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Daniel Perez

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On the persistent homology of almost surely $C^0([0, t], \mathbb{R})$ stochastic processes

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Abstract

This paper investigates the properties of the persistence diagrams stemming from almost surely continuous random processes on $[0, t]$. We focus our study on two variables which together characterize the barcode : the number of points of the persistence diagram inside a rectangle $]-\infty, x] \times [x + \varepsilon, \infty[$, $N^{x, x+\varepsilon}$ and the number of bars of length $\geq \varepsilon$, N^ε . For processes with the strong Markov property, we show both of these variables admit a moment generating function and in particular moments of every order. Switching our attention to semimartingales, we show the asymptotic behaviour of N^ε and $N^{x, x+\varepsilon}$ as $\varepsilon \rightarrow 0$ and of N^ε as $\varepsilon \rightarrow \infty$. Finally, we study the repercussions of the classical stability theorem of barcodes and illustrate our results with some examples, most notably Brownian motion and empirical functions converging to the Brownian bridge.

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1 Introduction

1.1 State of the art

One of the questions of interest in the theory of persistent homology is the following: given a random function on some topological space X , what can we say about the barcode $\mathcal{B}(X)$ of this process? The study of the topology of (super)level-sets of random functions has been a subject of interest in probability theory for a long time [1, 2, 13, 17, 19, 26]. Most prominently for this paper, by Le Gall and Duquesne, who gave a construction of a tree from any continuous function $f : [0, 1] \rightarrow \mathbb{R}$ [10], and who interpreted different properties of these trees to give fine results about Lévy processes [11]. Picard later linked the upper-box dimension of these trees to the regularity of the function f [24]. In essence, these trees have proved to be a fruitful and natural setting from which many results regarding the topology of the superlevel sets of the function f stem [8, 9, 12, 20, 22].

A natural question is whether, or indeed how, these results are applicable to the persistent homology of stochastic processes. The answer turns out to be total: the study of barcodes and trees are completely equivalent in degree 0 of homology. This has been established in [22], in which a dictionary between H_0 -barcodes and the trees of Le Gall and Duquesne was constructed.

In this paper, we focus mainly on two barcode-related quantities, namely the number of bars in the barcode of f of length $\geq \varepsilon$, which we will denote N_f^ε and the number of bars in the barcode of f containing the interval $[x, x + \varepsilon]$, which we will denote $N_f^{x, x+\varepsilon}$. Whenever the function is implicit from the context, we may omit the subscript f .

With the dictionary established in [22], it is possible to interpret the results previously obtained by Picard in the context of trees. Most notably, Picard showed that the regularity of functions is closely related to their small-bar asymptotics.

Theorem 1.1 (Picard, §3 [24]). *Given a continuous function $f : [0, 1] \rightarrow \mathbb{R}$,*

$$\mathcal{V}(f) = \overline{\dim} T_f = \limsup_{\varepsilon \rightarrow 0} \frac{\log N_f^\varepsilon}{\log(1/\varepsilon)} \vee 1 \quad (1.1)$$

where $\overline{\dim}$ denotes the upper-box dimension, $a \vee b = \max\{a, b\}$, T_f denotes the tree constructed from f (cf. equation 2.19) and

$$\mathcal{V}(f) := \inf\{p \geq 1 \mid \|f\|_{p\text{-var}} < \infty\}, \quad (1.2)$$

where $\|\cdot\|_{p\text{-var}}$ denotes the (true) p -variation defined in definition 2.13.

Moreover, for processes which are self-similar in distribution, Picard [24] shows the almost sure small-bar asymptotics is closely related to this self-similarity (the results are shown for trees, but are immediately interpretable in terms of barcodes by the results of [22]). Throughout this paper we will adhere to the following convention regarding asymptotic relations.

Definition 1.2. Let f and g be any two functions. We define \sim to be the following equivalence relation

$$f(x) \sim g(x) \quad \text{as } x \rightarrow a \quad \iff \quad \lim_{x \rightarrow a} \frac{f(x)}{g(x)} = 1. \quad (1.3)$$

Theorem 1.3 (Picard, §3 [24]). *Let $X : [0, 1] \rightarrow \mathbb{R}$ be Brownian motion, a Lévy α -stable process or fractional Brownian motion, then, almost surely, there exists a constant C_H such that*

$$N_X^\varepsilon \sim \frac{C_H}{\varepsilon^{1/H}} \quad \text{as } \varepsilon \rightarrow 0, \quad (1.4)$$

where H is the self-similarity index, i.e. the H for which X satisfies $X_{\lambda t} = \lambda^H X_t$ for every $\lambda > 0$ and $t \geq 0$ in distribution.

Parallel to these developments, some results regarding the persistent homology of Brownian motion have also been provided by the topological data analysis (TDA) community. In particular, for Brownian motion B Chazal and Divol gave a formula for the distribution of the number of points $N_B^{x,y}$ lying inside a given rectangle $]-\infty, x] \times [y, \infty[$ in the persistence diagram of Brownian motion, $\text{Dgm}(B)$ [7].

Proposition 1.4 (Chazal, Divol, [7]). For $0 < x < y$, the distribution of $N_B^{x,y}$ is

$$\mathbb{P}(N_B^{x,y} \geq k) = \int_{\Sigma_{2k-1}} \psi(x, t_1) \psi(y-x, s_1) \psi(y-x, t_2) \cdots \psi(y-x, t_k) \prod_{i=1}^{k-1} dt_i \prod_{j=1}^{k-2} ds_j, \quad (1.5)$$

where Σ_{2k-1} denotes the domain bounded by the $(2k-1)$ -simplex and we note a vector in \mathbb{R}^{2k-1} by $(t_1, s_1, \dots, s_{k-2}, t_{k-1})$ and

$$\psi(x, t) := \frac{x}{\sqrt{2\pi t^3}} e^{-\frac{x^2}{2t}}. \quad (1.6)$$

Using this result Chazal and Divol established that $\mathbb{E}[N^{x,y}]$ was C^1 in x and y . In the context of our own results, we will show that this expectation is in fact an analytic function of x and y and give an explicit expression for $N^{x,x+\varepsilon}$ for $x > 0$ and $\varepsilon > 0$.

Other results regarding $\mathbb{E}[N^{x,y}]$ were obtained by Baryshnikov [3], who computed exactly $\mathbb{E}[N_{B^\mu}^{x,y}]$ for the Brownian motion with a strictly positive linear drift $B_t^\mu := \mu t + B_t$ for $\mu > 0$ over the whole ray $[0, \infty[$.

Proposition 1.5 (Baryshnikov, [3]). The expected value of $N_{B^\mu}^{x,x+\varepsilon}$ for $x > 0$ and $\mu > 0$ over the whole ray $[0, \infty[$ is given by

$$\mathbb{E}\left[N_{B^\mu}^{x,x+\varepsilon}\right] = \frac{1}{e^{2\mu\varepsilon} - 1}. \quad (1.7)$$

In particular, as $\varepsilon \rightarrow 0$, the following asymptotic relation holds

$$\mathbb{E}\left[N_{B^\mu}^{x,x+\varepsilon}\right] \sim \frac{1}{2\mu\varepsilon} - \frac{1}{2} + \frac{1}{6}\mu\varepsilon + O(\mu^3\varepsilon^3) \quad \text{as } \varepsilon \rightarrow 0. \quad (1.8)$$

In this paper, we will be concerned with almost surely C^0 processes, but it is noteworthy that the study of smooth random fields (and their topology) is currently an open field of research in probability theory. A good introduction to the smooth setting is provided by the celebrated books of Adler and Taylor [1], and that of Azaïs and Wschebor [2].

Whenever possible and necessary, we give the definitions and most important results necessary to make the proofs in this paper self-contained. However, for the sake of brevity, we do not introduce *all* the probabilistic concepts necessary for this paper (most notably, we do not define Brownian motion or stochastic processes, filtrated probability spaces, nor stopping times and the strong Markov property). We kindly refer the reader unfamiliar with these concepts to Le Gall's [17] and Revuz and Yor's [25] books on stochastic calculus for a comprehensive introduction to the subject.

1.2 Our contribution

Our contribution can be summarized along the following points.

Theorem 1. Let X be a non-constant stochastic process on $[0, t]$ defined on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$ allowing an almost surely continuous modification and satisfying the strong Markov property. If we denote

$$R_t := \sup_{[0,t]} X - \inf_{[0,t]} X, \quad (1.9)$$

then,

$$\mathbb{P}(N_X^\varepsilon \geq k) \leq \mathbb{P}(R_t \geq \varepsilon)^{2(k-1)}. \quad (1.10)$$

Denote ε^* the infimal value such that for all $\varepsilon > \varepsilon^*$, $\mathbb{P}(R_t \geq \varepsilon) < 1$. Then, for every $\varepsilon > \varepsilon^*$ and every $x \in \mathbb{R}$, all the moments of the random variables N^ε and $N_X^{x, x+\varepsilon}$ are finite and their moment generating functions $M(\lambda)$ converge uniformly and absolutely for every $\lambda \in \mathbb{C}$ such that $\operatorname{Re}(\lambda) < -2\log(\mathbb{P}(R_t \geq \varepsilon))$.

This theorem answers the question of the existence of moments (and their quantification) for the quantities N^ε and $N_X^{x, x+\varepsilon}$ in the context of almost surely continuous Markov processes. It also allows us to show the behaviour of large bars of the barcode of such a process in expectation.

Corollary 1. Under the same assumptions,

$$\mathbb{E}[N_X^\varepsilon] \sim \mathbb{P}(R_t \geq \varepsilon) \quad \text{as } \varepsilon \rightarrow \infty. \quad (1.11)$$

Having partially answered the question of large bars in this very general context, we switch our attention to the behaviour of small bars. Indeed, as the following result shows, the latter is very regular and closely related to the nature of the noise at hand.

Theorem 2. Let $X = M + A$ be a continuous semimartingale on $[0, t]$ and suppose that for $s \geq 1$

$$\mathbb{E} \left[[M]_t^{s/2} + \left(\int_0^t |dA|_\tau \right)^s \right] < \infty, \quad (1.12)$$

where $[M]_t$ denotes the quadratic variation of M (cf. definition 3.2) and $\int_0^t |dA|_\tau$ the total variation of A . Then, in $L^s(\Omega)$,

$$\begin{aligned} N_X^{x, x+\varepsilon} &\sim \frac{L_X^x(t)}{2\varepsilon} + O(1) \quad \text{as } \varepsilon \rightarrow 0 \\ N_X^\varepsilon &\sim \frac{[X]_t}{2\varepsilon^2} + O(\varepsilon^{-3/2}) \quad \text{as } \varepsilon \rightarrow 0, \end{aligned}$$

where $L_X^x(t)$ denotes the local time of X on $[0, t]$ at x .

Remark 1.6. The condition expressed in equation 1.12 is a technical one and is often employed in the theory of stochastic integration (cf. [25]). It is noteworthy that this condition is satisfied for a very wide range of processes. In fact, the space \mathcal{S}^s of continuous semimartingales $X = M + A$ for which the quantity

$$\|X\|_{\mathcal{S}^s} := \left\| [M]_t^{1/2} + \int_0^t |dA|_\tau \right\|_{L^s(\Omega)} < \infty \quad (1.13)$$

is a vector space that we may equip with the seminorm $\|\cdot\|_{\mathcal{S}^s}$. When $s > 1$, the quotient of \mathcal{S}^s by processes indistinguishable from zero is a Banach space which contains the space of continuous martingales such that $\mathbb{E} \left[\|X\|_{L^\infty([0, t])}^s \right] < \infty$.

For particular instances of local martingales, we have a full description of the barcode of these processes, which depends on their quadratic variation.

Theorem 3. For any continuous local martingale M on $[0, t]$ having deterministic and strictly increasing quadratic variation $[M]_t$ such that $[M]_\infty = \infty$,

$$\begin{aligned} \mathbb{E}[N_M^\varepsilon] &= 4 \sum_{k \geq 1} (2k-1) \operatorname{erfc} \left(\frac{(2k-1)\varepsilon}{\sqrt{2[M]_t}} \right) - k \operatorname{erfc} \left(\frac{2k\varepsilon}{\sqrt{2[M]_t}} \right) \\ &= \frac{[M]_t}{2\varepsilon^2} + \frac{2}{3} + 2 \sum_{k \geq 1} (2(-1)^k - 1) \frac{e^{-\pi^2 k^2 [M]_t / 2\varepsilon^2} [M]_t}{\varepsilon^2} \left[1 + \frac{\varepsilon^2}{\pi^2 k^2 [M]_t} \right]. \end{aligned}$$

Moreover on $[0, t]$ and for $x > 0$,

$$\begin{aligned} \mathbb{E}\left[N_M^{x, x+\varepsilon}\right] &= \sum_{k=1}^{\infty} \operatorname{erfc}\left(\frac{x + (2k-1)\varepsilon}{\sqrt{2[M]t}}\right) \\ &\sim \frac{1}{2\varepsilon} \int_0^{[M]t} \varphi(x, s) ds + \sum_{k \geq 0} \frac{4(-2)^k (2^{2k+1} - 1) \zeta(2k+2)}{\pi^{2k+2}} \left[\frac{\partial^k}{\partial s^k} \Big|_{s=[M]t} \varphi(x, s) \right] \varepsilon^{2k+1} \text{ as } \varepsilon \rightarrow 0, \end{aligned}$$

where $\varphi(x, t)$ is the density of a centered Gaussian random variable of variance t and ζ denotes the Riemann zeta function.

Remark 1.7. Both expressions for $\mathbb{E}[N_M^\varepsilon]$ in the above theorem converge (this can be easily shown by recalling that $\operatorname{erfc}(x) \sim \frac{e^{-x^2}}{x\sqrt{\pi}}$ as $x \rightarrow \infty$ and applying the ratio test). The first expression converges very fast for large ε (this is easily seen given the decay properties of the complementary error function erfc) and the second one yields extremely fast convergence for small ε . Similarly, the first expression for $\mathbb{E}\left[N_M^{x, x+\varepsilon}\right]$ converges quickly for large ε . On the other hand, the second expression can only be considered as an asymptotic expansion for small ε .

This theorem allows us to explain exactly the experimental observations we made through simulation regarding the barcode of Brownian motion (*cf.* figures 1 and 2).

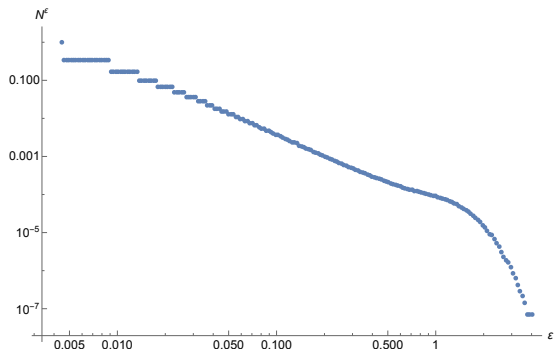


Figure 1: Numerical simulation of $\mathbb{E}[N_B^\varepsilon]$ using Rademacher walks.

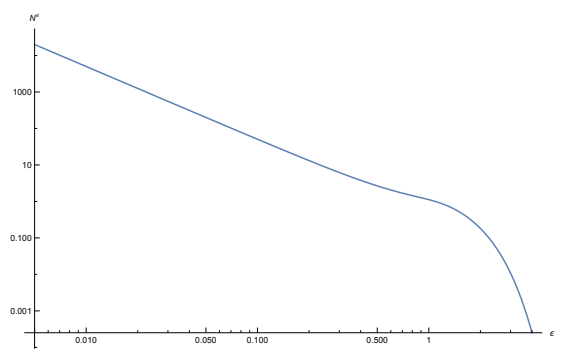


Figure 2: $\mathbb{E}[N_B^\varepsilon]$ as calculated using theorem 3.

We use the classical stability theorem of barcodes [6] and complete our description of random processes by examining sequences converging to the processes above, which constitute universal limits for many random processes (most notably, random walks and empirical processes).

Theorem 4. Let (M, d) be a compact Polish metric space and let X be an almost surely continuous stochastic process on M , defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $(X_n)_{n \in \mathbb{N}}$ be any sequence of continuous stochastic processes defined on the same probability space and suppose there exists a $1 \leq p \leq \infty$ such that

$$\delta_n := \|X - X_n\|_{L^p(\Omega, L^\infty(M, \mathbb{R}))} \xrightarrow{n \rightarrow \infty} 0. \quad (1.14)$$

Then, for every $\varepsilon \geq 2a\delta_n$,

$$|N_{X_n}^\varepsilon - N_X^\varepsilon| \leq N_X^{\varepsilon-2a\delta_n} - N_X^{\varepsilon+2a\delta_n}, \quad (1.15)$$

with probability $\geq 1 - \frac{1}{a^p}$. Suppose further that $\mathbb{E}[N_X^\varepsilon]$ is continuous in ε . Then,

$$\mathbb{E}\left[|N_{X_n}^\varepsilon - N_X^\varepsilon| \mid \|X - X_n\|_{L^\infty(M, \mathbb{R})} \leq a\delta_n\right] \leq \omega_\varepsilon(2a\delta_n). \quad (1.16)$$

where $\omega_\varepsilon(\delta) := \mathbb{E}\left[N_X^{\varepsilon-\delta} - N_X^{\varepsilon+\delta}\right]$. Moreover,

$$\mathbb{P}\left(|N_{X_n}^\varepsilon - N_X^\varepsilon| \geq k\right) \leq \frac{\omega_\varepsilon(2a\delta_n)}{k} + \frac{1}{a^p} \quad \text{and} \quad \mathbb{P}\left(|N_{X_n}^\varepsilon - N_X^\varepsilon| \geq k, \|X - X_n\|_{L^\infty(M, \mathbb{R})} \leq a\delta_n\right) \leq \frac{\omega_\varepsilon(2a\delta_n)}{k}. \quad (1.17)$$

If $p = \infty$ and still assuming $\mathbb{E}[N_X^\varepsilon] < \infty$ and is continuous, for any $\varepsilon \geq 2\delta_n$,

$$N_{X_n}^\varepsilon \xrightarrow[n \rightarrow \infty]{L^1} N_X^\varepsilon \quad \text{and} \quad \mathbb{E}[|N_X^\varepsilon - N_{X_n}^\varepsilon|] \leq \omega_\varepsilon(2\delta_n).$$

With analogous hypotheses, the same statement holds for $N_X^{x, x+\varepsilon}$.

Remark 1.8. The condition that $\mathbb{E}[N^\varepsilon]$ is often satisfied. By virtue of theorem 1 and the dominated convergence theorem, this continuity condition is satisfied if for all $\varepsilon > \varepsilon^*$ N_X^ε almost surely does not have a discontinuity at ε . That is, if the process $\varepsilon \mapsto N^\varepsilon$ is continuous in probability. More concretely, the condition also holds as soon as the distributions of the stopping times constructed in definition 2.11 are themselves continuous in ε . Some immediate examples of processes where the continuity condition is satisfied are continuous local martingale M with deterministic and strictly increasing quadratic variation with $[M]_t < \infty$, as implied by theorem 3.

Finally, we use the previous results to give statements about well-known processes, such as Brownian motion, Itô processes and some limiting processes (most notably the Fourier decomposition of Brownian motion and the convergence of empirical processes to the Brownian bridge), cf. section 5.

2 Some generalities about H_0 homology and trees

Let us briefly recall the construction of a tree from a continuous function $f : M \rightarrow \mathbb{R}$ detailed in [22].

Definition 2.1. Let M denote a connected, locally path-connected, compact topological space, $f : M \rightarrow \mathbb{R}$ be a continuous function and let $x, y \in X$, then the function

$$d_f(x, y) := f(x) + f(y) - 2 \sup_{\gamma: x \rightarrow y} \min_{s \in [0, t]} f(\gamma(s)), \quad (2.18)$$

where the supremum runs over all paths $\gamma : [0, 1] \rightarrow M$ is a pseudo-distance on M and the quotient metric space

$$T_f := M / \{x \sim y \iff d_f(x, y) = 0\} \quad (2.19)$$

equipped with the distance d_f is a rooted \mathbb{R} -tree, whose root coincides with the image in T_f of the point in $[0, 1]$ at which f achieves its infimum.

Definition 2.2. Let M denote a connected, locally path-connected, compact topological space and $f : M \rightarrow \mathbb{R}$ be a continuous function. We denote $\pi_f : M \rightarrow T_f$ the canonical projection.

The tree T_f has the particularity that its branches correspond to connected components of the superlevel sets of f , as illustrated by figure 3. To define N^ε on this tree, it is first necessary to introduce the so-called ε -simplified or ε -trimmed tree of T_f . This object is obtained by “giving a haircut” of length ε to T_f . More precisely, if we define a function $h : T_f \rightarrow \mathbb{R}$ which to a point $\tau \in T_f$ associates the distance from τ to the highest leaf above τ with respect to the filtration on T_f induced by f , then

Definition 2.3. Let $\varepsilon \geq 0$. An ε -trimming or ε -simplification of T_f is the metric subspace of T_f defined by

$$T_f^\varepsilon := \{\tau \in T_f \mid h(\tau) \geq \varepsilon\} \quad (2.20)$$

With this definition, we can interpret N^ε geometrically as being equal to the number of leaves of T_f^ε . The reason for this is explicated in [22]. The idea is that, starting from T_f , we can look at the longest branch (starting from the root) of T_f . This branch corresponds to the longest bar of $\mathcal{B}(f)$, since branches of T_f correspond to connected components of the superlevel sets of f . Next, we erase this longest branch and, on the remaining (rooted) forest, look for the next longest branch. This will be the second longest bar of the barcode. Proceeding iteratively in this way, we retrieve $\mathcal{B}(f)$. An illustration of this can be found in figure 4.

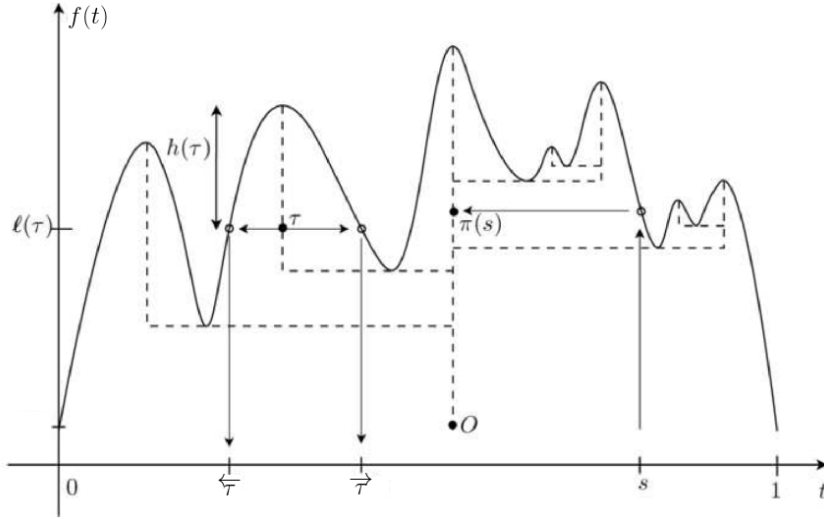


Figure 3: A function $f : [0, 1] \rightarrow \mathbb{R}$ and its associated tree T_f in dashed lines. Visually, T_f can be obtained by imagining that we put glue everywhere along the bottom of the curve, and then collapse the curve along the t -axis of the drawing above. Key quantities we have introduced such as a typical node of the tree τ , the value of its height function $h(\tau)$ and the projection $\pi : [0, 1] \rightarrow T_f$ are depicted, as well as the root of the tree O . Finally, some quantities relating to the preimages of nodes $\tau \in T_f$ from definition 2.7 are also shown.

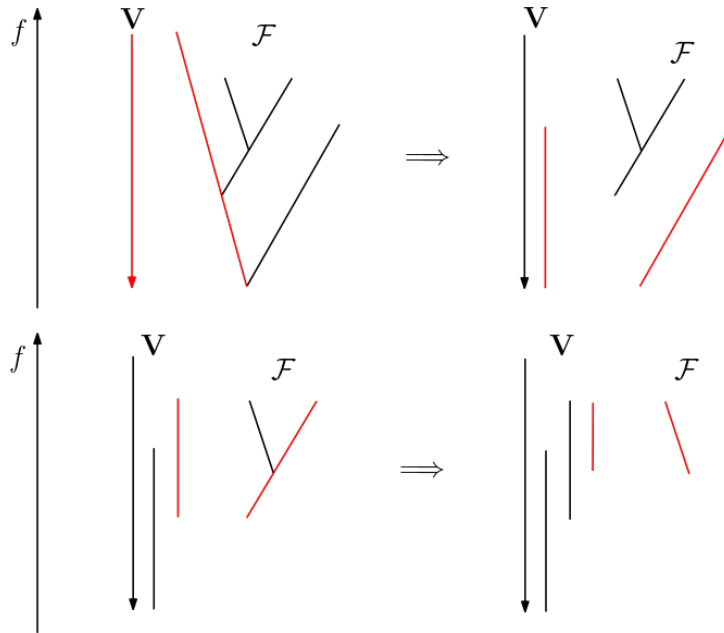


Figure 4: A depiction of the first steps of the algorithm which assigns a barcode $\mathcal{B}(f)$ to a tree T_f . We start by picking the longest branch in the tree, starting from the root. Its endpoints give us the endpoint values of the longest interval in the barcode (since this is the first bar and the interval is connected, we may extend this bar to $-\infty$). Next, we erase the branch on the tree, so that we are left with a forest \mathcal{F} and pick the largest branch in the remaining forest. We retain its endpoints, yielding the second bar, and erase the corresponding branch. We repeat this procedure for all subsequent branches until we have erased the entire tree.

Proposition 2.4. The number of bars containing x persisting through $x + \varepsilon$ (*i.e.* the bars containing the interval $[x, x + \varepsilon]$),

$$N_f^{x, x+\varepsilon} := \text{rank}(H_0(\{f \geq x + \varepsilon\} \rightarrow \{f \geq x\})) = \#\{\tau \in T_f \mid f(\tau) = x \text{ and } h(\tau) \geq \varepsilon\}. \quad (2.21)$$

Proof. By the results of [22], there is a correspondence between T_f and the barcode $\mathcal{B}(f)$. Every

element of the set

$$\{\tau \in T_f \mid f(\tau) = x \text{ and } h(\tau) \geq \varepsilon\}, \quad (2.22)$$

has a corresponding image in one and only one bar of the barcode which contains the interval $[x, x + \varepsilon]$. Conversely, for every bar in the barcode containing the interval $[x, x + \varepsilon]$, there is a unique image on the tree at height x , τ , and, since the bar persists a length $\geq \varepsilon$ after x , $h(\tau) \geq \varepsilon$. ■

Remark 2.5 (Link with traditional persistence diagrams and sublevel set filtrations). In the tree formalism, it is typical to consider **superlevel** filtrations as opposed to sublevel ones, as is typically done in traditional persistence theory. Considering one or the other poses no problem for us, as one can pass from one filtration to the other by switching f into $-f$. Diagrams (in the sense of collections of points (b, d) of moments of birth b and death d of bars of the barcode) of a filtration by superlevel sets lie *below* the diagonal, as the moment of birth occurs *higher* than the moment of death. Given the diagram of $-f$ as computed with the *superlevel* filtration (using for instance the tree), we can retrieve the diagram associated to *sublevels* of f by sending each point in the diagram (of the superlevel filtration of $-f$)

$$(b, d) \mapsto (-d, -b). \quad (2.23)$$

For most results, this subtlety makes little to no difference, as most of the examples considered (typically Brownian motion B) are processes which are symmetric, *i.e.* $B_t = -B_t$ in distribution for all t . This is notably the case for the results obtained by Chazal and Divol [7]. However, when considering asymmetric processes and for results where specific formulæ are given (such as in theorem 3), it is necessary to keep in mind the transformation above to correct the formulæ accordingly wherever necessary if one is considering sublevel sets.

Finally, with respect to the superlevel (resp. sublevel) set filtration, we will henceforth **always** consider that the infinite bars of the barcode are capped at $\inf(f)$ (resp. $\sup(f)$). Equivalently, in terms of barcodes we will always consider for any bar

$$b = b \cap [\inf f, \sup f]. \quad (2.24)$$

On a tree T_f , we can define a notion of integration by defining the unique atomless Borel measure λ which is characterized by the property that every geodesic segment on T_f has measure equal to its length. Formally, we can express λ in two ways [24]

$$\lambda = \int_{\mathbb{R}} dx \sum_{\substack{\tau \in T_f \\ f(\tau) = x}} \delta_{\tau} \quad \text{and} \quad \lambda = \int_0^{\infty} d\varepsilon \sum_{\substack{\tau \in T_f \\ h(\tau) = \varepsilon}} \delta_{\tau}. \quad (2.25)$$

Proposition 2.6.

$$\lambda(T_f^\varepsilon) = \int_{\varepsilon}^{\infty} N_f^a da = \int_{\mathbb{R}} N_f^{x, x+\varepsilon} dx \quad (2.26)$$

Proof. By using the second identity for λ ,

$$\lambda(T_f^\varepsilon) = \int_{\varepsilon}^{\infty} N_f^a da, \quad (2.27)$$

since every sum in the second expression is finite for all $\varepsilon > 0$ and has N^ε terms. Writing it using the first identity, we must restrict the sum in the identity to

$$\sum_{\substack{\tau \in T_f \\ f(\tau) = x \\ h(\tau) \geq \varepsilon}} \delta_{\tau}, \quad (2.28)$$

which is finite for all $\varepsilon > 0$. There are exactly $N_f^{x, x+\varepsilon}$ terms in this sum, therefore

$$\lambda(T_f^\varepsilon) = \int_{\mathbb{R}} N_f^{x, x+\varepsilon} dx. \quad (2.29)$$

■

2.1 Persistent homology of processes on $[0, t]$

Consider now $f : [0, t] \rightarrow \mathbb{R}$. Given the total order on \mathbb{R} , the preimages of T_f by $\pi_f : [0, t] \rightarrow T_f$ inherit a natural order structure. This allows us to define

Definition 2.7. The **right** (resp. **left**) **preimage** of $\tau \in T_f$ by π_f is

$$\vec{\tau} := \sup \pi_f^{-1}(\tau) \quad (\text{resp. } \overleftarrow{\tau} := \inf \pi_f^{-1}(\tau)). \quad (2.30)$$

See figure 3 for a depiction of these preimages.

Remark 2.8. One can have $\overleftarrow{\tau} = \vec{\tau}$.

Whenever $f : [0, t] \rightarrow \mathbb{R}$, we can compute N_f^ε is by counting the number of times we go up by at least ε from a local minimum and down by at least ε from a local maximum. This idea can be formalized by the following sequence, originally introduced by Neveu *et al.* [20].

Definition 2.9. Setting $S_0^\varepsilon = T_0^\varepsilon = 0$, we define a sequence of times by induction

$$\begin{aligned} T_{i+1}^\varepsilon &:= \inf \left\{ s \geq S_i^\varepsilon \mid \sup_{[S_i^\varepsilon, s]} f - f(s) > \varepsilon \right\} \\ S_{i+1}^\varepsilon &:= \inf \left\{ s \geq T_{i+1}^\varepsilon \mid f(s) - \inf_{[T_{i+1}^\varepsilon, s]} f > \varepsilon \right\} \end{aligned}$$

Lemma 2.10. If $k \geq 2$, and $f : [0, t] \rightarrow \mathbb{R}$ is continuous,

$$N_f^\varepsilon \geq k \iff S_{k-1}^\varepsilon \leq t. \quad (2.31)$$

Proof. ($N^\varepsilon \geq k \implies S_{k-1}^\varepsilon \leq t$): We start by selecting all bars of length $\geq \varepsilon$, they are of the form $[b_i, a_i]$. For each bar, pick the point $a_i - \varepsilon$ and look at its corresponding node $\tau \in T_f$ (this is equivalent to considering the set of nodes τ_i on the tree such that $h(\tau_i) = \varepsilon$). Next, let us order the bars by the value of their left preimages of the τ_i by virtue of the total order on \mathbb{R} . By construction, since the τ_i all stem from different connected components of the superlevel set the $(\overleftarrow{\tau}_i, \vec{\tau}_i)$ satisfy

$$\vec{\tau}_i \leq \vec{\tau}_j \quad \text{for } i < j. \quad (2.32)$$

Since $N^\varepsilon \geq k$, there are at least $(k-1)$ right preimages and left preimages by π_f of leaves of T_f^ε stemming from the first $(k-1)$ bars, which we denote $\{(\overleftarrow{\tau}_i, \vec{\tau}_i)\}_{1 \leq i \leq k-1}$ satisfying

$$\vec{\tau}_1 = T_1^\varepsilon \leq S_1^\varepsilon \leq \vec{\tau}_2 = T_2^\varepsilon \leq \dots \leq \vec{\tau}_{k-1} = T_{k-1}^\varepsilon. \quad (2.33)$$

Note that $\vec{\tau}_1 > 0$ as soon as $k \geq 2$ and $\varepsilon > 0$. Since $N^\varepsilon \geq k$, there at least an extra bar unaccounted for. Consider taking the left preimages of the nodes in T_f corresponding to the bottom and top of this bar and note these preimages t_b and t_t respectively. Note that $t \geq t_b \geq T_{k-1}^\varepsilon$, as otherwise we would have $t_b < \vec{\tau}_i$ for some $i \leq k-1$, entailing that $\pi_f(t_b)$ belongs to the connected subtree $\pi_f([0, \vec{\tau}_i])$, which contradicts that $\pi_f(t_b)$ belongs to the k th bar, as this bar is by construction disjoint from $\pi_f([0, \vec{\tau}_i])$. Similarly $t \geq t_t \geq T_{k-1}^\varepsilon$. Moreover, $d_f(t_t, t_b) > \varepsilon$, which shows that $S_{k-1}^\varepsilon \in [t_b, t_t]$ and in particular that $S_{k-1}^\varepsilon \leq t$.

($N^\varepsilon \geq k \iff S_{k-1}^\varepsilon \leq t$): Since $S_{k-1}^\varepsilon > T_{k-1}^\varepsilon = \vec{\tau}_{k-1}$, there are at least $(k-1)$ distinct bars. We easily check that, by definition, $d_f(S_{k-1}^\varepsilon, T_{k-1}^\varepsilon) > 0$, implying that $\pi_f(S_{k-1}^\varepsilon)$ and $\pi_f(T_{k-1}^\varepsilon)$ lie on different branches of T_f , and therefore S_{k-1}^ε is a preimage of a distinct bar of length $\geq \varepsilon$, implying $N^\varepsilon \geq k$. ■

Definition 2.11. Let $f : [0, t] \rightarrow \mathbb{R}$ be a continuous function. Setting $US_0^{x,\varepsilon} = UT_0^{x,\varepsilon} = 0$, we define a sequence of times recursively

$$\begin{aligned} UT_{i+1}^{x,\varepsilon} &:= \inf \left\{ s \geq US_i^{x,\varepsilon} \mid f(s) \leq x \right\} \\ US_{i+1}^{x,\varepsilon} &:= \inf \left\{ s \geq UT_{i+1}^{x,\varepsilon} \mid f(s) \geq x + \varepsilon \right\}. \end{aligned}$$

The maximum i for which $US_i^{x,\varepsilon} \leq t$ is called the **number of upcrossings by f from x to $x + \varepsilon$** and we denote it $U_f^{x,x+\varepsilon}$. Similarly, setting $DS_0^{x,\varepsilon} = DT_0^{x,\varepsilon} = 0$ and defining

$$DT_{i+1}^{x,\varepsilon} := \inf \left\{ s \geq DS_{i+1}^{x,\varepsilon} \mid f(s) \leq x \right\}$$

$$DS_{i+1}^{x,\varepsilon} := \inf \left\{ s \geq DT_i^{x,\varepsilon} \mid f(s) \geq x + \varepsilon \right\},$$

we can define the **number of downcrossings by f from x to $x + \varepsilon$** and denote it $D_f^{x,x+\varepsilon}$ as the maximum i for which $DT_i^{x,\varepsilon} \leq t$.

Proposition 2.12. Let $f : [0, t] \rightarrow \mathbb{R}$ be a continuous function. Then,

$$N_f^{x,x+\varepsilon} = U_f^{x,x+\varepsilon} \vee D_f^{x,x+\varepsilon} \leq D_f^{x,x+\varepsilon} + 1. \quad (2.34)$$

If $x \geq 0$, and $f(0) = 0$, $N_f^{x,x+\varepsilon} = U_f^{x,x+\varepsilon}$.

Proof. Each bar alive at $x + \varepsilon$ persisting until x (seen as a branch of T_f) admits a left and right preimage (which can be sometimes equal), this shows $N^{x,x+\varepsilon} \geq U_f^{x,x+\varepsilon} \vee D_f^{x,x+\varepsilon}$. Conversely, every downcrossing (resp. upcrossing) generates a distinct homology class alive at $x + \varepsilon$ persisting until x , which shows $U_f^{x,x+\varepsilon} \vee D_f^{x,x+\varepsilon} \geq N^{x,x+\varepsilon}$. The last inequality is a consequence of the fact that, by definition $\left| U_f^{x,x+\varepsilon} - D_f^{x,x+\varepsilon} \right| \leq 1$. Finally, if $x \geq 0$, and $f(0) = 0$, by continuity of f , there is a bijective correspondence between upcrossings and the bars alive at $x + \varepsilon$ persisting until x . \blacksquare

With the above, we are now ready to prove theorem 1.

Proof of theorem 1. For simplicity and without loss of generality, let us set $t = 1$. The probability $\mathbb{P}(N^\varepsilon \geq k)$ can be written in terms of the stopping times T_i^ε and S_i^ε and their increments by the (strong) Markov property of X . By lemma 2.10, $\mathbb{P}(N^\varepsilon \geq k) = \mathbb{P}(S_{k-1}^\varepsilon \leq 1)$ for $k \geq 2$, so

$$\mathbb{P}(N^\varepsilon \geq k) = \int_{\Sigma_{2k-2}} \mathbb{P}(T_1^\varepsilon = t_1) \mathbb{P}(S_1 = s_1 \mid T_1 = t_1) \cdots \mathbb{P}(S_{k-1} = s_{k-1} \mid T_{k-1} = t_{k-1}) \prod_{i=1}^{k-1} ds_i dt_i. \quad (2.35)$$

where, abusing the notation, the $\mathbb{P}(S_i = s_i \mid T_i = t_i)$ and $\mathbb{P}(T_i = t_i \mid S_i = s_i)$ denote the corresponding conditional densities and where Σ_{2k-2} denotes the simplex

$$\Sigma_{2k-2} := \left\{ (t_1, s_1, \dots, s_{k-1}) \in \mathbb{R}^{2k-1} \mid 0 \leq t_1 \leq s_2 \leq \dots \leq s_{k-1} \leq 1 \right\}. \quad (2.36)$$

By the definition of these stopping times we know that

$$\mathbb{P}(s \leq T_i^\varepsilon \leq t \mid S_{i-1}^\varepsilon = s) = \mathbb{P} \left(\sup_{\tau \in [s,t]} \left[\sup_{[s,\tau]} X - X_\tau \right] \geq \varepsilon \right)$$

$$\mathbb{P}(t \leq S_i^\varepsilon \leq s \mid T_i^\varepsilon = t) = \mathbb{P} \left(\sup_{\tau \in [t,s]} \left[X_\tau - \inf_{[t,\tau]} X \right] \geq \varepsilon \right)$$

Both of these expressions are dominated by $\mathbb{P}(R_1 \geq \varepsilon)$. Indeed,

$$\mathbb{P} \left(\sup_{\tau \in [s,s']} \left[\sup_{[s,\tau]} X - X_\tau \right] \geq \varepsilon \right) \leq \mathbb{P} \left(\sup_{s \in [0,1]} \left[\sup_{[0,s]} X - X_s \right] \geq \varepsilon \right) \quad (2.37)$$

and the supremum on the right hand side is dominated by R_1 . Thus,

$$\mathbb{P} \left(\sup_{\tau \in [s,s']} \left[\sup_{[s,\tau]} X - X_\tau \right] \geq \varepsilon \right) \leq \mathbb{P}(R_1 \geq \varepsilon). \quad (2.38)$$

Integrating the expression as a nested integral of $\mathbb{P}(N^\varepsilon \geq k)$, the variable s_{k-1} between t_{k-1} and 1

$$\begin{aligned} \mathbb{P}(N^\varepsilon \geq k) &= \int_{\Sigma_{2k-3}} \mathbb{P}(T_1^\varepsilon = t_1) \mathbb{P}(S_1^\varepsilon = s_1 | T_1^\varepsilon = t_1) \cdots \mathbb{P}(t_{k-1} \leq S_{k-1}^\varepsilon \leq 1 | T_{k-1}^\varepsilon = t_{k-1}) dt_{k-1} \prod_{i=1}^{k-2} ds_i dt_i \\ &\leq \mathbb{P}(R \geq \varepsilon) \int_{\Sigma_{2k-3}} \mathbb{P}(T_1^\varepsilon = t_1) \mathbb{P}(S_1^\varepsilon = s_1 | T_1^\varepsilon = t_1) \cdots \mathbb{P}(T_{k-1}^\varepsilon = t_{k-1} | S_{k-2}^\varepsilon = s_{k-2}) dt_{k-1} \prod_{i=1}^{k-2} ds_i dt_i \end{aligned}$$

Carrying out the subsequent $2k - 3$ integrations and by repeated use of the inequality given in equation 2.38, we obtain the result

$$\mathbb{P}(N^\varepsilon \geq k) \leq \mathbb{P}(R_1 \geq \varepsilon)^{2k-2}. \quad (2.39)$$

By the hypothesis of the theorem, for all $\varepsilon \geq \varepsilon^*$, $\mathbb{P}(R_1 \geq \varepsilon) < 1$ so the above condition guarantees the summability (and absolute and uniform convergence) of the series $\mathbb{E}[e^{\lambda N^\varepsilon}]$ on the half plane $\text{Re}(\lambda) < -2 \log(\mathbb{P}(R_1 \geq \varepsilon))$. Moreover the same summability condition holds for $N^{x, x+\varepsilon}$, since the latter is dominated by N^ε . ■

2.2 *A priori* estimates on the asymptotic behaviour of small and large bars

If f is a continuous function, the asymptotic behaviour of N^ε is closely related to the regularity of f . For functions $f : [0, t] \rightarrow \mathbb{R}$, the correct notion of regularity to look at is the p -variation, of which we recall the definition.

Definition 2.13. Let $f : [0, t] \rightarrow \mathbb{R}$ be a function. The (**true**) p -variation of f is defined as

$$\|f\|_{p\text{-var}} := \sup_{\mathcal{P}} \left[\sum_{t_k \in \mathcal{P}} |f(t_k) - f(t_{k-1})|^p \right]^{1/p} \quad (2.40)$$

where the supremum ranges over all finite partitions of $[0, t]$.

The p -variation can be used to infer something about the asymptotic behaviour of N_f^ε , as shown by the following theorem.

Theorem 2.14 (Picard, §3 [24]). *Given a continuous function $f : [0, 1] \rightarrow \mathbb{R}$,*

$$\mathcal{V}(f) = \overline{\dim} T_f = \limsup_{\varepsilon \rightarrow 0} \frac{\log N^\varepsilon}{\log(1/\varepsilon)} \vee 1 \quad (2.41)$$

where $\overline{\dim}$ denotes the upper-box dimension, $a \vee b = \max\{a, b\}$,

$$\mathcal{V}(f) := \inf\{p \geq 1 \mid \|f\|_{p\text{-var}} < \infty\}. \quad (2.42)$$

Remark 2.15. A general version of this theorem, applicable on more general metric spaces exists [22].

Corollary 2.16. *For any deterministic function $f : [0, 1] \rightarrow \mathbb{R}$ and for every $\delta > 0$,*

$$N_f^\varepsilon = O(\varepsilon^{-\mathcal{V}(f)-\delta}) \quad \text{as } \varepsilon \rightarrow 0. \quad (2.43)$$

Having characterized the behaviour of N^ε for small ε , we can characterize the behaviour of large bars (in expectation) using theorem 1.

Proof of corollary 1. Note $p_\varepsilon = \mathbb{P}(R_t \geq \varepsilon)$. From theorem 1, we deduce,

$$p_\varepsilon \leq \mathbb{E}[N^\varepsilon] \leq p_\varepsilon + \frac{p_\varepsilon^2}{1 - p_\varepsilon^2} \implies \mathbb{E}[N^\varepsilon] \sim p_\varepsilon \text{ as } \varepsilon \rightarrow \infty. \quad (2.44)$$

■

3 Continuous semimartingales, local times and asymptotic behaviour of barcodes

In the previous section, we quantified the asymptotics of $N^{x, x+\varepsilon}$ and N^ε solely based on the regularity of the functions considered. For stochastic processes on $[0, t]$, we can refine this analysis by focusing on (continuous) semimartingales. Semimartingales constitute the largest class of processes with respect to which the Itô and Stratonovich integrals can be defined. In other words, they are a class of processes rich enough to be worthy of particular attention. For a comprehensive introduction to these objects and the probabilistic concepts included in this section, we kindly refer the reader to classical references on stochastic calculus [17, 25].

Definition 3.1 (Local martingales and semimartingales). Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a filtered probability space and let $\mathcal{F}_* := (\mathcal{F}_t)_{t \geq 0}$ be the filtration of \mathcal{F} . An \mathcal{F}_* -adapted process $M : [0, \infty) \times \Omega \rightarrow \mathbb{R}$ is an \mathcal{F}_* -**local martingale** if there exists a sequence of stopping times τ_k such that

1. The τ_k are a.s. increasing, *i.e.* $\mathbb{P}(\tau_k < \tau_{k+1}) = 1$;
2. The τ_k are a.s. divergent, *i.e.* $\mathbb{P}(\lim_{k \rightarrow \infty} \tau_k = \infty) = 1$;
3. The process $M_{t \wedge \tau_k}$ is an \mathcal{F}_* -martingale, *i.e.* for all $s < t$,

$$\mathbb{E}[M_{t \wedge \tau_k} | \mathcal{F}_s] = M_{s \wedge \tau_k}. \quad (3.45)$$

A process $(X_t)_{t \geq 0}$ is a **continuous semimartingale** if it can be written in the form

$$X_t = M_t + A_t \quad (3.46)$$

where A_t is a process of finite variation and M_t is a continuous local martingale.

Throughout this section, we will use two key concepts stemming from the theory of stochastic integration: the local time of a (continuous) semimartingale and the quadratic variation. The latter is the easiest to define, given our introduction of the **true** quadratic variation in the previous section.

3.1 Quadratic variation and *a priori* bounds

Definition 3.2 (Quadratic variation, Theorem 4.9 [17]). Suppose that $(X_t)_t$ is a \mathbb{R} -valued stochastic process indexed by \mathbb{R}_+ . The **quadratic variation of X** is the process defined as

$$[X]_t := \lim_{\|\mathcal{P}\| \rightarrow 0} \sum_{k=1}^n (X_{t_k} - X_{t_{k-1}})^2, \quad (3.47)$$

where \mathcal{P} ranges over the set of finite partitions of the interval $[0, t]$ and $\|\mathcal{P}\|$ the mesh of the partition \mathcal{P} . If it exists, the limit is taken in the sense of convergence in probability. Whenever X is a continuous semimartingale, the quadratic variation always exists.

Remark 3.3. This definition is strictly weaker than the **true** 2-variation, which is defined as a supremum over the set of all finite partitions (*cf.* definition 2.13). In fact, for Brownian motion B , it is possible to show that $[B]_t = t$, but on the interval $[0, t]$, $\|B\|_{2\text{-var}} = \infty$ almost surely.

Understanding the quadratic variation is in some sense enough to understand local martingales, as the following theorem shows.

Theorem 3.4 (Dambis-Dubins-Schwarz). *Let M be a continuous local martingale vanishing at 0 such that $[M]_\infty = \infty$, then there exists a Brownian motion B such that, a.s. for all $t \geq 0$,*

$$M_t = B_{[M]_t}. \quad (3.48)$$

This theorem allows us to give *a priori* bounds on the small bar asymptotics of the barcode of a semimartingale. Since Brownian motion is a.s. $(\frac{1}{2} - \delta)$ -Hölder continuous for every $\delta > 0$, the p -variation of any semimartingale is almost surely finite as soon as $p > 2$ (since p -variation does not depend on parametrization). In particular we expect

$$N_X^\varepsilon = O(\varepsilon^{-2-\delta}) \quad \text{as } \varepsilon \rightarrow 0. \quad (3.49)$$

for every $\delta > 0$, by virtue of Picard's theorem. We may further refine this result by introducing the local time.

3.2 The local time and sharp asymptotics

The local time of a continuous semimartingale on an interval $[0, t]$ can be informally understood as the “time spent” by the process X around a level $x \in \mathbb{R}$, in an equation

$$L_X^x(t) = \int_0^t \delta(x - X_s) d[X]_s. \quad (3.50)$$

The above equation is informal and ill-defined, but gives an intuitive insight on what the local time represents and how it behaves. Formally, we define the local time as follows.

Proposition 3.5 (Proposition 9.2 [17]). Let X be a continuous semimartingale and $x \in \mathbb{R}$. There exists an increasing process $(L_X^x(t))_{t \geq 0}$ such that the three following identities hold

$$\begin{aligned} |X_t - x| &= |X_0 - x| + \int_0^t \operatorname{sgn}(X_s - x) dX_s + L_X^x(t) \\ (X_t - x)_+ &= (X_0 - x)_+ + \int_0^t \mathbf{1}_{X_s > x} dX_s + \frac{1}{2} L_X^x(t) \\ (X_t - x)_- &= (X_0 - x)_- - \int_0^t \mathbf{1}_{X_s \leq x} dX_s + \frac{1}{2} L_X^x(t) \end{aligned}$$

Definition 3.6. The increasing process $(L_X^x(t))_{t \geq 0}$ is called the **local time of X at level x** . Furthermore, for every stopping time T , the local time at x of the stopped process X at T , $L_{X^T}^x(t) = L_X^x(t \wedge T)$.

The local time of a process is useful, as it allows us to exchange time and space in integrations. A first useful result in this direction, which we will later use, is the following proposition.

Proposition 3.7 (Density occupation formula, Corollary 9.7 [17]). Almost surely, for every non-negative, measurable function ϕ on \mathbb{R} ,

$$\int_0^t \phi(X_s) d[X]_s = \int_{\mathbb{R}} \phi(a) L_X^a(t) da. \quad (3.51)$$

This result can be informally derived using the informal definition of the local time we gave earlier. For a rigorous proof, we refer the reader to the cited reference. From the informal description we gave about the local time as the time spent by the process around a level x , we could expect the local time to be related to $N_X^{x, x+\varepsilon}$ for small ε . This turns out to be a well-known fact.

Proposition 3.8 (Approximation of the local time by downcrossings, §VI, Theorem 1.10 [25]). For every $t \geq 0$ and $s \geq 1$, if $X = M + A$ is a continuous semimartingale on $[0, t]$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and suppose that for $s \geq 1$

$$\mathbb{E} \left[[M]_t^{s/2} + \left(\int_0^t |dA|_s \right)^s \right] < \infty. \quad (3.52)$$

Then, then number of downcrossings from $x + \varepsilon$ to x , $D_X^{x, x+\varepsilon}$ satisfies

$$\varepsilon D_X^{x, x+\varepsilon} \xrightarrow[\varepsilon \rightarrow 0]{L^s} \frac{1}{2} L_X^x(t). \quad (3.53)$$

Moreover, in $L^s(\Omega)$,

$$D_X^{x, x+\varepsilon} \sim \frac{L_X^x(t)}{2\varepsilon} + O(1) \quad \text{as } \varepsilon \rightarrow 0. \quad (3.54)$$

Remark 3.9. An analogous statement can be proven for upcrossings.

With the above in mind, we proceed to prove theorem 2.

Proof of theorem 2. Since $0 \leq N_X^{x, x+\varepsilon} - D_X^{x, x+\varepsilon} \leq 1$, the statement of proposition 3.8 is applicable to $N_X^{x, x+\varepsilon}$ yielding the first result. It follows that in $L^s(\Omega)$,

$$\left| 2\varepsilon \lambda(T_X^\varepsilon) - \int_{\mathbb{R}} L_X^x(t) dx \right| = O(\varepsilon). \quad (3.55)$$

Note that while the integral carries over \mathbb{R} , its support is contained within the compact interval $[\inf X, \sup X]$. We now use the density occupation formula for the local time where $\phi = 1$, so

$$2\varepsilon \lambda(T_X^\varepsilon) = [X]_t + O(\varepsilon) \quad \text{as } \varepsilon \rightarrow 0 \text{ in } L^s(\Omega) \quad (3.56)$$

We now use the fact that

$$\lambda(T_X^\varepsilon) = \int_\varepsilon^\infty N_X^a da. \quad (3.57)$$

By monotonicity of N_X^ε , for every $\delta > 0$, almost surely,

$$N_X^{\varepsilon(1+\delta)} \leq \frac{\lambda(T_X^\varepsilon) - \lambda(T_X^{\varepsilon(1+\delta)})}{\delta\varepsilon} = \frac{1}{\delta\varepsilon} \int_\varepsilon^{\varepsilon(1+\delta)} N_X^a da \leq N_X^\varepsilon. \quad (3.58)$$

Thus, for $\delta > 0$ small enough,

$$\frac{2\varepsilon}{\delta} [\lambda(T_X^\varepsilon) - \lambda(T_X^{\varepsilon(1+\delta)})] \leq 2\varepsilon^2 N^\varepsilon \leq \frac{2\varepsilon}{\delta} [\lambda(T_X^{\varepsilon(1-\delta)}) - \lambda(T_X^\varepsilon)] \quad (3.59)$$

$$\frac{1}{\delta} [2\varepsilon \lambda(T_X^\varepsilon) - \frac{2\varepsilon(1+\delta)}{1+\delta} \lambda(T_X^{\varepsilon(1+\delta)})] \leq 2\varepsilon^2 N^\varepsilon \leq \frac{1}{\delta} [\frac{2\varepsilon(1-\delta)}{1-\delta} \lambda(T_X^{\varepsilon(1-\delta)}) - 2\varepsilon \lambda(T_X^\varepsilon)] \quad (3.60)$$

It follows that in $L^s(\Omega)$, for every $\delta > 0$ small enough,

$$\frac{1}{\delta} \left[\frac{\delta}{1+\delta} [X]_t + O(\varepsilon) \right] \leq 2\varepsilon^2 N^\varepsilon \leq \frac{1}{\delta} \left[\frac{\delta}{1-\delta} [X]_t + O(\varepsilon) \right] \quad \text{as } \varepsilon \rightarrow 0 \quad (3.61)$$

$$\frac{[X]_t}{1+\delta} + O(\varepsilon/\delta) \leq 2\varepsilon^2 N_X^\varepsilon \leq \frac{[X]_t}{1-\delta} + O(\varepsilon/\delta) \quad \text{as } \varepsilon \rightarrow 0. \quad (3.62)$$

Taking $\delta \rightarrow 0$ at a rate $\delta = O(\varepsilon^{1/2})$, we obtain the desired result. ■

4 Limiting processes

An interesting point of view relative to these stability results and the results of this paper is to consider these irregular processes as almost sure C^0 -limits of smooth processes, which have traditionally been more difficult to study. In this way, we can make affirmations about the barcodes of smooth processes up to some (small) error. This way of thinking is inspired by the study of trees, ultralimits and asymptotic cones in geometric group theory [27].

Lemma 4.1. Let (M, d) be a compact Polish metric space and let X be an almost surely continuous stochastic process on M , defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $(X_n)_{n \in \mathbb{N}}$ be any sequence of continuous stochastic processes defined on the same probability space and suppose

$$\delta_n := \|X - X_n\|_{L^\infty(\Omega, L^\infty(M, \mathbb{R}))} \xrightarrow{n \rightarrow \infty} 0. \quad (4.63)$$

If for all $\varepsilon > 0$, $\mathbb{E}[N_X^\varepsilon] < \infty$ and is continuous in ε , then for any $\varepsilon \geq 2\delta_n$,

$$N_{X_n}^\varepsilon \xrightarrow[n \rightarrow \infty]{L^1} N_X^\varepsilon,$$

Moreover,

$$\mathbb{E}[|N_{X_n}^\varepsilon - N_X^\varepsilon|] \leq \omega_\varepsilon(\delta_n), \quad (4.64)$$

where $\omega_\varepsilon(\delta) := \mathbb{E}[N_X^{\varepsilon-\delta} - N_X^{\varepsilon+\delta}]$. With analogous hypotheses, the same statement holds for $N_X^{x, x+\varepsilon}$.

Proof. The L^∞ -stability of barcodes with respect to the L^∞ -distance tells us that δ_n controls the bottleneck distance between the two barcodes [6, 21]. This implies that there exists a δ_n -matching between the barcodes of X_n and that of X , therefore:

- If $\alpha \in \mathcal{B}(X)$ has length $|\alpha| \geq 2\delta_n$, then $\exists! \beta \in \mathcal{B}(X_n)$ such that (α, β) are matched and the difference $||\alpha| - |\beta|| \leq 2\delta_n$;
- If $\beta \in \mathcal{B}(X_n)$ has length $|\beta| \geq 2\delta_n$, then $\exists! \alpha \in \mathcal{B}(X)$ such that (α, β) are matched and the difference $||\alpha| - |\beta|| \leq 2\delta_n$;
- If $\alpha \in \mathcal{B}(X)$ or $\beta \in \mathcal{B}(X_n)$ is unmatched, then they have length $\leq 2\delta_n$.

It follows that for $\varepsilon \geq 2\delta_n$ we have inequalities

$$\begin{aligned} N_{X_n}^{\varepsilon+2\delta_n} &\leq N_X^\varepsilon \\ N_X^{\varepsilon+2\delta_n} &\leq N_{X_n}^\varepsilon, \end{aligned}$$

from which we obtain bounds on $N_{X_n}^\varepsilon$

$$N_X^{\varepsilon+2\delta_n} \leq N_{X_n}^\varepsilon \leq N_X^{\varepsilon-2\delta_n}. \quad (4.65)$$

These inequalities imply

$$|N_{X_n}^\varepsilon - N_X^\varepsilon| \leq \left| N_X^{\varepsilon+2\delta_n} - N_X^\varepsilon \right| \vee \left| N_X^{\varepsilon-2\delta_n} - N_X^\varepsilon \right|.$$

Taking expectations of both sides, we have

$$\begin{aligned} \mathbb{E}[|N_{X_n}^\varepsilon - N_X^\varepsilon|] &\leq \mathbb{E}\left[\left| N_X^{\varepsilon+2\delta_n} - N_X^\varepsilon \right| \vee \left| N_X^{\varepsilon-2\delta_n} - N_X^\varepsilon \right|\right] \\ &\leq \mathbb{E}\left[\left| N_X^{\varepsilon+2\delta_n} - N_X^\varepsilon \right|\right] + \mathbb{E}\left[\left| N_X^{\varepsilon-2\delta_n} - N_X^\varepsilon \right|\right] \\ &= \mathbb{E}\left[N_X^\varepsilon - N_X^{\varepsilon+2\delta_n}\right] + \mathbb{E}\left[N_X^{\varepsilon-2\delta_n} - N_X^\varepsilon\right] \\ &\leq \omega_\varepsilon(2\delta_n) \end{aligned}$$

by monotonicity of N_X^ε . The right hand side of the inequality tends to 0 as $n \rightarrow \infty$ by continuity of $\mathbb{E}[N_X^\varepsilon]$, so $N_{X_n}^\varepsilon \xrightarrow[n \rightarrow \infty]{L^1} N_X^\varepsilon$. ■

Remark 4.2. The condition of uniform convergence over Ω can be quite restrictive, but covers some cases of distributions with compact supports in $C^0(X, \mathbb{R})$. Moreover, it covers the case of processes derived from random point clouds P (the filtration function is given by $d(-, P)$) stemming from a distribution with compact support on any Polish metric space M .

We can adapt the proof of the above proposition to a setting relaxing the $L^\infty(\Omega, L^\infty(M, \mathbb{R}))$ convergence condition.

Proof of theorem 4. Notice lemma 4.1 shows the result for $p = \infty$. The rest of the proof goes as that of lemma 4.1 with a few exceptions. To obtain the probable bound on $\|X - X_n\|_{L^\infty(M, \mathbb{R})}$, we apply the Markov inequality. Doing so yields

$$\mathbb{P}(\|X_n - X\|_{L^\infty(M, \mathbb{R})} \geq a\delta_n) \leq \frac{1}{a^p}. \quad (4.66)$$

The rest of the proof follows by first noticing that

$$\mathbb{P}(A) = \mathbb{P}(A \cap B) + \mathbb{P}(A \cap B^c) \leq \mathbb{P}(A \cap B) + \mathbb{P}(B^c) \quad (4.67)$$

and taking the events A and B to be $A = \{|N_{X_n}^\varepsilon - N_X^\varepsilon| \geq k\}$ and $B = \{\|X_n - X\|_{L^\infty(M, \mathbb{R})} \leq a\delta_n\}$ respectively, since

$$\mathbb{P}(A \cap B) = \mathbb{P}(A | B)\mathbb{P}(B) \leq \mathbb{P}(A | B) \leq \frac{\mathbb{E}\left[\|X - X_n\|_{L^\infty(M, \mathbb{R})} \mid B\right]}{k}. \quad (4.68)$$

■

Remark 4.3. By an application of the Borel-Cantelli lemma, if X_n tends to X in $L^p(\Omega, L^\infty(M, \mathbb{R}))$ for $p > 1$ at a rate $O(r(n))$ (where r is a function tending to 0 at infinity), then almost surely $X_n \xrightarrow[n \rightarrow \infty]{L^\infty} X$ at a rate $O(r(n))$ as well.

5 Applications

5.1 Brownian motion and local martingales with deterministic strictly increasing quadratic variation

For Brownian motion, it is possible to compute our quantities of interest exactly.

Theorem 5.1 (Perez, Proposition 4.4 [23]). *For Brownian motion B on $[0, t]$, $\mathbb{E}[N_B^\varepsilon]$ admits the following series representations which converge quickly for large and small ε respectively*

$$\mathbb{E}[N_B^\varepsilon] = 4 \sum_{k \geq 1} (2k - 1) \operatorname{erfc}\left(\frac{(2k - 1)\varepsilon}{\sqrt{2t}}\right) - k \operatorname{erfc}\left(\frac{2k\varepsilon}{\sqrt{2t}}\right) \quad (5.69)$$

$$= \frac{t}{2\varepsilon^2} + \frac{2}{3} + 2 \sum_{k \geq 1} (2(-1)^k - 1) \frac{e^{-\pi^2 k^2 t / 2\varepsilon^2}}{\varepsilon^2} \left[1 + \frac{\varepsilon^2}{\pi^2 k^2 t}\right]. \quad (5.70)$$

Remark 5.2. In particular, $\mathbb{E}[N_B^\varepsilon]$ is analytic in ε on an open wedge around the positive real axis in the complex plane, and so in particular C^∞ in ε , thereby confirming and extending Divol and Chazal's results.

Proof. We start by writing

$$\mathbb{E}[N_B^\varepsilon] = \sum_{k=1}^{\infty} \mathbb{P}(N_B^\varepsilon \geq k) = \mathbb{P}(R_t \geq \varepsilon) + \sum_{k=2}^{\infty} \mathbb{P}(S_{k-1}^\varepsilon \leq t) \quad (5.71)$$

Using the scale invariance of Brownian motion,

$$\mathbb{E}[N_B^\varepsilon] = \mathbb{P}(R_1 \geq t^{-\frac{1}{2}}\varepsilon) + \sum_{k=2}^{\infty} \mathbb{P}(S_{k-1}^1 \leq t/\varepsilon^2) \quad (5.72)$$

We now notice that the stopping times

$$S_{k-1}^\varepsilon = \sum_{i=1}^{k-1} (S_i^\varepsilon - T_i^\varepsilon) + (T_i^\varepsilon - S_{i-1}^\varepsilon) \quad (5.73)$$

and that the increments $(S_i^\varepsilon - T_i^\varepsilon)$ and $(T_i^\varepsilon - S_{i-1}^\varepsilon)$ are independent and identically distributed. Moreover, in distribution,

$$\sup_{[0,t]} B - B_t = |B_t|, \quad (5.74)$$

so that the $(S_i^\varepsilon - T_i^\varepsilon)$ and $(T_i^\varepsilon - S_{i-1}^\varepsilon)$ are distributed like the first hitting time of ε by $|B_t|$. It is a classical result [4, p.641] that this hitting time H^ε satisfies

$$\mathbb{E}\left[e^{\lambda H^\varepsilon}\right] = \operatorname{sech}(\varepsilon\sqrt{2\lambda}). \quad (5.75)$$

Similarly, it is also well-known that the range of Brownian motion satisfies

$$\mathcal{L}_t(\mathbb{P}(R_t \geq \varepsilon))(\lambda) = \frac{\operatorname{sech}^2(\varepsilon\sqrt{\frac{\lambda}{2}})}{\lambda} \quad (5.76)$$

We now take the Laplace transform with respect to the time variable (which we will denote \mathcal{L}_t) of $\mathbb{E}[N_B^\varepsilon]$

$$\begin{aligned} \mathcal{L}_t(\mathbb{E}[N_B^\varepsilon])(\lambda) &= \frac{\operatorname{sech}^2(\varepsilon\sqrt{\frac{\lambda}{2}})}{\lambda} + \frac{1}{\lambda} \sum_{k=2}^{\infty} \mathbb{E}\left[e^{-\lambda\varepsilon^2 S_{k-1}^1}\right] \\ &= \frac{\operatorname{sech}^2(\varepsilon\sqrt{\frac{\lambda}{2}})}{\lambda} + \frac{1}{\lambda} \sum_{k=2}^{\infty} \mathbb{E}\left[e^{-\lambda\varepsilon^2 H^1}\right]^{2(k-1)}, \end{aligned}$$

where the last equality holds by virtue of i.i.d. character of the increments $(S_i^\varepsilon - T_i^\varepsilon)$ and $(T_i^\varepsilon - S_{i-1}^\varepsilon)$. Replacing the value of $\mathbb{E}[e^{\lambda H^\varepsilon}]$,

$$\mathcal{L}_t(\mathbb{E}[N_B^\varepsilon])(\lambda) = \frac{\operatorname{sech}^2(\varepsilon\sqrt{\frac{\lambda}{2}}) + \operatorname{csch}^2(\varepsilon\sqrt{2\lambda})}{\lambda}$$

The result is obtained by taking the inverse Laplace transform. ■

Proof of theorem 3. Persistent homology is invariant under reparametrization. In particular if $\lambda : [0, t] \rightarrow [0, \lambda(t)]$ is an increasing bijection, $N_{f \circ \lambda}^\varepsilon$ as calculated on $[0, \lambda(t)]$ is exactly equal to N_f^ε as calculated on $[0, t]$. Invoking the Dambis-Dubins-Schwarz theorem (theorem 3.4), for every continuous local martingale M (such that $M_0 = 0$) with deterministic and strictly increasing quadratic variation $[M]_t$, almost surely,

$$N_M^\varepsilon[0, t] = N_B^\varepsilon[0, [M]_t]. \quad (5.77)$$

We can perform a similar calculation for $N_B^{x, x+\varepsilon}$ when $x > 0$,

Proposition 5.3 (Perez, Proposition 4.7 [23]). For Brownian motion on $[0, t]$ and $x > 0$,

$$\begin{aligned} \mathbb{E}\left[N_B^{x, x+\varepsilon}\right] &= \sum_{k=1}^{\infty} \operatorname{erfc}\left(\frac{x + (2k-1)\varepsilon}{\sqrt{2t}}\right) \\ &\sim \frac{1}{2\varepsilon} \int_0^t \varphi(x, s) ds + \sum_{k \geq 0} \frac{4(-2)^k (2^{2k+1} - 1) \zeta(2k+2)}{\pi^{2k+2}} \left[\frac{\partial^k}{\partial t^k} \varphi(x, t)\right] \varepsilon^{2k+1} \text{ as } \varepsilon \rightarrow 0, \end{aligned}$$

where $\varphi(x, t)$ is the density of a centered Gaussian random variable of variance t and ζ denotes the Riemann zeta function.

■

Remark 5.4. In the appropriate limits, we retrieve the results established for semimartingales. By virtue of the Dambis-Dubins-Schwarz theorem and by the invariance under reparametrization of persistent homology, we may also immediately extend the results and formulæ of [23] to local martingales with deterministic and strictly increasing quadratic variation by replacing $t \mapsto [M]_t$.

5.2 Itô processes

Definition 5.5. An **Itô process** is the solution to a stochastic differential equation of the form

$$dX_t = \mu_t dt + \sigma_t dB_t \quad (5.78)$$

where μ_t and σ_t are adapted processes.

Every Itô process of this form has the strong Markov property and moreover, it is a semimartingale. We deduce that

$$N_X^\varepsilon \sim \frac{[X]_t}{2\varepsilon^2} + O(\varepsilon^{-1}) \text{ as } \varepsilon \rightarrow 0, \quad (5.79)$$

and an analogous expression for $N_X^{x, x+\varepsilon}$ in terms of the local time of X . In expectation, we may always say something about $N_X^{x, x+\varepsilon}$ by virtue of the following proposition.

Proposition 5.6. Let X be an Itô process with deterministic quadratic variation, then

$$\mathbb{E}[L_X^x(t)] = \int_0^t \varphi_X(x, s) d[X]_s, \quad (5.80)$$

where $\varphi_X(-, s)$ is the density function of the random variable X_s (which itself is a solution of the Fokker-Planck PDE associated to the SDE).

Proof. We take the expectation of both sides of the occupation density formula with $\phi(a) = e^{-i\lambda a}$.

$$\int_0^t \mathbb{E}\left[e^{-i\lambda X_s}\right] d[X]_s = \int_{\mathbb{R}} e^{-i\lambda a} \mathbb{E}[L_X^a(t)] da \quad (5.81)$$

The result follows from taking the inverse Fourier transform (with respect to λ) of both sides. ■

Proposition 5.7 (Asymptotics of barcodes of Itô processes). Let X be an Itô process with deterministic quadratic variation on $[0, t]$, then,

$$\mathbb{E}\left[N_X^{x, x+\varepsilon}\right] \sim \frac{1}{2\varepsilon} \int_0^t \varphi_X(x, s) d[X]_s + O(1) \text{ as } \varepsilon \rightarrow 0. \quad (5.82)$$

where $\varphi_X(-, s)$ is the density function of the random variable X_s . ■

As a Markov processes, Itô processes also satisfy

$$\mathbb{E}[N_X^\varepsilon] \sim \mathbb{P}(R_t \geq \varepsilon) \text{ as } \varepsilon \rightarrow \infty. \quad (5.83)$$

Finally, when $\mu_t = 0$ and $\sigma_t > 0$ for all $t > 0$ and is deterministic, an Itô process is a local martingale with quadratic variation

$$[X]_t = \int_0^t \sigma_s^2 ds, \quad (5.84)$$

and, in particular, its barcode is then known completely.

5.3 Limiting processes

The Karhunen-Loève theorem [14,18] shows that many well-known C^0 -processes can be seen as limits of smooth processes and that the logic of studying smooth objects with their more irregular C^0 limits is a logic which can also be applicable in higher dimensions. Brownian motion itself can be seen as such a limit [17,19]. Indeed, Paul Lévy showed that if $(\xi_k)_{k \in \mathbb{N}}$ is a sequence of i.i.d. standard normal variables, the series

$$\xi_0 t + \frac{\sqrt{2}}{\pi} \sum_{k=1}^{\infty} \xi_k \frac{\sin(\pi k t)}{k} \quad (5.85)$$

almost surely uniformly converges to the standard Brownian motion on $[0,1]$. Noting

$$S_n B := \xi_0 t + \frac{\sqrt{2}}{\pi} \sum_{k=1}^{n-1} \xi_k \frac{\sin(\pi k t)}{k}, \quad (5.86)$$

it can be shown (for instance using [13, Chapter 15, Theorem 4]) that for $p \geq 1$ there exists a finite constant C_p such that

$$\|B - S_n B\|_{L^p(\Omega, L^\infty([0,1], \mathbb{R}))} \leq C_p n^{-\frac{1}{2}} \log^{\frac{1}{2}}(n). \quad (5.87)$$

Applying the results of theorem 4 and optimizing in a , we have

$$\mathbb{P}(|N_{S_n B}^\varepsilon - N_B^\varepsilon| \geq k) = O\left(\left[\frac{C_p n^{-\frac{1}{2}} \log^{\frac{1}{2}}(n)}{p \varepsilon^3 k}\right]^{\frac{p}{p+1}}\right) \quad \text{as } \varepsilon \rightarrow 0. \quad (5.88)$$

In particular, we know that through this approximation yields a curve on the log-chart which doesn't stray far away from the Brownian motion's for $\varepsilon \gtrsim (n^{-1} \log(n))^{\frac{1}{2}}$. Brownian motion can also be approximated by random walks. The Komlós–Major–Tusnády (KMT) theorem provides a sharp estimate of the rate of convergence of these empirical processes to the Brownian bridge (which we will denote W_t).

Definition 5.8. Let $(X_n)_{n \in \mathbb{N}^*}$ be a sequence of (reduced, centered) i.i.d random variables. The **empirical process** defined by X is the process

$$\alpha_n^X(t) := \left[\frac{1}{n} \sum_{k=1}^n \mathbf{1}_{]-\infty, t]}(X_k) \right] - t. \quad (5.89)$$

Theorem 5.9 (KMT Theorem, [15,16]). *Let $(U_n)_{n \in \mathbb{N}^*}$ be an i.i.d. sequence of uniform random variables on $[0,1]$. Then, there exists a Brownian bridge $(W_t)_{1 \geq t \geq 0}$ such that for all $n \in \mathbb{N}^*$ and all $x > 0$*

$$\mathbb{P}\left(\|\alpha_n^U(t) - W_t\|_{L^\infty} > n^{-\frac{1}{2}}(C \log(n) + x)\right) \leq L e^{-\lambda x} \quad (5.90)$$

for some universal positive constants C, L and λ which are explicitly known [5].

The process W is a semimartingale with quadratic variation $[W]_t = t$ on $[0,1]$, as it can be obtained as the solution to the SDE

$$dW_t = \frac{-W_t}{1-t} dt + dB_t. \quad (5.91)$$

We deduce that $N_W^\varepsilon \sim \frac{1}{2\varepsilon^2}$ as $\varepsilon \rightarrow 0$. The KMT theorem and the same reasoning behind theorem 4 imply

$$\mathbb{P}\left(|N_{\alpha_n^U}^\varepsilon - N_W^\varepsilon| \geq k\right) = O\left(\frac{n^{-\frac{1}{2}} \log(n)}{\varepsilon^3 k}\right) \quad \text{as } \varepsilon \rightarrow 0. \quad (5.92)$$

In particular, this approximation yields the curve $N_{\alpha_n^U}^\varepsilon$ on the log-chart which doesn't stray far away from N_W^ε 's for $\varepsilon \gtrsim n^{-\frac{1}{2}} \log(n)$.

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