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► **To cite this version:**

Karim Kassab. Negative and positive controllability results for coupled systems of second and fourth order parabolic equations. 2020. hal-03080968

**HAL Id: hal-03080968**

**<https://hal.science/hal-03080968>**

Preprint submitted on 18 Dec 2020

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# Negative and positive controllability results for coupled systems of second and fourth order parabolic equations

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October 23, 2020

## Abstract

In this paper, we consider a fourth order parabolic equation in a bounded smooth domain  $\Omega$  with homogeneous Dirichlet boundary conditions on the solution and the laplacian of the solution. The first main result we establish is the existence of insensitizing controls for this equation in an arbitrarily small open set  $\omega$  included in  $\Omega$ . The second main result we establish is the existence of many coupled systems of second and fourth order parabolic equations where the approximate controllability is not satisfied.

**Keywords:** Fourth order parabolic equation, global Carleman estimate, insensitizing controls.

## 1 Introduction

In the present paper, we consider  $\Omega \subset \mathbb{R}^N$  with ( $N \geq 2$ ) a bounded connected open set whose boundary  $\partial\Omega$  is regular enough. Let  $\omega \subset \Omega$  be a (small) nonempty open subset and  $O \subset \Omega$  be a (small) nonempty open subset. We will use the notation  $Q = (0, T) \times \Omega$  and  $\Sigma = (0, T) \times \partial\Omega$  and we will denote by  $\vec{n}(x)$  the outward unit normal vector to  $\Omega$  at the point  $x \in \partial\Omega$ . On the other hand, we will denote by  $C_0$  a generic positive constant which depends on  $\Omega$  and  $\omega$  but not on  $T$ .

Let us introduce the following control system :

$$\begin{cases} \partial_t w + \Delta^2 w + f(w, \nabla w, \nabla^2 w) = \zeta + \chi_\omega v & \text{in } Q, \\ w = \Delta w = 0 & \text{on } \Sigma, \\ w(0, \cdot) = y_0(\cdot) + \tau \tilde{y}_0 & \text{in } \Omega, \end{cases} \quad (1)$$

where  $y_0 \in L^2(\Omega)$  is the initial condition,  $\tau \in \mathbb{R}$  unknown and small enough,  $f$  is a  $C^1$  globally Lipschitz-continuous function defined on  $\mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^{N^2}$ ,  $\tilde{y}_0 \in L^2(\Omega)$  is unknown,  $\|\tilde{y}_0\|_{L^2(\Omega)} = 1$ ,  $v \in L^2(Q)$  is the control function and  $\zeta \in L^2(Q)$ . In this paper we suppose that

$$\omega \cap O \neq \emptyset. \quad (2)$$

Our objective is to establish the existence of insensitizing controls for this equation. Let us introduce the following two functionals :

$$\phi_1(w) = \frac{1}{2} \iint_{(0,T) \times O} |w|^2 dx dt \quad (3)$$

and

$$\phi_2(w) = \frac{1}{2} \iint_{(0,T) \times O} |\nabla w|^2 dx dt. \quad (4)$$

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**Definition 1.1.** Let  $y_0 \in L^2(\Omega)$  and  $\zeta \in L^2(Q)$ . We say that the control  $v$  insensitizes  $\phi_1$ , respectively  $\phi_2$ , if for all  $\tilde{y}_0 \in L^2(\Omega)$  such that  $\|\tilde{y}_0\|_{L^2(\Omega)} = 1$ , we have

$$\left| \frac{\partial \phi_1(w(x, t, v, \tau))}{\partial \tau} \Big|_{\tau=0} \right| = 0 \quad (5)$$

respectively

$$\left| \frac{\partial \phi_2(w(x, t, v, \tau))}{\partial \tau} \Big|_{\tau=0} \right| = 0. \quad (6)$$

In fact, the original problem was introduced by Jacques Louis Lions in [19] for the heat equation. In order to guarantee the insensitivity for all  $y_0 \in L^2(\Omega)$ , the authors in [3] introduced the  $\epsilon$ -insensitivity.

**Definition 1.2.** Let  $\epsilon > 0$ ,  $y_0 \in L^2(\Omega)$  and  $\zeta \in L^2(Q)$ . We say that the control  $v_\epsilon$   $\epsilon$ -insensitizes  $\phi_1$  respectively  $\phi_2$ , if for all  $\tilde{y}_0 \in L^2(\Omega)$  such that  $\|\tilde{y}_0\|_{L^2(\Omega)} = 1$ , we have

$$\left| \frac{\partial \phi_1(w(x, t, v_\epsilon, \tau))}{\partial \tau} \Big|_{\tau=0} \right| \leq \epsilon \quad (7)$$

respectively

$$\left| \frac{\partial \phi_2(w(x, t, v_\epsilon, \tau))}{\partial \tau} \Big|_{\tau=0} \right| \leq \epsilon. \quad (8)$$

Before presenting our results, we will cite some physical motivations which are related to the system under view.

In [16], the authors studied the epitaxial growth of nanoscale thin films, which is modeled by the following system :

$$\begin{cases} \partial_t u + \Delta^2 u - \nabla \cdot (f(\nabla u)) = g & \text{in } \tilde{Q}, \\ \frac{\partial u}{\partial \tilde{n}} = \frac{\partial \Delta u}{\partial \tilde{n}} = 0 & \text{on } \tilde{\Sigma}, \\ u(0, \cdot) = u_0(\cdot) & \text{in } \tilde{\Omega}, \end{cases} \quad (9)$$

where  $\tilde{\Omega} = (0, L)^2$ ,  $\tilde{Q} = (0, T) \times \tilde{\Omega}$ ,  $\tilde{\Sigma} = (0, T) \times \partial \tilde{\Omega}$ ,  $u_0 \in L^2(\tilde{\Omega})$ ,  $f \in C^1(\mathbb{R}^N, \mathbb{R}^N)$  and  $g \in L^2((0, T) \times \tilde{\Omega})$ . In this context,  $u$  is the scaled film height, the term  $\Delta^2 u$  represents the capillarity-driven surface diffusion and  $g$  denotes the deposition flux, while  $\nabla \cdot (f(\nabla u))$  describes the upward hopping of atoms.

Furthermore, in [14] the authors studied the following system :

$$\begin{cases} \partial_t u + \nabla \cdot (|\nabla \Delta u|^{p(x)-2} \nabla \Delta u) = f(x, u) & \text{in } Q, \\ u = \Delta u = 0 & \text{on } \Sigma, \\ u(0, \cdot) = u_0(\cdot) & \text{in } \Omega, \end{cases} \quad (10)$$

where  $p$  and  $f$  are specific functions and  $u_0$  is an initial data. The previous model may describe some properties of medical magnetic resonance images in space and time. When the nonlinear source  $f(x, u)$  is equal to  $\eta(x, t)$ , then the functions  $u(x, t)$  and  $\eta(x, t)$ , respectively, represent the pixel intensity value of a digital image and a random noise. On the other hand, the author in [17] studied a fourth order parabolic system similar to (1) that models the long range effect of insects dispersal. Moreover, the authors in [6] were interested by a fourth order parabolic system where the solution describes the height of a viscous droplet spreading on a plain. For more details about this subject, see for instance [18], [9], [1], [2], [20], [21].

Before presenting our main results let us introduce the following space :

$$L^{2,\gamma}(Q) = \left\{ g \in L^2(Q); \iint_Q e^{\frac{\gamma}{\sqrt{t}}} |g|^2 dxdt < \infty \right\}, \quad (11)$$

for  $\gamma > 0$ . Let us now introduce the first main result :

**Theorem 1.3.** *Assume that  $y_0 = 0$ ,  $f$  is a  $C^1$  globally Lipschitz function such that  $f(0,0,0) = 0$  and (2) holds. Then, there exists a constant  $\bar{C}_2 = C(\Omega, \omega, O, T, f)$  such that for any  $\zeta \in L^{2,\bar{C}_2}(Q)$ , there exists a control  $v \in L^2((0,T) \times \omega)$  such that (5) is satisfied. Moreover, the control  $v$  satisfies the following estimate :*

$$\|v\|_{L^2((0,T) \times \omega)} \leq e^{\bar{C}_2} \left( \iint_Q e^{\frac{\bar{C}_2}{\sqrt{t}}} |\zeta|^2 dxdt \right)^{1/2}, \quad (12)$$

where

$$\bar{C}_2 = C_0(\Omega, \omega) \left( 1 + \frac{1}{\sqrt{T}} + (1+T)(1 + \|f\|_{W_\infty^1}^2) \right). \quad (13)$$

Concerning  $\phi_2$ , we will suppose that  $f$  satisfies

$$f(w, \nabla w, \nabla^2 w) = aw + B \cdot \nabla w + E : \nabla^2 w, \quad (14)$$

where

$$a \in L^\infty(0,T), \quad B \in L^\infty(0,T)^N, \quad E \in L^\infty(0,T)^{N^2}. \quad (15)$$

Then, the second main result is the following one :

**Theorem 1.4.** *Assume that  $y_0 = 0$ ,  $f$  is given by (14) and (2) and (15) hold. Then, there exists a constant  $\bar{C}_3 = C(\Omega, \omega, O, T, f)$  such that for any  $\zeta \in L^{2,\bar{C}_3}(Q)$ , there exists a control  $v \in L^2((0,T) \times \omega)$  such that (6) is satisfied. Moreover, the control  $v$  satisfies the following estimate :*

$$\|v\|_{L^2((0,T) \times \omega)} \leq e^{\bar{C}_3} \left( \iint_Q e^{\frac{\bar{C}_3}{\sqrt{t}}} |\zeta|^2 dxdt \right)^{1/2}. \quad (16)$$

Concerning our third main result, we are going to present some negative results related to the heat and fourth order parabolic equation. Before we continue, let us suppose that :

$$a \in L^\infty(Q) \text{ and } B \in L^\infty(Q)^N. \quad (17)$$

• Let us start by the heat operator. Let us introduce the following coupled system :

$$\begin{cases} \partial_t y - \Delta y + a(t,x)y + B(t,x) \cdot \nabla y = \chi_\omega v & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0, \cdot) = y_0 & \text{in } \Omega, \end{cases} \quad (18)$$

$$\begin{cases} -\partial_t u - \Delta u + a(t,x)u - \nabla \cdot (B(t,x)u) = \nabla \cdot (\nabla y \mathbb{1}_O) & \text{in } Q, \\ u = 0 & \text{on } \Sigma, \\ u(T, \cdot) = u_T & \text{in } \Omega, \end{cases} \quad (19)$$

where  $y_0, u_T \in L^2(\Omega)$ . Before we continue let us present the definition of the approximate controllability of the previous coupled system :

**Definition 1.5.** *Let  $y_0, u_T \in L^2(\Omega)$  and (17) holds true. We say that system (18)-(19) is approximately controllable, if for all  $\varepsilon > 0$ , there exists  $v_\varepsilon \in L^2(Q)$  such that the corresponding solution  $(y_\varepsilon, u_\varepsilon)$  of (18)-(19) for  $v = v_\varepsilon$  satisfies*

$$\|u_\varepsilon(0, \cdot)\|_{L^2(\Omega)} \leq \varepsilon. \quad (20)$$

On the other hand, the approximate controllability of (18)-(19) holds true if and only if for all  $\varphi_0 \in L^2(\Omega)$ , the solution  $(\psi, \varphi)$  of the following system :

$$\begin{cases} -\partial_t \psi - \Delta \psi + a(t, x)\psi - \nabla \cdot (B(t, x)\psi) = \nabla \cdot (\nabla \varphi \mathbf{1}_O) & \text{in } Q, \\ \partial_t \varphi - \Delta \varphi + a(t, x)\varphi + B(t, x) \cdot \nabla \varphi = 0 & \text{in } Q, \\ \psi = 0 & \text{on } \Sigma, \\ \varphi = 0 & \text{on } \Sigma, \\ \psi(T, \cdot) = 0, \varphi(0, \cdot) = \varphi_0(\cdot) & \text{in } \Omega, \end{cases} \quad (21)$$

satisfies the following *Unique Continuation Principle* :

$$\psi = 0 \text{ in } (0, T) \times \omega \implies \psi = \varphi = 0 \text{ in } Q. \quad (22)$$

When we started our study, we expected that the *Unique Continuation Principle* holds true for all  $\Omega \subset \mathbb{R}^N$ ,  $\omega \subset\subset \Omega$ ,  $O \subset\subset \Omega$  such that (2) holds,  $a \in L^\infty(Q)$  and  $B \in L^\infty(Q)^N$ . We tried to prove a *Carleman estimate* of the form

$$\int_0^T \int_\Omega \mathfrak{P}^2(t, x) (|\psi(t, x)|^2 + |\varphi(t, x)|^2) dt dx \leq C \int_0^T \int_\omega \mathfrak{P}^2(t, x) |\psi(t, x)|^2 dt dx,$$

for  $C > 0$  and where  $\mathfrak{P}$  is a weight function. We tried several tracks but none succeeded. Then, we were convinced that this Carleman does not hold in all cases. So, we present the following result :

**Lemma 1.6.** *Let  $T > 0$ . Then, there exist a bounded connected set  $\Omega \subset \mathbb{R}^N$ , an open non empty subset  $O \subset\subset \Omega$ ,  $\varphi_0 \in L^2(\Omega)$ ,  $a \in L^\infty(Q)$  and  $B \in L^\infty(Q)^N$  such that the corresponding solution  $(\psi, \varphi)$  of (21) does not satisfy the *Unique Continuation Principle* given in (22) for any  $\omega \subset \Omega$ .*

From what we said above and Lemma 1.6, we deduce the following Theorem :

**Theorem 1.7.** *There exist a bounded connected set  $\Omega \subset \mathbb{R}^N$ , an open non empty subset  $O \subset\subset \Omega$ ,  $a \in L^\infty(Q)$  and  $B \in L^\infty(Q)^N$ ,  $y_0, u_T \in L^2(Q)$  such that for any open  $\omega \subset \Omega$ , the coupled system (18)-(19) is not approximately controllable.*

Moreover, we can extend our result as follows :

**Remark 1.8.** *Let us replace (20) by*

$$\|y_\varepsilon(T, \cdot)\|_{L^2(\Omega)} + \|u_\varepsilon(0, \cdot)\|_{L^2(\Omega)} \leq \varepsilon. \quad (23)$$

*Then, Theorem 1.7 holds true.*

• Let us now present our result concerning the fourth order parabolic equation. Let us introduce following coupled system :

$$\begin{cases} \partial_t y + \Delta^2 y + V \cdot \Delta \nabla y + E : \nabla^2 y + B \cdot \nabla y + ay = \chi_\omega v & \text{in } Q, \\ y = \Delta y = 0 & \text{on } \Sigma, \\ y(0, \cdot) = y_0 & \text{in } \Omega, \end{cases} \quad (24)$$

$$\begin{cases} -\partial_t u + \Delta^2 u - \nabla \cdot \Delta(Vu) + \nabla^2 : (Eu) - \nabla \cdot (Bu) + au = \nabla \cdot (\nabla y \mathbf{1}_O) & \text{in } Q, \\ u = \Delta u = 0 & \text{on } \Sigma, \\ u(T, \cdot) = u_T & \text{in } \Omega, \end{cases} \quad (25)$$

where  $y_0, u_T \in L^2(\Omega)$ ,  $a \in L^\infty(Q)$ ,  $E \in L^\infty(Q)^{N^2}$  and  $B, V \in L^\infty(Q)^N$ .

**Definition 1.9.** Let  $y_0, u_T \in L^2(\Omega)$ . We say that system (24)-(25) is approximately controllable, if for all  $\varepsilon > 0$ , there exists  $v_\varepsilon \in L^2(Q)$  such that the corresponding solution  $(y_\varepsilon, u_\varepsilon)$  of (24)-(25) for  $v = v_\varepsilon$  satisfies

$$\|u_\varepsilon(0, \cdot)\|_{L^2(\Omega)} \leq \varepsilon. \quad (26)$$

It is well known that the approximate controllability of (24)-(25) holds true if and only if for all  $\varphi_0 \in L^2(\Omega)$ , the solution  $(\psi, \varphi)$  of the following system :

$$\begin{cases} -\partial_t \psi + \Delta^2 \psi - \nabla \cdot \Delta(V\psi) + \nabla^2 : (E\psi) - \nabla \cdot (B\psi) + a\psi = \nabla \cdot (\nabla \varphi \mathbf{1}_O) & \text{in } Q, \\ \partial_t \varphi + \Delta^2 \varphi + V \cdot \nabla \Delta \varphi + E : \nabla^2 \varphi + B \cdot \nabla \varphi + a\varphi = 0 & \text{in } Q, \\ \psi = \Delta \psi = 0 & \text{on } \Sigma, \\ \varphi = \Delta \varphi = 0 & \text{on } \Sigma, \\ \psi(T, \cdot) = 0, \varphi(0, \cdot) = \varphi_0(\cdot) & \text{in } \Omega, \end{cases} \quad (27)$$

satisfies the following *Unique Continuation Principle* :

$$\psi = 0 \text{ in } (0, T) \times \omega \implies \psi = \varphi = 0 \text{ in } Q. \quad (28)$$

So, we present the following result :

**Lemma 1.10.** Let  $N \in \{1, 2\}$  and  $T > 0$ . Then, there exist a bounded connected set  $\Omega \subset \mathbb{R}^N$ , an open non empty subset  $O \subset \subset \Omega$ ,  $\varphi_0 \in L^2(\Omega)$ ,  $a \in L^\infty(Q)$ ,  $B, V \in L^\infty(Q)^N$  and  $E \in L^\infty(Q)^{N^2}$  such that the corresponding solution  $(\psi, \varphi)$  of (27) does not satisfy the Unique Continuation Principle given in (28) for any open non empty subset  $\omega \subset \subset \Omega$ .

From what we said above and Lemma 1.10, we deduce the following Theorem :

**Theorem 1.11.** Let  $N \in \{1, 2\}$ . Then, there exist a bounded connected set  $\Omega \subset \mathbb{R}^N$ , an open non empty subset  $O \subset \subset \Omega$ ,  $a \in L^\infty(Q)$ ,  $B, V \in L^\infty(Q)^N$  and  $E \in L^\infty(Q)^{N^2}$  such that for any open non empty subset  $\omega \subset \subset \Omega$ , the coupled system (24)-(25) is not approximately controllable.

Moreover, we can extend our result as follows:

**Remark 1.12.** Let us replace (26) by

$$\|y_\varepsilon(T, \cdot)\|_{L^2(\Omega)} + \|u_\varepsilon(0, \cdot)\|_{L^2(\Omega)} \leq \varepsilon. \quad (29)$$

Then, Theorem 1.11 holds true.

Let us now present some results concerning the existence of insensitizing controls for parabolic equations.

• We start by the heat equation. In order to guarantee the insensitivity for all  $y_0 \in L^2(\Omega)$ , the authors in [3] introduced the  $\varepsilon$ -insensitivity and studied the following system :

$$\begin{cases} \partial_t y - \Delta y + f(y) = \xi + \chi_\omega v & \text{in } Q, \\ y = g + \tau_1 \hat{g} & \text{on } \Sigma, \\ y(0, \cdot) = y_0(\cdot) + \tau_0 \hat{y}_0 & \text{in } \Omega, \end{cases} \quad (30)$$

where  $y_0 \in L^2(\Omega)$ ,  $g \in L^2(\Sigma)$  are given,  $\tau_0 \in \mathbb{R}$  and  $\tau_1 \in \mathbb{R}$  are unknown and small enough,  $f$  is a  $C^1$  globally Lipschitz-continuous function defined on  $\mathbb{R}$ ,  $\hat{y}_0 \in L^2(\Omega)$  and  $\hat{g} \in L^2(\Sigma)$  are unknown such that  $\|\hat{y}_0\|_{L^2(\Omega)} = \|\hat{g}\|_{L^2(\Sigma)} = 1$ ,  $v \in L^2(Q)$  is the control function and  $\xi$  belongs to a specific space. On the other hand, author in [19] studied this system where  $f$  was also a  $C^1$  globally Lipschitz-continuous function defined

on  $\mathbb{R}$ . Furthermore, the author in [7] studied the existence of controls that  $\varepsilon$ -insensitize the norm of the solution of (30) for  $g \equiv 0$ ,  $\tau_1 = 0$  and where  $\Omega$  is an open unbounded domain set of class  $C^2$ . On the other hand, author in [8] studied also the system (30) for  $y_0 \equiv 0$ ,  $g \equiv 0$ ,  $\tau_1 = 0$  and  $f(0) = 0$  and proved the existence of an initial data  $y_0 \in L^2(\Omega)$  where  $\Omega \setminus \bar{\omega} \neq \emptyset$  such that the insensitivity does not hold. Moreover, the authors in [5] studied the system (30) for  $y_0 \equiv 0$ ,  $g \equiv 0$ ,  $\tau_1 = 0$  and  $f = f(y, \nabla y)$  where  $f$  is a  $C^1$  globally Lipschitz-continuous function defined on  $\mathbb{R} \times \mathbb{R}^N$  and  $f(0, 0) = 0$ . Also in this article, authors studied the insensitivity for the same system with Fourier boundary condition. Furthermore, the authors in [4] studied the existence of insensitizing controls for system (30) where  $y_0 \equiv 0$ ,  $g \equiv 0$ ,  $\tau_1 = 0$ ,  $f \in C^2(\mathbb{R})$  and  $f(0) = 0$  with a super-linear nonlinearity of the form

$$\lim_{|s| \rightarrow \infty} \frac{|f'(s)|}{\log(1 + |s|)} = 0. \quad (31)$$

Concerning the functional  $\phi_2$  defined in (4), the author in [12] proved the existence of controls that insensitize the functional  $\phi_2$  for the following heat equation :

$$\begin{cases} \partial_t y - \Delta y + ay + B \cdot \nabla y = \xi + \chi_\omega v & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0, \cdot) = y_0(\cdot) + \tau_0 \hat{y}_0 & \text{in } \Omega, \end{cases} \quad (32)$$

where  $y_0 \equiv 0$ ,  $\hat{y}_0 \in L^2(\Omega)$  is unknown such that  $\|\hat{y}_0\|_{L^2(\Omega)} = 1$ ,  $\tau_0 \in \mathbb{R}$  is unknown and small enough,  $a \in L^\infty(0, T)$ ,  $B \in L^\infty(0, T)^N$ ,  $v \in L^2(Q)$  is the control function and  $\xi$  belongs to a specific space.

- Concerning fourth order parabolic equations, there has been limited publication on this subject. In fact, the only previous insensitivity result is [11], where the case of  $N = 1$  and a non-linear term  $f(y)$  is treated (only depending on  $y$ ). On other words, the author in [11] studied the following system :

$$\begin{cases} \partial_t y + \partial_{xxxx} y + f(y) = \zeta + \chi_\omega v & \text{in } (0, T) \times (0, 1), \\ y(\cdot, 0) = y(\cdot, 1) = 0 & \text{in } (0, T), \\ \partial_x y(\cdot, 0) = \partial_x y(\cdot, 1) = 0 & \text{in } (0, T), \\ y(0, \cdot) = y_0(\cdot) + \tau z_0 & \text{in } (0, 1), \end{cases}$$

where  $y_0 \equiv 0$ ,  $\zeta \in L^2((0, T) \times (0, 1))$  are given,  $\tau_0 \in \mathbb{R}$  is unknown and small enough,  $z_0 \in L^2(0, 1)$  is unknown such that  $\|z_0\|_{L^2(0, 1)} = 1$ ,  $v \in L^2((0, T) \times (0, 1))$  is the control function and where  $f$  is a  $C^1$  function defined on  $\mathbb{R}$  such that  $\frac{d^2 f}{dx^2} \in L_{loc}^\infty(\mathbb{R})$  and

$$\lim_{|s| \rightarrow \infty} \frac{|f'(s)|}{\log(1 + |s|)} = 0. \quad (33)$$

For  $N \in \mathbb{N}$ , authors in [13] proved a Carleman estimate to study the null controllability of a fourth order parabolic equation in the linear case. Later, improvements have been done in [15] for the semi-linear case.

Concerning the new tools used in this paper, first we prove a new Carleman inequality for a coupled fourth-order parabolic system. Secondly, we study the existence of insensitizing controls for (4) where  $f$  is linear and where the coefficients depend only on time. At the end, we prove the existence of some systems where the functional  $\phi_2$  given in (4) cannot be  $\varepsilon$ -insensitized (see (8)). This result, obliged us to treat only the case where the coefficients depend only on time (for more details see Section 4).

The rest of this paper is organised as follows. The next section is devoted for technical and previous results. In the third section, we prove Theorem 1.3. The fourth one concerns the proof of Theorem 1.4. At the end, in the last section, we introduce some systems where where the functional  $\phi_2$  given in (4) cannot be  $\varepsilon$ -insensitized.

## 2 Technical results

Before we start let us introduce the following linear system :

$$\begin{cases} \partial_t z + \Delta^2 z + az + B \cdot \nabla z + D : \nabla^2 z = F & \text{in } Q, \\ z = \Delta z = 0 & \text{on } \Sigma, \\ z(0, \cdot) = z_0(\cdot) & \text{in } \Omega. \end{cases} \quad (34)$$

with  $a \in L^\infty(Q)$ ,  $B \in L^\infty(Q)^N$  and  $D \in L^\infty(Q)^{N^2}$ . Let us denote :

$$X_2 = L^2(Q), \quad X_1 = L^2(0, T; H^{-1}(\Omega)), \quad X_0 = L^2(0, T; (H^2(\Omega) \cap H_0^1(\Omega))'), \quad (35)$$

$$\begin{aligned} Y_2 &= H^1(0, T; L^2(\Omega)) \cap L^2(0, T; H^4(\Omega)), \\ Y_1 &= L^2(0, T; H^3(\Omega)) \cap H^1(0, T; H^{-1}(\Omega)), \\ Y_0 &= H^1(0, T; (H^2(\Omega) \cap H_0^1(\Omega))') \cap L^2(0, T; H^2(\Omega)), \end{aligned} \quad (36)$$

and

$$Z_2 = H^2(\Omega) \cap H_0^1(\Omega), \quad Z_1 = H_0^1(\Omega), \quad Z_0 = L^2(Q). \quad (37)$$

Let us now present a first result :

**Lemma 2.1.** *Let  $\Lambda$  be the map which associates  $z$ , the solution of (34) to each  $(F, z_0)$ . Then,  $\Lambda$  is continuous from  $X_i \times Z_i$  to  $Y_i$  for  $i = 0, 1, 2$ .*

Moreover, there exists  $C_0(\Omega)$  such that

$$\|z\|_{Y_i} \leq e^{C_0(\Omega)\bar{C}_1} \left( \|F\|_{X_i} + \|z_0\|_{Z_i} \right) \quad (38)$$

for all  $(F, z_0) \in X_i \times Z_i$  ( $i = 0, 1, 2$ ) and where

$$\bar{C}_1 = 1 + T(\|a\|_\infty + \|B\|_\infty^{4/3} + \|D\|_\infty^2) + \|a\|_\infty^2 + \|B\|_\infty^2 + \|D\|_\infty^2. \quad (39)$$

*Proof.* Let us start our proof for  $i = 0$ . Let  $(F, z_0) \in X_0 \times Z_0$  and let us prove the following estimate :

$$\sup_{t \in [0, T]} \|z(t)\|_{L^2(\Omega)} \leq \exp[C_0(1 + T(\|a\|_\infty + \|B\|_\infty^{4/3} + \|D\|_\infty^2))] \left( \|F\|_{X_0} + \|z_0\|_{Z_0} \right). \quad (40)$$

By multiplying (34)<sub>1</sub> by  $z$ , integrating by parts, we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_\Omega |z(t)|^2 dx + \int_\Omega |\Delta z(t)|^2 dx - \varepsilon \|z\|_{H^2(\Omega)}^2 \\ & \leq C_{0,\varepsilon} \left( \|F(t)\|_{(H^2(\Omega) \cap H_0^1(\Omega))'}^2 + (\|a\|_\infty + \|B\|_\infty^{4/3} + \|D\|_\infty^2) \int_\Omega |z(t)|^2 dx \right), \end{aligned} \quad (41)$$

for  $\varepsilon > 0$ . In fact, we have used Young's inequality in order to deduce

$$\left| \int_\Omega B \cdot \nabla z z dx \right| \leq \|B\|_\infty \|z\|_{H^1(\Omega)} \|z\|_{L^2(\Omega)} \leq \|B\|_\infty \|z\|_{L^2(\Omega)}^{3/2} \|z\|_{H^2(\Omega)}^{1/2} \leq C_{0,\varepsilon} \|B\|_\infty^{4/3} \|z\|_{L^2(\Omega)}^2 + \varepsilon \|z\|_{H^2(\Omega)}^2.$$

Let us notice that there exists  $\lambda > 0$  such that for any  $u \in H^2(\Omega) \cap H_0^1(\Omega)$ , we have

$$\int_\Omega |\Delta u|^2 dx \geq \lambda \|u\|_{H^2(\Omega)}^2. \quad (42)$$

Combining this with (41), we deduce

$$\frac{d}{dt} \left( \exp[-tC_0(\|a\|_\infty + \|B\|_\infty^{4/3} + \|D\|_\infty^2)] \int_\Omega |z(t)|^2 dx \right) \leq C_0 \|F(t)\|_{(H^2(\Omega) \cap H_0^1(\Omega))'}^2.$$

Integrating in  $(0, t)$ , we have

$$\|z(t)\|_{L^2(\Omega)}^2 \leq \exp[C_0(1 + T(\|a\|_\infty + \|B\|_\infty^{4/3} + \|D\|_\infty^2))] \left( \|F\|_{X_0}^2 + \|z_0\|_{Z_0}^2 \right). \quad (43)$$

So, we deduce (40) by taking  $\sup_{t \in [0, T]}$ . On the other hand, combining (41) with (40) and (42), we deduce

$$\|z\|_{L^2(0, T; H^2(\Omega))} \leq \exp[C_0(1 + T(1 + \|a\|_\infty + \|B\|_\infty^{4/3} + \|D\|_\infty^2))] \left( \|F\|_{X_0} + \|z_0\|_{Z_0} \right). \quad (44)$$

By using the last estimate with the fact that for any  $u \in C^0(\bar{\Omega})$  such that  $u = 0$  on  $\partial\Omega$ , we have  $\Delta u \in (H^2(\Omega) \cap H_0^1(\Omega))'$ , we can deduce that

$$\|z\|_{H^1(0, T; (H^2(\Omega) \cap H_0^1(\Omega))')} \leq \exp[C_0(1 + T(1 + \|a\|_\infty + \|B\|_\infty^{4/3} + \|D\|_\infty^2))] \left( \|F\|_{X_0} + \|z_0\|_{Z_0} \right). \quad (45)$$

Indeed, it suffices to remark that

$$\partial_t z = F - (\Delta^2 z + az + B \cdot \nabla z + D : \nabla^2 z) \quad \text{in } Q$$

and

$$F - (\Delta^2 z + az + B \cdot \nabla z + D : \nabla^2 z) \in L^2(0, T; (H^2(\Omega) \cap H_0^1(\Omega))').$$

This finishes the proof for  $i = 0$ .

Concerning  $i = 2$ , let  $(F, z_0) \in X_2 \times Z_2$ . By applying the case  $i = 0$ , we have

$$\|z\|_{L^2(0, T; H^2(\Omega))} \leq \exp[C_0(1 + T(1 + \|a\|_\infty + \|B\|_\infty^{4/3} + \|D\|_\infty^2))] \left( \|F\|_{X_0} + \|z_0\|_{Z_0} \right). \quad (46)$$

To simplify our computations, let us denote  $\tilde{F} = F - (az + B \cdot \nabla z + D : \nabla^2 z)$ . By multiplying (34)<sub>1</sub> by  $\partial_t z$  and integrating by parts, we have

$$(1 - \varepsilon) \int_\Omega |\partial_t z(t)|^2 dx + \frac{1}{2} \frac{d}{dt} \int_\Omega |\Delta z(t)|^2 dx \leq C_{0, \varepsilon} \|\tilde{F}(t)\|_{L^2(\Omega)}^2. \quad (47)$$

Integrating in  $(0, T)$ , we deduce that

$$\|z\|_{H^1(0, T; L^2(\Omega))}^2 \leq C_{0, \varepsilon} \left( \|\tilde{F}\|_{X_2}^2 + \|z_0\|_{Z_2}^2 \right). \quad (48)$$

Let us notice that

$$\|\tilde{F}\|_{X_2}^2 \leq \left( \|F\|_{X_2}^2 + (\|a\|_\infty^2 + \|B\|_\infty^2 + \|D\|_\infty^2) \|z\|_{L^2(0, T; H^2(\Omega))}^2 \right). \quad (49)$$

Combining the last estimate with (48) and (46), we deduce

$$\|z\|_{H^1(0, T; L^2(\Omega))}^2 \leq e^{C_0 \bar{C}_1} \left( \|\tilde{F}\|_{X_2}^2 + \|z_0\|_{Z_2}^2 \right). \quad (50)$$

To finish the proof, from (34)<sub>1</sub>, we have

$$\Delta^2 z = F - (\partial_t z + az + B \cdot \nabla z + D : \nabla^2 z) \quad \text{in } Q.$$

Combining this with (50) and (46), we deduce that

$$\|z\|_{Y_2} \leq e^{C_0 \bar{C}_1} \left( \|F\|_{X_2} + \|z_0\|_{Z_2} \right).$$

Finally, to prove the case  $i = 1$ , it suffices to use an interpolation argument between the  $X_0$ ,  $X_2$  spaces,  $Z_0$ ,  $Z_2$  spaces and  $Y_0$ ,  $Y_2$  spaces.  $\square$

**Remark 2.2.** Let us notice that Lemma 2.1 holds true for forward systems. Indeed, it suffices to apply Lemma 2.1 to  $\tilde{z}(t, \cdot) = z(T - t, \cdot)$  for all  $t \in [0, T]$  where  $z$  is the solution of (34).

Now, we will introduce some weight function:

$$\alpha(x, t) = \frac{e^{4\lambda\|\eta\|_\infty} - e^{\lambda(2\|\eta\|_\infty + \eta(x))}}{t^{1/2}(T-t)^{1/2}}, \quad \xi(x, t) = \frac{e^{\lambda(2\|\eta\|_\infty + \eta(x))}}{t^{1/2}(T-t)^{1/2}},$$

where  $\eta$  satisfies:

$$\eta \in C^4(\bar{\Omega}), \quad \eta|_{\partial\Omega} = 0, \quad |\nabla\eta| \geq C_0 > 0 \quad \text{in } \Omega \setminus \bar{\omega}_0, \quad (51)$$

with  $\omega_0 \subset\subset \omega \cap O$  an open set. For the existence of  $\eta$ , see [10]. Let us notice some essential properties on the weight functions :

**Remark 2.3.** We have

$$\nabla\xi = \lambda\xi\nabla\eta \quad \text{in } Q, \quad \xi^{-1} \leq \frac{T}{2} \quad \text{in } Q, \quad \nabla\eta = \frac{\partial\eta}{\partial\bar{n}} \bar{n} \quad \text{on } \Sigma.$$

The second result was proved in [13] :

**Lemma 2.4.** Let  $\tilde{\omega}$  such that  $\omega_0 \subset\subset \tilde{\omega}$ . There exists a positive constant  $C_0 = C_0(\Omega, \tilde{\omega})$  such that

$$\begin{aligned} & \iint_Q e^{-2s\alpha} \left( s^6 \lambda^8 \xi^6 |\varphi|^2 + s^4 \lambda^6 \xi^4 |\nabla\varphi|^2 + s^3 \lambda^4 \xi^3 |\Delta\varphi|^2 \right. \\ & \quad \left. + s^2 \lambda^4 \xi^2 |\nabla^2\varphi|^2 + s \lambda^2 \xi |\nabla\Delta\varphi|^2 + s^{-1} \xi^{-1} (|\partial_t\varphi|^2 + |\Delta^2\varphi|^2) \right) dxdt \\ & \leq C_0 \left( s^7 \lambda^8 \iint_{\tilde{\omega} \times (0, T)} e^{-2s\alpha} \xi^7 |\varphi|^2 dxdt + \iint_Q e^{-2s\alpha} |\tilde{f}|^2 dxdt \right) \end{aligned} \quad (52)$$

for any  $\lambda \geq C_0$  and any  $s \geq C_0(T^{1/2} + T)$  and where  $\varphi$  solution of

$$\begin{cases} -\partial_t\varphi + \Delta^2\varphi = \tilde{f} & \text{in } Q, \\ \varphi = \Delta\varphi = 0 & \text{on } \Sigma, \\ \varphi(T, \cdot) = \varphi_0(\cdot) & \text{in } \Omega, \end{cases} \quad (53)$$

where  $\varphi_0 \in L^2(\Omega)$  and  $\tilde{f} \in L^2(Q)$ .

On the other hand, from [15], by doing some modifications on boundary conditions, we have the following Lemma :

**Lemma 2.5.** Let  $\tilde{\omega}$  such that  $\omega_0 \subset\subset \tilde{\omega}$ . There exists a positive constant  $C_0 = C_0(\Omega, \tilde{\omega})$  such that

$$\begin{aligned} s^6 \lambda^8 \iint_Q e^{-2s\alpha} \xi^6 |q|^2 dxdt &\leq C_0 \left( s^7 \lambda^8 \iint_{\tilde{\omega} \times (0, T)} e^{-2s\alpha} \xi^7 |q|^2 dxdt + \iint_Q e^{-2s\alpha} |F_0|^2 dxdt \right. \\ &\quad + s^2 \lambda^2 \iint_Q e^{-2s\alpha} \xi^2 |F_1|^2 dxdt + s^4 \lambda^4 \iint_Q e^{-2s\alpha} \xi^4 \sum_{i,j=1}^N |\hat{F}_{ij}|^2 dxdt \\ &\quad \left. + s^3 \lambda^3 \iint_{\Sigma} e^{-2s\alpha} \xi^3 |f_1|^2 d\sigma dt + s^7 \lambda^7 \iint_{\Sigma} e^{-2\alpha} \xi^7 |f_0|^2 d\sigma dt \right), \end{aligned} \quad (54)$$

for any  $\lambda \geq C_0$  and any  $s \geq C_0(T^{1/2} + T)$  and where  $q$  fulfills

$$\begin{cases} \partial_t q - \Delta^2 q = F_0 + \nabla \cdot F_1 + \sum_{i,j=1}^N \partial_{ij} \hat{F}_{ij} & \text{in } Q, \\ q = f_0, \quad \Delta q = f_1 & \text{on } \Sigma, \\ q(T, \cdot) = q_0(\cdot) & \text{in } \Omega. \end{cases} \quad (55)$$

Before we finish this section, we give the following remark :

**Remark 2.6.** Let us notice that, Lemmas 2.4 and 2.5 hold true for forward system. Indeed, it suffices to apply Lemma 2.4 (resp. Lemma 2.5) to  $\tilde{\varphi}(t, \cdot) = \varphi(T - t, \cdot)$  (resp.  $\tilde{q}(t, \cdot) = q(T - t, \cdot)$ ) for all  $t \in [0, T]$ .

### 3 Insensitizing controls for $\phi_1$ .

We know that the existence of a control  $v$  such that (5) holds true is equivalent to the null controllability of a coupled system. This result is given in the following lemma :

**Lemma 3.1.** There exists a control  $v$  such that (5) holds true if and only if the following system :

$$\begin{cases} \partial_t y + \Delta^2 y + f(y, \nabla y, \nabla^2 y) = \zeta + \chi_\omega v & \text{in } Q, \\ y = \Delta y = 0 & \text{on } \Sigma, \\ y(0, \cdot) = y_0(\cdot) & \text{in } \Omega, \end{cases} \quad (56)$$

$$\begin{cases} -\partial_t u + \Delta^2 u + \partial_s f(y, \nabla y, \nabla^2 y)u - \nabla \cdot (\partial_p f(y, \nabla y, \nabla^2 y)u) & \text{in } Q, \\ \quad + \nabla^2 : (\partial_q f(y, \nabla y, \nabla^2 y)u) = \chi_O y & \\ u = \Delta u = 0 & \text{on } \Sigma, \\ u(T, \cdot) = 0 & \text{in } \Omega, \end{cases} \quad (57)$$

verifies

$$u(0, \cdot) = 0 \quad \text{in } \Omega. \quad (58)$$

*Proof.* Let us notice that

$$\frac{\partial \phi(w(x, t, h, \tau))}{\partial \tau} \Big|_{\tau=0} = \iint_{(0, T) \times O} y_\tau y dxdt, \quad (59)$$

where  $y_\tau$  verifies

$$\begin{cases} \partial_t y_\tau + \Delta^2 y_\tau + \partial_s f(y, \nabla y, \nabla^2 y) y_\tau + \partial_p f(y, \nabla y, \nabla^2 y) \cdot \nabla y_\tau + \partial_q f(y, \nabla y, \nabla^2 y) : \nabla^2 y_\tau = 0 & \text{in } Q, \\ y_\tau = \Delta y_\tau = 0 & \text{on } \Sigma, \\ y_\tau(0, \cdot) = \tilde{y}_0 & \text{in } \Omega. \end{cases} \quad (60)$$

By multiplying (57)<sub>1</sub> by  $y_\tau$  and integrating by parts, we deduce

$$\frac{\partial \phi(w(x, t, h, \tau))}{\partial \tau} \Big|_{\tau=0} = \iint_{(0, T) \times \mathcal{O}} y_\tau y \, dx dt = \langle \tilde{y}_0, u(0, \cdot) \rangle_{L^2(\Omega)}.$$

□

The next subsection is devoted to the study of the linear system.

### 3.1 The linear case.

In this section we are going to treat the linear case. So let us consider the following linear system :

$$\begin{cases} \partial_t y + \Delta^2 y + ay + B \cdot \nabla y + D : \nabla^2 y = \zeta + \chi_\omega v & \text{in } Q, \\ y = \Delta y = 0 & \text{on } \Sigma, \\ y(0, \cdot) = y_0 & \text{in } \Omega, \end{cases} \quad (61)$$

$$\begin{cases} -\partial_t u + \Delta^2 u + \tilde{a}u - \nabla \cdot (\tilde{B}u) + \sum_{i,j=1}^N \partial_{ij}(\tilde{D}_{ij}u) = \chi_{\mathcal{O}} y & \text{in } Q, \\ u = \Delta u = 0 & \text{on } \Sigma, \\ u(T, \cdot) = 0 & \text{in } \Omega, \end{cases} \quad (62)$$

where

$$\begin{aligned} a &\in L^\infty(Q), \quad B \in L^\infty(Q)^N, \quad D \in L^\infty(Q)^{N^2} \\ \tilde{a} &\in L^\infty(Q), \quad \tilde{B} \in L^\infty(Q)^N, \quad \tilde{D} \in L^\infty(Q)^{N^2} \end{aligned} \quad (63)$$

and  $y_0 \in L^2(\Omega)$ . Before we start, let us denote  $C_0(\Omega, \omega)$  a constant which only depends on  $\Omega$  and  $\omega$  and

$$C_1 \triangleq \max(C_2, C_3), \quad (64)$$

where

$$C_2 = C_0(\Omega, \omega) \left( 1 + C_4 + \frac{1}{\sqrt{T}} + T(1 + \|a\|_\infty + \|B\|_\infty^2 + \|D\|_\infty^2 + \|\tilde{a}\|_\infty + \|\tilde{B}\|_\infty^2 + \|\tilde{D}\|_\infty^2) \right), \quad (65)$$

$$C_3 = C_0(\Omega, \omega) \left( 1 + \sqrt{T} C_4 \right) \quad (66)$$

and

$$C_4 = 1 + \|a\|_\infty^{1/3} + \|B\|_\infty^{1/2} + \|D\|_\infty + \|\tilde{a}\|_\infty^{1/3} + \|\tilde{B}\|_\infty^{1/2} + \|\tilde{D}\|_\infty. \quad (67)$$

The goal of this subsection is to prove the following approximate controllability result :

**Proposition 3.2.** *Assume that*

$$\iint_Q e^{\frac{C_1}{\sqrt{t}}} |\zeta|^2 \, dx dt < \infty,$$

where  $C_1$  is given in (64). Then, for every  $T > 0$  and  $\varepsilon > 0$ , there exists a control  $v^\varepsilon \in L^2((0, T) \times \omega)$  such that  $(y_\varepsilon, u_\varepsilon)$  the corresponding solution of (61)-(62) for  $v = v^\varepsilon$  satisfies

$$\|u_\varepsilon(0, \cdot)\|_{L^2(\Omega)} \leq \varepsilon. \quad (68)$$

In addition, we have the following estimate :

$$\|v^\varepsilon\|_{L^2((0, T) \times \omega)} \leq e^{C_1} \left( \iint_Q e^{\frac{C_2}{\sqrt{t}}} |\zeta|^2 \, dx dt \right)^{1/2},$$

where  $C_1$  is given in (64).

**Remark 3.3.** *Let us notice that we can also prove that for every  $T > 0$ , there exists a control  $\tilde{v} \in L^2((0, T) \times \omega)$  such that  $(y, u)$  the corresponding solution of (61)-(62) for  $v = \tilde{v}$  satisfies  $u(0, \cdot) \equiv 0$  in  $\Omega$ , but this result will not be useful for the rest of this section.*

So, let us consider the following adjoint system :

$$\begin{cases} -\partial_t z + \Delta^2 z + az - \nabla \cdot (Bz) + \sum_{i,j=1}^N \partial_{ij}(D_{ij}z) = \chi_{OP} & \text{in } Q, \\ z = \Delta z = 0 & \text{on } \Sigma, \\ z(T, \cdot) = 0 & \text{in } \Omega, \end{cases} \quad (69)$$

$$\begin{cases} \partial_t p + \Delta^2 p + \tilde{a}y + \tilde{B} \cdot \nabla p + \tilde{D} : \nabla^2 p = 0 & \text{in } Q, \\ p = \Delta p = 0 & \text{on } \Sigma, \\ p(0, \cdot) = p_0 & \text{in } \Omega, \end{cases} \quad (70)$$

where  $p_0 \in L^2(\Omega)$ . In order to prove Proposition 3.2, we are going to prove a Carleman estimate for the solutions of (69)-(70) :

**Proposition 3.4.** *Let (2) holds true. Then, there exists a positive constant  $C_0 > 0$  such that for every solution  $(z, p)$  of (69)-(70) with initial data  $p_0 \in L^2(\Omega)$  the following inequality holds true:*

$$s^6 \lambda^8 \iint_Q e^{-2s\alpha} \xi^6 (|z|^2 + |p|^2) dxdt \leq C_0 s^{16} \lambda^{16} \iint_{(0,T) \times \omega} e^{-2s\alpha} \xi^{16} |z|^2 dxdt, \quad (71)$$

for  $\lambda \geq C_0$  and  $s \geq C_0(TC_4 + T^{1/2})$  and where  $C_4$  is given in (67).

From the previous Proposition, we can deduce a new estimate for the solutions of (69)-(70):

**Proposition 3.5.** *Let (2) holds true. Then, there exists a constant  $C_0(\Omega, \omega) > 0$  such that every solution  $(z, p)$  of (69)-(70) with initial data  $p_0 \in L^2(\Omega)$  satisfies*

$$\iint_Q e^{-\frac{C_1}{\sqrt{t}}} |z|^2 dxdt \leq e^{C_1} \iint_{(0,T) \times \omega} |z|^2 dxdt,$$

where  $C_1$  is given in (64).

*Proof.* Let us denote

$$\begin{aligned} I &\triangleq \iint_Q e^{-\frac{C_1}{\sqrt{t}}} |z|^2 dxdt \\ &= \iint_{(0,T/2) \times \Omega} e^{-\frac{C_1}{\sqrt{t}}} |z|^2 dxdt + \iint_{(T/2,T) \times \Omega} e^{-\frac{C_1}{\sqrt{t}}} |z|^2 dxdt \\ &= I_1 + I_2. \end{aligned} \quad (72)$$

We will start by proving the following estimate :

$$I_1 \leq e^{C_1} \iint_{(0,T) \times \omega} |z|^2 dxdt. \quad (73)$$

At first let us notice that

$$\begin{cases} e^{-2s\alpha\xi^6} \geq C_0 T^{-6} \exp[-\frac{C_0 s}{\sqrt{T}t}], & \forall t \in (0, \frac{T}{2}), \\ e^{-2s\alpha\xi^{16}} \leq C_0 T^{-16} \exp[C_1], & \forall (t, x) \in Q, \\ e^{-2s\alpha\xi^6} \geq C_0 T^{-6} \exp[-2C_1], & \forall (t, x) \in (\frac{T}{4}, \frac{3T}{4}) \times \Omega. \end{cases} \quad (74)$$

Combining the last estimates with Proposition 3.4, we deduce

$$I_1 \leq C_0 T^6 \iint_{(0,T) \times \Omega} e^{-2s\alpha\xi^6} |z|^2 dx dt \leq e^{C_1} \iint_{(0,T) \times \omega} |z|^2 dx dt. \quad (75)$$

Now we are going to treat  $I_2$ . We will prove at first the following estimate on  $p$  the solution of (70) :

$$\int_{\Omega} |p(t + T/4)|^2 dx \leq e^{C_1} \int_{\Omega} |p(t)|^2 dx, \quad \forall t \in (\frac{T}{4}, \frac{3T}{4}). \quad (76)$$

By multiplying (70)<sub>1</sub> by  $p$  and integrating by parts, we have

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} |p(t, x)|^2 dx + \int_{\Omega} |\Delta p(t, x)|^2 dx - \varepsilon \|p(t)\|_{H^2(\Omega)}^2 \leq C(\varepsilon)(1 + \|\tilde{a}\|_{\infty} + \|\tilde{B}\|_{\infty}^2 + \|\tilde{D}\|_{\infty}^2) \int_{\Omega} |p(t, x)|^2 dx,$$

for  $\varepsilon > 0$ . By using the fact that there exists  $\lambda > 0$  such that for any  $p \in H^2(\Omega) \cap H_0^1(\Omega)$ , we have

$$\int_{\Omega} |\Delta p|^2 dx \geq \lambda \|p\|_{H^2(\Omega)}^2, \quad (77)$$

we deduce

$$\frac{d}{dt} \left( \exp(-C_0(1 + \|\tilde{a}\|_{\infty} + \|\tilde{B}\|_{\infty}^2 + \|\tilde{D}\|_{\infty}^2)t) \int_{\Omega} |p(t, x)|^2 dx \right) \leq 0.$$

Then by integrating in  $(t, t + \frac{T}{4})$  for  $t \in (\frac{T}{4}, \frac{3T}{4})$ , we deduce (76). Now, we are going to prove the following estimate on  $z$  the solution of (69) :

$$\int_{\Omega} |z(t, x)|^2 dx \leq C_0 \int_t^T \int_{\Omega} |p(s, x)|^2 dt dx, \quad \forall t \in (0, T). \quad (78)$$

By multiplying (69)<sub>1</sub> by  $z$  and integrating by parts, we have

$$\begin{aligned} & -\frac{1}{2} \frac{d}{dt} \int_{\Omega} |z(t, x)|^2 dx + \int_{\Omega} |\Delta z(t, x)|^2 dx - \varepsilon \|z(t)\|_{H^2(\Omega)}^2 \\ & \leq C(\varepsilon)(1 + \|a\|_{\infty} + \|B\|_{\infty}^2 + \|D\|_{\infty}^2) \int_{\Omega} |z(t, x)|^2 dx + C_0 \int_{\Omega} |p(t)|^2 dx, \end{aligned}$$

for  $\varepsilon > 0$ . By using (77) for  $z$ , we deduce

$$-\frac{d}{dt} \left( \exp(C_0(1 + \|\tilde{a}\|_{\infty} + \|\tilde{B}\|_{\infty}^2 + \|\tilde{D}\|_{\infty}^2)t) \int_{\Omega} |z(t, x)|^2 dx \right) \leq C_0 \int_{\Omega} |p(t)|^2 dx.$$

Then by integrating in  $(t, T)$  for  $t \in (0, T)$ , we deduce (78). Now, from the definition of  $I_2$ , we have

$$I_2 \leq \int_{T/2}^T \int_{\Omega} |z|^2 dx dt.$$

Using the last estimate with (76), we have

$$I_2 \leq e^{C_1} \int_{T/4}^{3T/4} \int_{\Omega} |p|^2 dx dt.$$

Combining the last estimate with (74)<sub>3</sub>, we deduce

$$I_2 \leq e^{C_1} \int_{T/4}^{3T/4} \int_{\Omega} e^{-2s\alpha} \xi^6 |p|^2 dx dt \leq e^{C_1} \int_0^T \int_{\Omega} e^{-2s\alpha} \xi^6 |p|^2 dx dt.$$

From the last estimate combined with Proposition 3.4 and (74)<sub>2</sub>, we deduce,

$$I_2 \leq e^{C_1} \iint_{(0,T) \times \omega} |z|^2 dx dt. \quad (79)$$

From (73) combined with (79), we deduce our aim result.  $\square$

Now, we give the proof of Proposition 3.4.

**Proof of Proposition 3.4.** Let us consider  $\omega_0 \subset\subset \omega' \subset\subset \omega \cap O$ . By applying Lemma 2.5 to  $z$  the solution of (69) for  $\tilde{\omega} = \omega$ ,  $F_0 = az$ ,  $F_1 = -Bz$ ,  $\hat{F}_{i,j} = D_{i,j}z$  and  $f_0 = f_1 = q_0 = 0$ , we have

$$\begin{aligned} s^6 \lambda^8 \iint_Q e^{-2s\alpha} \xi^6 |z|^2 dx dt &\leq C_0 \left( s^7 \lambda^8 \iint_{(0,T) \times \omega} e^{-2s\alpha} \xi^7 |z|^2 dx dt + \iint_Q e^{-2s\alpha} |az|^2 dx dt \right. \\ &\quad + s^2 \lambda^2 \iint_Q e^{-2s\alpha} \xi^2 |Bz|^2 dx dt + s^4 \lambda^4 \iint_Q e^{-2s\alpha} \xi^4 |Dz|^2 dx dt \\ &\quad \left. + \iint_{(0,T) \times O} e^{-2s\alpha} |p|^2 dx dt \right), \end{aligned} \quad (80)$$

for any  $\lambda \geq C_0$  and any  $s \geq C_0(T^{1/2} + T)$ . It is clear that

$$\iint_Q e^{-2s\alpha} \left( |az|^2 + s^2 \xi^2 |Bz|^2 + s^4 \xi^4 |Dz|^2 \right) dx dt \leq s^6 \iint_Q e^{-2s\alpha} \xi^6 |z|^2 dx dt$$

for  $\lambda \geq C_0$  and  $s \geq C_0 \left( T(\|a\|_{\infty}^{1/3} + \|B\|_{\infty}^{1/2} + \|D\|_{\infty}) + T^{1/2} \right)$ . Then, we deduce

$$s^6 \lambda^8 \iint_Q e^{-2s\alpha} \xi^6 |z|^2 dx dt \leq C_0 \left( s^7 \lambda^8 \iint_{(0,T) \times \omega} e^{-2s\alpha} \xi^7 |z|^2 dx dt + \iint_{(0,T) \times O} e^{-2s\alpha} |p|^2 dx dt \right), \quad (81)$$

for  $\lambda \geq C_0$  and  $s \geq C_0 \left( T(\|a\|_{\infty}^{1/3} + \|B\|_{\infty}^{1/2} + \|D\|_{\infty})T^{1/2} \right)$ . In order to estimate the last term in the right-hand side of (81), we apply Lemma 2.4 (see Remark 2.6) to  $p$  the solution of (70) for  $\tilde{\omega} = \omega'$ , we have

$$\begin{aligned} &\iint_Q e^{-2s\alpha} \left( s^6 \lambda^8 \xi^6 |p|^2 + s^4 \lambda^6 \xi^4 |\nabla p|^2 + s^3 \lambda^4 \xi^3 |\Delta p|^2 \right. \\ &\quad \left. + s^2 \lambda^4 \xi^2 |\nabla^2 p|^2 + s \lambda^2 \xi |\nabla \Delta p|^2 + s^{-1} \xi^{-1} (|\partial_t p|^2 + |\Delta^2 p|^2) \right) dx dt \\ &\leq C_0 \left( s^7 \lambda^8 \iint_{(0,T) \times \omega'} e^{-2s\alpha} \xi^7 |p|^2 dx dt + \iint_Q e^{-2s\alpha} \left( |\tilde{a}p|^2 + |\tilde{B} \cdot \nabla p|^2 + |\tilde{D} : \nabla^2 p|^2 \right) \right) dx dt, \end{aligned} \quad (82)$$

for  $\lambda \geq C_0$  and  $s \geq C_0(T + T^{1/2})$ . It is clear that

$$\iint_Q e^{-2s\alpha} \left( |\tilde{a}p|^2 + |\tilde{B} \cdot \nabla p|^2 + |\tilde{D} : \nabla^2 p|^2 \right) dxdt \leq s^2 \iint_Q e^{-2s\alpha} \xi^2 \left( s^4 \xi^4 |p|^2 + s^2 \xi^2 |\nabla p|^2 + |\nabla^2 p|^2 \right) dxdt,$$

for  $s \geq C_0 \left( T(\|\tilde{a}\|_\infty^{1/3} + \|\tilde{B}\|_\infty^{1/2} + \|\tilde{D}\|_\infty) + T^{1/2} \right)$ . Combining the last estimate with (82), we deduce

$$\begin{aligned} & \iint_Q e^{-2s\alpha} \left( s^6 \lambda^8 \xi^6 |p|^2 + s^4 \lambda^6 \xi^4 |\nabla p|^2 + s^3 \lambda^4 \xi^3 |\Delta p|^2 \right. \\ & \quad \left. + s^2 \lambda^4 \xi^2 |\nabla^2 p|^2 + s \lambda^2 \xi |\nabla \Delta p|^2 + s^{-1} \xi^{-1} (|\partial_t p|^2 + |\Delta^2 p|^2) \right) dxdt \\ & \leq C_0 s^7 \lambda^8 \iint_{(0,T) \times \omega'} e^{-2s\alpha} \xi^7 |p|^2 dxdt, \end{aligned} \tag{83}$$

for  $s \geq C_0(TC_4 + T^{1/2})$  ( $C_4$  is given in (67)).

To treat the local term in the right-hand side of (83), let us introduce  $\theta \in C_0^\infty(\Omega)$  such that

$$0 \leq \theta \leq 1, \quad \theta(x) = 1 \text{ if } x \in \omega', \quad \theta(x) = 0 \text{ if } x \in \Omega \setminus \overline{\omega \cap \Omega}.$$

By multiplying (69)<sub>1</sub> by  $s^7 \lambda^8 \xi^7 e^{-2s\alpha} \theta^8 p$  and integrating par parts, we have

$$\begin{aligned} I & \triangleq s^7 \lambda^8 \iint_{(0,T) \times \omega} \theta^8 e^{-2s\alpha} \xi^7 |p|^2 dxdt \\ & = s^7 \lambda^8 \iint_{(0,T) \times \omega} z \left( \partial_t (e^{-2s\alpha} \xi^7 \theta^8 p) + \Delta^2 (e^{-2s\alpha} \xi^7 \theta^8 p) \right. \\ & \quad \left. + e^{-2s\alpha} \xi^7 \theta^8 a p + B \cdot \nabla (e^{-2s\alpha} \xi^7 \theta^8 p) + D : \nabla^2 (e^{-2s\alpha} \xi^7 \theta^8 p) \right) dxdt. \end{aligned}$$

By using the fact that

$$\begin{aligned} \Delta^2 (e^{-2s\alpha} \xi^7 \theta^8 p) & = \Delta^2 (e^{-2s\alpha} \xi^7 \theta^8) p + e^{-2s\alpha} \xi^7 \theta^8 \Delta^2 p + 4 \nabla \Delta (e^{-2s\alpha} \xi^7 \theta^8) \cdot \nabla p + 4 \nabla \Delta p \cdot \nabla (e^{-2s\alpha} \xi^7 \theta^8) \\ & \quad + 2 \Delta (e^{-2s\alpha} \xi^7 \theta^8) \Delta p + 4 \nabla^2 (e^{-2s\alpha} \xi^7 \theta^8) : \nabla^2 p \\ & \leq C_0 e^{-2s\alpha} \xi^7 \theta \left( |\Delta^2 p| + s^4 \lambda^4 \xi^4 |p| + s^3 \lambda^3 \xi^3 |\nabla p| + s^2 \lambda^2 \xi^2 |\nabla^2 p| + s \lambda \xi |\nabla \Delta p| \right), \end{aligned}$$

combined with (83) and by applying Young's inequality, we have

$$I \leq C_0 s^{16} \lambda^{16} \iint_{(0,T) \times \omega} e^{-2\alpha} \xi^{16} |z|^2 dxdt,$$

for  $\lambda \geq C_0$  and  $s \geq C_0(TC_4 + T^{1/2})$  and where  $C_4$  is given in (67). Combining the last estimate with (83), we deduce

$$\begin{aligned} & \iint_Q e^{-2s\alpha} \left( s^6 \lambda^8 \xi^6 |p|^2 + s^4 \lambda^6 \xi^4 |\nabla p|^2 + s^3 \lambda^4 \xi^3 |\Delta p|^2 \right. \\ & \quad \left. + s^2 \lambda^4 \xi^2 |\nabla^2 p|^2 + s \lambda^2 \xi |\nabla \Delta p|^2 + s^{-1} \xi^{-1} (|\partial_t p|^2 + |\Delta^2 p|^2) \right) dxdt \\ & \leq C_0 s^{16} \lambda^{16} \iint_{(0,T) \times \omega} e^{-2s\alpha} \xi^{16} |z|^2 dxdt, \end{aligned} \tag{84}$$

for  $\lambda \geq C_0$  and  $s \geq C_0(TC_4 + T^{1/2})$  and where  $C_4$  is given in (67). Combining the last estimate with (81), we deduce our aim result.  $\square$

Now we give the proof of Proposition 3.2.

### Proof of Proposition 3.2.

We give a sketch of the proof.

For  $\varepsilon > 0$ , we consider the following functional defined on  $L^2(\Omega)$  :

$$\mathfrak{J}_\varepsilon(p_0) = \frac{1}{2} \iint_{(0,T) \times \omega} |z|^2 dx dt + \varepsilon \|p_0\|_{L^2(\Omega)} + \iint_Q \zeta z dx dt, \quad (85)$$

where  $(z, p)$  is the solution of (69)-(70) with initial data  $p_0 \in L^2(\Omega)$ . It is easy to see that  $\mathfrak{J}_\varepsilon$  is a continuous and strictly convex functional. Moreover, from the fact that

$$\liminf_{\|p_0\|_{L^2(\Omega)} \rightarrow +\infty} \frac{\mathfrak{J}_\varepsilon(p_0)}{\|p_0\|_{L^2(\Omega)}} \geq \varepsilon,$$

we deduce that  $\mathfrak{J}_\varepsilon$  is coercive. So, we deduce the existence of a unique minimum  $p_0^\varepsilon \in L^2(\Omega)$ . By taking  $v^\varepsilon = z^\varepsilon \chi_\omega$  where  $(z^\varepsilon, p^\varepsilon)$  is the corresponding solution of (69)-(70) with initial data  $p_0 = p_0^\varepsilon$ , we deduce easily (68).  $\square$

## 3.2 A null controllability result for the semi-linear system.

This section is devoted to the proof of Theorem 1.3.

At first, from the fact that  $f$  is  $C^1$  globally Lipschitz function and  $f(0, 0, 0) = 0$ , we have the following decomposition :

$$f(s, p, q) = g(s, p, q)s + G(s, p, q) \cdot p + E(s, p, q) : q, \quad \forall (s, p, q) \in \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^{N^2},$$

where  $g : \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^{N^2} \rightarrow \mathbb{R}$ ,  $G : \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^{N^2} \rightarrow \mathbb{R}^N$  and  $E : \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^{N^2} \rightarrow \mathbb{R}^{N^2}$ . Furthermore,

$$g(s, p, q) = \int_0^1 \partial_s f(\lambda s, \lambda p, \lambda q) d\lambda, \quad G(s, p, q) = \int_0^1 \partial_p f(\lambda s, \lambda p, \lambda q) d\lambda, \quad E(s, p, q) = \int_0^1 \partial_q f(\lambda s, \lambda p, \lambda q) d\lambda. \quad (86)$$

Let us set  $Z = L^2(0, T; H^2 \cap H_0^1(\Omega))$ . For each  $\mathfrak{z} \in Z$ , we consider the following system :

$$\begin{cases} \partial_t y_{\mathfrak{z}} + \Delta^2 y_{\mathfrak{z}} + g(\mathfrak{z}, \nabla \mathfrak{z}, \nabla^2 \mathfrak{z}) y_{\mathfrak{z}} + G(\mathfrak{z}, \nabla \mathfrak{z}, \nabla^2 \mathfrak{z}) \cdot \nabla y_{\mathfrak{z}} + E(\mathfrak{z}, \nabla \mathfrak{z}, \nabla^2 \mathfrak{z}) : \nabla^2 y_{\mathfrak{z}} = \zeta + \chi_\omega v_{\mathfrak{z}} & \text{in } Q, \\ y_{\mathfrak{z}} = \Delta y_{\mathfrak{z}} = 0 & \text{on } \Sigma, \\ y_{\mathfrak{z}}(0, \cdot) = 0 & \text{in } \Omega, \end{cases} \quad (87)$$

$$\begin{cases} -\partial_t u_{\mathfrak{z}} + \Delta^2 u_{\mathfrak{z}} + \partial_s f(\mathfrak{z}, \nabla \mathfrak{z}, \nabla^2 \mathfrak{z}) u_{\mathfrak{z}} - \nabla \cdot (\partial_p f(\mathfrak{z}, \nabla \mathfrak{z}, \nabla^2 \mathfrak{z}) u_{\mathfrak{z}}) \\ \quad + \nabla^2 (\partial_q f(\mathfrak{z}, \nabla \mathfrak{z}, \nabla^2 \mathfrak{z}) u_{\mathfrak{z}}) = \chi_\omega v_{\mathfrak{z}} & \text{in } Q, \\ u_{\mathfrak{z}} = \Delta u_{\mathfrak{z}} = 0 & \text{on } \Sigma, \\ u_{\mathfrak{z}}(T, \cdot) = 0 & \text{in } \Omega. \end{cases} \quad (88)$$

Let us set for each  $\mathfrak{z} \in Z$

$$\begin{cases} a_{\mathfrak{z}} = g(\mathfrak{z}, \nabla \mathfrak{z}, \nabla^2 \mathfrak{z}) & \in L^\infty(Q), \\ B_{\mathfrak{z}} = G(\mathfrak{z}, \nabla \mathfrak{z}, \nabla^2 \mathfrak{z}) & \in L^\infty(Q)^N, \\ D_{\mathfrak{z}} = E(\mathfrak{z}, \nabla \mathfrak{z}, \nabla^2 \mathfrak{z}) & \in L^\infty(Q)^{N^2}, \\ \tilde{a}_{\mathfrak{z}} = \partial_s f(\mathfrak{z}, \nabla \mathfrak{z}, \nabla^2 \mathfrak{z}) & \in L^\infty(Q), \\ \tilde{B}_{\mathfrak{z}} = \partial_p f(\mathfrak{z}, \nabla \mathfrak{z}, \nabla^2 \mathfrak{z}) & \in L^\infty(Q)^N, \\ \tilde{D}_{\mathfrak{z}} = \partial_q f(\mathfrak{z}, \nabla \mathfrak{z}, \nabla^2 \mathfrak{z}) & \in L^\infty(Q)^{N^2}. \end{cases} \quad (89)$$

From the definition of  $f$ , we have

$$\|a_z\|_\infty + \|B_z\|_\infty + \|D_z\|_\infty + \|\tilde{a}_z\|_\infty + \|\tilde{B}_z\|_\infty + \|\tilde{D}_z\|_\infty \leq M, \quad \forall z \in Z, \quad (90)$$

where  $M = \|f\|_{W_\infty^1}$ . Then, by applying Proposition 3.2, we deduce that there exists a control  $v^\varepsilon = v_{\mathfrak{z}}^\varepsilon \in L^2((0, T) \times \omega)$  such that the corresponding solution  $(y_{\mathfrak{z}}^\varepsilon, u_{\mathfrak{z}}^\varepsilon)$  of (87)-(88) satisfies

$$\|u_{\mathfrak{z}}^\varepsilon(0, \cdot)\|_{L^2(\Omega)} \leq \varepsilon$$

and where

$$\|v_{\mathfrak{z}}^\varepsilon\|_{L^2((0, T) \times \omega)} \leq e^{\bar{C}_2} \left( \iint_Q e^{\frac{\bar{C}_2}{\sqrt{t}}} |\zeta|^2 dx dt \right)^{1/2}, \quad (91)$$

where  $\bar{C}_2$  is given in (13). Here, we used (90).

Now, the idea is to prove the existence of at least one fixed point  $y^\varepsilon$  of the following mapping :

$$\mathfrak{z} \mapsto \Lambda(\mathfrak{z}) = y_{\mathfrak{z}}^\varepsilon,$$

where  $y_{\mathfrak{z}}^\varepsilon$  is the corresponding solution of (87) for  $v_{\mathfrak{z}} = v_{\mathfrak{z}}^\varepsilon$ . Thus, let us recall the Schauder fixed point theorem :

**Theorem 3.6.** *Let  $Z$  be a Banach space and  $K \subset Z$  is a compact set. If  $\Lambda : K \rightarrow K$  is continuous, then  $\Lambda$  has a fixed point.*

Let us check that Schauder fixed point theorem can be applied to  $\Lambda$ . First, it is clear that  $Z$  is a Banach space. On the other hand, by applying Lemma 2.1 to  $y_{\mathfrak{z}}^\varepsilon$  the solution of (87), for  $a = a_{\mathfrak{z}}$ ,  $B = B_{\mathfrak{z}}$ ,  $D = D_{\mathfrak{z}}$  and  $F = \zeta + \chi_\omega v_{\mathfrak{z}}^\varepsilon$  we have

$$\|y_{\mathfrak{z}}^\varepsilon\|_Y \leq e^{C_0(\Omega)\bar{C}_1} \|\zeta + \chi_\omega v_{\mathfrak{z}}^\varepsilon\|_{L^2(Q)}, \quad (92)$$

where  $\bar{C}_1$  is given in (39) and  $Y = H^1(0, T; L^2(\Omega)) \cap L^2(0, T; H^4(\Omega)) \cap C^0([0, T]; H^2(\Omega))$ . So let us introduce the subset  $K$  :

$$K = \left\{ y \in Y; \|y\|_Y \leq e^{\bar{C}_2} \left( \iint_Q e^{\frac{\bar{C}_2}{\sqrt{t}}} |\zeta|^2 dx dt \right)^{1/2} \right\},$$

where  $\bar{C}_2$  is given in (13). Then, we deduce that  $\Lambda(Z)$  is embedded in  $K$ .

Now, let us prove that  $\Lambda$  is continuous. Let  $(\mathfrak{z}_n)_{n \in \mathbb{N}}$  be such that  $\mathfrak{z}_n \rightarrow \mathfrak{z}$  in  $Z$ . We have

$$\begin{cases} a_{\mathfrak{z}_n} \rightharpoonup a_{\mathfrak{z}}, \tilde{a}_{\mathfrak{z}_n} \rightharpoonup \tilde{a}_{\mathfrak{z}} & \text{weakly } * \text{ in } L^\infty(Q), \\ B_{\mathfrak{z}_n} \rightharpoonup B_{\mathfrak{z}}, \tilde{B}_{\mathfrak{z}_n} \rightharpoonup \tilde{B}_{\mathfrak{z}} & \text{weakly } * \text{ in } L^\infty(Q)^N, \\ D_{\mathfrak{z}_n} \rightharpoonup D_{\mathfrak{z}}, \tilde{D}_{\mathfrak{z}_n} \rightharpoonup \tilde{D}_{\mathfrak{z}} & \text{weakly } * \text{ in } L^\infty(Q)^{N^2}. \end{cases} \quad (93)$$

Let us denote  $\hat{p}_0^\varepsilon$  respectively  $\hat{p}_{0,n}^\varepsilon$  for  $n \in \mathbb{N}$  the unique minimizer of the functional  $\mathfrak{J}$  defined in (85) where  $a = a_{\mathfrak{z}}$ ,  $B = B_{\mathfrak{z}}$ ,  $D = D_{\mathfrak{z}}$ ,  $\tilde{a} = \tilde{a}_{\mathfrak{z}}$ ,  $\tilde{B} = \tilde{B}_{\mathfrak{z}}$  and  $\tilde{D} = \tilde{D}_{\mathfrak{z}}$  respectively  $a = a_{\mathfrak{z}_n}$ ,  $B = B_{\mathfrak{z}_n}$ ,  $D = D_{\mathfrak{z}_n}$ ,  $\tilde{a} = \tilde{a}_{\mathfrak{z}_n}$ ,  $\tilde{B} = \tilde{B}_{\mathfrak{z}_n}$  and  $\tilde{D} = \tilde{D}_{\mathfrak{z}_n}$ . Using the fact that  $\mathfrak{J}_\varepsilon$  is uniformly coercive (by using (90)), we deduce that  $(\hat{p}_{0,n}^\varepsilon)_{n \in \mathbb{N}}$  is uniformly bounded. By extracting subsequences, we have

$$\hat{p}_{0,n}^\varepsilon \rightharpoonup \hat{p}_0^\varepsilon \text{ weakly in } L^2(\Omega). \quad (94)$$

Let us now denote  $(\hat{z}^\varepsilon, \hat{p}^\varepsilon)$  respectively  $(\hat{z}_n^\varepsilon, \hat{p}_n^\varepsilon)$  for  $n \in \mathbb{N}$  the solution of (69)-(70) for  $a = a_{\mathfrak{z}}$ ,  $B = B_{\mathfrak{z}}$ ,  $D = D_{\mathfrak{z}}$ ,  $\tilde{a} = \tilde{a}_{\mathfrak{z}}$ ,  $\tilde{B} = \tilde{B}_{\mathfrak{z}}$ ,  $\tilde{D} = \tilde{D}_{\mathfrak{z}}$  and  $p_0 = \hat{p}_0^\varepsilon$  respectively  $a = a_{\mathfrak{z}_n}$ ,  $B = B_{\mathfrak{z}_n}$ ,  $D = D_{\mathfrak{z}_n}$ ,  $\tilde{a} = \tilde{a}_{\mathfrak{z}_n}$ ,  $\tilde{B} = \tilde{B}_{\mathfrak{z}_n}$ ,  $\tilde{D} = \tilde{D}_{\mathfrak{z}_n}$  and  $p_0 = \hat{p}_{0,n}^\varepsilon$ . From (93) and (94), we deduce that

$$\hat{p}_n^\varepsilon \rightarrow \hat{p}^\varepsilon \text{ in } L^2(Q) \quad (95)$$

and

$$\hat{z}_n^\varepsilon \rightarrow \hat{z}^\varepsilon \text{ in } L^2(Q). \quad (96)$$

By denoting  $\hat{v}^\varepsilon = \hat{z}^\varepsilon \chi_\omega$  and  $\hat{v}_n^\varepsilon = \hat{z}_n^\varepsilon \chi_\omega$ , we deduce from the last two convergence

$$\hat{v}_n^\varepsilon \rightarrow \hat{v}^\varepsilon \text{ in } L^2(Q). \quad (97)$$

From the last convergence and (93), we deduce that  $\Lambda(\mathfrak{z}_n) \rightarrow \Lambda(\mathfrak{z})$  in  $Z$ . This finishes the proof of the continuity of  $\Lambda$ . So, we deduce that  $\Lambda$  has at least one fixed point. In other words, we proved that, for any  $\varepsilon > 0$ , there exists  $v^\varepsilon \in L^2(Q)$  such that the solution  $(y^\varepsilon, u^\varepsilon)$  of

$$\begin{cases} \partial_t y^\varepsilon + \Delta^2 y^\varepsilon + f(y^\varepsilon, \nabla y^\varepsilon, \nabla^2 y^\varepsilon) = \zeta + \chi_\omega v^\varepsilon & \text{in } Q, \\ y^\varepsilon = \Delta y^\varepsilon = 0 & \text{on } \Sigma, \\ y^\varepsilon(0, \cdot) = 0 & \text{in } \Omega, \end{cases} \quad (98)$$

and

$$\begin{cases} -\partial_t u^\varepsilon + \Delta^2 u^\varepsilon + \partial_s f(y^\varepsilon, \nabla y^\varepsilon, \nabla^2 y^\varepsilon) u^\varepsilon - \nabla \cdot (\partial_p f(y^\varepsilon, \nabla y^\varepsilon, \nabla^2 y^\varepsilon) u^\varepsilon) & \text{in } Q, \\ \quad + \nabla^2 : (\partial_q f(y^\varepsilon, \nabla y^\varepsilon, \nabla^2 y^\varepsilon) u^\varepsilon) = \chi_\omega y^\varepsilon & \\ u^\varepsilon = \Delta u^\varepsilon = 0 & \text{on } \Sigma, \\ u^\varepsilon(T, \cdot) = 0 & \text{in } \Omega, \end{cases} \quad (99)$$

satisfies

$$\|u^\varepsilon(0, \cdot)\|_{L^2(\Omega)} \leq \varepsilon. \quad (100)$$

To finish the proof, it suffices to pass to the limit in (98), (99) and (100). First, let us notice that  $v^\varepsilon$  satisfies

$$\|v^\varepsilon\|_{L^2((0,T) \times \omega)} \leq e^{C_0 \bar{C}_2} \left( \iint_Q e^{\frac{C_0 \bar{C}_2}{\sqrt{t}}} |\zeta|^2 dx dt \right)^{1/2}, \quad (101)$$

where  $\bar{C}_2$  is given in (13). Combining this with (90), we deduce that the sequence  $(y^\varepsilon, u^\varepsilon)_{\varepsilon > 0}$  is bounded in  $Y \times \left( L^2(0, T; H^2(\Omega)) \cap H^1(0, T; H^{-2}(\Omega)) \right)$ . By extracting a subsequence, we deduce that

$$v^\varepsilon \rightharpoonup v \text{ weakly in } L^2(Q), \quad (102)$$

$$y^\varepsilon \rightarrow y \text{ in } L^2(0, T; H^2(\Omega)), \quad (103)$$

$$u^\varepsilon \rightarrow u \text{ in } L^2(Q), \quad (104)$$

and

$$u^\varepsilon(0, \cdot) \rightarrow u(0, \cdot) \text{ in } L^2(\Omega). \quad (105)$$

By passing to the limit in (98), (99) and (100), we deduce that there exists  $v \in L^2(Q)$  such that the corresponding solution of (56)-(57), satisfies (58). Moreover, we deduce that the control  $v$  satisfies (12). This finish the proof of Theorem 1.3.

## 4 Insensitizing controls for $\phi_2$ .

In this section, we are going to prove Theorem 1.4. We only prove a new Carleman estimate which leads us to prove this Theorem by using the same ideas as in Section 3. Let us introduce the following control system :

$$\begin{cases} \partial_t w + \Delta^2 w + aw + B \cdot \nabla w + E : \nabla^2 w = \chi_\omega v & \text{in } Q , \\ w = \Delta w = 0 & \text{on } \Sigma , \\ w(0, \cdot) = y_0(\cdot) + \tau \tilde{y}_0 & \text{in } \Omega , \end{cases} \quad (106)$$

where  $a \in L^\infty(0, T)$ ,  $B \in L^\infty(0, T)^N$  and  $E \in L^\infty(0, T)^{N^2}$ . As we said in the previous section, the existence of a control  $v$  such that (1) holds is equivalent to the null controllability of a coupled system. So, we introduce the following coupled system :

$$\begin{cases} \partial_t y + \Delta^2 y + ay + B \cdot \nabla y + E : \nabla^2 y = \chi_\omega v & \text{in } Q , \\ y = \Delta y = 0 & \text{on } \Sigma , \\ y(0, \cdot) = y_0(\cdot) & \text{in } \Omega , \end{cases} \quad (107)$$

$$\begin{cases} -\partial_t u + \Delta^2 u + au - B \cdot \nabla u + E : \nabla^2 u = \nabla \cdot (\nabla y \mathbf{1}_O) & \text{in } Q , \\ u = \Delta u = 0 & \text{on } \Sigma , \\ u(T, \cdot) = 0 & \text{in } \Omega . \end{cases} \quad (108)$$

Let us consider the following adjoint parabolic system :

$$\begin{cases} -\partial_t \psi + \Delta^2 \psi + a\psi - B \cdot \nabla \psi + E : \nabla^2 \psi = \nabla \cdot (\nabla \varphi \mathbf{1}_O) & \text{in } Q , \\ \partial_t \varphi + \Delta^2 \varphi + a\varphi + B \cdot \nabla \varphi + E : \nabla^2 \varphi = 0 & \text{in } Q , \\ \psi = \Delta \psi = 0 & \text{on } \Sigma , \\ \varphi = \Delta \varphi = 0 & \text{on } \Sigma , \\ \psi(T, \cdot) = 0 , \varphi(0, \cdot) = \varphi_0(\cdot) & \text{in } \Omega , \end{cases} \quad (109)$$

where  $\varphi_0 \in L^2(\Omega)$ . Before we start our study, we introduce the following weight functions:

$$\begin{aligned} \tilde{\alpha}(t) &= \min_{x \in \Omega} \alpha, \quad \tilde{\xi}(t) = \max_{x \in \Omega} \xi, \\ \alpha^* &= \alpha|_{\partial\Omega} = \max_{x \in \Omega} \alpha, \quad \xi^* = (\xi)|_{\partial\Omega} = \min_{x \in \Omega} \xi, \end{aligned} \quad (110)$$

$\eta$  is given in (51) and where  $\omega_0 \subset\subset O' \cap \omega$ . In order to shorten the formulas , we define

$$\begin{aligned} J(\psi) \triangleq & s^6 \lambda^8 \iint_Q e^{-\frac{5}{2}s\alpha} \xi^6 |\psi|^2 dxdt + \lambda^8 \int_0^T e^{-\frac{5}{2}s\alpha^*} \left( (s\xi^*)^4 \|\psi\|_{H^1(\Omega)}^2 + (s\xi^*)^2 \|\psi\|_{H^2(\Omega)}^2 \right. \\ & \left. + \|\psi\|_{H^3(\Omega)}^2 \right) dt \end{aligned} \quad (111)$$

and

$$\begin{aligned} I(\varphi) \triangleq & \lambda^8 \int_0^T e^{-2s\alpha^*} \left( (s\xi^*)^3 \|\varphi\|_{H^3(\Omega)}^2 + \|\varphi\|_{H^4(\Omega)}^2 + \|\partial_t \varphi\|_{L^2(\Omega)}^2 \right) dt \\ & + s^6 \lambda^8 \iint_Q e^{-2s\alpha} \xi^6 |\Delta \varphi|^2 dxdt. \end{aligned} \quad (112)$$

In this section, let us denote  $C$  a positive constant depending on  $\Omega$ ,  $\omega$ ,  $\|a\|_\infty$ ,  $\|B\|_\infty$  and  $\|E\|_\infty$ .

We are going to prove the following Proposition :

**Proposition 4.1.** *There exists  $C = C(\Omega, \omega, \|a\|_\infty, \|B\|_\infty, \|E\|_\infty)$  such that*

$$J(\psi) + I(\varphi) \leq C \iint_{\omega \times (0, T)} e^{-\frac{1}{4}s\alpha} |\psi|^2 dxdt, \quad (113)$$

for any  $\lambda \geq C$  and any  $s \geq C(T^{1/2} + T)$ , where  $(\psi, \varphi)$  is the solution of (109).

We divide the proof in several subsections, the first one concerns the proof of a new Carleman estimate for  $\varphi$ . In the second subsection we apply a Carleman estimate for  $\psi$  and we conclude.

#### 4.0.1 New Carleman estimate for $\varphi$ .

Let us recall the system satisfied by  $\varphi$  :

$$\begin{cases} \partial_t \varphi + \Delta^2 \varphi + a\varphi + B \cdot \nabla \varphi + E : \nabla^2 \varphi = 0 & \text{in } Q, \\ \varphi = \Delta \varphi = 0 & \text{on } \Sigma, \\ \varphi(0, \cdot) = \varphi_0(\cdot) & \text{in } \Omega, \end{cases} \quad (114)$$

where  $\varphi_0 \in L^2(\Omega)$ . For this system we can prove the following Carleman estimate :

**Lemma 4.2.** *There exists  $C > 0$  such that for all  $\varphi_0 \in L^2(\Omega)$ , we have*

$$I(\varphi) \leq Cs^7 \lambda^8 \iint_{\omega' \times (0, T)} e^{-2s\alpha} \xi^7 |\Delta \varphi|^2 dxdt, \quad (115)$$

for any  $\lambda \geq C$ , any  $s \geq C(T^{1/2} + T)$  and where  $\omega_0 \subset\subset \omega' \subset\subset \omega \cap O'$ .

*Proof.* We divide it in several steps. Before we start let us introduce the following remark :

**Remark 4.3.** *In all this section, we prove Lemma 4.2 by using a density argument. Indeed, it suffices to consider  $a_n, B_n, E_n$  and  $\varphi_{0,n}$  are regular enough and prove the following estimate :*

$$I(\varphi_n) \leq C_n s^7 \lambda^8 \iint_{\omega' \times (0, T)} e^{-2s\alpha} \xi^7 |\Delta \varphi_n|^2 dxdt, \quad (116)$$

where  $C_n = C(\Omega, \omega, \|a_n\|_\infty, \|B_n\|_\infty, \|D_n\|_\infty)$  and  $\varphi_n$  is the corresponding solution of (114) for  $a = a_n, B = B_n, E = E_n$  and  $\varphi_0 = \varphi_{0,n}$ . Then, by passing to the limit in (116), we deduce our aim result.

Let us now start with the first step of the proof.

*Step 1. Estimate for  $\Delta \varphi$ .*

Let us notice that  $\Delta \varphi$  verifies

$$\begin{cases} \partial_t \Delta \varphi + \Delta^2 \Delta \varphi + \Delta(a\varphi + B \cdot \nabla \varphi + E : \nabla^2 \varphi) = 0 & \text{in } Q, \\ \Delta \varphi = 0 & \text{on } \Sigma, \\ \Delta \varphi(0, \cdot) = \Delta \varphi_0(\cdot) & \text{in } \Omega. \end{cases} \quad (117)$$

Then by applying Lemma 2.5 to  $\Delta\varphi$  with  $\tilde{\omega} = \omega'$ ,  $F_0 = -a\Delta\varphi$ ,  $F_1 = -B \cdot \Delta\varphi$  and  $\hat{F}_{ij} = -E_{i,j}\Delta\varphi$  and taking into consideration Remark 2.6, we deduce

$$\begin{aligned}
s^6\lambda^8 \iint_Q e^{-2s\alpha}\xi^6|\Delta\varphi|^2 dxdt &\leq C\left(s^7\lambda^8 \iint_{\omega' \times (0,T)} e^{-2s\alpha}\xi^7|\Delta\varphi|^2 dxdt \right. \\
&\quad + s^3\lambda^3 \iint_{\Sigma} e^{-2s\alpha}\xi^3|\Delta^2\varphi|^2 dxdt \\
&\quad + \iint_Q e^{-2s\alpha}\left(|a\Delta\varphi|^2 + s^2\lambda^2\xi^2|B|^2|\Delta\varphi|^2 \right. \\
&\quad \left. + s^4\lambda^4\xi^2|E|^2|\Delta\varphi|^2 dxdt\right), \tag{118}
\end{aligned}$$

for any  $\lambda \geq C$  and any  $s \geq C(T^{1/2} + T)$ . By taking  $\lambda \geq C$  and  $C(T^{1/2} + T)$ , we deduce

$$\begin{aligned}
s^6\lambda^8 \iint_Q e^{-2s\alpha}\xi^6|\Delta\varphi|^2 dxdt &\leq C\left(s^7\lambda^8 \iint_{\omega' \times (0,T)} e^{-2s\alpha}\xi^7|\Delta\varphi|^2 dxdt \right. \\
&\quad \left. + s^3\lambda^3 \iint_{\Sigma} e^{-2s\alpha}\xi^3|\Delta^2\varphi|^2 dxdt\right). \tag{119}
\end{aligned}$$

*Step 2. Treating the boundary term and adding other terms.*

To treat the boundary term in the right-hand side of (119), let us prove the following Lemma :

**Lemma 4.4.** *Let  $\varphi$  fulfil (114). Then, there exists  $C > 0$  such that*

$$I(\varphi) \leq Cs^6\lambda^8 \iint_Q e^{-2s\alpha}\xi^6|\Delta\varphi|^2 dxdt, \tag{120}$$

for any  $\lambda \geq C$ , any  $s \geq C(T^{1/2} + T)$ , where  $I(\varphi)$  was defined in (112).

*Proof.* Before we start, let us remark that

$$|\partial_t\alpha^*| + |\partial_t\xi^*| \leq CT(\xi^*)^3. \tag{121}$$

Let us set  $\vartheta = e^{-s\alpha^*}\varphi$ . It is easy to verify that  $\vartheta$  fulfills

$$\begin{cases} \partial_t\vartheta + \Delta^2\vartheta + a\vartheta + B \cdot \nabla\vartheta + E : \nabla^2\vartheta = \partial_t e^{-s\alpha^*}\varphi & \text{in } Q, \\ \vartheta = \Delta\vartheta = 0 & \text{on } \Sigma, \\ \vartheta(0, \cdot) = 0 & \text{in } \Omega. \end{cases} \tag{122}$$

By using the fact that  $\partial_t(e^{-s\alpha^*})\varphi \in L^2(Q)$ , Lemma 2.1 for  $F = \partial_t(e^{-s\alpha^*})\varphi$  and  $D = E$  and estimate (121), we can easily deduce

$$\lambda^8 \int_0^T e^{-2s\alpha^*} (\|\varphi\|_{H^4(\Omega)}^2 + \|\partial_t\varphi\|_{L^2(\Omega)}^2) dt \leq Cs^6\lambda^8 \iint_Q e^{-2s\alpha}\xi^6|\Delta\varphi|^2 dxdt, \tag{123}$$

for any  $\lambda \geq C$ , any  $s \geq C(T^{1/2} + T)$ . Finally, by using interpolation inequalities, we can add the term

$$s^3\lambda^8 \int_0^T e^{-2s\alpha^*} (\xi^*)^3 \|\varphi\|_{H^3(\Omega)}^2 dt,$$

to the left-hand side of (123).

□

Coming back to (119), we deduce

$$\begin{aligned}
I(\varphi) &\leq C \left( s^7 \lambda^8 \iint_{\omega' \times (0, T)} e^{-2s\alpha} \xi^7 |\Delta\varphi|^2 dxdt \right. \\
&\quad \left. + s^3 \lambda^3 \iint_{\Sigma} e^{-2s\alpha} \xi^3 |\Delta^2\varphi|^2 dxdt \right), \tag{124}
\end{aligned}$$

for  $\lambda \geq C$  and  $s \geq C(T^{1/2} + T)$ . To absorb the boundary term in the right-hand side of (124), it suffices to see that from (114) :

$$\Delta^2\varphi = -\partial_t\varphi - a\varphi - B \cdot \nabla\varphi - E : \nabla^2\varphi \quad \text{in } Q.$$

Then, we deduce that

$$\Delta^2\varphi = -B \cdot \nabla\varphi - E : \nabla^2\varphi \quad \text{on } \Sigma.$$

So, we have the following estimate :

$$\begin{aligned}
s^3 \lambda^3 \iint_{\Sigma} e^{-2s\alpha^*} (\xi^*)^3 |\Delta^2\varphi|^2 dxdt &= s^3 \lambda^3 \iint_{\Sigma} e^{-2s\alpha^*} (\xi^*)^3 |-B \cdot \nabla\varphi - E : \nabla^2\varphi|^2 dxdt \\
&\leq C s^3 \lambda^3 \int_0^T e^{-2s\alpha^*} (\xi^*)^3 \|\varphi\|_{H^{5/2+\varepsilon}(\Omega)}^2 dt, \tag{125}
\end{aligned}$$

for  $\lambda \geq C$  and  $\varepsilon > 0$ . So we can deduce that the boundary term in the right-hand side of (124) is absorbed by a  $H^3(\Omega)$  term that appears in the definition of  $I(\varphi)$  for  $\lambda \geq C$  and  $\varepsilon > 0$  small enough. This finishes the proof of Lemma 4.2.  $\square$

#### 4.0.2 New Carleman estimate for $\psi$ .

Let us recall the system verified by  $\psi$  for a function  $\varphi$  given

$$\begin{cases} -\partial_t\psi + \Delta^2\psi + a\psi - B \cdot \nabla\psi + E : \nabla^2\psi = \nabla \cdot (\nabla\varphi \mathbb{1}_O) & \text{in } Q = (0, T) \times \Omega, \\ \psi = \Delta\psi = 0 & \text{on } \Sigma, \\ \psi(T, \cdot) = 0 & \text{in } \Omega. \end{cases} \tag{126}$$

By applying Lemma 2.5 to  $\psi$  for  $\tilde{\omega} = \omega$ ,  $F_0 = a\psi$ ,  $F_1 = \nabla\varphi - B\psi$  and  $\tilde{F} = E\psi$  with the small weight  $e^{-\frac{5}{2}s\alpha}$ , we have

$$\begin{aligned}
s^6 \lambda^8 \iint_Q e^{-\frac{5}{2}s\alpha} \xi^6 |\psi|^2 dxdt &\leq C \left( s^7 \lambda^8 \iint_{\omega \times (0, T)} e^{-\frac{5}{2}s\alpha} \xi^7 |\psi|^2 dxdt \right. \\
&\quad + \iint_Q e^{-\frac{5}{2}s\alpha} (|a\psi|^2 + s^2 \lambda^2 \xi^2 |B\psi|^2 + s^4 \lambda^4 \xi^4 |E\psi|^2) dxdt \\
&\quad \left. + s^2 \lambda^2 \iint_Q e^{-\frac{5}{2}s\alpha} \xi^2 |\nabla\varphi|^2 dxdt \right), \tag{127}
\end{aligned}$$

for  $\lambda \geq C$  and  $s \geq C(T^{1/2} + T)$ . To treat the last term in the right-hand side of (127), let us notice that

$$\begin{aligned}
s^2 \lambda^2 \iint_Q e^{-\frac{5}{2}s\alpha} \xi^2 |\nabla\varphi|^2 dxdt &\leq s^2 \lambda^2 \iint_Q e^{-\frac{5}{2}s\tilde{\alpha}} \tilde{\xi}^2 |\nabla\varphi|^2 dxdt \\
&\leq C s^2 \lambda^2 \int_0^T e^{-\frac{5}{2}s\tilde{\alpha}} \tilde{\xi}^2 \|\Delta\varphi\|_{L^2(\Omega)}^2 dt \\
&\leq C s^2 \lambda^2 \iint_Q e^{-2s\alpha} \xi^2 |\Delta\varphi|^2 dt, \tag{128}
\end{aligned}$$

for  $\lambda \geq C$  and  $s \geq C(T^{1/2} + T)$ . We recall that  $\tilde{\alpha}(t) = \min_{x \in \bar{\Omega}} \alpha$  and  $\tilde{\xi}(t) = \max_{x \in \bar{\Omega}} \alpha$  (see (110)). Here we used that  $\varphi = 0$  on  $\partial\Omega$ .

Let us now prove some estimates on the first, second and third order derivatives of  $\psi$ . Let us set  $\tilde{\psi} = \lambda^4 e^{-\frac{5}{4}s\alpha^*} \psi$ . It is easy to verify that  $\tilde{\psi}$  fulfills

$$\begin{cases} -\partial_t \tilde{\psi} + \Delta^2 \tilde{\psi} + a\tilde{\psi} - B \cdot \nabla \tilde{\psi} + E : \nabla^2 \tilde{\psi} = \tilde{f} & \text{in } Q, \\ \tilde{\psi} = \Delta \tilde{\psi} = 0 & \text{on } \Sigma, \\ \tilde{\psi}(0, \cdot) = 0 & \text{in } \Omega, \end{cases} \quad (129)$$

where  $\tilde{f} = \lambda^4 \nabla \cdot (e^{-\frac{5}{4}s\alpha^*} \nabla \varphi \mathbf{1}_O) - \lambda^4 \partial_t (e^{-\frac{5}{4}s\alpha^*}) \psi$ . Let us notice that  $\tilde{f} \in L^2(0, T; H^{-1}(\Omega))$ . Indeed, it suffices to remark that

$$\begin{aligned} \|\tilde{f}\|_{L^2(0, T; H^{-1}(\Omega))}^2 &\leq \lambda^8 \int_0^T e^{-\frac{5}{2}s\alpha^*} \|\nabla \varphi\|_{L^2(\Omega)}^2 dt + s^6 \lambda^8 \iint_Q e^{-\frac{5}{2}s\alpha^*} (\xi^*)^6 |\psi|^2 dx dt \\ &\leq C \lambda^8 \int_0^T e^{-\frac{5}{2}s\alpha^*} \|\Delta \varphi\|_{L^2(\Omega)}^2 dt + s^6 \lambda^8 \iint_Q e^{-\frac{5}{2}s\alpha} \xi^6 |\psi|^2 dx dt \\ &\leq CI(\varphi) + s^6 \lambda^8 \iint_Q e^{-\frac{5}{2}s\alpha} \xi^6 \lambda^8 |\psi|^2 dx dt \end{aligned} \quad (130)$$

for  $\lambda \geq C$  and  $s \geq C(T^{1/2} + T)$ . By applying Lemma 2.1, by taking in consideration Remark 2.2, we deduce

$$\begin{aligned} \|\tilde{\psi}\|_{L^2(0, T; H^3(\Omega))}^2 &\leq C \|\tilde{f}\|_{L^2(0, T; H^{-1}(\Omega))}^2 \\ &\leq C \left( I(\varphi) + s^6 \lambda^8 \iint_Q e^{-\frac{5}{2}s\alpha} \xi^6 |\psi|^2 dx dt \right), \end{aligned} \quad (131)$$

for  $\lambda \geq C$  and  $s \geq C(T^{1/2} + T)$ .

Here we proved an estimate for the third derivatives of  $\psi$ . To add the first and second order derivatives of  $\psi$ , it suffices to use interpolation argument between  $H^3(\Omega)$  and  $L^2(\Omega)$  spaces.

Combining the last inequality with (115), (127) and (128), we deduce

$$J(\psi) + I(\varphi) \leq C \left( s^7 \lambda^8 \iint_{\omega \times (0, T)} e^{-\frac{5}{2}s\alpha} \xi^7 |\psi|^2 dx dt + s^7 \lambda^8 \iint_{\omega' \times (0, T)} e^{-2s\alpha} \xi^7 |\Delta \varphi|^2 dx dt \right), \quad (132)$$

for any  $\lambda \geq C$  and any  $s \geq C(T^{1/2} + T)$ .

#### 4.0.3 Last arrangements and conclusion.

In this subsection, we treat the last term on the right-hand side of (132). Let us introduce a function  $\theta$  such that

$$0 \leq \theta \leq 1, \quad \theta(x) = 1 \text{ if } x \in \omega', \quad \theta(x) = 0 \text{ if } x \in \Omega \setminus \bar{\omega}.$$

Let us notice that from (109), we have

$$-\partial_t \psi + \Delta^2 \psi + a\psi - B \cdot \nabla \psi + E : \nabla^2 \psi = \Delta \varphi \quad \text{in } \omega'.$$

Recall that  $\omega'$  was defined in Lemma 115. Then we deduce that

$$|\Delta \varphi|^2 = \Delta \varphi (-\partial_t \psi + \Delta^2 \psi + a\psi - B \cdot \nabla \psi + E : \nabla^2 \psi) \quad \text{in } \omega'.$$

So, we have

$$s^7 \lambda^8 \iint_{\omega' \times (0, T)} e^{-2s\alpha} \xi^7 |\Delta\varphi|^2 dxdt \leq s^7 \lambda^8 \iint_{\omega \times (0, T)} e^{-2s\alpha} \xi^7 \theta^4 \Delta\varphi (-\partial_t \psi + \Delta^2 \psi + a\psi - B \cdot \nabla \psi + E : \nabla^2 \psi) dxdt. \quad (133)$$

By integrating by parts and using the fact that  $e^{-2s\alpha} \leq e^{-2s\tilde{\alpha}}$  and  $\xi \leq \tilde{\xi}$ , we deduce

$$\begin{aligned} s^7 \lambda^8 \iint_{\omega \times (0, T)} e^{-2s\tilde{\alpha}} (\tilde{\xi})^7 \theta^4 |\Delta\varphi|^2 dxdt &= s^7 \lambda^8 \iint_{\omega \times (0, T)} \theta^4 \partial_t (e^{-2s\tilde{\alpha}} (\tilde{\xi})^7 \Delta\varphi) \psi dxdt \\ &+ s^7 \lambda^8 \iint_{\omega \times (0, T)} e^{-2s\tilde{\alpha}} (\tilde{\xi})^7 \Delta(\theta^4 \Delta\varphi) \Delta\psi dxdt \\ &+ s^7 \lambda^8 \iint_{\omega \times (0, T)} e^{-2s\tilde{\alpha}} (\tilde{\xi})^7 \theta^4 \Delta\varphi a \psi dxdt \\ &- s^7 \lambda^8 \iint_{\omega \times (0, T)} e^{-2s\tilde{\alpha}} (\tilde{\xi})^7 \theta^4 \Delta\varphi B \cdot \nabla \psi dxdt \\ &+ s^7 \lambda^8 \iint_{\omega \times (0, T)} e^{-2s\tilde{\alpha}} (\tilde{\xi})^7 \theta^4 \Delta\varphi E : \nabla^2 \psi dxdt \\ &= I_1 + I_2 + I_3 + I_4 + I_5. \end{aligned} \quad (134)$$

To shorten the formulas below, let us denote

$$\mathfrak{J}_\omega(\psi) = s^{14} \lambda^8 \iint_{\omega \times (0, T)} e^{-4s\tilde{\alpha} + 2s\alpha^*} (\tilde{\xi})^{14} (|\psi|^2 + \theta |\nabla \psi|^2 + \theta^2 |\nabla^2 \psi|^2) dxdt.$$

We use the fact that

$$\partial_t (e^{-2s\tilde{\alpha}} (\tilde{\xi})^7) \leq C s^3 e^{-2s\tilde{\alpha}} (\tilde{\xi})^{10}$$

for  $s \geq C(T^{1/2} + T)$ , we integrate by parts and we deduce

$$\begin{aligned} I_1 &= s^7 \lambda^8 \iint_{\omega \times (0, T)} \theta^4 \partial_t (e^{-2s\tilde{\alpha}} (\tilde{\xi})^7) \Delta\varphi \psi dxdt + s^7 \lambda^8 \iint_{\omega \times (0, T)} \left| e^{-2s\tilde{\alpha}} \partial_t \varphi (\tilde{\xi})^7 \Delta(\theta^4 \psi) \right| dxdt \\ &\leq \varepsilon I(\varphi) + C(\varepsilon) \mathfrak{J}_\omega(\psi), \end{aligned}$$

for  $\varepsilon > 0$ ,  $\lambda \geq C$  and  $s \geq C(T^{1/2} + T)$ . On the other hand, we have

$$\begin{aligned} I_2 &= s^7 \lambda^8 \iint_{\omega \times (0, T)} e^{-2s\tilde{\alpha}} (\tilde{\xi})^7 (\Delta(\theta^4) \Delta\varphi + 2\nabla(\theta^4) \cdot \nabla \Delta\varphi + \theta^4 \Delta^2 \varphi) \Delta\psi dxdt \\ &\leq \varepsilon I(\varphi) + C(\varepsilon) \mathfrak{J}_\omega(\psi), \end{aligned} \quad (135)$$

for  $\varepsilon > 0$ ,  $\lambda \geq C$  and  $s \geq C(T^{1/2} + T)$ . Moreover, it is easy to see that

$$I_3 + I_4 + I_5 \leq \varepsilon I(\varphi) + C(\varepsilon) \mathfrak{J}_\omega(\psi),$$

for  $\varepsilon > 0$ ,  $\lambda \geq C$  and  $s \geq C(T^{1/2} + T)$ . Combining the last computations with (132), we deduce

$$J(\psi) + I(\varphi) \leq C \mathfrak{J}_\omega(\psi), \quad (136)$$

for  $\lambda \geq C$  and  $s \geq C(T^{1/2} + T)$ . To finish the proof, it suffices to treat the term on the right-hand side of (136). We will treat only the last term of  $\mathfrak{J}_\omega(\psi)$  because it is more complicated than the others. Let us denote

$$\mathfrak{J}_{3,\omega}(\psi) \triangleq s^{14}\lambda^8 \iint_{\omega \times (0,T)} e^{-4s\bar{\alpha} + 2s\alpha^*} (\tilde{\xi})^{14} \theta^2 |\nabla^2 \psi|^2 dxdt.$$

By integrating by parts, we have

$$\begin{aligned} \mathfrak{J}_{3,\omega}(\psi) &\leq C \left( s^{14}\lambda^8 \iint_{\omega \times (0,T)} e^{-4s\bar{\alpha} + 2s\alpha^*} (\tilde{\xi})^{14} \theta^2 |\nabla \Delta \psi| |\nabla \psi| dxdt \right. \\ &\quad \left. + s^{14}\lambda^8 \iint_{\omega \times (0,T)} e^{-4s\bar{\alpha} + 2s\alpha^*} (\tilde{\xi})^{14} \theta |\nabla^2 \psi| |\nabla \psi| dxdt \right) \\ &\leq \varepsilon J(\psi) + C_\varepsilon s^{28}\lambda^8 \iint_{\omega \times (0,T)} e^{-8s\bar{\alpha} + \frac{13}{2}s\alpha^*} (\tilde{\xi})^{28} \theta |\nabla \psi|^2 dxdt \end{aligned} \quad (137)$$

for  $\varepsilon > 0$ ,  $\lambda \geq C$  and  $s \geq C(T^{1/2} + T)$ . To treat the last term on the right-hand side of (137), by integrating by parts for the last time, we deduce

$$\mathfrak{J}_{3,\omega}(\psi) \leq \varepsilon J(\psi) + C_\varepsilon s^{54}\lambda^8 \iint_{\omega \times (0,T)} e^{-16s\bar{\alpha} + \frac{31}{2}s\alpha^*} (\tilde{\xi})^{54} |\psi|^2 dxdt, \quad (138)$$

for  $\varepsilon > 0$ ,  $\lambda \geq C$  and  $s \geq C(T^{1/2} + T)$ . Combining this with (136), We deduce

$$J(\psi) + I(\varphi) \leq C s^{54}\lambda^8 \iint_{\omega \times (0,T)} e^{-16s\bar{\alpha} + \frac{31}{2}s\alpha^*} (\tilde{\xi})^{54} |\psi|^2 dxdt, \quad (139)$$

for  $\lambda \geq C$  and  $s \geq C(T^{1/2} + T)$ . Using the fact that for  $\lambda \geq C$  and  $s \geq C(T^{1/2} + T)$ ,

$$s^{54}\lambda^8 e^{-16s\bar{\alpha} + \frac{31}{2}s\alpha^*} (\tilde{\xi})^{54} \leq e^{-\frac{1}{4}s\alpha},$$

we deduce

$$J(\psi) + I(\varphi) \leq C \iint_{\omega \times (0,T)} e^{-\frac{1}{4}s\alpha} |\psi|^2 dxdt, \quad (140)$$

for  $\lambda \geq C$  and  $s \geq C(T^{1/2} + T)$ . This finish the proof of Proposition 4.1

## 5 Negative controllability results for coupled systems of second and fourth order parabolic equations in dimension $N$ .

This section is devoted for the proof of Lemma 1.6 and Lemma 1.10. We divide the proofs of these two Lemmas into several parts depending on the dimension  $N \in \mathbb{N}$ .

### Proof of Lemma 1.6.

- Dimension  $N = 1$ .

Let us denote  $\Omega = ]0, 2\pi[$ ,  $O = ]\frac{\pi}{2}, \frac{3\pi}{2}[$  and let us consider the following system :

$$\begin{cases} L\varphi = -\frac{d^2\varphi}{dx^2} + a(x)\varphi + B(x)\frac{d\varphi}{dx} = 0 & \text{in } \Omega , \\ \varphi(0) = \varphi(2\pi) = 0 , \end{cases} \quad (141)$$

where  $a \in L^\infty(0, 2\pi)$  and  $B \in L^\infty(0, 2\pi)$ . To find a counter example, the main idea is to take some functions  $a$  and  $B$  vanishing in  $O$ . So let us define the function  $a$  as follows :

$$a(x) = \begin{cases} -1 & \text{if } x \in [0, \frac{\pi}{2}] \cup [\frac{3\pi}{2}, 2\pi] , \\ 0 & \text{if } x \in ]\frac{\pi}{2}, \frac{3\pi}{2}[ \end{cases}$$

and  $B \equiv 0$ . It is not hard to see that a solution for this system is the following one :

$$\tilde{\varphi}(x) = \begin{cases} \sin(x) & \text{if } x \in [0, \frac{\pi}{2}] \cup [\frac{3\pi}{2}, 2\pi] \\ 1 & \text{if } x \in ]\frac{\pi}{2}, \frac{3\pi}{2}[ , \\ -\sin(x) & \text{if } x \in [\frac{3\pi}{2}, 2\pi] \end{cases}$$

and  $\tilde{\varphi} \in H^2(\Omega)$ . By taking  $\varphi_0 = \tilde{\varphi}$  in (21), we deduce that the corresponding solution  $(\psi, \varphi)$  of (21) does not satisfies *Unique Continuation Principle* given in (22). Indeed, let us remark that the corresponding solution of (21) verifies  $\psi \equiv 0$ ,  $\varphi(t, x) = \tilde{\varphi}(x)$  for  $(t, x) \in Q$  and we have

$$\psi = 0 \text{ in } (0, T) \times \omega \text{ but } \varphi \neq 0 \text{ in } Q, \quad (142)$$

for any  $\omega \subset \Omega$ .

- *Dimension*  $N = 2$  .

Let us denote  $O = \{(x, y) \in \mathbb{R}^2; x^2 + y^2 < e^{-\pi}\}$ ,  $\Omega = \{(x, y) \in \mathbb{R}^2; x^2 + y^2 < 1\}$  and let us consider the following system :

$$\begin{cases} L\varphi = -\Delta\varphi + a(x, y)\varphi + B(x, y) \cdot \nabla\varphi = 0 & \text{in } \Omega , \\ \varphi = 0 & \text{on } \partial\Omega \end{cases} \quad (143)$$

where

$$a(x, y) = \begin{cases} -\frac{1}{x^2 + y^2} & \text{if } (x, y) \in \bar{\Omega} \setminus O , \\ 0 & \text{if } (x, y) \in O, \end{cases}$$

and  $B \equiv 0$ . So, we deduce a solution :

$$\tilde{\varphi}(x, y) = \begin{cases} \sin(\log(\sqrt{x^2 + y^2})) & \text{if } (x, y) \in \{(x, y) \in \mathbb{R}^2; e^{-\pi} \leq x^2 + y^2 \leq 1\} , \\ -1 & \text{if } (x, y) \in O \end{cases}$$

and  $\tilde{\varphi} \in H^2(\Omega)$ . By taking  $\varphi_0 = \tilde{\varphi}$  in (21), we deduce that the corresponding solution  $(\psi, \varphi)$  of (21) verifies  $\psi \equiv 0$ ,  $\varphi(t, x) = \tilde{\varphi}(x)$  for  $(t, x) \in Q$  and we have

$$\psi = 0 \text{ in } (0, T) \times \omega \text{ but } \varphi \neq 0 \text{ in } Q, \quad (144)$$

for any  $\omega \subset \Omega$ .

- Dimension  $N > 2$ .

Let us denote  $O = \left\{ x = (x_1, x_2, \dots, x_N) \in \mathbb{R}^N; \|x\|_2^2 = \sum_{i=1}^N x_i^2 < e^{-\pi} \right\}$ ,  $\Omega = \left\{ x = (x_1, x_2, \dots, x_N) \in \mathbb{R}^N; \|x\|_2^2 < 1 \right\}$  and let us consider the following system :

$$\begin{cases} L\varphi = -\Delta\varphi + a(x)\varphi + B \cdot \nabla\varphi = 0 & \text{in } \Omega, \\ \varphi = 0 & \text{on } \partial\Omega \end{cases} \quad (145)$$

where  $a \in L^\infty(\Omega)$  and  $B \in L^\infty(\Omega)^N$ . Let us denote

$$a(x) = \begin{cases} -\frac{1}{\|x\|_2^2} & \text{if } x \in \overline{\Omega} \setminus \omega, \\ 0 & \text{if } x \in O \end{cases}$$

and

$$B(x) = \begin{cases} \frac{N-2}{\|x\|_2^2} & \text{if } x \in \overline{\Omega} \setminus \omega, \\ 0 & \text{if } x \in O. \end{cases}$$

Then, we deduce the following solution :

$$\tilde{\varphi}(x) = \begin{cases} \sin(\log(\|x\|_2)) & \text{if } x \in \Omega \setminus \overline{\omega}, \\ -1 & \text{if } x \in O \end{cases}$$

and  $\tilde{\varphi} \in H^2(\Omega)$ . As we did above it is not hard to see that the corresponding solution  $(\psi, \varphi) = (0, \tilde{\varphi})$  of (21) for  $\varphi_0 = \tilde{\varphi}$  does not satisfies *Unique Continuation Principle* given in (22) for any  $\omega \subset \Omega$ .  $\square$

Let us now give the proof of Lemma 1.10.

### Proof of Lemma 1.10.

- Dimension  $N = 1$ .

Let us denote  $\Omega = ]0, 6\pi[$ ,  $O = ]\frac{3\pi}{2}, \frac{9\pi}{2}[$  and let us consider the following system

$$\begin{cases} L\varphi = \frac{d^4\varphi}{dx^4} + a(x)\varphi + b(x)\frac{d^2\varphi}{dx^2} = 0 & \text{in } \Omega, \\ \varphi(0) = \varphi(6\pi) = 0, \\ \frac{d^2\varphi}{dx^2}(0) = \frac{d^2\varphi}{dx^2}(6\pi) = 0, \end{cases} \quad (146)$$

where  $a, b \in L^\infty(0, 6\pi)$ . As we did above, the same idea will be applied here. We consider some functions  $a$  and  $b$  vanishing in  $O$ . So let us define two function  $a$  and  $b$

$$a(x) = \begin{cases} \frac{1}{9} & \text{if } x \in [0, \frac{3\pi}{2}] \cup [\frac{9\pi}{2}, 6\pi] , \\ 0 & \text{if } x \in ]\frac{3\pi}{2}, \frac{9\pi}{2}[ \end{cases}$$

and

$$b(x) = \begin{cases} \frac{10}{9} & \text{if } x \in [0, \frac{3\pi}{2}] \cup [\frac{9\pi}{2}, 6\pi] , \\ 0 & \text{if } x \in ]\frac{3\pi}{2}, \frac{9\pi}{2}[ . \end{cases}$$

It is not hard to see that a solution for this system is the following one :

$$\tilde{\varphi}(x) = \begin{cases} \frac{1}{8} \sin(x) + \frac{9}{8} \sin(\frac{x}{3}) & \text{if } x \in [0, \frac{3\pi}{2}] , \\ 1 & \text{if } x \in ]\frac{3\pi}{2}, \frac{9\pi}{2}[ , \\ -\frac{1}{8} \sin(x) - \frac{9}{8} \sin(\frac{x}{3}) & \text{if } x \in [\frac{9\pi}{2}, 6\pi] \end{cases}$$

and  $\tilde{\varphi} \in H^4(\Omega)$ . Let us notice that the corresponding solution  $(\psi, \varphi) = (0, \tilde{\varphi})$  of (27) for  $\varphi_0 = \tilde{\varphi}$  does not satisfies *Unique Continuation Principle* given in (28) for any  $\omega \subset \Omega$ .

- Dimension  $N = 2$ .

In dimension  $N = 2$ , let us denote  $O = \{(x, y) \in \mathbb{R}^2; x^2 + y^2 < e^{-3\pi}\}$ ,  $\Omega = \{(x, y) \in \mathbb{R}^2; x^2 + y^2 < 1\}$  and let us consider the following system :

$$\begin{cases} L\varphi = \Delta^2\varphi + 4V \cdot \nabla\Delta\varphi + B \Delta\varphi + a(x, y)\varphi = 0 & \text{in } \Omega , \\ \varphi = \Delta\varphi = 0 & \text{on } \partial\Omega , \end{cases} \quad (147)$$

where

$$a(x, y) = \begin{cases} \frac{1}{9(x^2 + y^2)^2} & \text{if } (x, y) \in \bar{\Omega} \setminus O , \\ 0 & \text{if } (x, y) \in O , \end{cases}$$

$$B(x, y) = \begin{cases} -\frac{46}{9(x^2 + y^2)} & \text{if } (x, y) \in \bar{\Omega} \setminus O , \\ 0 & \text{if } (x, y) \in O \end{cases}$$

and

$$V(x, y) = \begin{cases} (\frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2}) & \text{if } (x, y) \in \bar{\Omega} \setminus O , \\ (0, 0) & \text{if } (x, y) \in O . \end{cases}$$

We can deduce a solution of the previous system :

$$\tilde{\varphi}(x, y) = \begin{cases} \frac{1}{9} \sin(\log(\sqrt{x^2 + y^2})) + \sin(\frac{1}{3} \log(\sqrt{x^2 + y^2})) & \text{if } (x, y) \in \bar{\Omega} \setminus O, \\ \frac{8}{-9} & \text{if } (x, y) \in O \end{cases}$$

and  $\tilde{\varphi} \in H^4(\Omega)$ . At the end, we can deduce that for  $\varphi_0 = \tilde{\varphi}$ , the corresponding solution  $(\psi, \varphi) = (0, \tilde{\varphi})$  of (27) does not satisfies *Unique Continuation Principle* given in (28) for any  $\omega \subset \Omega$ .  $\square$

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