Uniform controllability of a transport equation in zero fourth order equation-dispersion limit

Karim Kassab

Uniform controllability of a transport equation in zero fourth order equation-dispersion limit

K.Kassab[*]

November 13, 2020

Abstract

In this paper, we study the cost of a transport equation perturbed by small diffusion and dispersion terms. When the control time is large enough, we prove that this cost decreases exponentially to zero as the diffusion and dispersion coefficients of the equation vanishes. When the control time is small, on the contrary, we prove that this cost increases exponentially to infinity.

Keywords: Fourth order parabolic equation, Carleman estimates, cost of null controllability, boundary control, uniform null controllability, diffusion-dispersion limit.

1 Introduction

In the present paper, we consider $\Omega =]0, L[\subset \mathbb{R}$. We will use the notation $Q = (0,T) \times \Omega$. On the other hand, we will denote by C_0 a generic positive constant which depends on Ω and ω but not on T.

Let us introduce the following control system :

$$\begin{cases} \partial_t y + \varepsilon y_{xxxx} - \delta y_{xxx} + M y_x = 0 & \text{in } Q\ , \\ y(t,0) = v_1, \quad y(t,L) = 0, & t \in (0,T)\ , \\ y_x(t,0) = v_2, \quad y_x(t,L) = 0, & t \in (0,T)\ , \\ y(0,\cdot) = y_0(\cdot) & \text{in } \Omega\ , \end{cases} \tag{1}$$

where $y_0 \in L^2(\Omega)$ is the initial condition, $\varepsilon > 0$, $\delta > 0$ and $M > 0$ and v_1, v_2 are the control functions. The purpose of this paper is to study the cost of null controllability of equation (1) given by the following formula :

$$C_y(\varepsilon,\delta) = \sup_{y_0 \in L^2(\Omega),\ y_0 \neq 0} \ \min_{(v_1,v_2) \in L^2(0,T)^2,\ y(T,\cdot)=0} \frac{\|v_1\|^2_{L^2(0,T)} + \|v_2\|^2_{L^2(0,T)}}{\|y_0\|^2_{L^2(0,L)}}. \tag{2}$$

Being more precise, we are interested to know about its behavior with respect to the diffusion and dispersion coefficient, and in particular, to know what happens as $\varepsilon \to 0^+$ and $\delta \to 0^+$. The motivation for studying the dissipation-dispersion mechanism comes from continuum mechanics. These terms may represent viscosity and capillarity-driven surface diffusion. These are particularly important in the theory of nonclassical shock waves (see the book of LeFloch [16]). Nonclassical shock waves are shock waves for conservation laws with nonconvex flux, which are selected through perturbative terms such as the ones of (1). Before we continue, let us consisder the following unperturbed transport equation :

$$\partial_t y + M y_x = 0 \text{ in } \ ,$$

[*]Laboratoire JACQUES-LOUIS LIONS, Université PIERRE ET MARIE CURIE, 75005 PARIS-FRANCE,
E-mail: kassab@ljll.math.upmc.fr

where $y(0,\cdot)=y_0\in L^2(0,L)$, controlled from the boundary $y_{x=0}=u(\cdot)$ if $M>0$ and $y_{x=L}=u(\cdot)$ if $M<0$ and where $u\in L^2(0,T)$. It is known that this equation is null controllable if $T>L/|M|$. Indeed, it suffices to take $u=0$ to deduce that $y(T,\cdot)=0$ and we have a null cost. On the other hand, if $T<L/|M|$, it is easy to see that this equation is not null controllable. Our objective is to prove that it is possible for $T\gtrsim L/|M|$ to control (1) at a uniform cost as ε and δ tend to 0. On the other hand, it is to expect that for times $T\lesssim L/|M|$, the cost of null-controllability will dramatically increase.

Let us now present some interesting results related to our work. This kind of problems was initially considered in [6] for the case of the heat equation with vanishing viscosity coefficient. (see also Refs. [14] and [10]). Later, improvements have been done in [9], [18] and [17]. On the other hand, authors in [1], [11] and [12] studied many problems related to the linear Korteweg de Vries equation with vanishing dispersion coefficient.

Concerning unperturbed fourth-order parabolic equation, many results were proved in [13],[7],[20],[15],[5] and [8] for internal controllability and [3],[4] for boundary controllability.

Up to our knowledge, there exist two works addressing this kind of problem for a perturbed fourth-order parabolic equation where in the first one [2], the authors treated this problem for $\delta=0$ with different boundary conditions and in the second work [19], the authors treated this problem for $\delta=2\varepsilon^{2/3}M^{1/3}$ with different boundary conditions. Let us start with our first result :

Theorem 1.1. *There exists $C>0$ such that, for any $y_0\in L^2(0,L)$, there exists v_1, $v_2\in L^2(0,T)$ driving $y_0\in L^2(0,L)$ to zero and which can be estimated as follows :*

$$\|v_1\|_{L^2(0,T)}+\|v_2\|_{L^2(0,T)}\leq \exp\left\{\frac{C}{\min\{\delta^{1/2},\,\varepsilon^{1/3}\}}\right\}\|y_0\|_{L^2(0,L)}.$$

The proof of this result is based on the controllability-observability duality (see also Proposition 3.3). Nevertheless, this information does not allow us to say anything about the behavior of the cost when $\varepsilon\to 0^+$ and $\delta\to 0^+$. In the following result, we establish the uniform null controllability, with respect to the diffusion coefficient, of equation (1) when the control time is large enough and the initial condition is in $L^2(0,L)$:

Theorem 1.2. *There exists a constant $C>0$ such that, for any $y_0\in L^2(0,L)$, $M>0$ and $T>C\,L/M$, there exist two constants $\hat{C}>0$ and $\hat{c}>0$ depending on T, such that for any $(\varepsilon,\delta)\in(0,1]\times[0,1]$, there exists v_1, $v_2\in L^2(0,T)$ driving $y_0\in L^2(0,L)$ to zero and which can be estimated as follows :*

$$\|v_1\|_{L^2(0,T)}+\|v_2\|_{L^2(0,T)}\leq \frac{\hat{C}}{\sqrt{M\varepsilon}}\exp\left\{-\frac{\hat{c}M^{1/2}}{\max\{\delta^{1/2},\,(M^{1/2}\varepsilon)^{1/3}\}}\right\}\|y_0\|_{L^2(0,L)}.$$

In the following result, we give a lower bound for the norms of the controls when the control time is small and the initial condition is in $L^2(0,L)$:

Theorem 1.3. *Let $M>0$ and $T>0$ such that*

$$T<\frac{L}{M}.$$

Then there exist $y_0\in L^2(0,L)$, $c>0$ and $l\in\mathbb{N}$ independent of $\varepsilon\in(0,1]$ and $\delta\in[0,1]$, such that for any $v_1,v_2\in L^2(0,T)$ driving y_0 to 0 are estimated from below as follows :

$$\|v_1\|_{L^2(0,T)}+\|v_2\|_{L^2(0,T)}\geq c\varepsilon^l\exp\left\{\frac{c}{\max(\delta^{1/2},\varepsilon^{1/3})}\right\}\|y_0\|_{L^2(0,L)}. \tag{3}$$

This paper is organized as follows. In Section 2, we study the existence of solution for system (1). In Section 3, we prove Theorem 1.2 which states the null controllability of equation (1), by using a new Carleman estimate and a new exponential dissipation result. In Section 4, we prove Theorem 1.3 which gives a lower bound for the norms of the controls when the control time is small and the initial condition is in $L^2(0,L)$. Finally, in Appendix A, we prove the new Carleman estimate.

2 Cauchy problem.

In this section, we present the well-posedness results needed for the study of equation (1). To this end, let us consider the following adjoint system :

$$\begin{cases} -w_t + \varepsilon w_{xxxx} + \delta w_{xxx} - M w_x = f & \text{in } Q\ , \\ w(t,0) = w(t,L) = 0, & t \in (0,T)\ , \\ w_x(t,0) = w_x(t,L) = 0, & t \in (0,T)\ , \\ w(T,\cdot) = 0 & \text{in } \Omega\ , \end{cases} \tag{4}$$

where $f \in L^2((0,T)\times(0,L))$. The solutions of system (1) are to be understood in the sense of transposition. Before we continue, let us set

$$X := C([0,T]; H^{-2}(0,L)) \cap L^2((0,T)\times(0,L)).$$

Definition 2.1. *Given $T>0$, $y_0 \in H^{-2}(0,L)$ and v_1, $v_2 \in L^2(0,T)$, we call y a solution of (1), a function $y \in X$ satisfying for all $f \in L^2((0,T)\times(0,L))$*

$$\int_0^T\int_0^L y\, f\, dxdt = (y_0, w(0,\cdot))_{H^{-2}(0,L)\times H_0^2(0,L)} - \varepsilon \int_0^T v_1(t) w_{xxx}(t,0)dt + \int_0^T (\varepsilon v_2(t) - \delta v_1(t)) w_{xx}(t,0)dt, \tag{5}$$

where w is the corresponding solution of (4).

It is easy to see that any regular solution of (1) is a solution in the above sense. Indeed, it suffices to use integration by parts.

Proposition 2.2. *Let $M \in \mathbb{R}$, $\varepsilon \in (0,1]$, $\delta \in [0,1]$, $T>0$, $y_0 \in H_0^{-2}(0,L)$ and v_1, $v_2 \in L^2(0,T)$. Then, there exists a unique solution of transposition (5). Moreover, there exists $C>0$ independent of ε and δ such that*

$$\|y\|_X \le \frac{C}{\varepsilon^3}\left(\|y_0\|_{H_0^{-2}(0,L)} + \|v_1\|_{L^2(0,T)} + \|v_2\|_{L^2(0,T)}\right).$$

Proof. To prove this proposition, first we are going to demonstrate that for $f \in L^2(0,L)$, we have $w \in C^0([0,T]; H_0^2(0,L))$ and $w_{xx}(\cdot,0)$, $w_{xxx}(\cdot,0) \in L^2(0,T)$. Then, by using the Riesz representation theorem, we deduce the existence and uniqueness of $y \in L^2((0,T)\times(0,L))$ the solution of (1). At the end, we use the equation verified by y to prove that $y \in C([0,T]; H^{-2}(0,L))$. Now, by multiplying $(4)_1$ by w and integrating by parts, we obtain

$$-\frac{1}{2}\frac{d}{dt}\int_0^L |w(t,x)|^2 dx + \varepsilon \int_0^L |w_{xx}(t,x)|^2 dx = \int_0^L f(t,x) w(t,x) dx. \tag{6}$$

Integrating the last equality between t and T, we deduce

$$\begin{aligned} \int_0^L |w(t,x)|^2 dx + \varepsilon \int_0^T\int_0^L |w_{xx}(t,x)|^2 dxdt \le \frac{C}{\varepsilon}\int_0^T\int_0^L |f(t,x)|^2 dx \\ + \frac{\varepsilon}{2}\int_0^T\int_0^L |w_{xx}(t,x)|^2 dxdt, \end{aligned} \tag{7}$$

for $0 \leq t \leq T$. Then, we deduce

$$\|w\|^2_{C^0([0,T];L^2(0,L))\cap L^2(0,T;H^2_0(0,L))} \leq \frac{C}{\varepsilon^2}\|f\|^2_{L^2(0,T;L^2(0,L))}, \tag{8}$$

for $C > 0$. Now, if we multiply $(4)_1$ by w_{xxxx} and integrating by parts, we have

$$\begin{aligned} -\frac{1}{2}\frac{d}{dt}\int_0^L |w_{xx}(t,x)|^2dx + \varepsilon\int_0^L |w_{xxxx}(t,x)|^2dx \leq \int_0^L f(t,x)w_{xxxx}(t,x)dx \\ +\delta\left|\int_0^L w_{xxxx}(t,x)w_{xxx}(t,x)dx\right| + M\int_0^L w_x\, w_{xxx}dx. \end{aligned} \tag{9}$$

To treat the terms on the right-hand side of (9), let us notice that

$$\begin{aligned} &\left|\int_0^L w_{xxxx}(t,x)w_{xxx}(t,x)dx\right| + \int_0^L f(t,x)w_{xxxx}(t,x)dx + M\int_0^L w_x\, w_{xxx}dx \\ &\leq \frac{\varepsilon}{2}\int_0^L |w_{xxxx}(t,x)|^2dx + C\left(\frac{\delta^4+M^2}{\varepsilon^3}\int_0^L |w_{xx}(t,x)|^2dx + \frac{1}{\varepsilon}\int_0^L |f(t,x)|^2dx\right), \end{aligned} \tag{10}$$

where $C > 0$. Here we used the fact

$$\begin{aligned} \delta\left|\int_0^L w_{xxxx}(t,x)w_{xxx}(t,x)dx\right| &\leq \delta\|w\|^{1/4}_{L^2(0,T;H^2(0,L))}\|w\|^{3/4}_{L^2(0,T;H^4(0,L))} \\ &\leq \frac{C\delta^4}{\varepsilon^3}\|w\|^2_{L^2(0,T;H^2(0,L))} + \frac{\varepsilon}{4}\|w\|^2_{L^2(0,T;H^4(0,L))} \end{aligned}$$

and that there exists $\lambda > 0$ such that for any $u \in H^4(\Omega)\cap H^2_0(\Omega)$, we have

$$\int_\Omega |\Delta^2 u|^2\, dx \geq \lambda\, \|u\|^2_{H^4(\Omega)}. \tag{11}$$

Combining (10) with (9) and integrating between t and T, we deduce

$$\begin{aligned} \int_0^L |w_{xx}(t,x)|^2dx + \varepsilon\int_0^T\int_0^L |w_{xxxx}(t,x)|^2dx \leq C\Bigg(\frac{\delta^4+M^2}{\varepsilon^3}\|w\|^2_{L^2(0,T;H^2_0(0,L))} \\ +\frac{1}{\varepsilon}\int_0^T\int_0^L |f(t,x)|^2dx\Bigg), \end{aligned} \tag{12}$$

where $C > 0$ and for $0 \leq t \leq T$. From (8) and (12), we deduce

$$\|w\|^2_{C^0([0,T];H^2_0(0,L))\cap L^2(0,T;H^4_0(0,L))} \leq \frac{C}{\varepsilon^6}\|f\|^2_{L^2(0,T;L^2(0,L))}, \tag{13}$$

for $C > 0$. We can deduce by using the last estimate with (5)

$$\|y\|_{L^2((0,T)\times(0,L))} \leq \frac{C}{\varepsilon^3}\left(\|y_0\|_{H_0^{-2}(0,L)} + \|v_1\|_{L^2(0,T)} + \|v_2\|_{L^2(0,T)}\right).$$

Now, from $(1)_1$ combined with the previous estimate, we deduce

$$\partial_t y = -\varepsilon y_{xxxx} + \delta y_{xxx} - My_x \in L^2(0,T;H^{-4}(0,L)).$$

Then, we deduce that $y \in L^2(0,T)\times L^2(0,L))\cap H^1(0,T;H^{-4}(0,L))$. By using an interpolation argument, we finish the proof of Proposition 2.2. □

3 Proof of Theorem 1.2.

3.1 Carleman estimate.

Let us consider the following adjoint system :

$$\begin{cases} -\partial_t\varphi + \varepsilon\varphi_{xxxx} + \delta\varphi_{xxx} - M\varphi_x = 0 & \text{in } Q\ , \\ \varphi(t,0) = \varphi(t,L) = 0, & t \in (0,T)\ , \\ \varphi_x(t,0) = \varphi_x(t,L) = 0, & t \in (0,T)\ , \\ \varphi(T,\cdot) = \varphi_0(\cdot) & \text{in } \Omega\ , \end{cases} \tag{14}$$

where $\varphi_0 \in L^2(0,T)$. The objective of this section is to state a Carleman inequality for the solutions of this system. Let us introduce the weight function :

$$\alpha(t,x) = \frac{-\frac{1}{2}x^2 + 8Lx + \frac{L}{2}}{t^\mu(T-t)^\mu}, \tag{15}$$

where $\mu \in [\frac{1}{3}, \frac{1}{2}]$.

Remark 3.1. *We have*

$$\alpha^{-1} \le CT^{2\mu}\,, \quad C_1\alpha \le \alpha_x \le C_2\alpha\,, \quad C_1\alpha \le -\alpha_{xx} \le C_2\alpha \quad \textit{in } (0,T)\times(0,L),$$

and

$$|\alpha_t| + |\alpha_{tx}| + |\alpha_{txx}| \le CT\alpha^{1+1/\mu}\,, \quad |\alpha_{tt}| \le CT^2\alpha^{1+2/\mu} \textit{ in } (0,T)\times(0,L),$$

where C, C_1, C_2 are positive constants indenpendent of T.

Proposition 3.2. *Let $\mu \in [\frac{1}{3}, \frac{1}{2}]$. Then, there exists a positive constant C independent of $T > 0$, $\varepsilon > 0$, $\delta \ge 0$, $M > 0$ such that, for any $\varphi_0 \in L^2(\Omega)$, we have*

$$\begin{aligned} &\varepsilon^2 s^7 \iint_Q e^{-2s\alpha}\alpha^7|\varphi|^2dxdt + s^5\delta^2 \iint_Q e^{-2s\alpha}\alpha^5|\varphi|^2dxdt \\ &\le C\bigg(\int_0^T e^{-2s\alpha(t,0)}(\varepsilon^2 s\alpha(t,0) + \varepsilon\delta)|\varphi_{xxx}(t,0)|^2dt \\ &+ \int_0^T (e^{-2s\alpha(0)}(\varepsilon^2 s^3\alpha^3(t,0) + \delta^2 s\alpha(t,0))|\varphi_{xx}(t,0)|^2dt\bigg), \end{aligned} \tag{16}$$

for $s \ge C_0T^\mu\left(T^\mu + \dfrac{1+T^\mu M^\mu}{\varepsilon^{1-2\mu}\ \delta^{3\mu-1}}\right)$ and where φ is the corresponding solution of (14).

Since the proof of Proposition 3.2 is very technical, we postpone it to an Appendix, at the end of the paper. We can deduce from the Carleman estimate an observability inequality for (14), as follows :

Proposition 3.3. *There exists a positive constant C such that for any $\varphi_0 \in L^2(0,L)$, we have*

$$\|\varphi(0,\cdot)\|^2_{L^2(0,L)} \le \exp\left\{\frac{C(1+M^\mu)}{\varepsilon^{1-2\mu}\ \delta^{3\mu-1}}\right\}\bigg(\|\varphi_{xx}(\cdot,0)\|^2_{L^2(0,T)} + \|\varphi_{xxx}(\cdot,0)\|^2_{L^2(0,T)}\bigg). \tag{17}$$

Proof. Before we start, let us notice that we can add the following term :

$$s^{2/\mu+1}\ \varepsilon^{2/\mu-4}\ \delta^{6-2/\mu}\iint_Q e^{-2s\alpha}\alpha^{2/\mu+1}\ |\varphi|^2 dxdt, \tag{18}$$

in the left-hand side of (16). Indeed, it suffices to use an interpolation argument between the two terms in the left-hand side of (16).

Let us fix s as follows :

$$s = C_0 T^\mu\left(T^\mu + \frac{1+T^\mu M^\mu}{\varepsilon^{1-2\mu}\ \delta^{3\mu-1}}\right). \tag{19}$$

Now, from (16) and (18), we have

$$\begin{aligned} &s^{2/\mu+1}\ \varepsilon^{2/\mu-4}\ \delta^{6-2/\mu}\iint_Q e^{-2s\alpha}\alpha^{2/\mu+1}\ |\varphi|^2 dxdt \\ &\leq C\bigg(\int_0^T e^{-2s\alpha(t,0)}\Big(\varepsilon^2 s\alpha(t,0)+\varepsilon\delta\Big)|\varphi_{xxx}(t,0)|^2 dt \\ &+\int_0^T e^{-2s\alpha(t,0)}\Big(\varepsilon^2 s^3\alpha^3(t,0)+\delta^2 s\alpha(t,0)\Big)|\varphi_{xx}(t,0)|^2 dt\bigg), \end{aligned} \tag{20}$$

for some $C>0$ independent of δ, ϵ and M. Combining the definition of α (see (15)) with Remark 3.1, we deduce

$$\begin{aligned} \frac{s^{2/\mu+1}\ \varepsilon^{2/\mu-4}\ \delta^{6-2/\mu}}{T^{4+2\mu}}e^{-\frac{C_2 s}{T^{2\mu}}}\int_{T/4}^{3T/4}\int_0^L |\varphi|^2 dxdt \leq C_3\bigg(e^{-\frac{C_3 s}{T^{2\mu}}}\left(\frac{\varepsilon^2 s}{T^{2\mu}}+\varepsilon\delta\right)\int_0^T |\varphi_{xxx}(t,0)|^2 dt \\ +e^{-\frac{C_3 s}{T^{2\mu}}}\left(\frac{\varepsilon^2 s^3}{T^{6\mu}}+\frac{\delta^2 s}{T^{2\mu}}\right)\int_0^T |\varphi_{xx}(t,0)|^2 dt\bigg), \end{aligned} \tag{21}$$

for some $C_2>0$ and $C_3>0$. We use the following energy estimate :

$$\int_0^L |\varphi(t_1,x)|^2 dx \leq \int_0^L |\varphi(t_2,x)|^2 dx, \tag{22}$$

for $0\leq t_1\leq t_2\leq T$, combined with (21) and the fact that $\dfrac{s^{2/\mu+1}}{T^{4+2\mu}}\geq 1$, we deduce

$$\int_0^L |\varphi(0,x)|^2 dxdt \leq \tilde{C}\bigg(\int_0^T |\varphi_{xxx}(t,0)|^2 dt + \int_0^T |\varphi_{xx}(t,0)|^2 dt\bigg), \tag{23}$$

where

$$\tilde{C} = \exp\left\{\frac{C(1+M^\mu)}{\varepsilon^{1-2\mu}\ \delta^{3\mu-1}}\right\}, \tag{24}$$

with $C>0$ independent of ε, δ, and M (C depends on T).

□

3.2 Exponential dissipation result.

Let us consider $\varepsilon>0$, $\delta\geq 0$ and $M>0$. In this subsection, we are going to prove an exponential dissipation result to compensate the observability constant found in Theorem 1.1. Let us introduce K as the smallest constant such that

$$\int_0^L |\varphi(t_1,x)|^2 dx \leq K\int_0^L |\varphi(t_2,x)|^2 dx, \tag{25}$$

where φ is the solution of (14) and $0 \le t_1 \le t_2 \le T$. We will prove that, whenever the time passed $t_2 - t_1$ is larger than L/M, the constant of the dissipation result can be dramatically improved. It behaves like :

$$\exp\left\{ - \frac{C}{\max(\delta^{1/2}, \varepsilon^{1/2})} \right\}$$

where C is positive.

Proposition 3.4. *There exists $C > 0$ such that for any $T > 0$, $\varepsilon > 0$, $\delta > 0$, $M > 0$, $0 \le t_1 < t_2 \le T$ such that $t_2 - t_1 \ge L/M$ we have the following decay properties for the solution of (14):*

- *If* $\delta^3 \le \frac{4 \cdot 5^2}{9} \varepsilon^2 (M - L/(t_2 - t_1))$, *then*

$$K \le \exp\left\{ - C \frac{(M(t_2 - t_1) - L)^{4/3}}{(\varepsilon(t_2 - t_1))^{1/3}} \right\}. \tag{26}$$

- *If* $\delta^3 \ge \frac{4 \cdot 5^2}{9} \varepsilon^2 (M - L/(t_2 - t_1))$, *then*

$$K \le \exp\left\{ - C \frac{(M(t_2 - t_1) - L)^{3/2}}{(\delta(t_2 - t_1))^{1/2}} \right\}. \tag{27}$$

Proof of Proposition 3.4.

By multiplying $(14)_1$ by $\exp\{r(M(T-t)+x)\}\varphi$ where r is a positive constant which will be chosen below, then integrating in $(0, L)$ and integrating by parts with respect to x, we deduce

$$-\frac{1}{2}\frac{d}{dt}\int_0^L \exp\{r(M(T-t)+x)\}|\varphi|^2 dx + \varepsilon \int_0^L (\exp\{r(M(T-t)+x)\}\varphi)_{xx}\varphi_{xx} dx$$
$$+\delta \int_0^L (\exp\{r(M(T-t)+x)\}\varphi)_{xx}\varphi_x dx = 0.$$

Here we used that, $\varphi(t,0) = \varphi(t,L) = \varphi_x(t,0) = \varphi_x(t,L) = 0$ for $t \in (0,T)$. Then, we deduce

$$-\frac{1}{2}\frac{d}{dt}\int_0^L \exp\{r(M(T-t)+x)\}|\varphi|^2 dx + \varepsilon r^2 \int_0^L (\exp\{r(M(T-t)+x)\}\varphi\varphi_{xx} dx$$
$$+\varepsilon r \int_0^L (\exp\{r(M(T-t)+x)\}(|\varphi_x|^2)_x dx + \varepsilon \int_0^L (\exp\{r(M(T-t)+x)\}|\varphi_{xx}|^2 dx$$
$$+\delta\frac{r^2}{2}\int_0^L (\exp\{r(M(T-t)+x)\}(|\varphi|^2)_x dx + 2\delta r \int_0^L (\exp\{r(M(T-t)+x)\}|\varphi_x|^2 dx$$
$$+\delta\frac{r}{2}\int_0^L (\exp\{r(M(T-t)+x)\}(|\varphi_x|^2)_x dx = 0.$$

By integrating by parts with respect to x, we deduce

$$-\frac{1}{2}\frac{d}{dt}\int_0^L \exp\{r(M(T-t)+x)\}|\varphi|^2 dx - 2\varepsilon r^2 \int_0^L (\exp\{r(M(T-t)+x)\}|\varphi_x|^2 dx$$
$$+\frac{\varepsilon}{2} r^4 \int_0^L (\exp\{r(M(T-t)+x)\}|\varphi|^2 dx + \varepsilon \int_0^L (\exp\{r(M(T-t)+x)\}|\varphi_{xx}|^2 dx \tag{28}$$
$$-\frac{r^3}{2}\delta \int_0^L (\exp\{r(M(T-t)+x)\}|\varphi|^2 dx + \frac{3r}{2}\delta \int_0^L (\exp\{r(M(T-t)+x)\}|\varphi_x|^2 dx = 0.$$

By using the fact that

$$\begin{aligned}\varepsilon r^2\int_0^L(\exp\{r(M(T-t)+x)\}|\varphi_x|^2dx &\leq \varepsilon r^4\int_0^L(\exp\{r(M(T-t)+x)\}|\varphi|^2dx\\ &\quad+\varepsilon r^2\int_0^L(\exp\{r(M(T-t)+x)\}\varphi_{xx}\varphi dx\\ &\leq \frac{3}{2}\varepsilon r^4\int_0^L(\exp\{r(M(T-t)+x)\}|\varphi|^2dx\\ &\quad+\frac{\varepsilon}{2}\int_0^L(\exp\{r(M(T-t)+x)\}|\varphi_{xx}|^2dx,\end{aligned}$$

combined with (28), we deduce

$$\begin{aligned}-\frac{1}{2}\frac{d}{dt}\int_0^L\exp\{r(M(T-t)+x)\}|\varphi|^2dx-\frac{5\varepsilon}{2}r^4\int_0^L(\exp\{r(M(T-t)+x)\}|\varphi|^2dx\\ -\frac{r^3}{2}\delta\int_0^L(\exp\{r(M(T-t)+x)\}|\varphi|^2dx\leq 0.\end{aligned}\tag{29}$$

Then, we deduce

$$-\frac{d}{dt}\left(\exp\{(-5\varepsilon r^4-\delta r^3)(T-t)\}\int_0^L\exp\{r(M(T-t)+x)\}|\varphi(t,x)|^2dx\right)\leq 0,\tag{30}$$

for $t\in(0,T)$. Integrating between t_1 and t_2, we have

$$\int_0^L|\varphi(t_1,x)|^2dx\leq K\int_0^L|\varphi(t_2,x)|^2dx,\tag{31}$$

where

$$K=\exp\{5\varepsilon(t_2-t_1)r^4+\delta(t_2-t_1)r^3+(L-M(t_2-t_1))r\}.\tag{32}$$

Now, we are going to minimise K. Let us denote $\frac{\alpha}{4}:=5\varepsilon(t_2-t_1)$, $\frac{\beta}{3}:=\delta(t_2-t_1)$ and $\gamma:=L-M(t_2-t_1)$.

Case 1. $\frac{\beta^3}{27\alpha^2}\leq\frac{-\gamma}{4}$**.**

In this case, we have

$$\beta\leq\frac{3}{4^{1/3}}(-\gamma)^{1/3}\alpha^{2/3}.$$

From (32), we deduce

$$K\leq\exp\left\{\frac{\alpha}{4}r^4+\frac{1}{4^{1/3}}(-\gamma)^{1/3}\alpha^{2/3}r^3+\gamma r\right\}.$$

By taking $r^*=\frac{(-\gamma)^{1/3}}{\alpha^{1/3}}$, we deduce

$$K\leq\exp\left\{\left(\frac{1}{4}+\frac{1}{4^{1/3}}-1\right)\frac{(-\gamma)^{4/3}}{\alpha^{1/3}}\right\}.$$

Case 2. $\frac{\beta^3}{27\alpha^2}\geq\frac{-\gamma}{4}$**.**

In this regime, let us notice that

$$\alpha \leq \frac{2}{3\sqrt{3}} \frac{\beta^{3/2}}{(-\gamma)^{1/2}}. \tag{33}$$

Using the previous estimate with the definition of K given in (32), we deduce that

$$K \leq \exp\left\{ \frac{1}{6\sqrt{3}} \frac{\beta^{3/2}}{(-\gamma)^{1/2}} r^4 + \frac{\beta}{3} r^3 + \gamma r \right\}. \tag{34}$$

By taking $r = \dfrac{(-\gamma)^{1/2}}{\beta^{1/2}} > 0$, we can deduce that

$$\begin{aligned} K &\leq \exp\left\{ \frac{1}{6\sqrt{3}} \frac{\beta^{3/2}}{(-\gamma)^{1/2}} \frac{(\gamma)^2}{\beta^2} + \frac{\beta}{3} \frac{(-\gamma)^{3/2}}{\beta^{3/2}} + \gamma \frac{(-\gamma)^{1/2}}{\beta^{1/2}} \right\} \\ &\leq \exp\left\{ \left(\frac{1}{6\sqrt{3}} + \frac{1}{3} - 1 \right) \frac{(-\gamma)^{3/2}}{\beta^{1/2}} \right\}. \end{aligned} \tag{35}$$

□

3.3 Proof of Theorem 1.2.

The proof is divided in two steps.

Step 1. Observability inequality.

Let us first deduce an observability inequality from the Carleman inequality (16). Let us consider φ a regular solution of (14) and use the Proposition 3.2 for a time $T_1 = \dfrac{1}{M}$ and we denote $Q_1 = [0, T_1] \times (0, L)$ and $\tilde{Q}_1 = [T_1/3, 2T_1/3] \times (0, L)$.

By applying the same ideas as in the proof of Proposition 3.3 but this time in $[0, T_1]$, where we fix s as follows :

$$s = C_0 T_1^{\mu} \left(T_1^{\mu} + \frac{1 + T_1^{\mu} M^{\mu}}{\varepsilon^{1-2\mu}\ \delta^{3\mu-1}} \right). \tag{36}$$

we deduce

$$\int_0^L |\varphi(0,x)|^2 dx dt \leq \tilde{C} \left(\int_0^{T_1} |\varphi_{xxx}(t,0)|^2 dt + \int_0^{T_1} |\varphi_{xx}(t,0)|^2 dt \right), \tag{37}$$

where

$$\tilde{C} = \exp\left\{ \frac{C_1 M^{\mu}}{\varepsilon^{1-2\mu}\ \delta^{3\mu-1}} \right\}, \tag{38}$$

with $C_1 > 0$ independent of ε and δ.

Step 2. Combination between the observability inequality and energy estimate.

Let $T \geq C_0/M$ with C_0 is large enough. By applying the same ideas as before, between times $T - T_1$ and T, we deduce

$$\int_0^L |\varphi(T - 1/M, x)|^2 dx \leq \tilde{C} \left(\int_{T-1/M}^{T} |\varphi_{xxx}(t,0)|^2 dt + \int_{T-1/M}^{T} |\varphi_{xx}(t,0)|^2 dt \right), \tag{39}$$

where $\tilde{C}$ is given in (38). Applying Proposition 3.4 for $t_2 = T - 1/M$ and $t_1 = 1/M$, combined with (37), we deduce

$$\int_0^L |\varphi(0,x)|^2 dx \leq K\tilde{C}\Big(\int_0^{T_1} |\varphi_{xxx}(t,0)|^2 dt + \int_0^{T_1} |\varphi_{xx}(t,0)|^2 dt \Big), \tag{40}$$

where $\tilde{C}$ is given in (38) and where K is estimated in (26)-(27). By taking C_0 large enough, the observability constant $K\tilde{C}$ can be estimated in the following way :

$$K\tilde{C} \leq \begin{cases} C_2 \exp\Big\{ - \dfrac{C_3 M^{1/3}}{\varepsilon^{1/3}} \Big\} & \text{when } \delta^3 \leq M\varepsilon^2 \ , \\ C_2 \exp\Big\{ - \dfrac{C_3 M^{1/2}}{\delta^{1/2}} \Big\} & \text{when } \delta^3 \geq M\varepsilon^2 \ , \end{cases} \tag{41}$$

where $C_2 > 0$ and $C_3 > 0$ are independent of ε, δ. Here, we take $\mu = 1/3$ in the first regime and $\mu = 1/2$ in the second one.

At the end, it is classical to prove that for any $y_0 \in L^2(0,L)$, there exists a control v_1, $v_2 \in L^2(0,T)$ such that the solution $y \in L^2(0,T;H^2(0,L))$ of (1) satisfies $y(T,\cdot) = 0$ in $(0,L)$ where v is estimated as follows :

$$\|v_1\|^2_{L^2(0,T)} + \|v_2\|^2_{L^2(0,T)} \leq \frac{K\tilde{C}}{\varepsilon^2} \|y_0\|^2_{L^2(0,L)}. \tag{42}$$

Then, we deduce the following estimate on $\dfrac{K\tilde{C}}{\varepsilon^2}$:

$$\frac{K\tilde{C}}{\varepsilon^2} \leq \begin{cases} \dfrac{C_4}{\varepsilon} \exp\Big\{ - \dfrac{C_5 M^{1/3}}{\varepsilon^{1/3}} \Big\} & \text{in the first regime} \ , \\ \dfrac{C_4}{\varepsilon} \exp\Big\{ - \dfrac{C_5 M^{1/2}}{\delta^{1/2}} \Big\} & \text{in the second regime} \ , \end{cases} \tag{43}$$

where $C_4 > 0$ and $C_5 > 0$. This concludes the proof of Theorem 1.2.

□

4 Proof of Theorem 1.3.

In this proof, we adapt the ideas used in [10]. Let us introduce R

$$0 < R < \frac{L - MT}{2}. \tag{44}$$

Let us consider $\hat{\varphi}_0 \in C^\infty(0,L)$ such that

$$\begin{cases} Supp(\hat{\varphi}_0) \subset (R, 2R), \\ \hat{\varphi}_0 \geq 0, \\ \displaystyle\int_0^L |\hat{\varphi}_0|^2 = 1. \end{cases} \tag{45}$$

Let us denote $\hat{\varphi}$ the corresponding solution of (14) for $\varphi(T,\cdot) = \hat{\varphi}_0$. The rest of the proof is divided in three steps.

Step 1. Estimate of $\|\hat{\varphi}(0,\cdot)\|_{L^2(0,L)}$.

Let us introduce $\beta(t,x)$ the solution of

$$\begin{cases} \partial_t \beta + M\beta_x = 0 & \text{in } Q \ , \\ \beta(T,\cdot) = \hat{\varphi}_0 & \text{in } (0,L) \ . \end{cases} \tag{46}$$

Let us notice that from (44) and (45), we have

$$Supp(\beta(t,\cdot)) \subset (0,1), \quad t \in [0,T].$$

By mutiplying $(14)_1$ (where $\varphi(T,\cdot) = \hat{\varphi}_0$) by β and integrating by parts, we deduce

$$\int_0^L \beta(0,x)\hat{\varphi}(0,x)dx - \|\hat{\varphi}_0\|^2_{L^2(0,L)} + \int_0^T\int_0^L \Big(\varepsilon\beta_{xxxx} - \delta\beta_{xxx}\Big)\hat{\varphi}dx = 0.$$

By taking $\delta > 0$ and $\varepsilon > 0$ small enough we deduce

$$\int_0^L \beta(0,x)\hat{\varphi}(0,x)dx \geq C\|\hat{\varphi}_0\|^2_{L^2(0,L)} \geq C > 0, \tag{47}$$

where $C > 0$ is independent of ε, δ and M. Then, we can easily deduce that

$$\|\hat{\varphi}(0,\cdot)\|^2_{L^2(0,L)} \geq C > 0. \tag{48}$$

Step 2. Estimate of $\|\hat{\varphi}_{xx}(\cdot,0)\|^2_{L^2(0,T)} + \|\hat{\varphi}_{xxx}(\cdot,0)\|^2_{L^2(0,T)}$.

By mutiplying $(14)_1$ (where $\varphi(T,\cdot) = \hat{\varphi}_0$) by $(R/4-x)^4\hat{\varphi}$ and integrating by parts, we deduce

$$-\frac{1}{2}\frac{d}{dt}\Big(\int_0^{\frac{R}{4}}(\frac{R}{4}-x)^4|\hat{\varphi}|^2dx\Big) + \varepsilon\int_0^{\frac{R}{4}}(\frac{R}{4}-x)^4|\hat{\varphi}_{xx}|^2dx + \varepsilon\int_0^{\frac{R}{4}}(\frac{R}{4}-x)^2|\hat{\varphi}_x|^2dx + 12\varepsilon\int_0^{\frac{R}{4}}|\hat{\varphi}|^2dx$$
$$+12\delta\int_0^{\frac{R}{4}}(\frac{R}{4}-x)|\hat{\varphi}|^2dx = 6\delta\int_0^{\frac{R}{4}}(\frac{R}{4}-x)^3|\hat{\varphi}_x|^2dx + 2M\int_0^{\frac{R}{4}}(\frac{R}{4}-x)^3|\hat{\varphi}|^2dx + 25\varepsilon\int_0^{\frac{R}{4}}(\frac{R}{4}-x)^2|\hat{\varphi}_x|^2dx. \tag{49}$$

Let us notice that

$$25\varepsilon\int_0^{\frac{R}{4}}(\frac{R}{4}-x)^2|\hat{\varphi}_x|^2dx \leq C\varepsilon\int_0^{\frac{R}{4}}|\hat{\varphi}|^2dx + \frac{\varepsilon}{3}\int_0^{\frac{R}{4}}(\frac{R}{4}-x)^4|\hat{\varphi}_{xx}|^2dx$$

and

$$6\delta\int_0^{\frac{R}{4}}(\frac{R}{4}-x)^3|\hat{\varphi}_x|^2dx \leq C(\delta\frac{R}{4} + \frac{\delta^2R^2}{\varepsilon})\int_0^{\frac{R}{4}}|\hat{\varphi}|^2dx + \frac{\varepsilon}{3}\int_0^{\frac{R}{4}}(\frac{R}{4}-x)^4|\hat{\varphi}_{xx}|^2dx.$$

Combining the last two estimates with (49), integrating between 0 and T and using the fact that $\hat{\varphi}_0(x) = 0$ for $x \in (0,R)$, we deduce

$$\begin{aligned}\int_0^{\frac{R}{4}}(\frac{R}{4}-x)^4|\hat{\varphi}(0,x)|^2dx + \varepsilon\int_0^T\int_0^{\frac{R}{4}}(\frac{R}{4}-x)^4|\hat{\varphi}_{xx}|^2dxdt + \varepsilon\int_0^T\int_0^{\frac{R}{4}}(\frac{R}{4}-x)^2|\hat{\varphi}_x|^2dxdt \\ \leq C(\varepsilon,M,\delta)\int_0^{\frac{R}{4}}|\hat{\varphi}|^2dx.\end{aligned} \tag{50}$$

for some constant $C(\varepsilon,M,\delta)$ whose growth in $\frac{1}{\varepsilon}$, M, R, δ is at most polynomial. Let us notice that, we can add the following term to the left-hand side of (50) :

$$\varepsilon R^4\|\hat{\varphi}\|^2_{L^2(0,T;H^2(0,R/16))}, \tag{51}$$

By mutiplying $(14)_1$ by $(\frac{R}{4}-x)^8\hat{\varphi}_{xxxx}$ and integrating by parts, we deduce

$$\begin{aligned}-\frac{1}{2}\frac{d}{dt}\Big(\int_0^{\frac{R}{4}}(\frac{R}{4}-x)^8|\hat{\varphi}_{xx}|^2dx\Big) + \varepsilon\int_0^{\frac{R}{4}}(\frac{R}{4}-x)^8|\hat{\varphi}_{xxxx}|^2dx + 4\delta\int_0^{\frac{R}{4}}(\frac{R}{4}-x)^7|\hat{\varphi}_{xxx}|^2dx \\ \leq C(\varepsilon,M,\delta)\Big(\int_0^{\frac{R}{4}}|\hat{\varphi}|^2dx + |\tilde{\varphi}_{xxx}(t,0)|^2\Big) + 16\int_0^{\frac{R}{4}}(\frac{R}{4}-x)^7\hat{\varphi}_t\hat{\varphi}_{xxx}dx \\ +56\int_0^{\frac{R}{4}}(\frac{R}{4}-x)^6\hat{\varphi}_t\hat{\varphi}_{xx}dx + \frac{\varepsilon}{4}\int_0^{\frac{R}{4}}(\frac{R}{4}-x)^8|\hat{\varphi}_{xxxx}|^2dx.\end{aligned} \tag{52}$$

Before we continue let us notice that if we want to add the term

$$\varepsilon R^8 \|\hat{\varphi}(t,\cdot)\|^2_{H^4(0,R/16)}, \tag{53}$$

in the left-hand side of (52), it suffices to add

$$C(\delta,\varepsilon,M)\Big(\int_0^{\frac{R}{4}}(\frac{R}{4}-x)^6|\hat{\varphi}_{xxx}|^2dx+\|\hat{\varphi}(t,\cdot)\|^2_{H^2(0,R/16)}\Big), \tag{54}$$

in the right-hand side of (52). Furthermore, to add the following term

$$\frac{1}{\varepsilon}\int_0^{\frac{R}{4}}(\frac{R}{4}-x)^8|\hat{\varphi}_t|^2dx, \tag{55}$$

in the left-hand side of (52), it suffices to add

$$C(\delta,\varepsilon,M)\Big(\int_0^{\frac{R}{4}}(\frac{R}{4}-x)^6|\hat{\varphi}_{xxx}|^2dx+\int_0^{\frac{R}{4}}(\frac{R}{4}-x)^4|\hat{\varphi}_x|^2dx\Big),$$

in the right-hand side of (52). Indeed, it suffices, to use $-\partial_t\varphi=-(\varepsilon\varphi_{xxxx}+\delta\varphi_{xxx}-M\varphi_x)$ in Ω. By applying Cauchy-Schwartz's inequality, we deduce that the terms in the right-hand side of (52) can be estimated by

$$\begin{aligned}C(\varepsilon,M,\delta)\Big(\int_0^{\frac{R}{4}}|\hat{\varphi}|^2dx+\int_0^{\frac{R}{4}}(\frac{R}{4}-x)^4|\hat{\varphi}_{xx}|^2dx+\int_0^{\frac{R}{4}}(\frac{R}{4}-x)^6|\hat{\varphi}_{xxx}|^2dx+\|\hat{\varphi}(t,\cdot)\|^2_{H^2(0,R/16)}\Big)\\ +\frac{\varepsilon}{4}\int_0^{\frac{R}{4}}(\frac{R}{4}-x)^8|\hat{\varphi}_{xxxx}|^2dx+\frac{\varepsilon}{4}R^8\|\hat{\varphi}(t,\cdot)\|^2_{H^4(0,R/16)}+\frac{1}{2\varepsilon}\int_0^{\frac{R}{4}}(\frac{R}{4}-x)^8|\hat{\varphi}_t|^2dx,\end{aligned} \tag{56}$$

even if we add the two terms (55)-(53) in the left-hand side of (52). Here, we used that :

$$C(\varepsilon,M,\delta)\Big(|\hat{\varphi}_{xxx}(t,0)|^2+|\hat{\varphi}_{xx}(t,0)|^2\Big)\leq\frac{\varepsilon}{4}R^8\|\hat{\varphi}(t,\cdot)\|^2_{H^4(0,R/16)}+\tilde{C}(\varepsilon,M,\delta)\|\hat{\varphi}(t,\cdot)\|^2_{H^2(0,R/16)}. \tag{57}$$

for some constant $\tilde{C}(\varepsilon,M,\delta)$ whose growth in $\frac{1}{\varepsilon}$, M, R, δ is at most polynomial. To finish this part, let us notice that

$$\begin{aligned}C(\varepsilon,M,\delta)\int_0^{\frac{R}{4}}(\frac{R}{4}-x)^6|\hat{\varphi}_{xxx}|^2dx\leq\tilde{C}(\varepsilon,M,\delta)\Big(\int_0^{\frac{R}{4}}(\frac{R}{4}-x)^4|\hat{\varphi}_{xx}|^2dx+\|\hat{\varphi}(t,\cdot)\|^2_{H^2(0,R/16)}\Big)\\ +\frac{\varepsilon}{8}\int_0^{\frac{R}{4}}(\frac{R}{4}-x)^8|\hat{\varphi}_{xxxx}|^2dx+\frac{\varepsilon}{4}R^8\|\hat{\varphi}(t,\cdot)\|^2_{H^4(0,R/16)}.\end{aligned}$$

Here we used (57). Combining the last estimate with (56)-(52) by taking into consideration that we add the terms (55)-(53), then integrating between 0 and T and using (51) with (50) we deduce

$$\varepsilon R^8\|\hat{\varphi}\|^2_{L^2(0,T;H^4(0,R/16))}\leq\tilde{C}(\varepsilon,M,\delta)\int_0^T\int_0^{\frac{R}{4}}|\hat{\varphi}|^2dx. \tag{58}$$

for some constant $\tilde{C}(\varepsilon,M,\delta)$ whose growth in $\frac{1}{\varepsilon}$, M, R, δ is at most polynomial. From the last estimate, we deduce

$$\|\hat{\varphi}_{xx}(\cdot,0)\|^2_{L^2(0,T)}+\|\hat{\varphi}_{xxx}(\cdot,0)\|^2_{L^2(0,T)}\leq\tilde{C}(\varepsilon,M,\delta)\int_0^{\frac{R}{4}}|\hat{\varphi}|^2dx, \tag{59}$$

for some constant $\tilde{C}(\varepsilon,M,\delta)$ whose growth in $\frac{1}{\varepsilon}$, M, δ is at most polynomial.

Step 3. Last computations.

Let us introduce $\psi \in C^\infty(\mathbb{R})$ as follows :

$$\begin{cases} \psi = 0 & \text{in } [R, +\infty), \\ \psi = 1 & \text{in } (-\infty, R/2], \\ \psi' \leq 0, \end{cases} \tag{60}$$

We apply the same ideas used in the proof of Proposition 3.4. Let us denote

$$E(t) = \exp\{-(\varepsilon r^4 + \delta r^3)(T-t)\} \int_0^L \exp\{r(M(T-t)-x)\}\psi(x - M(T-t))|\hat{\varphi}|^2 dx.$$

By multiplying $(14)_1$ by $\psi(x - M(T-t))\exp\{r(M(T-t)-x)\}\hat{\varphi}$ where r is a positive constant which will be chosen below, we deduce

$$\begin{aligned} -\frac{d}{dt}E(t) \leq C(r) \int_{R/2+M(T-t)}^{R+M(T-t)} & (\|\psi'\|_{L^\infty(0,L)} + \|\psi''\|_{L^\infty(0,L)} + \|\psi'''\|_{L^\infty(0,L)} + \|\psi''''\|_{L^\infty(0,L)}) \\ & \exp\{r(M(T-t)-x)\}|\hat{\varphi}|^2 dx, \end{aligned} \tag{61}$$

where $C(r)$ depends on r in a polynomial way. Then, we deduce

$$-\frac{d}{dt}E(t) \leq C(r)\exp\{-\frac{rR}{2}\} \int_0^L |\hat{\varphi}|^2 dx. \tag{62}$$

Before we continue, let us estimate the right-hand side of (62). By multiplying $(14)_1$ by $\hat{\varphi}$ and integrating by parts, we deduce

$$-\frac{1}{2}\frac{d}{dt}\int_0^L |\hat{\varphi}|^2 dx + \varepsilon \int_0^L |\hat{\varphi}_{xx}|^2 dx = 0.$$

Integrating the last inequality over (t, T), we deduce

$$\int_0^L |\hat{\varphi}(t,x)|^2 dx + \varepsilon \int_0^T \int_0^L |\hat{\varphi}_{xx}|^2 dx dt = \int_0^L |\hat{\varphi}(0,x)|^2 dx.$$

Combing the last estimate with (62), we deduce

$$-\frac{d}{dt}E(t) \leq C(r)\exp\{-\frac{rR}{2}\} \int_0^L |\hat{\varphi}_0|^2 dx. \tag{63}$$

Integrating over (t, T), for $0 \leq t \leq T$, we deduce

$$\begin{aligned} & \int_0^L \exp\{r(M(T-t)-x)\}\psi(x - M(T-t))|\hat{\varphi}\}|^2 dx \\ & \leq C\exp\left\{\varepsilon(T-t)r^4 + \delta(T-t)r^3 - \frac{rR}{2}\right\} \int_0^L |\hat{\varphi}_0|^2 dx. \end{aligned} \tag{64}$$

By using the fact that $\psi(x - M(T-t)) = 1$ for $(t, x) \in (0, T) \times (0, \frac{R}{4})$, we deduce

$$\begin{aligned} & \exp\left\{r(M(T-t) - \frac{R}{4})\right\} \int_0^{\frac{R}{4}} |\hat{\varphi}(t,x)|^2 dx \\ & \leq \int_0^L \exp\{r(M(T-t)-x)\}\psi(x - M(T-t))|\hat{\varphi}|^2 dx. \end{aligned} \tag{65}$$

Combining (64) with (65), we deduce

$$\begin{aligned}&\exp\left\{r(M(T-t)-\frac{R}{4})\right\}\int_0^{\frac{R}{4}}|\hat{\varphi}(t,x)|^2dx\\&\leq C\exp\left\{\varepsilon(T-t)r^4+\delta(T-t)r^3-\frac{rR}{2}\right\}\int_0^L|\hat{\varphi}_0|^2dx.\end{aligned}\tag{66}$$

At the end, we deduce

$$\int_0^{\frac{R}{4}}|\hat{\varphi}(t,x)|^2dx\leq C\exp\left\{\varepsilon Tr^4+\delta Tr^3-\frac{rR}{2}\right\}\int_0^L|\hat{\varphi}_0|^2dx.\tag{67}$$

By choosing

$$r=\frac{1}{3}\min\left\{\left(\frac{R}{\delta T}\right)^{1/2},\left(\frac{R}{\varepsilon T}\right)^{1/3}\right\},$$

we deduce

$$\int_0^{\frac{R}{4}}|\hat{\varphi}(t,x)|^2dx\leq C\exp\left\{-\frac{R^2}{\max((R\delta T)^{1/2},(R^2\varepsilon T)^{1/3})}\right\}\int_0^L|\hat{\varphi}_0|^2dx,\tag{68}$$

where $C>0$. Combining the last estimate with (59), we deduce

$$\|\hat{\varphi}_{xx}(\cdot,0)\|^2_{L^2(0,T)}+\|\hat{\varphi}_{xxx}(\cdot,0)\|^2_{L^2(0,T)}\leq C(\varepsilon)\exp\left\{-\frac{c}{\max(\delta^{1/2},\varepsilon^{1/3})}\right\}\int_0^L|\hat{\varphi}_0|^2dx,\tag{69}$$

for some constant $C(\varepsilon)$ whose growth in $\frac{1}{\varepsilon}$ is at most polynomial and $c>0$. Combining the last estimate with (48), we finish the proof of Theorem 1.3.

□

5 Appendix A.

Proof of Proposition 3.2.

Let us set

$$\psi(t,x)=e^{-s\alpha(t,x)}\varphi(t,x),\ \forall(t,x)\in Q.\tag{70}$$

By replacing φ by $e^{s\alpha}\psi$ in the equation $-\partial_t\varphi+\varepsilon\varphi_{xxxx}+\delta\varphi_{xxx}-M\varphi_x$, we have

$$L_1\psi+L_2\psi=L_3\psi\tag{71}$$

where

$$\begin{aligned}L_1\psi&=L_{1,\varepsilon}\psi+L_{1,\delta}\psi+\tilde{L}_1\psi,\\L_2\psi&=L_{2,\varepsilon}\psi+L_{2,\delta}\psi+\tilde{L}_2\psi,\end{aligned}\tag{72}$$

where

$$\begin{aligned}L_{1,\varepsilon}\psi&=4\varepsilon s^3\alpha_x^3\psi_x+4\varepsilon s\alpha_x\psi_{xxx}+6\varepsilon s^3\alpha_x^2\alpha_{xx}\psi,\\L_{1,\delta}\psi&=\delta\psi_{xxx}+3\delta s^2\alpha_x^2\psi_x,\\\tilde{L}_1\psi&=-\psi_t-M\psi_x,\\L_{2,\varepsilon}\psi&=\varepsilon s^4\alpha_x^4\psi+6\varepsilon s^2\alpha_x^2\psi_{xx}+\varepsilon\psi_{xxxx}+12\varepsilon s^2\alpha_x\alpha_{xx}\psi_x,\\L_{2,\delta}\psi&=\delta s^3\alpha_x^3\psi+3\delta s\alpha_x\psi_{xx}+3\delta s\alpha_{xx}\psi_x,\\\tilde{L}_2\psi&=-s\alpha_t\psi-Ms\alpha_x\psi\end{aligned}\tag{73}$$

and

$$L_3\psi = -3\varepsilon s^2\alpha_{xx}^2\psi - 6\varepsilon s\alpha_{xx}\psi_{xx} - 3s^2\delta\alpha_{xx}\alpha_x\psi. \tag{74}$$

Moreover,

$$\|L_1\psi\|^2_{L^2(Q)} + \|L_2\psi\|^2_{L^2(Q)} + 2(L_1\psi, L_2\psi)_{L^2(Q)} = \|L_3\psi\|^2_{L^2(Q)}. \tag{75}$$

$$\begin{aligned}(L_1\psi, L_2\psi)_{L^2(Q)} &= (L_{1,\varepsilon}\psi, L_{2,\varepsilon}\psi)_{L^2(Q)} + (L_{1,\delta}\psi, L_{2,\delta}\psi)_{L^2(Q)} + (L_{1,\delta}\psi, L_{2,\varepsilon}\psi)_{L^2(Q)}\\ &+(L_{2,\delta}\psi, L_{1,\varepsilon}\psi)_{L^2(Q)} + (L_{1,\varepsilon}\psi, \tilde{L}_2\psi)_{L^2(Q)} + (\tilde{L}_1\psi, \tilde{L}_2\psi)_{L^2(Q)}\\ &+(L_{1,\delta}\psi, \tilde{L}_2\psi)_{L^2(Q)} + (L_{2,\delta}\psi, \tilde{L}_1\psi)_{L^2(Q)} + (L_{2,\varepsilon}\psi, \tilde{L}_1\psi)_{L^2(Q)}.\end{aligned} \tag{76}$$

Step 1. Computation of $(L_{1,\delta}\psi, L_{2,\delta}\psi)_{L^2(Q)}$ and first main estimate.

In this step, we will compute $\iint_Q L_{1,\delta}\psi L_{2,\delta}dxdt$ under the form $\sum_{i=1}^{2}\sum_{j=1}^{3} I_{ij}^{\delta}$ where I_{ij}^{δ} is the scalar product in $L^2(Q)$ of the i-th term of $L_{1,\delta}\psi$ with the j-th term of $L_{2,\delta}\psi$.

By integration by parts we have,

$$\begin{aligned}I_{11}^{\delta} &= \delta^2 s^3\iint_Q \alpha_x^3\psi_{xxx}\psi dxdt\\ &= -\delta^2 s^3\iint_Q \alpha_x^3\psi_x\psi_{xx}dxdt - 3\delta^2 s^3\iint_Q \alpha_x^2\alpha_{xx}\psi\psi_{xx}dxdt\\ &\geq -\frac{1}{2}\delta^2 s^3\iint_Q \alpha_x^3(|\psi_x|^2)_x dxdt + 3\delta^2 s^3\iint_Q \alpha_x^2\alpha_{xx}|\psi_x|^2dxdt + C_0\delta^2 s^3 T^{4\mu}\iint_Q \alpha_x^5|\psi|^2dxdt\\ &\geq \frac{9}{2}\delta^2 s^3\iint_Q \alpha_x^2\alpha_{xx}|\psi_x|^2dxdt - C_0\delta^2 s^3 T^{4\mu}\iint_Q \alpha^5|\psi|^2dxdt.\end{aligned} \tag{77}$$

On the other hand

$$I_{21}^{\delta} = 3\delta^2 s^5\iint_Q \alpha_x^5\psi_x\psi dxdt = \frac{3}{2}\delta^2 s^5\iint_Q \alpha_x^5(|\psi|^2)_x dxdt = -\frac{15}{2}\delta^2 s^5\iint_Q \alpha_x^4\alpha_{xx}|\psi|^2dxdt. \tag{78}$$

Moreover,

$$\begin{aligned}I_{12}^{\delta} &= 3\delta^2 s\iint_Q \alpha_x\psi_{xx}\psi_{xxx}dxdt = \frac{3}{2}\delta^2 s\iint_Q \alpha_x(|\psi_{xx}|^2)_x dxdt\\ &= -\frac{3}{2}\delta^2 s\iint_Q \alpha_{xx}|\psi_{xx}|^2dxdt + \frac{3}{2}\delta^2 s\int_0^T(\alpha_x|\psi_{xx}|^2)_{|_{x=0,L}}dxdt.\end{aligned} \tag{79}$$

Furthermore,

$$\begin{aligned}I_{22}^{\delta} &= 9\delta^2 s^3\iint_Q \alpha_x^3\psi_{xx}\psi_x dxdt = \frac{9}{2}\delta^2 s^3\iint_Q \alpha_x^3(|\psi_x|^2)_x dxdt\\ &= -\frac{27}{2}\delta^2 s^3\iint_Q \alpha_x^2\alpha_{xx}|\psi_x|^2dxdt.\end{aligned} \tag{80}$$

Moreover,

$$I_{13}^{\delta} = 3\delta^2 s\iint_Q \alpha_{xx}\psi_{xxx}\psi_x dxdt = -3\delta^2 s\iint_Q \alpha_{xx}|\psi_{xx}|^2dxdt. \tag{81}$$

On the other hand,

$$I_{23}^{\delta} = 9\delta^2 s^3\iint_Q \alpha_{xx}\alpha_x^2|\psi_x|^2dxdt. \tag{82}$$

Combining the previous estimates, we can easily deduce

$$\boxed{\begin{array}{rcl}(L_{1,\delta}\psi, L_{2,\delta}\psi)_{L^2(Q)} & \geq & -\dfrac{15}{2}\delta^2 s^5 \displaystyle\iint_Q \alpha_x^4 \alpha_{xx}|\psi|^2 dxdt - \dfrac{9}{2}\delta^2 s \displaystyle\iint_Q \alpha_{xx}|\psi_{xx}|^2 dxdt \\ & & -\dfrac{3}{2}\delta^2 s \displaystyle\int_0^T (\alpha_x|\psi_{xx}|^2)_{x=0} dt - C_0\delta^2 s^3 T^{4\mu} \displaystyle\iint_Q \alpha^5|\psi|^2 dxdt.\end{array}} \tag{83}$$

Step 2. Computation of $(L_{1,\varepsilon}\psi, L_{2,\varepsilon}\psi)_{L^2(Q)}$ and first main estimate.

In this step, we will compute $\displaystyle\iint_Q L_{1,\varepsilon}\psi L_{2,\varepsilon} dxdt$ under the form $\displaystyle\sum_{i=1}^{3}\sum_{j=1}^{4} I_{ij}^{\varepsilon}$ where I_{ij}^{ε} is the scalar product in $L^2(Q)$ of the i-th term of $L_{1,\varepsilon}\psi$ with the j-th term of $L_{2,\varepsilon}\psi$.

By integration by parts we have,

$$I_{11}^{\varepsilon} = 4\varepsilon^2 s^7 \iint_Q \alpha_x^7 \psi_x \psi dxdt = 2\varepsilon^2 s^7 \iint_Q \alpha_x^7 (|\psi|^2)_x dxdt = -14\varepsilon^2 s^7 \iint_Q \alpha_x^6 \alpha_{xx}|\psi|^2 dxdt. \tag{84}$$

On the other hand, we have

$$\begin{array}{rcl} I_{21}^{\varepsilon} & = & 4\varepsilon^2 s^5 \displaystyle\iint_Q \alpha_x^5 \psi_{xxx}\psi dxdt = -20\varepsilon^2 s^5 \iint_Q \alpha_x^4 \alpha_{xx}\psi\psi_{xx} dxdt - 2\varepsilon^2 s^5 \iint_Q \alpha_x^5 (|\psi_x|^2)_x dxdt \\ & \geq & 30\varepsilon^2 s^5 \displaystyle\iint_Q \alpha_x^4 \alpha_{xx}|\psi_x|^2 dxdt - C_0\varepsilon^2 s^5 T^{4\mu} \iint_Q \alpha^7|\psi|^2 dxdt \end{array} \tag{85}$$

.

Moreover,

$$I_{31}^{\varepsilon} = 6\varepsilon^2 s^7 \iint_Q \alpha_x^6 \alpha_{xx}|\psi|^2 dxdt. \tag{86}$$

Furthermore,

$$I_{12}^{\varepsilon} = 12\varepsilon^2 s^5 \iint_Q \alpha_x^5 (|\psi_x|^2)_x dxdt = -60\varepsilon^2 s^5 \iint_Q \alpha_x^4 \alpha_{xx}|\psi_x|^2 dxdt. \tag{87}$$

On the other hand, we have

$$\begin{array}{rcl} I_{22}^{\varepsilon} & = & 12\varepsilon^2 s^3 \displaystyle\iint_Q \alpha_x^3 (|\psi_{xx}|^2)_x dxdt = -36\varepsilon^2 s^3 \iint_Q \alpha_x^2 \alpha_{xx}|\psi_{xx}|^2 dxdt \\ & & +12\varepsilon^2 s^3 \displaystyle\int_0^T (\alpha_x^3|\psi_{xx}|^2)_{|_{x=0,L}} dt. \end{array} \tag{88}$$

Moreover, we have

$$I_{32}^{\varepsilon} = 36\varepsilon^2 s^5 \iint_Q \alpha_x^5 \psi_{xx}\psi dxdt \geq -36\varepsilon^2 s^5 \iint_Q \alpha_x^4 \alpha_{xx}|\psi_x|^2 dxdt - C_0\varepsilon^2 s^5 T^{4\mu} \iint_Q \alpha^7|\psi|^2 dxdt. \tag{89}$$

On the other hand, we have

$$\begin{array}{rcl} I_{13}^{\varepsilon} & = & 4\varepsilon^2 s^3 \displaystyle\iint_Q \alpha_x^3 \psi_{xxxx}\psi_x dxdt = -2\varepsilon^2 s^3 \iint_Q \alpha_x^3 (|\psi_{xx}|^2)_x dxdt - 12\varepsilon^2 s^3 \iint_Q \alpha_x^2 \alpha_{xx}\psi_{xxx}\psi_x dxdt \\ & \geq & 18\varepsilon^2 s^3 \displaystyle\iint_Q \alpha_x^2 \alpha_{xx}|\psi_{xx}|^2 dxdt - 2\varepsilon^2 s^3 \int_0^T (\alpha_x^3|\psi_{xx}|^2)_{|_{x=0,L}} dt - C_0\varepsilon^2 s^3 T^{4\mu} \iint_Q \alpha^5|\psi_x|^2 dxdt. \end{array} \tag{90}$$

Moreover, we have

$$I_{23}^{\varepsilon} = 2\varepsilon^2 s \iint_Q \alpha_x (|\psi_{xxx}|^2)_x dxdt = -2\varepsilon^2 s \iint_Q \alpha_{xx} |\psi_{xxx}|^2 dxdt + 2\varepsilon^2 s \int_0^T (\alpha_x |\psi_{xxx}|^2)_{|_{x=0,L}} dt. \tag{91}$$

Furthermore, we have

$$\begin{aligned} I_{33}^{\varepsilon} &= 6\varepsilon^2 s^3 \iint_Q \alpha_x^2 \alpha_{xx} \psi_{xxxx} \psi dxdt \\ &= -12\varepsilon^2 s^3 \iint_Q \alpha_x \alpha_{xx}^2 \psi_{xxx} \psi dxdt - 6\varepsilon^2 s^3 \iint_Q \alpha_x^2 \alpha_{xx} \psi_{xxx} \psi_x dxdt \\ &\geq 6\varepsilon^2 s^3 \iint_Q \alpha_x^2 \alpha_{xx} |\psi_{xx}|^2 dxdt - C_0 \varepsilon^2 s^3 T^{8\mu} \iint_Q \alpha^7 |\psi|^2 dxdt \\ &\quad - C_0 \varepsilon^2 s^3 T^{4\mu} \iint_Q \alpha^5 |\psi_x|^2 dxdt. \end{aligned} \tag{92}$$

Moreover, we have

$$I_{14}^{\varepsilon} = 48\varepsilon^2 s^5 \iint_Q \alpha_x^4 \alpha_{xx} |\psi_x|^2 dxdt. \tag{93}$$

Furthermore, we have

$$\begin{aligned} I_{24}^{\varepsilon} &= 48\varepsilon^2 s^3 \iint_Q \alpha_x^2 \alpha_{xx} \psi_x \psi_{xxx} dxdt \\ &\geq -48\varepsilon^2 s^3 \iint_Q \alpha_x^2 \alpha_{xx} |\psi_{xx}|^2 dxdt - C_0 \varepsilon^2 s^3 T^{4\mu} \iint_Q \alpha^5 |\psi_x|^2 dxdt. \end{aligned} \tag{94}$$

At the end, we have

$$I_{34}^{\varepsilon} \geq -C_0 \varepsilon^2 s^5 T^{4\mu} \iint_Q \alpha^7 |\psi|^2 dxdt. \tag{95}$$

Combining the previous computations, we deduce

$$\boxed{\begin{aligned} (L_{1,\varepsilon}\psi, L_{2,\varepsilon}\psi)_{L^2(Q)} &\geq -8\varepsilon^2 s^7 \iint_Q \alpha_x^6 \alpha_{xx} |\psi|^2 - 18\varepsilon^2 s^5 \iint_Q \alpha_x^4 \alpha_{xx} |\psi_x|^2 dxdt \\ &\quad -60\varepsilon^2 s^3 \iint_Q \alpha_x^2 \alpha_{xx} |\psi_{xx}|^2 dxdt - 2\varepsilon^2 s \iint_Q \alpha_{xx} |\psi_{xxx}|^2 dxdt \\ &\quad -2\varepsilon^2 s \int_0^T (\alpha_x |\psi_{xxx}|^2)_{x=0} dt - 10\varepsilon^2 s^3 \int_0^T (\alpha_x^3 |\psi_{xx}|^2)_{x=0} dt \\ &\quad -C_0 \varepsilon^2 (s^5 T^{4\mu} + s^3 T^{8\mu}) \iint_Q \alpha^7 |\psi|^2 dxdt - C_0 \varepsilon^2 s^3 T^{4\mu} \iint_Q \alpha^5 |\psi_x|^2 dxdt. \end{aligned}} \tag{96}$$

Step 3. Computation of the left scalar products..

In this step, we will compute the rest of the scalar products. Let us start with $\iint_Q L_{1,\varepsilon}\psi L_{2,\delta} dxdt$, that we write under the form $\sum_{i=1}^{3}\sum_{j=1}^{3} I_{ij}^{\varepsilon,\delta}$ where $I_{ij}^{\varepsilon,\delta}$ is the scalar product in $L^2(Q)$ of the i-th term of $L_{1,\varepsilon}\psi$ with the j-th term of $L_{2,\delta}\psi$.

By integrating by parts, we have

$$I_{11}^{\varepsilon,\delta} = 2\varepsilon\delta s^6 \iint_Q \alpha_x^6 (|\psi|^2)_x dxdt = -12\varepsilon\delta s^6 \iint_Q \alpha_x^5 \alpha_{xx} |\psi|^2 dxdt. \tag{97}$$

On the other hand, we have

$$\begin{aligned} I_{21}^{\varepsilon,\delta} &= 4\varepsilon\delta s^4 \iint_Q \alpha_x^4 \psi_{xxx} \psi dxdt = -16\varepsilon\delta s^4 \iint_Q \alpha_x^3 \alpha_{xx} \psi_{xx} \psi dxdt - 2\varepsilon\delta s^4 \iint_Q \alpha_x^4 (|\psi_x|^2)_x dxdt \\ &= 24\varepsilon\delta s^4 \iint_Q \alpha_x^2 \alpha_{xx}^2 (|\psi|^2)_x dxdt + 24\varepsilon\delta s^4 \iint_Q \alpha_x^3 \alpha_{xx} |\psi_x|^2 dxdt \\ &= -48\varepsilon\delta s^4 \iint_Q \alpha_x \alpha_{xx}^3 |\psi|^2 dxdt + 24\varepsilon\delta s^4 \iint_Q \alpha_x^3 \alpha_{xx} |\psi_x|^2 dxdt. \end{aligned} \tag{98}$$

Moreover, we have

$$I_{31}^{\varepsilon,\delta} = 6\varepsilon\delta s^6 \iint_Q \alpha_x^5 \alpha_{xx} |\psi|^2 dxdt. \tag{99}$$

Furthermore, we have

$$I_{12}^{\varepsilon,\delta} = 6\varepsilon\delta s^4 \iint_Q \alpha_x^4 (|\psi_x|^2)_x dxdt = -24\varepsilon\delta s^4 \iint_Q \alpha_x^3 \alpha_{xx} |\psi_x|^2 dxdt. \tag{100}$$

On the other hand, we have

$$I_{22}^{\varepsilon,\delta} = 6\varepsilon\delta s^2 \iint_Q \alpha_x^2 (|\psi_{xx}|^2)_x dxdt = -12\varepsilon\delta s^2 \iint_Q \alpha_x \alpha_{xx} |\psi_{xx}|^2 dxdt + 6\varepsilon\delta s^2 \int_0^T (\alpha_x^2 |\psi_{xx}|^2)_{|_{x=0,L}} dt. \tag{101}$$

Moreover, we have

$$\begin{aligned} I_{32}^{\varepsilon,\delta} &= 18\varepsilon\delta s^4 \iint_Q \alpha_x^3 \alpha_{xx} \psi_{xx} \psi dxdt = -27\varepsilon\delta s^4 \iint_Q \alpha_x^2 \alpha_{xx}^2 (|\psi|^2)_x dxdt - 18\varepsilon\delta s^4 \iint_Q \alpha_x^3 \alpha_{xx} |\psi_x|^2 dxdt \\ &= 54\varepsilon\delta s^4 \iint_Q \alpha_x \alpha_{xx}^3 |\psi|^2 dxdt - 18\varepsilon\delta s^4 \iint_Q \alpha_x^3 \alpha_{xx} |\psi_x|^2 dxdt. \end{aligned} \tag{102}$$

Furthermore, we have

$$I_{13}^{\varepsilon,\delta} = 12\varepsilon\delta s^4 \iint_Q \alpha_x^3 \alpha_{xx} |\psi_x|^2 dxdt. \tag{103}$$

On the other hand, we have

$$\begin{aligned} I_{23}^{\varepsilon,\delta} &= 12\varepsilon\delta s^2 \iint_Q \alpha_x \alpha_{xx} \psi_{xxx} \psi_x dxdt = -6\varepsilon\delta s^2 \iint_Q \alpha_{xx}^2 (|\psi_x|^2)_x dxdt - 12\varepsilon\delta s^2 \iint_Q \alpha_x \alpha_{xx} |\psi_{xx}|^2 dxdt \\ &= -12\varepsilon\delta s^2 \iint_Q \alpha_x \alpha_{xx} |\psi_{xx}|^2 dxdt. \end{aligned} \tag{104}$$

Here we used the fact that $\alpha_{xxx} = 0$. Moreover, we have

$$I_{33}^{\varepsilon,\delta} = 9\varepsilon\delta s^4 \iint_Q \alpha_x^2 \alpha_{xx}^2 (|\psi|^2)_x dxdt = -18\varepsilon\delta s^4 \iint_Q \alpha_x \alpha_{xx}^3 |\psi|^2 dxdt. \tag{105}$$

Combining the previous computations, we deduce

$$\begin{array}{rcl}(L_{1,\varepsilon}\psi, L_{2,\delta}\psi)_{L^2(Q)} & \geq & -6\varepsilon\delta s^6 \displaystyle\iint_Q \alpha_x^5\alpha_{xx}|\psi|^2dxdt - C_0\varepsilon\delta s^4 T^{4\mu}\iint_Q \alpha^6|\psi|^2dxdt \\ & & -6\varepsilon\delta s^4\displaystyle\iint_Q \alpha_x^3\alpha_{xx}|\psi_x|^2dxdt - 24\varepsilon\delta s^2\iint_Q \alpha_x\alpha_{xx}|\psi_{xx}|^2dxdt \\ & & -6\varepsilon\delta s^2\displaystyle\int_0^T(\alpha_x^2|\psi_{xx}|^2)_{x=0}dt.\end{array} \tag{106}$$

Now, we study the term $\displaystyle\iint_Q L_{1,\delta}\psi L_{2,\varepsilon}dxdt$, that we write under the form $\displaystyle\sum_{i=1}^{2}\sum_{j=1}^{4} I_{ij}^{\delta,\varepsilon}$ where $I_{ij}^{\delta,\varepsilon}$ is the scalar product in $L^2(Q)$ of the i-th term of $L_{1,\delta}\psi$ with the j-th term of $L_{2,\varepsilon}\psi$.

By integrating by parts, we have

$$\begin{array}{rcl}I_{11}^{\delta,\varepsilon} & = & \varepsilon\delta s^4\displaystyle\iint_Q \alpha_x^4\psi_{xxx}\psi dxdt = -\frac{1}{2}\delta\varepsilon s^4\iint_Q \alpha_x^4(|\psi_x|^2)_x dxdt - 4\varepsilon\delta s^4\iint_Q \alpha_x^3\alpha_{xx}\psi_{xx}\psi dxdt \\ & = & 2\delta\varepsilon s^4\displaystyle\iint_Q \alpha_x^3\alpha_{xx}|\psi_x|^2dxdt + 4\varepsilon\delta s^4\iint_Q \alpha_x^3\alpha_{xx}|\psi_x|^2dxdt + 6\varepsilon\delta s^4\iint_Q \alpha_x^2\alpha_{xx}^2(|\psi|^2)_x dxdt \\ & = & 6\delta\varepsilon s^4\displaystyle\iint_Q \alpha_x^3\alpha_{xx}|\psi_x|^2dxdt - 12\varepsilon\delta s^4\iint_Q \alpha_x\alpha_{xx}^3|\psi|^2dxdt.\end{array} \tag{107}$$

On the other hand, we have

$$I_{12}^{\delta,\varepsilon} = 3\varepsilon\delta s^2\iint_Q \alpha_x^2(|\psi_{xx}|^2)_x dxdt = -6\varepsilon\delta s^2\iint_Q \alpha_x\alpha_{xx}|\psi_{xx}|^2dxdt + 3\delta\varepsilon s^2\int_0^T \alpha_x^2(|\psi_{xx}|^2)_{x=0,L}dt. \tag{108}$$

Moreover, we have

$$I_{13}^{\delta,\varepsilon} = \frac{1}{2}\varepsilon\delta\iint_Q(|\psi_{xxx}|^2)_x dxdt = \frac{1}{2}\varepsilon\delta\int_0^T(|\psi_{xxx}|^2)_{x=0,L}dt. \tag{109}$$

Furthermore, we have

$$\begin{array}{rcl}I_{14}^{\delta,\varepsilon} & = & 12\varepsilon\delta s^2\displaystyle\iint_Q \alpha_x\alpha_{xx}\psi_{xxx}\psi_x dxdt = -6\varepsilon\delta s^2\iint_Q \alpha_{xx}^2(|\psi_x|^2)_x dxdt \\ & & -12\varepsilon\delta s^2\displaystyle\iint_Q \alpha_x\alpha_{xx}|\psi_{xx}|^2dxdt = -12\varepsilon\delta s^2\iint_Q \alpha_x\alpha_{xx}|\psi_{xx}|^2dxdt.\end{array} \tag{110}$$

Here we used the fact that $\alpha_{xxx} = 0$. On the other hand, we have

$$I_{21}^{\delta,\varepsilon} = \frac{3}{2}\varepsilon\delta s^6\iint_Q \alpha_x^6(|\psi|^2)_x dxdt = -9\varepsilon\delta s^6\iint_Q \alpha_x^5\alpha_{xx}|\psi|^2dxdt. \tag{111}$$

Furthermore, we have

$$I_{22}^{\delta,\varepsilon} = 9\varepsilon\delta s^4\iint_Q \alpha_x^4(|\psi_x|^2)_x dxdt = -36\varepsilon\delta s^4\iint_Q \alpha_x^3\alpha_{xx}|\psi_x|^2dxdt. \tag{112}$$

Moreover, we have

$$\begin{aligned} I_{23}^{\delta,\varepsilon} &= 3\varepsilon\delta s^2\iint_Q \alpha_x^2\psi_{xxxx}\psi_x dxdt = -\frac{3}{2}\varepsilon\delta s^2\iint_Q \alpha_x^2(|\psi_{xx}|^2)_x dxdt - 6\varepsilon\delta s^2\iint_Q \alpha_x\alpha_{xx}\psi_{xxx}\psi_x dxdt \\ &= 9\varepsilon\delta s^2\iint_Q \alpha_x\alpha_{xx}|\psi_{xx}|^2 dxdt - \frac{3}{2}\varepsilon\delta s^2\int_0^T(\alpha_x^2|\psi_{xx}|^2)_{x=0,L}dxdt + 3\varepsilon\delta s^2\iint_Q \alpha_{xx}^2(|\psi_x|^2)_x dxdt \\ &= 9\varepsilon\delta s^2\iint_Q \alpha_x\alpha_{xx}|\psi_{xx}|^2 dxdt - \frac{3}{2}\varepsilon\delta s^2\int_0^T(\alpha_x^2|\psi_{xx}|^2)_{x=0,L}dt. \end{aligned}$$

(113)

On the other hand, we have

$$I_{24}^{\delta,\varepsilon} = 36\varepsilon\delta s^4\iint_Q \alpha_x^3\alpha_{xx}|\psi_x|^2 dxdt. \tag{114}$$

Combining the previous computations, we deduce

$$\begin{aligned} (L_{1,\delta}\psi, L_{2,\varepsilon}\psi)_{L^2(Q)} \geq\ & 6\delta\varepsilon s^4\iint_Q \alpha_x^3\alpha_{xx}|\psi_x|^2 dxdt - C_0\varepsilon\delta s^4T^{4\mu}\iint_Q \alpha^6|\psi|^2 dxdt \\ & -9\varepsilon\delta s^6\iint_Q \alpha_x^5\alpha_{xx}|\psi|^2 dxdt - 9\varepsilon\delta s^2\iint_Q \alpha_x\alpha_{xx}|\psi_{xx}|^2 dxdt \\ & -\frac{1}{2}\varepsilon\delta\int_0^T(|\psi_{xxx}|^2)_{x=0}dt - \frac{3}{2}\varepsilon\delta s^2\int_0^T(\alpha_x^2|\psi_{xx}|^2)_{x=0}dt. \end{aligned} \tag{115}$$

Combining (115) with (106), we deduce

$$\begin{aligned} (L_{1,\delta}\psi, L_{2,\varepsilon}\psi)_{L^2(Q)} + (L_{1,\varepsilon}\psi, L_{2,\delta}\psi)_{L^2(Q)} \geq\ & -C_0\varepsilon\delta s^4T^{4\mu}\iint_Q \alpha^6|\psi|^2 dxdt - 15\varepsilon\delta s^6\iint_Q \alpha_x^5\alpha_{xx}|\psi|^2 dxdt \\ & -33\varepsilon\delta s^2\iint_Q \alpha_x\alpha_{xx}|\psi_{xx}|^2 dxdt - \frac{1}{2}\varepsilon\delta\int_0^T(|\psi_{xxx}|^2)_{x=0}dt \\ & -\frac{21}{2}\varepsilon\delta s^2\int_0^T(\alpha_x^2|\psi_{xx}|^2)_{x=0}dt. \end{aligned} \tag{116}$$

Now, we study the term $\iint_Q L_{1,\varepsilon}\psi\tilde{L}_2 dxdt$, that we write under the form $\sum_{i=1}^{3}\sum_{j=1}^{2} I_{ij}^{\varepsilon,2}$ where $I_{ij}^{\varepsilon,2}$ is the scalar product in $L^2(Q)$ of the i-th term of $L_{1,\varepsilon}\psi$ with the j-th term of $\tilde{L}_2\psi$.

By integrating by parts, we have

$$I_{11}^{\varepsilon,2} = -2\varepsilon s^4\iint_Q \alpha_x^3\alpha_t(|\psi|^2)_x dxdt = 2\varepsilon s^4\iint_Q (\alpha_x^3\alpha_t)_x|\psi|^2 dxdt. \tag{117}$$

Moreover, we have

$$I_{12}^{\varepsilon,2} = -2M\varepsilon s^4 \iint_Q \alpha_x^4 (|\psi|^2)_x dxdt = 8M\varepsilon s^4 \iint_Q \alpha_x^3 \alpha_{xx} |\psi|^2 dxdt. \tag{118}$$

On the other hand, we have

$$\begin{aligned} I_{21}^{\varepsilon,2} &= -4\varepsilon s^2 \iint_Q \alpha_x \alpha_t \psi_{xxx} \psi dxdt = 4\varepsilon s^2 \iint_Q (\alpha_x \alpha_t)_x \psi_{xx} \psi dxdt + 2\varepsilon s^2 \iint_Q \alpha_x \alpha_t (|\psi_x|^2)_x dxdt \\ &= -2\varepsilon s^2 \iint_Q (\alpha_x \alpha_t)_{xx} (|\psi|^2)_x dxdt - 4\varepsilon s^2 \iint_Q (\alpha_x \alpha_t)_x |\psi_x|^2 dxdt - 2\varepsilon s^2 \iint_Q (\alpha_x \alpha_t)_x |\psi_x|^2 dxdt \\ &= 2\varepsilon s^2 \iint_Q \alpha_{xx} \alpha_{txx} |\psi|^2 dxdt - 6\varepsilon s^2 \iint_Q (\alpha_x \alpha_t)_x |\psi_x|^2 dxdt. \end{aligned} \tag{119}$$

Furthermore, we have

$$\begin{aligned} I_{22}^{\varepsilon,2} &= -4M\varepsilon s^2 \iint_Q \alpha_x^2 \psi_{xxx} \psi dxdt = 8M\varepsilon s^2 \iint_Q \alpha_x \alpha_{xx} \psi_{xx} \psi dxdt + 2M\varepsilon s^2 \iint_Q \alpha_x^2 (|\psi_x|^2)_x dxdt \\ &= -4M\varepsilon s^2 \iint_Q \alpha_{xx}^2 (|\psi|^2)_x dxdt - 12M\varepsilon s^2 \iint_Q \alpha_x \alpha_{xx} |\psi_x|^2 dxdt \\ &= -12M\varepsilon s^2 \iint_Q \alpha_x \alpha_{xx} |\psi_x|^2 dxdt. \end{aligned} \tag{120}$$

Here we used the fact that $\alpha_{xxx} = 0$. Moreover, we have

$$I_{31}^{\varepsilon,2} = -6\varepsilon s^4 \iint_Q \alpha_x^2 \alpha_{xx} \alpha_t |\psi|^2 dxdt. \tag{121}$$

On the other hand, we have

$$I_{32}^{\varepsilon,2} = -6M\varepsilon s^4 \iint_Q \alpha_x^3 \alpha_{xx} |\psi|^2 dxdt. \tag{122}$$

Combining the previous computations, we deduce

$$\boxed{\begin{aligned} (L_{1,\varepsilon}\psi, \tilde{L}_2\psi)_{L^2(Q)} &= \varepsilon \iint_Q \Big(-6s^4 \alpha_x^2 \alpha_{xx} \alpha_t + 2s^2 \alpha_{xx} \alpha_{txx} + 2\varepsilon s^4 (\alpha_x^3 \alpha_t)_x \Big) |\psi|^2 dxdt \\ &\quad + 2\varepsilon M s^4 \iint_Q \alpha_x^3 \alpha_{xx} |\psi|^2 dxdt - 6\varepsilon s^2 \iint_Q (\alpha_x \alpha_t)_x |\psi_x|^2 dxdt \\ &\quad - 12M\varepsilon s^2 \iint_Q \alpha_x \alpha_{xx} |\psi_x|^2 dxdt. \end{aligned}} \tag{123}$$

Now, we study the term $\iint_Q L_{1,\delta} \psi \tilde{L}_2 dxdt$, that we write under the form $\sum_{i=1}^{2} \sum_{j=1}^{2} I_{ij}^{\delta,2}$ where $I_{ij}^{\delta,2}$ is the scalar product in $L^2(Q)$ of the i-th term of $L_{1,\delta}\psi$ with the j-th term of $\tilde{L}_2\psi$.

By integrating by parts, we have

$$\begin{aligned} I_{11}^{\delta,2} &= -\delta s \iint_Q \alpha_t \psi_{xxx}\psi dxdt = \delta s \iint_Q \alpha_{tx}\psi_{xx}\psi dxdt + \frac{1}{2}\delta s \iint_Q \alpha_t (|\psi_x|^2)_x dxdt \\ &= -\frac{1}{2}\delta s \iint_Q \alpha_{txx}(|\psi|^2)_x dxdt - \frac{3}{2}\delta s \iint_Q \alpha_{tx}|\psi_x|^2 dxdt \\ &= -\frac{3}{2}\delta s \iint_Q \alpha_{tx}|\psi_x|^2 dxdt. \end{aligned} \tag{124}$$

Moreover, we have

$$\begin{aligned} I_{12}^{\delta,2} &= -\delta M s \iint_Q \alpha_x \psi_{xxx}\psi dxdt = \delta M s \iint_Q \alpha_{xx}\psi_{xx}\psi dxdt + \frac{1}{2}\delta M s \iint_Q \alpha_x (\psi_x|^2)_x dxdt \\ &= -\frac{3}{2}\delta M s \iint_Q \alpha_{xx}|\psi_x|^2 dxdt. \end{aligned} \tag{125}$$

Furthermore, we have

$$I_{21}^{\delta,2} = -\frac{3}{2}\delta s^3 \iint_Q \alpha_x^2 \alpha_t (|\psi|^2)_x dxdt = \frac{3}{2}\delta s^3 \iint_Q (\alpha_x^2\alpha_t)_x |\psi|^2 dxdt. \tag{126}$$

On the other hand, we have

$$I_{22}^{\delta,2} = -\frac{3}{2}M\delta s^3 \iint_Q \alpha_x^3 (|\psi|^2)_x dxdt = \frac{9}{2}M\delta s^3 \iint_Q \alpha_x^2\alpha_{xx}|\psi|^2 dxdt. \tag{127}$$

Combining the previous computations, we deduce

$$\boxed{\begin{aligned} (L_{1,\delta}\psi, \tilde{L}_2\psi)_{L^2(Q)} &= \frac{3}{2}\delta s^3 \iint_Q (\alpha_x^2\alpha_t)_x |\psi|^2 dxdt - \frac{3}{2}\delta M s \iint_Q \alpha_{xx}|\psi_x|^2 dxdt \\ &\quad -\frac{3}{2}\delta s \iint_Q \alpha_{tx}|\psi_x|^2 dxdt + \frac{9}{2}M\delta s^3 \iint_Q \alpha_x^2\alpha_{xx}|\psi|^2 dxdt. \end{aligned}} \tag{128}$$

Now, we study the term $\iint_Q \tilde{L}_1\psi L_{2,\varepsilon} dxdt$, that we write under the form $\sum_{i=1}^{2}\sum_{j=1}^{4} I_{ij}^{1,\varepsilon}$ where $I_{ij}^{1,\varepsilon}$ is the scalar product in $L^2(Q)$ of the i-th term of $\tilde{L}_1\psi$ with the j-th term of $L_{\varepsilon,2}\psi$.

By integrating by parts, we have

$$I_{11}^{1,\varepsilon} = -\frac{1}{2}\varepsilon s^4 \iint_Q \alpha_x^4 (|\psi|^2)_t dxdt = 2\varepsilon s^4 \iint_Q \alpha_x^3\alpha_{tx}|\psi|^2 dxdt. \tag{129}$$

On the other hand, we have

$$\begin{aligned} I_{12}^{1,\varepsilon} &= -6\varepsilon s^2 \iint_Q \alpha_x^2 \psi_{xx}\psi_t dxdt = 12\varepsilon s^2 \iint_Q \alpha_x\alpha_{xx}\psi_x\psi_t dxdt + 3\varepsilon s^2 \iint_Q \alpha_x^2 (|\psi_x|^2)_t dxdt \\ &= 12\varepsilon s^2 \iint_Q \alpha_x\alpha_{xx}\psi_x\psi_t dxdt - 6\varepsilon s^2 \iint_Q \alpha_x\alpha_{tx}|\psi_x|^2 dxdt. \end{aligned} \tag{130}$$

Furthermore, we have

$$I_{13}^{1,\varepsilon} = -\varepsilon \iint_Q \psi_{xxxx}\psi_t dxdt = -\frac{1}{2}\varepsilon \iint_Q (|\psi_{xx}|^2)_t dxdt = 0. \tag{131}$$

Moreover, we have

$$I_{14}^{1,\varepsilon} = -12\varepsilon s^2 \iint_Q \alpha_x \alpha_{xx} \psi_x \psi_t dxdt. \tag{132}$$

Furthermore, we have

$$I_{21}^{1,\varepsilon} = -\frac{1}{2}\varepsilon M s^4 \iint_Q \alpha_x^4 (|\psi|^2)_x dxdt = 2\varepsilon M s^4 \iint_Q \alpha_x^3 \alpha_{xx} |\psi|^2 dxdt. \tag{133}$$

On the other hand, we have

$$I_{22}^{1,\varepsilon} = -3\varepsilon M s^2 \iint_Q \alpha_x^2 (|\psi_x|^2)_x dxdt = 6\varepsilon M s^2 \iint_Q \alpha_x \alpha_{xx} |\psi_x|^2 dxdt. \tag{134}$$

Moreover, we have

$$I_{23}^{1,\varepsilon} = -\varepsilon M \iint_Q \psi_{xxxx}\psi_x dxdt = \frac{1}{2}\varepsilon M \iint_Q (|\psi_{xx}|^2)_x dxdt = \frac{1}{2}\varepsilon M \int_0^T (|\psi_{xx}|^2)_{x=0,L} dt. \tag{135}$$

Furthermore, we have

$$I_{24}^{1,\varepsilon} = -12\varepsilon M s^2 \iint_Q \alpha_x \alpha_{xx} |\psi_x|^2 dxdt. \tag{136}$$

Combining the previous computations, we deduce

$$\begin{aligned} (\tilde{L}_1\psi, L_{\varepsilon,2}\psi)_{L^2(Q)} \geq\ & 2\varepsilon s^4 \iint_Q \alpha_x^3 \alpha_{tx} |\psi|^2 dxdt - 6\varepsilon s^2 \iint_Q \alpha_x \alpha_{tx} |\psi_x|^2 dxdt \\ & + 2\varepsilon M s^4 \iint_Q \alpha_x^3 \alpha_{xx} |\psi|^2 dxdt - 6\varepsilon M s^2 \iint_Q \alpha_x \alpha_{xx} |\psi_x|^2 dxdt \\ & - \frac{1}{2}\varepsilon M \int_0^T (|\psi_{xx}|^2)_{x=0} dt. \end{aligned} \tag{137}$$

Now, we study the term $\iint_Q \tilde{L}_1\psi L_{2,\delta} dxdt$, that we write under the form $\sum_{i=1}^{2}\sum_{j=1}^{3} I_{ij}^{1,\delta}$ where $I_{ij}^{1,\delta}$ is the scalar product in $L^2(Q)$ of the i-th term of $\tilde{L}_1\psi$ with the j-th term of $L_{\delta,2}\psi$.

By integrating by parts, we have

$$I_{11}^{1,\delta} = -\frac{1}{2}\delta s^3 \iint_Q \alpha_x^3 (|\psi|^2)_t dxdt = \frac{3}{2}\delta s^3 \iint_Q \alpha_x^2 \alpha_{tx} |\psi|^2 dxdt. \tag{138}$$

On the other hand, we have

$$\begin{aligned} I_{12}^{1,\delta} &= -3\delta s \iint_Q \alpha_x \psi_{xx} \psi_t dxdt = 3\delta s \iint_Q \alpha_{xx} \psi_x \psi_t dxdt + \frac{3}{2}\delta s \iint_Q \alpha_x (\psi_x|^2)_t dxdt \\ &= 3\delta s \iint_Q \alpha_{xx} \psi_x \psi_t dxdt - \frac{3}{2}\delta s \iint_Q \alpha_{tx} |\psi_x|^2 dxdt. \end{aligned} \tag{139}$$

Furthermore, we have

$$I_{13}^{1,\delta} = -3\delta s \iint_Q \alpha_{xx} \psi_x \psi_t dxdt. \tag{140}$$

In addition, we have

$$I_{21}^{1,\delta} = -\frac{1}{2}\delta M s^3 \iint_Q \alpha_x^3 (|\psi|^2)_x dxdt = \frac{3}{2}\delta M s^3 \iint_Q \alpha_x^2 \alpha_{xx} |\psi|^2 dxdt. \tag{141}$$

Additionally, we have

$$I_{22}^{1,\delta} = -\frac{3}{2}\delta M s \iint_Q \alpha_x (|\psi_x|^2)_x dxdt = \frac{3}{2}\delta M s \iint_Q \alpha_{xx} |\psi_x|^2 dxdt. \tag{142}$$

Moreover, we have

$$I_{23}^{1,\delta} = -3\delta M s \iint_Q \alpha_{xx} |\psi_x|^2 dxdt. \tag{143}$$

Combining the previous computations, we deduce

$$\boxed{\begin{aligned} (\tilde{L}_1\psi, L_{\delta,2}\psi)_{L^2(Q)} &= \frac{3}{2}\delta s^3 \iint_Q \alpha_x^2 \alpha_{tx} |\psi|^2 dxdt - \frac{3}{2}\delta s \iint_Q \alpha_{tx} |\psi_x|^2 dxdt \\ &\quad + \frac{3}{2}\delta M s^3 \iint_Q \alpha_x^2 \alpha_{xx} |\psi|^2 dxdt - \frac{3}{2}\delta M s \iint_Q \alpha_{xx} |\psi_x|^2 dxdt. \end{aligned}} \tag{144}$$

Now, we study the term $\iint_Q \tilde{L}_1\psi \tilde{L}_2 dxdt$, that we write under the form $\sum_{i=1}^{2}\sum_{j=1}^{2} I_{ij}^{1,2}$ where $I_{ij}^{1,2}$ is the scalar product in $L^2(Q)$ of the i-th term of $\tilde{L}_1\psi$ with the j-th term of $\tilde{L}_2\psi$.

By integrating by parts, we have

$$I_{11}^{1,2} = \frac{1}{2} s \iint_Q \alpha_t (|\psi|^2)_t dxdt = -\frac{1}{2} s \iint_Q \alpha_{tt} |\psi|^2 dxdt. \tag{145}$$

In addition, we have

$$I_{12}^{1,2} = \frac{1}{2} M s \iint_Q \alpha_x (|\psi|^2)_t dxdt = -\frac{1}{2} M s \iint_Q \alpha_{tx} |\psi|^2 dxdt. \tag{146}$$

Furthermore, we have

$$I_{21}^{1,2} = \frac{1}{2} M s \iint_Q \alpha_t (|\psi|^2)_x dxdt = -\frac{1}{2} M s \iint_Q \alpha_{tx} |\psi|^2 dxdt. \tag{147}$$

Additionally, we have

$$I_{22}^{1,2} = \frac{1}{2}M^2 s\iint_Q \alpha_x(|\psi|^2)_x dxdt = -\frac{1}{2}M^2 s\iint_Q \alpha_{xx}|\psi|^2 dxdt. \tag{148}$$

Combining the previous computations, we deduce

$$\boxed{\begin{array}{rcl}(\tilde{L}_1\psi, \tilde{L}_2\psi)_{L^2(Q)} &=& -\frac{1}{2}M^2 s\iint_Q \alpha_{xx}|\psi|^2 dxdt - Ms\iint_Q \alpha_{tx}|\psi|^2 dxdt \\ && -\frac{1}{2}s\iint_Q \alpha_{tt}|\psi|^2 dxdt.\end{array}} \tag{149}$$

Step 4. Last computations.

Combining (75)-(83)-(96)-(116)-(123)-(128)-(137)-(144)-(149) with Remark 3.1, we deduce

$$\begin{array}{c}\varepsilon^2\iint_Q \Big(s^7\alpha^7|\psi|^2 + s^5\alpha^5|\psi_x|^2 dxdt + s^3\alpha^3|\psi_{xx}|^2 dxdt + s\alpha|\psi_{xxx}|^2\Big)dxdt \\ +\delta^2\iint_Q \Big(s^5\alpha^5|\psi|^2 + s\alpha|\psi_{xx}|^2\Big)dxdt + \varepsilon\delta\iint_Q \Big(s^6\alpha^6|\psi|^2 + s^2\alpha^2|\psi_{xx}|^2\Big)dxdt \\ \leq C\Big(\varepsilon^2 s\int_0^T (\alpha_x|\psi_{xxx}|^2)_{x=0}dt + \varepsilon^2 s^3\int_0^T (\alpha_x^3|\psi_{xx}|^2)_{x=0}dt \\ +\delta^2 s\int_0^T (\alpha_x|\psi_{xx}|^2)_{|_{x=0}}dt + \varepsilon\delta\int_0^T (|\psi_{xxx}|^2)_{x=0}dt + \varepsilon\delta s^2\int_0^T (\alpha_x^2|\psi_{xx}|^2)_{x=0}dt \\ +\varepsilon M\int_0^T (|\psi_{xx}|^2)_{x=0}dt + \varepsilon s^4 T\iint_Q \alpha^{4+1/\mu}|\psi|^2 dxdt + T\varepsilon s^2\iint_Q \alpha^{2+1/\mu}|\psi_x|^2 dxdt \\ +\delta s^3 T\iint_Q \alpha^{3+1/\mu}|\psi|^2 dxdt + \delta sT\iint_Q \alpha^{1+1/\mu}|\psi_x|^2 dxdt \\ +M\delta s^3\iint_Q \alpha^3|\psi|^2 dxdt + \varepsilon M s^4\iint_Q \alpha^4|\psi|^2 dxdt + MsT\iint_Q \alpha^{1+1/\mu}|\psi|^2 dxdt \\ +sT^2\iint_Q \alpha^{1+2/\mu}|\psi|^2 dxdt + \|L_3\psi\|^2_{L^2(Q)}\Big),\end{array} \tag{150}$$

for $s \geq C_0 T^{2\mu}$.

To finish the proof, it suffices to treat the global terms in the right-hand side of (150). Let us notice that we can add the following terms :

$$s^{4+1/\mu}\ \varepsilon^{1/\mu-1}\ \delta^{3-1/\mu}\iint_Q \alpha^{4+1/\mu}|\psi|^2 dxdt + s^{3+1/\mu}\ \delta^{4-1/\mu}\ \varepsilon^{1/\mu-2}\iint_Q \alpha^{3+1/\mu}|\psi|^2 dxdt$$

and

$$s^{2+1/\mu}\ \varepsilon^{1/\mu-1}\ \delta^{3-1/\mu}\iint_Q \alpha^{2+1/\mu}|\psi_x|^2 dxdt + s^{1+1/\mu}\ \delta^{4-1/\mu}\ \varepsilon^{1/\mu-2}\iint_Q \alpha^{1+1/\mu}|\psi_x|^2 dxdt,$$

in the left-hand side of (150) for $s \geq C_0 T^{2\mu}$. By taking

$$s \geq C_0 T^\mu\Big(T^\mu + \frac{1+T^\mu M^\mu}{\varepsilon^{1-2\mu}\ \delta^{3\mu-1}}\Big),$$

we absorb the global terms in the right-hand side of (150).

Finally, we come back to φ by using the definition of $\psi = e^{-s\alpha}\varphi$ and the properties on the weight function α given in (15).

□

References

[1] Carreño, N., and Guerrero, S. On the non-uniform null controllability of a linear KdV equation. *Asymptot. Anal. 94*, 1-2 (2015), 33–69.

[2] Carreño, N., and Guzmán, P. On the cost of null controllability of a fourth-order parabolic equation. *J. Differential Equations 261*, 11 (2016), 6485–6520.

[3] Carreño, N., and Cerpa, E. Local controllability of the stabilized Kuramoto-Sivashinsky system by a single control acting on the heat equation. *J. Math. Pures Appl. (9) 106*, 4 (2016), 670–694.

[4] Cerpa, E., and Mercado, A. Local exact controllability to the trajectories of the 1-D Kuramoto-Sivashinsky equation. *J. Differential Equations 250*, 4 (2011), 2024–2044.

[5] Cerpa, E., Mercado, A., and Guzmán, P. On the control of the linear KuramotoâSivashinsky equation. *ESAIM Control Optim . Calc. Var* (2015), 1543–1568.

[6] Coron, J.-M., and Guerrero, S. Singular optimal control: a linear 1-D parabolic-hyperbolic example. *Asymptot. Anal. 44*, 3-4 (2005), 237–257.

[7] Gao, P. Insensitizing controls for the Cahn-Hilliard type equation. *Electron. J. Qual. Theory Differ. Equ.* (2014), No. 35, 22.

[8] Gao, P. A new global Carleman estimate for CahnâHilliard type equation and its applications. *Journal of Differential Equations 260* (2016), 427–444.

[9] Glass, O. A complex-analytic approach to the problem of uniform controllability of a transport equation in the vanishing viscosity limit. *J. Funct. Anal. 258*, 3 (2010), 852–868.

[10] Glass, O., and Guerrero, S. Some exact controllability results for the linear KdV equation and uniform controllability in the zero-dispersion limit. *Asymptot. Anal. 60*, 1-2 (2008), 61–100.

[11] Glass, O., and Guerrero, S. Some exact controllability results for the linear KdV equation and uniform controllability in the zero-dispersion limit. *Asymptot. Anal. 60*, 1-2 (2008), 61–100.

[12] Glass, O., and Guerrero, S. Uniform controllability of a transport equation in zero diffusion-dispersion limit. *Math. Models Methods Appl. Sci. 19*, 9 (2009), 1567–1601.

[13] Guerrero, S., and Kassab, K. Carleman estimate and null controllability of a fourth order parabolic equation in dimension $N \geq 2$. *Journal de Mathématiques Pures et Appliquées* (2017), 10.

[14] Guerrero, S., and Lebeau, G. Singular optimal control for a transport-diffusion equation. *Comm. Partial Differential Equations 32*, 10-12 (2007), 1813–1836.

[15] Kassab, K. Null controllability of semi-linear fourth order parabolic equations. *Journal de Mathématiques Pures et Appliquées 136* (2020), 279–312.

[16] LeFloch, P. G. Hyperbolic Systems of Conservation Laws: The Theory of Classical and Nonclassical Shock Waves. *Lectures in Mathematics (ETH Zurich)* (2002).

[17] Lissy, P. A link between the cost of fast controls for the 1-D heat equation and the uniform controllability of a 1-D transport-diffusion equation. *C. R. Math. Acad. Sci. Paris 350*, 11-12 (2012), 591–595.

[18] Lissy, P. An application of a conjecture due to Ervedoza and Zuazua concerning the observability of the heat equation in small time to a conjecture due to Coron and Guerrero concerning the uniform controllability of a convection-diffusion equation in the vanishing viscosity limit. *Systems Control Lett. 69* (2014), 98–102.

[19] López-García, M., and Mercado, A. Uniform null controllability of a fourth-order parabolic equation with a transport term.

[20] Zhou, Z. Observability estimate and null controllability for one-dimensional fourth order parabolic equation. *Taiwanese J. Math. 16*, 6 (2012), 1991–2017.