphi|^2 + s^4 \lambda^3 \xi^4 |\varphi|^2) dxdt \right), \tag{26}
\end{aligned}$$

for  $\lambda \geq C_0$  and  $s \geq C_0(T^{1/2} + T)$ . The next step will be to add lower and higher order terms in the left-hand side of (26).

First, we apply Remark 2.5 and we use the definition of  $\psi$  (see (14)),  $A_1$  and  $A_2$  (see (19)) :

$$\begin{aligned}
& s^6 \lambda^8 \iint_Q \xi^6 e^{-2s\alpha} |\varphi|^2 dxdt + s^4 \lambda^6 \iint_Q \xi^4 e^{-2s\alpha} |\nabla \varphi|^2 dxdt + s^3 \lambda^4 \iint_Q \xi^3 e^{-2s\alpha} |\Delta \varphi|^2 dxdt \\
& + s^{-1} \iint_Q \xi^{-1} e^{-2s\alpha} (|\partial_t \varphi|^2 + |\Delta^2 \varphi|^2) dxdt + s^5 \lambda^5 \iint_\Sigma \xi^5 e^{-2s\alpha} \left| \frac{\partial \varphi}{\partial \vec{n}} \right|^2 d\sigma dt \\
& \leq C_0 \left( s^3 \lambda^4 \iint_{\omega' \times (0,T)} \xi^3 (|\Delta \varphi|^2 + s^2 \lambda \xi^2 |\nabla \varphi|^2 + s^4 \lambda^4 \xi^4 |\varphi|^2) dxdt \right. \\
& \quad \left. + A_3 + B_3 + B_4 + \|e^{-s\alpha_*} \varphi\|_{L^2(0,T;H^4(\Omega))}^2 + \|e^{-s\alpha} f\|_{L^2(Q)}^2 \right), \tag{27}
\end{aligned}$$

for  $\lambda \geq C_0$  and  $s \geq C_0(T^{1/2} + T)$ . Next, we use Lemma 2.3 (see estimate (11)) to absorb  $B_3$ ,  $B_4$  and  $\|e^{-s\alpha_*} \varphi\|_{L^2(0,T;H^4(\Omega))}^2$  (see (19) and (20) for the definition of  $B_3$  and  $B_4$ ) :

$$\begin{aligned}
& s^6 \lambda^8 \iint_Q \xi^6 e^{-2s\alpha} |\varphi|^2 dxdt + s^4 \lambda^6 \iint_Q \xi^4 e^{-2s\alpha} |\nabla \varphi|^2 dxdt + s^3 \lambda^4 \iint_Q \xi^3 e^{-2s\alpha} |\Delta \varphi|^2 dxdt \\
& + s^{-1} \iint_Q \xi^{-1} e^{-2s\alpha} (|\partial_t \varphi|^2 + |\Delta^2 \varphi|^2) dxdt + s^5 \lambda^5 \iint_\Sigma \xi^5 e^{-2s\alpha} \left| \frac{\partial \varphi}{\partial \vec{n}} \right|^2 d\sigma dt \\
& \leq C_0 \left( s^3 \lambda^4 \iint_{\omega' \times (0,T)} \xi^3 (|\Delta \varphi|^2 + s^2 \lambda \xi^2 |\nabla \varphi|^2 + s^4 \lambda^4 \xi^4 |\varphi|^2) dxdt + A_3 + \|e^{-s\alpha} f\|_{L^2(Q)}^2 \right), \tag{28}
\end{aligned}$$

for  $\lambda \geq C_0$  and  $s \geq C_0(T^{1/2} + T)$ .

Let us now set  $\tilde{\varphi} := s\lambda^2 e^{-s\alpha} \xi \varphi$ . We observe that  $\|\Delta \tilde{\varphi}\|_{L^2(Q)}^2$  is bounded by the left-hand side of (28), which means that  $\|\tilde{\varphi}\|_{L^2(0,T;H^2(\Omega))}^2$  also is, since  $\tilde{\varphi} = 0$  on  $\Sigma$ . This allows us to add the term

$$s^2 \lambda^4 \iint_Q e^{-2s\alpha} \xi^2 |\nabla^2 \varphi|^2 dxdt \tag{29}$$

to the left-hand side of (28) and absorb  $A_3$ .

Furthermore, by integration par parts we deduce that

$$s\lambda^2 \iint_Q e^{-2s\alpha} \xi |\nabla \Delta \varphi|^2 dxdt \leq C_0 \left( s^3 \lambda^4 \iint_Q e^{-2s\alpha} \xi^3 |\Delta \varphi|^2 dxdt + s^{-1} \iint_Q e^{-2s\alpha} \xi^{-1} |\Delta^2 \varphi|^2 dxdt \right), \tag{30}$$

for  $s \geq C_0(T + T^{1/2})$  and  $\lambda \geq C_0$ . Plugging (29)-(30) into (28), we obtain

$$\begin{aligned}
& s^6 \lambda^8 \iint_Q e^{-2s\alpha \xi^6} |\varphi|^2 dxdt + s^4 \lambda^6 \iint_Q e^{-2s\alpha \xi^4} |\nabla \varphi|^2 dxdt + s^3 \lambda^4 \iint_Q \xi^3 e^{-2s\alpha} |\Delta \varphi|^2 dxdt \\
& + s^2 \lambda^4 \iint_Q e^{-2s\alpha \xi^2} |\nabla^2 \varphi|^2 dxdt + s \lambda^2 \iint_Q e^{-2s\alpha \xi} |\nabla \Delta \varphi|^2 dxdt + s^{-1} \iint_Q \xi^{-1} e^{-2s\alpha} (|\partial_t \varphi|^2 + |\Delta^2 \varphi|^2) dxdt \\
& \leq C_0 \left( s^3 \lambda^4 \iint_{\omega' \times (0, T)} \xi^3 (|\Delta \varphi|^2 + s^2 \lambda \xi^2 |\nabla \varphi|^2 + s^4 \lambda^4 \xi^4 |\varphi|^2) dxdt + \iint_Q e^{-2s\alpha} |f|^2 dxdt \right),
\end{aligned} \tag{31}$$

for  $s \geq C_0(T + T^{1/2})$  and  $\lambda \geq C_0$ .

We are now ready to estimate the local terms on  $\nabla \varphi$  and  $\Delta \varphi$ . To this end, let us introduce a function  $\theta = \theta(x)$ , with

$$\theta \in C_c^2(\omega), \quad \theta = 1 \text{ in } \omega' \text{ and } 0 \leq \theta \leq 1$$

and let us make some computations :

$$\begin{aligned}
& s^3 \lambda^4 \iint_{\omega' \times (0, T)} e^{-2s\alpha \xi^3} |\Delta \varphi|^2 dxdt + s^5 \lambda^5 \iint_{\omega' \times (0, T)} e^{-2s\alpha \xi^5} |\nabla \varphi|^2 dxdt \\
& \leq s^3 \lambda^4 \iint_{\omega \times (0, T)} e^{-2s\alpha \xi^3 \theta^2} |\Delta \varphi|^2 dxdt + s^5 \lambda^5 \iint_{\omega \times (0, T)} e^{-2s\alpha \xi^5 \theta} |\nabla \varphi|^2 dxdt \\
& \leq \epsilon s^3 \lambda^4 \iint_Q e^{-2s\alpha \xi^3} |\Delta \varphi|^2 dxdt + \epsilon s \lambda^2 \iint_Q e^{-2s\alpha \xi} |\nabla \Delta \varphi|^2 dxdt \\
& + C_0 s^5 \lambda^6 \iint_{\omega \times (0, T)} e^{-2s\alpha \xi^5 \theta} |\nabla \varphi|^2 dxdt \\
& \leq \epsilon s^3 \lambda^4 \iint_Q e^{-2s\alpha \xi^3} |\Delta \varphi|^2 dxdt + \epsilon s \lambda^2 \iint_Q e^{-2s\alpha \xi} |\nabla \Delta \varphi|^2 dxdt \\
& + C_0 s^7 \lambda^8 \iint_{\omega \times (0, T)} e^{-2s\alpha \xi^7} |\varphi|^2 dxdt
\end{aligned}$$

for a small enough constant  $\epsilon = \epsilon(\Omega, \omega) > 0$  and where we have used the fact that  $\lambda \geq 1$  and  $s \geq C_0 T$ . From (31) and this estimate, we deduce the desired estimate (3).  $\square$

### 3 Open problems

In this section, we are going to talk about some open problems. The first open problem concerns the study of the controllability of system (1) with different boundary conditions.

**Open problem 3.1.** *From the proof of Theorem 1.2, we will see that the Carleman inequality (3) also holds when  $\varphi$  satisfies the boundary conditions*

$$\varphi = \frac{\partial \varphi}{\partial \vec{n}} = 0 \text{ on } \Sigma.$$

Nevertheless, the situation where

$$\frac{\partial \varphi}{\partial \vec{n}} = \frac{\partial \Delta \varphi}{\partial \vec{n}} = 0 \text{ on } \Sigma$$

is an more complicated. This case will be studied in a forthcoming paper.

The next open problem concerns the study of semi-linear fourth order parabolic equations.

**Open problem 3.2.** A challenging problem is the study of the controllability of the following system

$$\begin{cases} \partial_t y + \Delta^2 y + f(y, \nabla y, \nabla^2 y) = \chi_\omega v & \text{in } Q, \\ y = \Delta y = 0 & \text{on } \Sigma, \\ y(0, \cdot) = y_0(\cdot) & \text{in } \Omega, \end{cases} \quad (32)$$

where  $f$  verifies the following conditions:

$$f(s, p, q) = g(s, p, q)s + G(s, p, q) \cdot p + E(s, p, q) : q, \quad s \in \mathbb{R}, p \in \mathbb{R}^N, q \in \mathbb{R}^N \times \mathbb{R}^N,$$

with  $g \in L_{loc}^\infty(\mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^{N^2})$ ,  $G \in L_{loc}^\infty(\mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^{N^2})^N$ ,  $E \in L_{loc}^\infty(\mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^{N^2})^{N^2}$ ,

$$\lim_{|s|, |p|, |q| \rightarrow \infty} \frac{|g(s, p, q)|}{\log^{\theta_g}(1 + |s| + |p| + |q|)} = 0,$$

$$\lim_{|s|, |p|, |q| \rightarrow \infty} \frac{|G(s, p, q)|}{\log^{\theta_G}(1 + |s| + |p| + |q|)} = 0$$

and

$$\lim_{|s|, |p|, |q| \rightarrow \infty} \frac{|E(s, p, q)|}{\log^{\theta_E}(1 + |s| + |p| + |q|)} = 0,$$

for some positive  $\theta_g$ ,  $\theta_G$  and  $\theta_E$ . This kind of problems has already been studied for systems associated to heat equations : for more details, see for instance [14] and [12]. This open problem will be studied in [20].

The next open problem concerns the study of the existence of insensitizing controls for fourth order parabolic equations with  $N \geq 2$ .

**Open problem 3.3.** Let  $O \subset \Omega$  be a (small) nonempty open subset such that  $O \cap \omega \neq \emptyset$ . Let us introduce the functional

$$\phi(w) = \frac{1}{2} \iint_{O \times (0, T)} |w|^2 dx dt \quad (33)$$

and the following control system :

$$\begin{cases} \partial_t w + \Delta^2 w = \chi_\omega h & \text{in } Q, \\ w = \Delta w = 0 & \text{on } \Sigma, \\ w(0, \cdot) = y_0(\cdot) + \tau \tilde{y}_0 & \text{in } \Omega, \end{cases} \quad (34)$$

where  $w_0 \in L^2(\Omega)$  is the initial condition,  $h \in L^2(Q)$  is the control function and  $\tau \in \mathbb{R}$  and  $\tilde{y}_0 \in L^2(\Omega)$  are unknown but  $\tau$  is small enough and  $\tilde{y}_0$  satisfies  $\|\tilde{y}_0\|_{L^2(\Omega)} = 1$ . The objective is to establish the existence of insensitizing controls for this equation. In other words, the goal is to find a control  $h$  which insensitizes  $\phi$ , i.e.,

$$\left| \frac{\partial \phi(w(x, t, h, \tau))}{\partial \tau} \Big|_{\tau=0} \right| = 0. \quad (35)$$

For more details about insensitizing controls for the heat equation, see for instance [10] and [4]. This open problem will be studied in [21].

The next open problem concerns the study of boundary controllability.

**Open problem 3.4.** An interesting open problem is the study of the boundary controllability. Let us consider the following system:

$$\begin{cases} \partial_t y + \Delta^2 y = 0 & \text{in } Q, \\ y = \chi_{\gamma_1} v_1 & \text{on } \Sigma, \\ \Delta y = \chi_{\gamma_2} v_2 & \text{on } \Sigma, \\ y(0, \cdot) = y_0(\cdot) & \text{in } \Omega. \end{cases} \quad (36)$$

The case where  $\gamma_1 \cap \gamma_2 \neq \emptyset$  can be treated directly from Theorem 1.5 by extending the domain (see page 28-29 in [15] for more details). When  $\gamma_1 \cap \gamma_2 = \emptyset$ , the idea would be to establish the following observability inequality for the solutions of system (2) with  $f \equiv 0$  :

$$\|\varphi(0)\|_{L^2(\Omega)}^2 \leq \tilde{C} \left( \iint_{[0,T] \times \gamma_1} \left| \frac{\partial \Delta \varphi}{\partial \vec{n}} \right|^2 d\sigma dt + \iint_{[0,T] \times \gamma_2} \left| \frac{\partial \varphi}{\partial \vec{n}} \right|^2 d\sigma dt \right).$$

This estimate seems hard to prove.

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