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APPLICATIVE PROPERTIES OF SINUSOIDAL FOURIER SERIES
EXPANSIONS OF MULTI-VARIABLE FUNCTIONS

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Abstract: In our previous research [1, 2], we have developed a method of solving differential equations by determining the coefficients of the expansion of unknown functions in sinusoidal (or non-sinusoidal) series. This has proven to be an extremely powerful method, applicable to all types of differential (ODEs) and integrodifferential equations, linear and nonlinear, to systems of such equations, valid over a closed interval, regardless of their order and regardless of the complexity of their coefficients. The conditions required for the application of this method are not very restrictive and are easy to fulfill, in particular for the situations encountered in physics and in engineering.

To apply this method also in the case of equations and systems of partial differential equations (for functions of several variables), it is necessary to generalize the applicatives formulas which describe the properties of monovariate functions (used in the case of ODEs) for functions depending on several variables (subject to partial differential equations PDEs), valid in a closed domain. As in the case of ODEs, the differentiation and the integration operations performed on the expansions in Fourier sinusoidal series (SFS) will turn into algebraic operations performed on the coefficients of the expansion. Therefore, solving PDEs will turn into solving some algebraic equations.

Keywords: periodic sinusoidal Fourier series, non-sinusoidal series, linear differential equations, differential equations with partial derivatives

1. Introduction

Research initiated by Fourier from the study of the diffusion of heat [3] and continued by a large number of mathematicians, showed that any real function, with real argument $f(x) : R \rightarrow R$, T -periodic over the interval (x_1, x_2) , where $T = x_2 - x_1$, continuous and piecewise differentiable, or square integrable over the interval T , i.e.: $\int_{x_1}^{x_2} |f(x)|^2 dx < +\infty$, can be represented by a weighted sum of simple trigonometric functions:

$$\bar{f}(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} [a_n \cos(\omega_n x) + b_n \sin(\omega_n x)] \quad \text{where, for } \forall n \in \mathbf{N}, \text{ we will define } \omega_n = n \frac{2\pi}{T} \quad (1)$$

The coefficients of this expansion can be calculated using Euler's formulas:

$$a_n = \frac{2}{T} \int_{x_1}^{x_2} f(x) \cos(\omega_n x) dx \quad n=0, 1, 2, 3, \dots \quad (2)$$

$$b_n = \frac{2}{T} \int_{x_1}^{x_2} f(x) \sin(\omega_n x) dx \quad n=1, 2, 3, \dots$$

This assertion has been extended to real functions with several variables [4-7]. Using vector notations, the function $f(\mathbf{x}) : R^d \rightarrow R$, is periodic, with period $\mathbf{T} = (T_1, T_2, \dots, T_d)$ if, for all d -tuple of integers $\mathbf{n} \in \mathbf{Z}^d$, $\mathbf{n} = (n_1, n_2, \dots, n_d)$, we have: $f(\mathbf{x} + \mathbf{n} \cdot \mathbf{T}) = f(\mathbf{x})$. Here, $\mathbf{n} \cdot \mathbf{T}$ is a dot product, and the set of points \mathbf{n} of \mathbf{R}^d which contains only integers is called the d -dimensional integer lattice \mathbf{Z}^d .

If we denote $L_i = T_i/2$, $i = 1, 2, \dots, d$, by appropriate changes of variables, we can arrive at a periodic function $f(\mathbf{x})$ on the prism $\mathbf{L}^d = \prod_{i=1}^d [-L_i, L_i]$. If this function is piecewise continuous and derivable with respect to any one of the variables x_i and if it is a square integrable on the prism \mathbf{L}^d , that is: $\int_{\mathbf{L}^d} |f(\mathbf{x})|^2 d\mathbf{x} < +\infty$ (it belongs to the space $L^2([\mathbf{L}^d])$), the formulas and the theorems valid in the space $L^2([-L_1, L_1])$ for the monovariate functions, are practically identical. The "bricks" for the Fourier series expansion of multivariable periodic functions will be (as in the case of monovariate ones) the one-dimensional complex exponentials:

$$(f(x_1, x_2, \dots, x_d))^{SFS} = \sum_{n_1, n_2, \dots, n_d} c_{n_1 n_2 \dots n_d} e^{i\omega_{n_1} x_1} e^{i\omega_{n_2} x_2} \dots e^{i\omega_{n_d} x_d}, \text{ where for } \forall n_i \in \mathbf{Z}^d, \text{ we define } \omega_{n_i} = n_i \frac{\pi}{L_i}$$

The coefficients of this development are calculated using Euler's formulas:

$$c_{n_1 n_2 \dots n_d} = \frac{1}{\prod_{i=1}^d T_i} \int_{-L_1}^{L_1} \int_{-L_2}^{L_2} \dots \int_{-L_d}^{L_d} e^{-i\omega_{n_1} x_1} e^{-i\omega_{n_2} x_2} \dots e^{-i\omega_{n_d} x_d} f(x_1, x_2, \dots, x_d) dx_1 dx_2 \dots dx_d$$

With vector notations, we can write:

$$\bar{f}(\mathbf{x}) = \sum_{\mathbf{Z}^d} c_{\mathbf{n}} \exp(\boldsymbol{\omega}_{\mathbf{n}} \cdot \mathbf{x}_{\mathbf{n}}), \text{ où } \boldsymbol{\omega}_{\mathbf{n}} = (\omega_{n_1}, \omega_{n_2}, \dots, \omega_{n_d}), \text{ et } c_{\mathbf{n}} = \left(2^d \prod_{i=1}^d L_i \right)^{-1} \int_{\mathbf{L}^d} e^{-i\boldsymbol{\omega}_{\mathbf{n}} \cdot \mathbf{x}_{\mathbf{n}}} f(\mathbf{x}) d\mathbf{x} \quad (3)$$

As in the one-dimensional case [2,5], once we know the coefficients of the Fourier series expansion of a multivariable function, we can find computational formulas for the Fourier series expansion of the product $f(\mathbf{x}) \cdot g(\mathbf{x})$ of two functions of several variables (for $f(\mathbf{x})=g(\mathbf{x})$, for the square of such a function). By successively and repeatedly applying these formulas, the Fourier coefficients of more complicated analytical expressions can be calculated.

In addition, under certain conditions, relation (3) can be integrated, or derived repeatedly, term by term. Both the differentiation and the integration with respect to a single variable (partial differentiation and, by extension, first-order "**partial integration**") are identical to the differentiation, respectively the integration of a monovariate function (keeping the other variables at a constant value). Consequently, the results obtained in the case of the development in SFS for the monovariate functions [2] remain valid also in the case of the partial derivatives and the partial integrals of first order, and by repetition, also on those of a higher order.

Let the function $z = f(\mathbf{x})$, be defined on the $(d+1)$ -dimensional domain $\mathbf{R} \times \mathbf{L}^d$. In this domain, $f(\mathbf{x})$ are hypersurfaces. For the differentiation with respect to one of the variables x_i (the partial derivative $\partial f / \partial x_i$, or f_{x_i}), first, we will define the function x_i -monovariate $f(C, x_i) = f(C_1, \dots, x_i, \dots, C_d)$, where C_j , $j=1, 2, \dots, d$, $j \neq i$, are real constants that do not change during differentiation. For each combination \mathbf{C}_0 of $d-1$ variables other than x_i , we obtain an x_i -monovariate curve on the hypersurface $f(\mathbf{x})$, and for each value $x_i = x_0$, we obtain the intersection of this hypersurface with the plane $x_i = x_0$.

For the differentiation of the x_i -monovariate function $f(C, x_i)$, we will consider that the existence of finites discontinuities of this function (frequent in particular on the lateral surfaces of the d -dimensional prism \mathbf{L}^d) introduced by this variable, leads to the modification of the coefficients of the odd components of its series expansion, as well as a change in the mean value of the derivative [1, 2]. For all x_j -monovariate functions, the values of these changes depend on the imprint of the intersection of the function $f(\mathbf{x})$ with the hypersurface bordering the domain of definition. For each higher order derivative, it is necessary to know this imprint of the lower order derivative function. In the case of periodic PDEs, these imprints are the boundary conditions of the equation (at the boundary of the domain \mathbf{L}^d).

For the integration with respect to one of the variables x_i (the partial integral $\partial^{-1}f/\partial x_i$, or $f^{-1}_{x_i}$, is the antiderivative of the x_i -monovariabile function $f(C, x_i)$) we will proceed from the same as in the case of functions of a single variable. In the formula of the development in SFS of a multivariable function with real variables (resulting from the formula (3)), the "partial integration" (with respect to x_i) of the even component (with respect to x_i) of the x_i -monovariabile function $f(C, x_i)$ generates an odd component (relative to this variable) of the integral, and the "partial integration" of an odd component (with respect to x_i) generates an even component (with respect to this variable) and supplementary, a constant component (the value of which becomes the mean value of the "primitive" of the x_i -monovariabile function). The partial integration (with respect to one of the variables) of the mean value of the x_i -monovariabile function generates (with respect to x_i) a ramp function, whose expansion in Fourier series leads to the change of the coefficients of the odd component of the expansion of the primitive.

With the obtained formulas and with those resulting from their combinations, we have the necessary tools to transform, similarly with the case of ODEs [2], a partial differential equation (PDE), for which we know enough of the boundary conditions, into a system of algebraic equations, in which the unknown values are the coefficients of the sinusoidal series expansion of the unknown periodic multivariable function $f(\mathbf{x})$. In the following sections, we will analyze their formation and we will establish calculation formulas for the expressions of the coefficients of the expansion in SFS of the integrals and of the derivatives of multivariable functions and of the product of two such functions, with exemplification in the case of the functions $y(x_1, x_2)$ of two variables, defined on the space $\mathbf{L}^2 = [-L_1, L_1] \times [-L_2, L_2]$.

The development in SFS of this function is [5]:

$$\begin{aligned} [y(x_1, x_2)]^{\text{SFS}} &= [y_{00} + y_{E0}(x_1) + y_{O0}(x_1) + y_{0E}(x_2) + y_{0O}(x_2) + y_{EE}(x_1, x_2) + y_{EO}(x_1, x_2) + y_{OE}(x_1, x_2) + y_{OO}(x_1, x_2)]^{\text{SFS}} = \\ &= Y_{00} + \left(\sum_{m=1}^{\infty} Y_{m0}^{E0} \cos \omega_{m1} x_1 \right) + \left(\sum_{m=1}^{\infty} Y_{m0}^{O0} \sin \omega_{m1} x_1 \right) + \left(\sum_{n=1}^{\infty} Y_{0n}^{0E} \cos \omega_{n2} x_2 \right) + \left(\sum_{n=1}^{\infty} Y_{0n}^{0O} \sin \omega_{n2} x_2 \right) + \\ &+ \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} Y_{mn}^{EE} \cos \omega_{m1} x_1 \cos \omega_{n2} x_2 \right) + \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} Y_{mn}^{EO} \cos \omega_{m1} x_1 \sin \omega_{n2} x_2 \right) + \\ &+ \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} Y_{mn}^{OE} \sin \omega_{m1} x_1 \cos \omega_{n2} x_2 \right) + \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} Y_{mn}^{OO} \sin \omega_{m1} x_1 \sin \omega_{n2} x_2 \right), \text{ where:} \end{aligned} \quad (4)$$

$$Y_{00} = \frac{1}{4L_1L_2} \int_{-L_1}^{L_1} \int_{-L_2}^{L_2} y(x_1, x_2) dx_1 dx_2$$

$$Y_{m0}^{E0} = \frac{1}{2L_1L_2} \int_{-L_1}^{L_1} \int_{-L_2}^{L_2} y(x_1, x_2) \cos \omega_{m1} x_1 dx_1 dx_2$$

$$Y_{m0}^{O0} = \frac{1}{2L_1L_2} \int_{-L_1}^{L_1} \int_{-L_2}^{L_2} y(x_1, x_2) \sin \omega_{m1} x_1 dx_1 dx_2$$

$$Y_{0n}^{0E} = \frac{1}{2L_1L_2} \int_{-L_1}^{L_1} \int_{-L_2}^{L_2} y(x_1, x_2) \cos \omega_{n2} x_2 dx_1 dx_2$$

$$Y_{0n}^{0O} = \frac{1}{2L_1L_2} \int_{-L_1}^{L_1} \int_{-L_2}^{L_2} y(x_1, x_2) \sin \omega_{n2} x_2 dx_1 dx_2$$

$$Y_{mn}^{EE} = \frac{1}{L_1L_2} \int_{-L_1}^{L_1} \int_{-L_2}^{L_2} y(x_1, x_2) \cos \omega_{m1} x_1 \cos \omega_{n2} x_2 dx_1 dx_2 \quad (4a)$$

$$Y_{mn}^{EO} = \frac{1}{L_1L_2} \int_{-L_1}^{L_1} \int_{-L_2}^{L_2} y(x_1, x_2) \cos \omega_{m1} x_1 \sin \omega_{n2} x_2 dx_1 dx_2$$

$$Y_{mn}^{OE} = \frac{1}{L_1 L_2} \int_{-L_1}^{L_1} \int_{-L_2}^{L_2} y(x_1, x_2) \sin \omega_{m1} x_1 \cos \omega_{n2} x_2 dx_1 dx_2$$

$$Y_{mn}^{OO} = \frac{1}{L_1 L_2} \int_{-L_1}^{L_1} \int_{-L_2}^{L_2} y(x_1, x_2) \sin \omega_{m1} x_1 \sin \omega_{n2} x_2 dx_1 dx_2$$

$$\omega_{m_1} = m\omega_1 = m \frac{\pi}{L_1}, \quad \omega_{n_2} = n\omega_2 = n \frac{\pi}{L_2}, \quad \text{where } m, n \in \mathbf{N}^+$$

Note that, in what follows, we will always consider that the odd components y_O , and the odd components y_E also, have the mean value equal to zero.

2. The expansion in SFS of the partial derivatives and partial integrals of multivariable functions

If in the SFS expansion of the function $f(\mathbf{x}) \rightarrow \mathbf{L}^d = \prod_{i=1}^d [-L_i, L_i]$, we separate the terms

which contain the functions $\cos \omega_{mi} x_i$, respectively $\sin \omega_{mi} x_i$, we get the expansion in SFS of the x_i -monovariate function $f(C, x_i)$:

$$[f(C, x_i)]^{SFS} = F_{0,i}^{C.O} + \sum_{m=1}^{\infty} F_{m,i}^{C.E} \cos \omega_{mi} x_i + \sum_{m=1}^{\infty} F_{m,i}^{C.O} \sin \omega_{mi} x_i = F_{0,i}^C + F_{E,i}^C + F_{O,i}^C, \quad m = 1, 2, \dots, \infty$$

where $F_{0,i}^C, F_{E,i}^C, F_{O,i}^C$ are the 3 components of the function (the mean value, the even component and the odd one, with respect to the variable x_i), and the coefficients $F_{0,i}^{C.O}, F_{m,i}^{C.E}$ and $F_{m,i}^{C.O}$, are functions dependent on the others $d-1$ variables. Obviously:

$$F_{E,i}^C(L_i) = F_{E,i}^C(-L_i) \quad \text{and} \quad F_{O,i}^C(L_i) = -F_{O,i}^C(-L_i) \quad (6)$$

Any function $f(\mathbf{x})$, piecewise continuous, square integrable, defined on d -dimensional prism $f(\mathbf{x}) \rightarrow \mathbf{L}^d = \prod_{i=1}^d [-L_i, L_i]$, can be extended over the entire space \mathbf{R}^d , by extending each monovariate function $f(C, x_i)$ over the entire \mathbf{R} axis, the result being a $2L_i$ -periodic function with respect to each variable x_i [1]. If the function $f(\mathbf{x})$ is continuous inside the prism with respect to each of the variables x_i and if $f(C, -L_i) = f(C, L_i)$, $i=1, 2, \dots, d$, the function $f(\mathbf{x})$ is *continuous* over the entire space \mathbf{R}^d .

Considering that the first order partial derivative, with respect to the variable x_i , of a multivariable function $f(\mathbf{x})$ has the same properties as any first order derivative of a monovariate function $f(C, x_i)$, we can extend over this type of functions the results obtained for the monovariate functions and for their SFS expansions. Therefore, we can affirm [5]:

Theorem 1. For any *continuous* function $f(\mathbf{x}) \rightarrow \mathbf{R}^d$, $2L_i$ -periodic with respect to each of the variables x_i , having all the partial derivatives absolutely integrable functions (but which can be undefined at a finite number of points), the Fourier series of its partial derivatives $\partial f / \partial x_i$ is obtained from the Fourier series of the function $f(\mathbf{x})$ by term-by-term differentiation, with respect to the variable x_i .

$$\text{For } [f(C, x_i)]^{SFS} = F_{0,i}^{C.O} + \sum_{m=1}^{\infty} F_{m,i}^{C.E} \cos \omega_{mi} x_i + \sum_{m=1}^{\infty} F_{m,i}^{C.O} \sin \omega_{mi} x_i, \text{ we obtain:}$$

$$[\partial f(\mathbf{x}) / \partial x_i]^{SFS} = [f'(C, x_i)]^{SFS} = \omega_{mi} \sum_{m=1}^{\infty} (F_{m,i}^{C.O} \cos \omega_{mi} x_i - F_{m,i}^{C.E} \sin \omega_{mi} x_i) \quad (6)$$

Since the function $f(C, x_i)$ is absolutely integrable, we can write:

$$\lim_{m \rightarrow \infty} m F_{m,i}^{C.E} = \lim_{m \rightarrow \infty} m F_{m,i}^{C.O} = 0,$$

what means that for $m \rightarrow \infty$, the series of the functions $F_{m.i}^{C.E}$ and $F_{m.i}^{C.O}$ converge to 0 faster than the serie $1/m$.

Since the function $\partial f/\partial x_i$ satisfies all the conditions imposed by Theorem 1, it can be differentiated again, in the same way, with respect to one of the other variables. Regarding the repeated differentiation with respect to the same variable, can be applied the

Theorem 2. For any continuous function $f(\mathbf{x}) \rightarrow \mathbf{R}^d$, $2L_i$ -periodic with respect to each of the variables x_i , and which, with respect to the variables x_i has $n-1$ continuous derivatives and the derivative of order n absolutely integrable (this derivative can be undefined in a finite number of points), the Fourier series of the partial derivatives of order r , $\partial^r f/\partial x_i^r$, $r < n$ are obtained from the Fourier series of the function $f(\mathbf{x})$ by differentiating it r times, term by term, with respect to the variable x_i . All the series obtained by these differentiations, with the possible exception of the derivative of order n , converge uniformly towards the corresponding derivative.

The Fourier coefficients of the function $f(\mathbf{x})$ satisfy the relation:

$$\lim_{m \rightarrow \infty} m^n F_{m.i}^{C.E} = \lim_{m \rightarrow \infty} m^n F_{m.i}^{C.O} = 0$$

And the partial derivative of order r , $\partial^r f/\partial x_i^r$ can be differentiated once with respect to the other variables, if $f(\mathbf{x})$ satisfies, with respect to this variable, the conditions of the Theorem 1, or of a repeated number of times, if it satisfies the conditions of the Theorem 2.

The extension of the discontinuous function $f_d(\mathbf{x})$, for which $f(C, -L_i) \neq f(C, L_i)$, for at least one of the variables x_i , poses the problem of the value of the function $f(C, x_i)$ at points $x_i = -L_i$ and $x_i = L_i$. There is a class of functions $f_r(x_i)$, "almost everywhere equal" [1], which fulfill the conditions to have an convergent SFS expansion, and which in the interval $[-L_i, L_i]$ are equal to $f(C, x_i)$, but at its ends (or/and at other points of discontinuity $x_{i.s}$) can, each of them, take different finite values. All these functions have the same SFS expansion and satisfy the equalities:

$$\bar{f}_r(-L_i) = \bar{f}_r(L_i) = \frac{1}{2} \left[\lim_{x_i \rightarrow (-L_i)^+} f(x_i) + \lim_{x_i \rightarrow (L_i)^-} f(x_i) \right] \text{ and } \bar{f}_r(x_{i.s}) = \frac{1}{2} \left[\lim_{x_i \rightarrow x_{i.s}^+} f(x) + \lim_{x_i \rightarrow x_{i.s}^-} f(x) \right].$$

The behavior of this whole class of functions $f_r(x_i)$, can be described by a *Fourier function* $f_F(x_i)$, which belongs to this class and which satisfies the condition $f_F(x_i) = \bar{f}_r(x_i)$. In the following, we will consider all the functions $f(C, x_i)$ which satisfy the conditions to have an convergent SFS expansion, as Fourier functions, and for which within the interval $[-L_i, L_i]$:

$$f(-L_i) = \lim_{h \rightarrow 0} f(-L_i + h), \text{ et } f(L_i) = \lim_{h \rightarrow 0} f(L_i - h).$$

Theorem 3. If the fonction $f(\mathbf{x}) \rightarrow \mathbf{L}^d = \prod_{i=1}^d [-L_i, L_i]$, for which $f(C, -L_i) \neq f(C, L_i)$, has

the partial derivative with respect to the variable x_i absolutely integrable (but which can be undefined at a finite number of points), it admits the development in SFS:

$$\left[\frac{\partial f(\mathbf{x})}{\partial x_i} \right]^{SFS} = [f'(C, x_i)]^{SFS} = f_{O.i}(C, L_i) + \sum_{m=1}^{\infty} \left[(\omega_m F_{m.i}^{C.O} + 2(-1)^n f_{O.i}(C, L_i)) \cos \omega_m x_i - \omega_m F_{m.i}^{C.E} \sin \omega_m x_i \right] \quad (7)$$

where: $[f(C, x_i)]^{SFS} = F_{0.i}^{C.O} + \sum_{m=1}^{\infty} (F_{m.i}^{C.E} \cos \omega_m x_i + F_{m.i}^{C.O} \sin \omega_m x_i)$, and $f_{O.i}(C, L_i) = \frac{1}{2L_i} [f(C, L_i) - f(C, -L_i)]$

Theorem 3 is a consequence of the similar theorem applied to monovariable functions: before the differentiation, in the domain \mathbf{L}^d , we have decomposed the function $F_{O.i}^C$ (the odd component with respect to the variable x_i of the function $f(\mathbf{x})$), into the sum of the function $f_{O.i.c}(\mathbf{x}) = F_{O.i}^C - f_o(C, L_i) \cdot x_i$, continues on \mathbf{R}^d , and the function

$f_{o,i}(\mathbf{x}) = f_o(C, L_i) \cdot x_i$, which contains a „Heaviside hyper-jump” in the plane $x_i = L_i$, between the „ridge-curves” $f_o(C, L_i)$ and $f_o(C, -L_i)$ (these jumps are the only cause of the generation of Gibbs-type phenomena).

Remark 1: If $f(\mathbf{x})$ is a Fourier function over the whole domain of definition \mathbf{R}^d , this property also belongs to the function $f_{o,i,c}(\mathbf{x})$ and to its SFS expansion as well. Moreover, the SFS expansion of the function $f_o(C, L_i) \cdot x_i$ is a Fourier function. Therefore, their sum enjoys the same properties. As in the case of monovariate functions, in natural phenomena, with limited energy, this type of function is more plausible than that of “hyper-ramp” functions.

Remark 2: for $f(C, L_i) = f(C, -L_i) \Rightarrow f_{o,i}(C, L_i) = 0$ and (7) becomes (6).

Remark 3: As for the monovariate functions: $f_{o,i}(C, L_i) = \lim_{m \rightarrow \infty} 2(-1)^{m+1} m F_{m,i}^{C,O}$. Here $f_{o,i}(C, L_i)$ (and also $F_{0,i}^{C,O}$, $F_{m,i}^{C,O}$ and $F_{m,i}^{C,E}$, $m \in \mathbf{N}^+$), is a function of $d-1$ variables.

Remark 4: This theorem applies in the same way to the partial derivatives of the function $f(\mathbf{x})$ with respect to the same or to another variable. In this case, the ridge-curves $f_{o,i}(C, L_i)$ are derived accordingly.

The differentiation is performed in the same way if the discontinuities with respect to any of the variables occur within the interval of definition. If the function $f_d(\mathbf{x}) \rightarrow \mathbf{L}^d$ is discontinuous with respect to the variable x_i , at a finite number of points, the function $f_{o,i}(\mathbf{x})$, odd with respect to the variable x_i , can be written as the sum of a continuous function $f_{o,i,c}(\mathbf{x})$, obtained by translating along the Oz axis the continuous pieces of the respective hypersurfaces, so that their ends coincide (and for the differentiation of which applies theorem 3) and a discontinuous function $f_{o,i,H}(\mathbf{x})$, which contains plane jumps between interior ridge-curves (and whose derivative is a sum of Dirac's “hyper-combs”).

Theorem 4. For any absolutely integrable function $f(\mathbf{x}) \rightarrow \mathbf{R}^d$, $2L_i$ -periodic with respect to each of the variables x_i , the definite partial integral, with respect to the variable x_i , is obtained from the Fourier series of the function $f(\mathbf{x})$ by integrating it term by term, in relation to the variable x_i , between all integration limits, indifferent if the series is convergent

or not. For $[f(C, x_i)]^{SFS} = F_{C,i}^{C,O} + \sum_{m=1}^{\infty} F_{C,i}^{C,E} \cos \omega_m x_i + \sum_{m=1}^{\infty} F_{C,i}^{C,O} \sin \omega_m x_i$, we obtain:

$$\left. \frac{\partial^{-1}}{\partial x_i} f(\mathbf{x}) \right|_a^b = \int_a^b f(C, x_i) dx_i = F_{C,i}^{C,O}(b-a) + \sum_{m=1}^{\infty} \frac{-F_{C,i}^{C,O}(\cos \omega_m b - \cos \omega_m a) + F_{C,i}^{C,E}(\sin \omega_m b - \sin \omega_m a)}{\omega_m}$$

Theorem 5. For any function $f(\mathbf{x}) \rightarrow \mathbf{R}^d$, $2L_i$ -periodic with respect to each of the variables x_i , having all the partial derivatives absolutely integrable functions, the Fourier series of the partial integral $\partial f^{-1}/\partial x_i$ is obtained by starting from the Fourier series of the function $f(\mathbf{x})$ by integrating it term by term with respect to the variable x_i , regardless of whether the series is convergent or not.

For $[f(C, x_i)]^{SFS} = F_{C,i}^{C,O} + \sum_{m=1}^{\infty} F_{C,i}^{C,E} \cos \omega_m x_i + \sum_{m=1}^{\infty} F_{C,i}^{C,O} \sin \omega_m x_i$, we obtain: (8)

$$\left[\frac{\partial^{-1} f(\mathbf{x})}{\partial x_i} \right]^{SFS} = \left[\int_0^x f(C, x_i) dx_i \right]^{SFS} = \sum_{m=1}^{\infty} \frac{F_{C,i}^{C,O}}{\omega_m} + \sum_{m=1}^{\infty} \frac{-F_{C,i}^{C,O} \cos \omega_m x_i + [F_{C,i}^{C,E} + 2(-1)^{m+1} F_{C,i}^{C,O}] \sin \omega_m x_i}{\omega_m}$$

Since the function $\partial^{-1} f/\partial x_i$ satisfies all the conditions imposed by the theorem 4, it can be partially integrated, in the same way, in relation to any of the other variables.

3. The coefficients of the SFS expansions of the derivatives and of the first order partial integrals of the two-variable functions

With the signification of the relations (4a), the SFS expansion of the two variables function $y(x_1, x_2)$ is:

$$\begin{aligned} [y(x_1, x_2)]^{SFS} &= Y_{00} + \left(\sum_{m=1}^{\infty} Y_{m0}^{E0} \cos \omega_{m1} x_1 \right) + \left(\sum_{m=1}^{\infty} Y_{m0}^{O0} \sin \omega_{m1} x_1 \right) + \left(\sum_{n=1}^{\infty} Y_{0n}^{O0} \cos \omega_{n2} x_2 \right) + \left(\sum_{n=1}^{\infty} Y_{0n}^{E0} \sin \omega_{n2} x_2 \right) + \\ &+ \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} Y_{mn}^{EE} \cos \omega_{m1} x_1 \cos \omega_{n2} x_2 \right) + \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} Y_{mn}^{EO} \cos \omega_{m1} x_1 \sin \omega_{n2} x_2 \right) + \\ &+ \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} Y_{mn}^{OE} \sin \omega_{m1} x_1 \cos \omega_{n2} x_2 \right) + \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} Y_{mn}^{OO} \sin \omega_{m1} x_1 \sin \omega_{n2} x_2 \right) \end{aligned} \quad (4)$$

The SFS expansions SFS of the two x_1 -monovariate and x_2 -monovariate functions are:

$$\begin{aligned} [y(C, x_1)]^{SFS} &= Y_{C.1}^{C.0}(x_2) + \sum_{m=1}^{\infty} Y_{m.1}^{C.E}(x_2) \cos \omega_{m1} x_1 + \sum_{m=1}^{\infty} Y_{m.1}^{C.O}(x_2) \sin \omega_{m1} x_1 = \\ &= \left(Y_{00} + \sum_{n=1}^{\infty} Y_{0n}^{O0} \cos \omega_{n2} x_2 + \sum_{n=1}^{\infty} Y_{0n}^{E0} \sin \omega_{n2} x_2 \right) + \sum_{m=1}^{\infty} \left(Y_{m0}^{E0} + \sum_{n=1}^{\infty} Y_{mn}^{EE} \cos \omega_{n2} x_2 + \sum_{n=1}^{\infty} Y_{mn}^{EO} \sin \omega_{n2} x_2 \right) \cos \omega_{m1} x_1 + \\ &+ \sum_{m=1}^{\infty} \left(Y_{m0}^{O0} + \sum_{n=1}^{\infty} Y_{mn}^{OE} \cos \omega_{n2} x_2 + \sum_{n=1}^{\infty} Y_{mn}^{OO} \sin \omega_{n2} x_2 \right) \sin \omega_{m1} x_1 \end{aligned} \quad (4.1)$$

$$\begin{aligned} [y(C, x_2)]^{SFS} &= Y_{C.2}^{C.0}(x_1) + \sum_{n=1}^{\infty} Y_{n.2}^{C.E}(x_1) \cos \omega_{n2} x_2 + \sum_{n=1}^{\infty} Y_{n.2}^{C.O}(x_1) \sin \omega_{n2} x_2 = \\ &= \left(Y_{00} + \sum_{m=1}^{\infty} Y_{m0}^{E0} \cos \omega_{m1} x_1 + \sum_{m=1}^{\infty} Y_{m0}^{O0} \sin \omega_{m1} x_1 \right) + \sum_{n=1}^{\infty} \left(Y_{0n}^{O0} + \sum_{m=1}^{\infty} Y_{mn}^{EE} \cos \omega_{m1} x_1 + \sum_{m=1}^{\infty} Y_{mn}^{EO} \sin \omega_{m1} x_1 \right) \cos \omega_{n2} x_2 + \\ &+ \sum_{n=1}^{\infty} \left(Y_{0n}^{E0} + \sum_{m=1}^{\infty} Y_{mn}^{OE} \cos \omega_{m1} x_1 + \sum_{m=1}^{\infty} Y_{mn}^{OO} \sin \omega_{m1} x_1 \right) \sin \omega_{n2} x_2 \end{aligned} \quad (4.2)$$

If, over the entire interval $\mathbf{L}^2 = [-L_1, L_1] \times [-L_2, L_2]$, the real, two variables periodic function $y(x_1, x_2)$ is continuous with respect to both variables, if it a the two first order partial derivatives piecewise smooth and if take place the equalities $y(-L_1, x_2) = y(L_1, x_2)$, for $\forall x_2 \in [-L_2, L_2]$ and $y(x_1, -L_2) = y(x_1, L_2)$, for $\forall x_1 \in [-L_1, L_1]$, the Fourier sinusoidal expansion of both partial derivatives can be obtained by differentiating term by term, with respect to the respective variable the SFS expansion of the function $y(x_1, x_2)$. The series obtained converges uniformly towards $\partial f / \partial x_i$, for $i=1, 2, \dots$, at all points of the domain.

With the notations of (4), we can write:

$$\begin{aligned} \left(\frac{\partial y}{\partial x_1} \right)^{SFS} &= \sum_{m=1}^{\infty} \omega_{m1} Y_{m0}^{O0} \cos \omega_{m1} x_1 - \sum_{m=1}^{\infty} \omega_{m1} Y_{m0}^{E0} \sin \omega_{m1} x_1 + \\ &+ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \omega_{m1} Y_{mn}^{OE} \cos \omega_{m1} x_1 \cos \omega_{n2} x_2 + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \omega_{m1} Y_{mn}^{OO} \cos \omega_{m1} x_1 \sin \omega_{n2} x_2 + \\ &- \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \omega_{m1} Y_{mn}^{EE} \sin \omega_{m1} x_1 \cos \omega_{n2} x_2 - \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \omega_{m1} Y_{mn}^{EO} \sin \omega_{m1} x_1 \sin \omega_{n2} x_2 \end{aligned} \quad (9)$$

$$\begin{aligned} \left(\frac{\partial y}{\partial x_2} \right)^{SFS} &= \sum_{n=1}^{\infty} \omega_{n2} Y_{0n}^{O0} \cos \omega_{n2} x_2 - \sum_{n=1}^{\infty} \omega_{n2} Y_{0n}^{E0} \sin \omega_{n2} x_2 + \\ &+ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \omega_{n2} Y_{mn}^{EO} \cos \omega_{m1} x_1 \cos \omega_{n2} x_2 - \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \omega_{n2} Y_{mn}^{EE} \cos \omega_{m1} x_1 \sin \omega_{n2} x_2 + \\ &+ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \omega_{n2} Y_{mn}^{OO} \sin \omega_{m1} x_1 \cos \omega_{n2} x_2 - \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \omega_{n2} Y_{mn}^{OE} \sin \omega_{m1} x_1 \sin \omega_{n2} x_2 \end{aligned} \quad (10)$$

The case where $y(-L_1, x_2) \neq y(L_1, x_2)$ and/or $y(x_1, -L_2) \neq y(x_1, L_2)$ is a special case of multivariable functions described in the previous section, that of univariable curves $y(C, x_2)$, respectively $y(x_1, C)$, for which C takes the particular values $C=x_1$ and $C_0=-L_1$, $C_0=L_1$, respectively $C=x_2$ and $C_0=-L_2$, $C_0=L_2$. In the case of the function $y(C, x_2)$, for each value $x_1=C_k$, the univariable function $y(C_k, x_2)$ located in the plane $x_1=C_k$ is associated with the line segment which unites the points $y(C_k, -L_2)$ and $y(C_k, L_2)$. All these segments (for all values of k) constitute the surface $y_{O.i} \cdot x_i$. For the computation of the first order partial derivatives, we will first determine the expression of these ridge-curves: $y(-L_1, x_2)$, $y(L_1, x_2)$, $y(x_1, -L_2)$ and $y(x_1, L_2)$ resulting from the intersection of the surface $y(x_1, x_2)$ with each of the planes $x_1=-L_1$, $x_1=L_1$, $x_2=-L_2$ and $x_2=L_2$ which border the prism of the domain.

The differences between the values taken, at the limit, by the function $y_F(x_1, x_2)$ in the vicinity of the boundaries (before and after the separation plane) generate between the boundary curves (the ridge-curves) the „stair riser” for each discontinuity of the function. The infinite jumps that occurs due to the differentiation of this discontinuities can be avoided by the introduction of a "stair riser" of reverse effect, generated by the ramp-surface $y_{O.i} \cdot x_i$.

The SFS expansion coefficients of the ridge-curves are involved in the calculation of the series expansion coefficients of the partial derivatives (we will assign, by extension, the index O , by analogy with the constants $f_O(L)$ from the analysis of the univariable functions, although the $y_{O.i}$ function can have all three components):

$$y_{O.1}(L_1, x_2) = \frac{y(L_1, x_2) - y(-L_1, x_2)}{2L_1} = Y_{00}^1 + \sum_{n=1}^{\infty} Y_{0n}^{1.0E} \cos \omega_{n2} x_2 + \sum_{n=1}^{\infty} Y_{0n}^{1.0O} \sin \omega_{n2} x_2 \quad (11)$$

$$y_{O.2}(x_1, L_2) = \frac{y(x_1, L_2) - y(x_1, -L_2)}{2L_2} = Y_{00}^2 + \sum_{m=1}^{\infty} Y_{m0}^{2.E0} \cos \omega_{m1} x_1 + \sum_{m=1}^{\infty} Y_{m0}^{2.O0} \sin \omega_{m1} x_1$$

By applying formula (7) to the relations (4.1), respectively (4.2), we obtain:

$$\begin{aligned} \left(\frac{\partial y}{\partial x_1} \right)^{SFS} &= \sum_{m=1}^{\infty} \omega_{m1} \left[Y_{m0}^{O0} + \sum_{n=1}^{\infty} Y_{mn}^{OE} \cos \omega_{n2} x_2 + \sum_{n=1}^{\infty} Y_{mn}^{OO} \sin \omega_{n2} x_2 \right] + 2(-1)^m y_{O.1}(L_1, x_2) \cos \omega_{m1} x_1 - \\ &- \sum_{m=1}^{\infty} \omega_{m1} \left(Y_{m0}^{E0} + \sum_{n=1}^{\infty} Y_{mn}^{EE} \cos \omega_{n2} x_2 + \sum_{n=1}^{\infty} Y_{mn}^{EO} \sin \omega_{n2} x_2 \right) \sin \omega_{m1} x_1 + y_{O.1}(L_1, x_2) = \\ &= Y_{00}^1 + \sum_{m=1}^{\infty} \left[\omega_{m1} Y_{m0}^{O0} + 2(-1)^m Y_{00}^1 \right] \cos \omega_{m1} x_1 - \sum_{m=1}^{\infty} \omega_{m1} Y_{m0}^{E0} \sin \omega_{m1} x_1 + \sum_{n=1}^{\infty} Y_{0n}^{1.0E} \cos \omega_{n2} x_2 + \sum_{n=1}^{\infty} Y_{0n}^{1.0O} \sin \omega_{n2} x_2 + \\ &+ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[\omega_{m1} Y_{mn}^{OE} + 2(-1)^m Y_{0n}^{1.0E} \right] \cos \omega_{m1} x_1 \cos \omega_{n2} x_2 + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[\omega_{m1} Y_{mn}^{OO} + 2(-1)^m Y_{0n}^{1.0O} \right] \cos \omega_{m1} x_1 \sin \omega_{n2} x_2 - \\ &- \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \omega_{m1} Y_{mn}^{EE} \sin \omega_{m1} x_1 \cos \omega_{n2} x_2 - \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \omega_{m1} Y_{mn}^{EO} \sin \omega_{m1} x_1 \sin \omega_{n2} x_2 \quad (12) \end{aligned}$$

$$\begin{aligned} \left(\frac{\partial y}{\partial x_2} \right)^{SFS} &= \sum_{n=1}^{\infty} \omega_{n2} \left[Y_{0n}^{O0} + \sum_{m=1}^{\infty} Y_{mn}^{EO} \cos \omega_{m1} x_1 + \sum_{m=1}^{\infty} Y_{mn}^{OO} \sin \omega_{m1} x_1 \right] + 2(-1)^n y_{O.2}(L_2, x_1) \cos \omega_{n2} x_2 - \\ &- \sum_{n=1}^{\infty} \omega_{n2} \left(Y_{0n}^{E0} + \sum_{m=1}^{\infty} Y_{mn}^{EE} \cos \omega_{m1} x_1 + \sum_{m=1}^{\infty} Y_{mn}^{OE} \sin \omega_{m1} x_1 \right) \sin \omega_{n2} x_2 + y_{O.2}(L_2, x_1) = \\ &= Y_{00}^2 + \sum_{m=1}^{\infty} Y_{m0}^{2.E0} \cos \omega_{m1} x_1 + \sum_{m=1}^{\infty} Y_{m0}^{2.O0} \sin \omega_{m1} x_1 + \sum_{n=1}^{\infty} \left[\omega_{n2} Y_{0n}^{O0} + 2(-1)^n Y_{00}^2 \right] \cos \omega_{n2} x_2 - \sum_{n=1}^{\infty} \omega_{n2} Y_{0n}^{E0} \sin \omega_{n2} x_2 + \\ &+ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[\omega_{n2} Y_{mn}^{EO} + 2(-1)^n Y_{m0}^{2.E0} \right] \cos \omega_{m1} x_1 \cos \omega_{n2} x_2 - \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \omega_{n2} Y_{mn}^{EE} \cos \omega_{m1} x_1 \sin \omega_{n2} x_2 + \\ &+ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[\omega_{n2} Y_{mn}^{OO} + 2(-1)^n Y_{m0}^{2.O0} \right] \sin \omega_{m1} x_1 \cos \omega_{n2} x_2 - \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \omega_{n2} Y_{mn}^{OE} \sin \omega_{m1} x_1 \sin \omega_{n2} x_2 \quad (13) \end{aligned}$$

Remark 1: for $y(+L_1, x_2) = y(-L_1, x_2) \Rightarrow y_{0,1}(x_2) = 0$ and (12) becomes (9).

Remark 2: As for the monovariate functions:

$$y_{0,1}(x_2) = \lim_{m \rightarrow \infty} 2(-1)^{m+1} \omega_{m1} \left(Y_{m0}^{00} + \sum_{n=1}^{\infty} Y_{mn}^{0E} \cos \omega_{n2} x_2 + \sum_{n=1}^{\infty} Y_{mn}^{0O} \sin \omega_{n2} x_2 \right) \quad (11.1)$$

Remark 3: The ridge-curves of the function $y(x_1, x_2)$ are special cases of the monovariate functions $y(C, x_i)$, which are obtained when C takes the particular values $C = \pm L_i$.

The ends of each of these ridge-curves coincide with one end of the ridge-curve corresponding to the other variable:

$$y(C, x_1) \Big|_{C=-L_2, x_1=-L_1} = y(C, x_2) \Big|_{C=-L_1, x_2=-L_2}, \quad y(C, x_1) \Big|_{C=L_2, x_1=L_1} = y(C, x_2) \Big|_{C=L_1, x_2=L_2}, \text{ etc}$$

This observation is also valid for the curves resulting from the differentiation, whatever the number of differentiations and the the number of variables with respect to which the differentiation is made.

It is easy to observe that both derivations can be made by term-by-term differentiation of expression (4), but by compensating the jump of the function at the periphery of the prism, each time we differentiated an odd term with respect to the differentiation variable. With the same initial data, by applying formula (8) to the same x_i -monovariate functions, we obtain the series expansions of the first order partial integrals of the function y :

$$\begin{aligned} \left[\frac{\partial^{-1} y}{\partial x_1} \right]^{SFS} &= \left[\int_0^{x_1} y(C, x_1) dx_1 \right]^{SFS} = \sum_{m=1}^{\infty} \frac{Y_{C,1}^{C,0}}{\omega_{m1}} + \sum_{m=1}^{\infty} \frac{-Y_{C,1}^{C,0} \cos \omega_{m1} x_1 + [Y_{C,1}^{C,E} + 2(-1)^{m+1} Y_{C,1}^{C,0}] \sin \omega_{m1} x_1}{\omega_{m1}} \\ &= \sum_{m=1}^{\infty} \frac{Y_{m0}^{00} + \sum_{n=1}^{\infty} Y_{mn}^{0E} \cos \omega_{n2} x_2 + \sum_{n=1}^{\infty} Y_{mn}^{0O} \sin \omega_{n2} x_2}{\omega_{m1}} - \sum_{m=1}^{\infty} \frac{Y_{m0}^{00} + \sum_{n=1}^{\infty} Y_{mn}^{0E} \cos \omega_{n2} x_2 + \sum_{n=1}^{\infty} Y_{mn}^{0O} \sin \omega_{n2} x_2}{\omega_{m1}} \cos \omega_{m1} x_1 + \\ &+ \sum_{m=1}^{\infty} \frac{Y_{m0}^{0E} + \sum_{n=1}^{\infty} Y_{mn}^{EE} \cos \omega_{n2} x_2 + \sum_{n=1}^{\infty} Y_{mn}^{EO} \sin \omega_{n2} x_2 - 2(-1)^m \left(Y_{00} + \sum_{n=1}^{\infty} Y_{0n}^{0E} \cos \omega_{n2} x_2 + \sum_{n=1}^{\infty} Y_{0n}^{0O} \sin \omega_{n2} x_2 \right)}{\omega_{m1}} \sin \omega_{m1} x_1 \\ \left[\frac{\partial^{-1} y}{\partial x_1} \right]^{SFS} &= \sum_{m=1}^{\infty} \frac{Y_{m0}^{00}}{\omega_{m1}} - \sum_{m=1}^{\infty} \frac{Y_{m0}^{00}}{\omega_{m1}} \cos \omega_{m1} x_1 + \sum_{m=1}^{\infty} \frac{Y_{m0}^{0E} - 2(-1)^m Y_{00}}{\omega_{m1}} \sin \omega_{m1} x_1 + \sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} \frac{Y_{mn}^{0E}}{\omega_{m1}} \right) \cos \omega_{n2} x_2 + \sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} \frac{Y_{mn}^{0O}}{\omega_{m1}} \right) \sin \omega_{n2} x_2 - \\ &- \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{Y_{mn}^{0E}}{\omega_{m1}} \cos \omega_{m1} x_1 \cos \omega_{n2} x_2 - \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{Y_{mn}^{0O}}{\omega_{m1}} \cos \omega_{m1} x_1 \sin \omega_{n2} x_2 + \\ &+ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{Y_{mn}^{EE} - 2(-1)^m Y_{0n}^{0E}}{\omega_{m1}} \sin \omega_{m1} x_1 \cos \omega_{n2} x_2 + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{Y_{mn}^{EO} - 2(-1)^m Y_{0n}^{0O}}{\omega_{m1}} \sin \omega_{m1} x_1 \sin \omega_{n2} x_2 \end{aligned} \quad (14)$$

$$\begin{aligned} \left[\frac{\partial^{-1} y}{\partial x_2} \right]^{SFS} &= \left[\int_0^{x_2} y(C, x_2) dx_2 \right]^{SFS} = \sum_{n=1}^{\infty} \frac{Y_{C,2}^{C,0}}{\omega_{n2}} + \sum_{n=1}^{\infty} \frac{-Y_{C,2}^{C,0} \cos \omega_{n2} x_2 + [Y_{C,2}^{C,E} + 2(-1)^{n+1} Y_{C,2}^{C,0}] \sin \omega_{n2} x_2}{\omega_{n2}} \\ &= \sum_{n=1}^{\infty} \frac{Y_{0n}^{00} + \sum_{m=1}^{\infty} Y_{mn}^{EO} \cos \omega_{m1} x_1 + \sum_{m=1}^{\infty} Y_{mn}^{OO} \sin \omega_{m1} x_1}{\omega_{n2}} - \sum_{n=1}^{\infty} \frac{Y_{0n}^{00} + \sum_{m=1}^{\infty} Y_{mn}^{EO} \cos \omega_{m1} x_1 + \sum_{m=1}^{\infty} Y_{mn}^{OO} \sin \omega_{m1} x_1}{\omega_{n2}} \cos \omega_{n2} x_2 + \\ &+ \sum_{n=1}^{\infty} \frac{Y_{0n}^{0E} + \sum_{m=1}^{\infty} Y_{mn}^{EE} \cos \omega_{m1} x_1 + \sum_{m=1}^{\infty} Y_{mn}^{EO} \sin \omega_{m1} x_1 - 2(-1)^n \left(Y_{00} + \sum_{m=1}^{\infty} Y_{m0}^{E0} \cos \omega_{m1} x_1 + \sum_{m=1}^{\infty} Y_{m0}^{O0} \sin \omega_{m1} x_1 \right)}{\omega_{n2}} \sin \omega_{n2} x_2 \\ \left[\frac{\partial^{-1} y}{\partial x_2} \right]^{SFS} &= \sum_{n=1}^{\infty} \frac{Y_{0n}^{00}}{\omega_{n2}} + \sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} \frac{Y_{mn}^{EO}}{\omega_{n2}} \right) \cos \omega_{m1} x_1 + \sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} \frac{Y_{mn}^{OO}}{\omega_{n2}} \right) \sin \omega_{m1} x_1 - \sum_{n=1}^{\infty} \frac{Y_{0n}^{00}}{\omega_{n2}} \cos \omega_{n2} x_2 + \sum_{n=1}^{\infty} \frac{Y_{0n}^{0E} - 2(-1)^n Y_{00}}{\omega_{n2}} \sin \omega_{n2} x_2 - \\ &- \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{Y_{mn}^{EO}}{\omega_{n2}} \cos \omega_{m1} x_1 \cos \omega_{n2} x_2 + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{Y_{mn}^{EE} - 2(-1)^n Y_{0n}^{0E}}{\omega_{n2}} \cos \omega_{m1} x_1 \sin \omega_{n2} x_2 + \end{aligned} \quad (15)$$

$$+ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{Y_{mn}^{OO}}{\omega_{n2}} \sin \omega_{m1} x_1 \cos \omega_{n2} x_2 + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{Y_{nm}^{EO} - 2(-1)^n Y_{m0}^{OO}}{\omega_{n2}} \sin \omega_{m1} x_1 \sin \omega_{n2} x_2$$

4. The coefficients of the SFS expansion of the partial derivatives and integrals of a higher order for the functions with two variables

The SFS expansions of the derivatives and of the higher-order partial integrals of the multivariable function $y(\mathbf{x})$, with respect to the variable x_i , are obtained by applying the formula (7), respectively (8) (the differentiation, respectively the integration, with respect to the variable x_i), to the x_i -monovariate functions resulting from the differentiation, respectively from the partial integration of a lower order of the function $y(\mathbf{x})$. For example, second order derivatives and second order partial integrals are obtained from the x_i -monovariate functions starting from the expressions (12), (13), respectively (14), (15).

The calculation of the partial derivatives (with respect to the variable x_i) requires the determination of the new conditions at the boundaries of the domain: if the function $y(\mathbf{x})$ has been derived with respect to the variable x_i , the expressions of the ridge-curves (generated by the intersection of the hypersurfaces $y(C, x_i)$ with each of the planes $x_i = -L_i$ and $x_i = L_i$), are also derived with respect to this variable; the new ridge-curves are generated by the function $\partial Y / \partial x_i$ (where Y can be one of the derivatives calculated previously, with respect to other variables and has known boundary conditions). For the function $y(x_1, x_2)$, the new ridge curves, which border the „stair riser” of the discontinuity of the function $\partial y / \partial x_i$ and for the coefficients of their SFS expansions we will obtain:

$$y'_{0.1} = \frac{1}{2L_1} \left[\frac{\partial y(x_1, x_2)}{\partial x_2} \Big|_{x_1=L_1} - \frac{\partial y(x_1, x_2)}{\partial x_2} \Big|_{x_1=-L_1} \right] = \frac{\partial}{\partial x_2} \left(Y_{00}^1 + \sum_{n=1}^{\infty} Y_{0n}^{1.0E} \cos \omega_{n2} x_2 + \sum_{n=1}^{\infty} Y_{0n}^{1.0O} \sin \omega_{n2} x_2 \right) =$$

$$= Y_{00}^{1.1} + \sum_{n=1}^{\infty} \left[\omega_{n2} Y_{0n}^{1.0O} + 2(-1)^n Y_{00}^{1.1} \right] \cos \omega_{n2} x_2 - \sum_{n=1}^{\infty} \omega_{n2} Y_{0n}^{1.0E} \sin \omega_{n2} x_2 \quad (16)$$

$$y'_{0.2}(x_1, L_2) = \frac{1}{2L_2} \left[\frac{\partial y(x_1, x_2)}{\partial x_1} \Big|_{x_2=L_2} - \frac{\partial y(x_1, x_2)}{\partial x_1} \Big|_{x_2=-L_2} \right] = \frac{\partial}{\partial x_1} \left(Y_{00}^2 + \sum_{m=1}^{\infty} Y_{m0}^{2.E0} \cos \omega_{m1} x_1 + \sum_{m=1}^{\infty} Y_{m0}^{2.O0} \sin \omega_{m1} x_1 \right) =$$

$$= Y_{00}^{2.2} + \sum_{m=1}^{\infty} \left[\omega_{m1} Y_{m0}^{2.O0} + 2(-1)^m Y_{00}^{2.2} \right] \cos \omega_{m1} x_1 - \sum_{m=1}^{\infty} \omega_{m1} Y_{m0}^{2.E0} \sin \omega_{m1} x_1 \quad (17)$$

We observe that:

$$Y_{00}^{1.1} = \frac{1}{2L_2} \int_{-L_2}^{L_2} \frac{\partial}{\partial x_2} y_{0.1}(L_1, x_2) dx_2 = \frac{y(L_1, L_2) - y(L_1, -L_2) - y(-L_1, L_2) + y(-L_1, -L_2)}{4L_1 L_2} \text{ and}$$

$$Y_{00}^{2.2} = \frac{1}{2L_1} \int_{-L_1}^{L_1} \frac{\partial}{\partial x_1} y_{0.2}(x_1, L_2) dx_1 = \frac{y(L_1, L_2) - y(L_1, -L_2) - y(-L_1, L_2) + y(-L_1, -L_2)}{4L L_2}$$

$$\text{So, } Y_{00}^{1.1} = Y_{00}^{2.2} \quad (18)$$

Each of these two first-order partial derivatives can be written in two ways as a function x_i -monovariate:

$$\left(\frac{\partial y}{\partial x_1}(C, x_1) \right)^{SFS} = \left(Y_{00}^1 + \sum_{n=1}^{\infty} Y_{0n}^{1.0E} \cos \omega_{n2} x_2 + \sum_{n=1}^{\infty} Y_{0n}^{1.0O} \sin \omega_{n2} x_2 \right) +$$

$$+ \sum_{m=1}^{\infty} \left\{ \left[\omega_{m1} Y_{m0}^{OO} + 2(-1)^m Y_{00}^1 \right] + \sum_{n=1}^{\infty} \left[\omega_{m1} Y_{mn}^{OE} + 2(-1)^m Y_{0n}^{1.0E} \right] \cos \omega_{n2} x_2 + \sum_{n=1}^{\infty} \left[\omega_{m1} Y_{mn}^{OO} + 2(-1)^m Y_{0n}^{1.0O} \right] \sin \omega_{n2} x_2 \right\} \cos \omega_{m1} x_1 +$$

$$- \sum_{m=1}^{\infty} \left(\omega_{m1} Y_{m0}^{EO} + \sum_{n=1}^{\infty} \omega_{m1} Y_{mn}^{EE} \cos \omega_{n2} x_2 + \sum_{n=1}^{\infty} \omega_{m1} Y_{mn}^{EO} \sin \omega_{n2} x_2 \right) \sin \omega_{m1} x_1 \quad (12.1)$$

$$\left(\frac{\partial y}{\partial x_1}\right)^{SFS} = \left\{ Y_{00}^1 + \sum_{m=1}^{\infty} [\omega_{m1} Y_{m0}^{00} + 2(-1)^m Y_{00}^1] \cos \omega_{m1} x_1 - \sum_{m=1}^{\infty} \omega_{m1} Y_{m0}^{E0} \sin \omega_{m1} x_1 \right\} + \quad (12.2)$$

$$\begin{aligned} &+ \sum_{n=1}^{\infty} \left\{ Y_{0n}^{1.0E} + \sum_{m=1}^{\infty} [\omega_{m1} Y_{mn}^{OE} + 2(-1)^m Y_{0n}^{1.0E}] \cos \omega_{m1} x_1 - \sum_{m=1}^{\infty} \omega_{m1} Y_{mn}^{EE} \sin \omega_{m1} x_1 \right\} \cos \omega_{n2} x_2 \\ &+ \sum_{n=1}^{\infty} \left\{ Y_{0n}^{1.0O} + \sum_{m=1}^{\infty} [\omega_{m1} Y_{mn}^{OO} + 2(-1)^m Y_{0n}^{1.0O}] \cos \omega_{m1} x_1 - \sum_{m=1}^{\infty} \omega_{m1} Y_{mn}^{EO} \sin \omega_{m1} x_1 \right\} \sin \omega_{n2} x_2 \\ \left(\frac{\partial y}{\partial x_2}\right)^{SFS} &= \left\{ Y_{00}^2 + \sum_{n=1}^{\infty} [\omega_{n2} Y_{0n}^{00} + 2(-1)^n Y_{00}^2] \cos \omega_{n2} x_2 - \sum_{n=1}^{\infty} \omega_{n2} Y_{0n}^{0E} \sin \omega_{n2} x_2 \right\} + \\ &+ \sum_{m=1}^{\infty} \left\{ Y_{m0}^{2.E0} + \sum_{n=1}^{\infty} [\omega_{n2} Y_{mn}^{EO} + 2(-1)^n Y_{m0}^{2.E0}] \cos \omega_{n2} x_2 - \sum_{n=1}^{\infty} \omega_{n2} Y_{mn}^{EE} \sin \omega_{n2} x_2 \right\} \cos \omega_{m1} x_1 \\ &+ \sum_{m=1}^{\infty} \left\{ Y_{m0}^{2.O0} + \sum_{n=1}^{\infty} [\omega_{n2} Y_{mn}^{OO} + 2(-1)^n Y_{m0}^{2.O0}] \cos \omega_{n2} x_2 - \sum_{n=1}^{\infty} \omega_{n2} Y_{mn}^{OE} \sin \omega_{n2} x_2 \right\} \sin \omega_{m1} x_1 \end{aligned} \quad (13.1)$$

$$\begin{aligned} \left(\frac{\partial y}{\partial x_2}\right)^{SFS} &= \left(Y_{00}^2 + \sum_{m=1}^{\infty} Y_{m0}^{2.E0} \cos \omega_{m1} x_1 + \sum_{m=1}^{\infty} Y_{m0}^{2.O0} \sin \omega_{m1} x_1 \right) + \\ &+ \sum_{n=1}^{\infty} \left\{ [\omega_{n2} Y_{0n}^{00} + 2(-1)^n Y_{00}^2] + \sum_{m=1}^{\infty} [\omega_{n2} Y_{mn}^{OO} + 2(-1)^n Y_{m0}^{2.O0}] \sin \omega_{m1} x_1 + \sum_{m=1}^{\infty} [\omega_{n2} Y_{mn}^{EO} + 2(-1)^n Y_{m0}^{2.E0}] \cos \omega_{m1} x_1 \right\} \cos \omega_{n2} x_2 \\ &- \sum_{n=1}^{\infty} \left(\omega_{n2} Y_{0n}^{0E} + \sum_{m=1}^{\infty} \omega_{n2} Y_{mn}^{EE} \cos \omega_{m1} x_1 + \sum_{m=1}^{\infty} \omega_{n2} Y_{mn}^{OE} \sin \omega_{m1} x_1 \right) \sin \omega_{n2} x_2 \end{aligned} \quad (13.2)$$

With the boundary conditions (16), by applying formula (7) to these x_i -monovariable functions, we obtain the series expansions for the partial derivatives of the second order:

$$\begin{aligned} \left(\frac{\partial^2 y}{\partial x_1^2}\right)^{SFS} &= \frac{\partial}{\partial x_1} \left[\frac{\partial y}{\partial x_1}(C, x_1) \right]^{SFS} = Y_{00}^{1.1} + \sum_{n=1}^{\infty} [\omega_{n2} Y_{0n}^{1.0O} + 2(-1)^n Y_{00}^{1.1}] \cos \omega_{n2} x_2 - \sum_{n=1}^{\infty} \omega_{n2} Y_{0n}^{1.0E} \sin \omega_{n2} x_2 - \\ &- \sum_{m=1}^{\infty} \left[\omega_{m1}^2 \left(Y_{m0}^{E0} + \sum_{n=1}^{\infty} Y_{mn}^{EE} \cos \omega_{n2} x_2 + \sum_{n=1}^{\infty} Y_{mn}^{EO} \sin \omega_{n2} x_2 \right) \right] \cos \omega_{m1} x_1 - \\ &- \sum_{m=1}^{\infty} 2(-1)^m \left\{ Y_{00}^{1.1} + \sum_{n=1}^{\infty} [\omega_{n2} Y_{0n}^{1.0O} + 2(-1)^n Y_{00}^{1.1}] \cos \omega_{n2} x_2 - \sum_{n=1}^{\infty} \omega_{n2} Y_{0n}^{1.0E} \sin \omega_{n2} x_2 \right\} \cos \omega_{m1} x_1 - \\ &- \sum_{m=1}^{\infty} \omega_{m1}^2 \left(Y_{m0}^{O0} + \sum_{n=1}^{\infty} Y_{mn}^{OE} \cos \omega_{n2} x_2 + \sum_{n=1}^{\infty} Y_{mn}^{OO} \sin \omega_{n2} x_2 \right) \sin \omega_{m1} x_1 - \\ &- \sum_{m=1}^{\infty} 2(-1)^m \omega_{m1} \left(Y_{00}^1 + \sum_{n=1}^{\infty} Y_{0n}^{1.0E} \cos \omega_{n2} x_2 + \sum_{n=1}^{\infty} Y_{0n}^{1.0O} \sin \omega_{n2} x_2 \right) \sin \omega_{m1} x_1 = \\ &= Y_{00}^{1.1} - \sum_{m=1}^{\infty} [\omega_{m1}^2 Y_{m0}^{E0} + 2(-1)^m Y_{00}^{1.1}] \cos \omega_{m1} x_1 - \sum_{m=1}^{\infty} [\omega_{m1}^2 Y_{m0}^{O0} + 2(-1)^m \omega_{m1} Y_{00}^1] \sin \omega_{m1} x_1 + \\ &+ \sum_{n=1}^{\infty} [\omega_{n2} Y_{0n}^{1.0O} + 2(-1)^n Y_{00}^{1.1}] \cos \omega_{n2} x_2 - \sum_{n=1}^{\infty} \omega_{n2} Y_{0n}^{1.0E} \sin \omega_{n2} x_2 - \\ &- \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} [\omega_{m1}^2 Y_{mn}^{EE} + 2(-1)^m \omega_{n2} Y_{0n}^{1.0O} + 4(-1)^{m+n} Y_{00}^{1.1}] \cos \omega_{m1} x_1 \cos \omega_{n2} x_2 - \\ &- \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} [\omega_{m1}^2 Y_{mn}^{EO} + 2(-1)^m \omega_{n2} Y_{0n}^{1.0E}] \cos \omega_{m1} x_1 \sin \omega_{n2} x_2 - \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} [\omega_{m1}^2 Y_{mn}^{OE} + 2(-1)^m \omega_{m1} Y_{0n}^{1.0E}] \sin \omega_{m1} x_1 \cos \omega_{n2} x_2 - \\ &- \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} [\omega_{m1}^2 Y_{mn}^{OO} + \sum_{n=1}^{\infty} 2(-1)^m \omega_{m1} Y_{0n}^{1.0O}] \sin \omega_{m1} x_1 \sin \omega_{n2} x_2 = \end{aligned} \quad (19)$$

$$\begin{aligned}
&= y'_{O,1}(L_1, x_2) + \sum_{m=1}^{\infty} \left[-\omega_m^2 Y_{m,1}^{C,E}(x_2) + 2(-1)^m y'_{O,1}(L_1, x_2) \right] \cos \omega_m x_1 - \sum_{m=1}^{\infty} \left[\omega_m^2 Y_{m,1}^{C,O}(x_2) + 2(-1)^m \omega_m y_{O,1} \right] \sin \omega_m x_1 \\
&\left[\frac{\partial^2 y}{\partial x_2^2} \right]^{SFS} = \frac{\partial}{\partial x_2} \left[\frac{\partial y}{\partial x_2}(C, x_2) \right]^{SFS} = Y_{00}^{\prime 2} + \sum_{m=1}^{\infty} \left[\omega_m Y_{m0}^{2,00} + 2(-1)^m Y_{00}^{\prime 2} \right] \cos \omega_m x_1 - \sum_{m=1}^{\infty} \omega_m Y_{m0}^{2,E0} \sin \omega_m x_1 - \\
&- \sum_{n=1}^{\infty} \omega_{n2}^2 \left(Y_{0n}^{0E} + \sum_{m=1}^{\infty} Y_{nm}^{EE} \cos \omega_m x_1 + \sum_{m=1}^{\infty} Y_{nm}^{OE} \sin \omega_m x_1 \right) \cos \omega_{n2} x_2 - \\
&- \sum_{n=1}^{\infty} 2(-1)^n \left\{ Y_{00}^{\prime 2} + \sum_{m=1}^{\infty} \left[\omega_m Y_{m0}^{2,00} + 2(-1)^m Y_{00}^{\prime 2} \right] \cos \omega_m x_1 - \sum_{m=1}^{\infty} \omega_m Y_{m0}^{2,E0} \sin \omega_m x_1 \right\} \cos \omega_{n2} x_2 - \\
&- \sum_{n=1}^{\infty} \omega_{n2}^2 \left[Y_{0n}^{0O} + \sum_{m=1}^{\infty} \left(Y_{nm}^{OO} \sin \omega_m x_1 + Y_{nm}^{EO} \cos \omega_m x_1 \right) \right] \sin \omega_{n2} x_2 - \\
&- \sum_{n=1}^{\infty} 2(-1)^n \omega_{n2} \left[Y_{00}^{\prime 2} + \sum_{m=1}^{\infty} \left(Y_{m0}^{2,00} \sin \omega_m x_1 + Y_{m0}^{2,E0} \cos \omega_m x_1 \right) \right] \sin \omega_{n2} x_2 = \\
&= Y_{00}^{\prime 2} + \sum_{m=1}^{\infty} \left[\omega_m Y_{m0}^{2,00} + 2(-1)^m Y_{00}^{\prime 2} \right] \cos \omega_m x_1 - \sum_{m=1}^{\infty} \omega_m Y_{m0}^{2,E0} \sin \omega_m x_1 - \sum_{n=1}^{\infty} \left[\omega_{n2}^2 Y_{0n}^{0E} + 2(-1)^n Y_{00}^{\prime 2} \right] \cos \omega_{n2} x_2 - \\
&- \sum_{n=1}^{\infty} \left[\omega_{n2}^2 Y_{0n}^{0O} + 2(-1)^n \omega_{n2} Y_{00}^{\prime 2} \right] \sin \omega_{n2} x_2 - \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[\omega_{n2}^2 Y_{nm}^{EE} + 2(-1)^n \omega_m Y_{m0}^{2,00} + 4(-1)^{m+n} Y_{00}^{\prime 2} \right] \cos \omega_m x_1 \cos \omega_{n2} x_2 - \\
&- \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[\omega_{n2}^2 Y_{nm}^{OE} + 2(-1)^n \omega_m Y_{m0}^{2,E0} \right] \sin \omega_m x_1 \cos \omega_{n2} x_2 - \tag{20} \\
&- \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[\omega_{n2}^2 Y_{nm}^{EO} + 2(-1)^n \omega_{n2} Y_{m0}^{2,E0} \right] \cos \omega_m x_1 \sin \omega_{n2} x_2 - \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[\omega_{n2}^2 Y_{nm}^{OO} + 2(-1)^n \omega_{n2} Y_{m0}^{2,00} \right] \sin \omega_m x_1 \sin \omega_{n2} x_2 = \\
&= y'_{O,2}(x_1, L_2) + \sum_{n=1}^{\infty} \left[-\omega_{n2}^2 Y_{n,2}^{C,E}(x_1) + 2(-1)^n y'_{O,2}(x_1, L_2) \right] \cos \omega_{n2} x_2 - \sum_{n=1}^{\infty} \left[\omega_{n2}^2 Y_{n,2}^{C,O}(x_1) + 2(-1)^n \omega_{n2} y_{O,2} \right] \sin \omega_{n2} x_2 \text{ I}
\end{aligned}$$

In all the two relations, $F_{C,i}^{C,0}$, $F_{C,i}^{C,E}$, $F_{C,i}^{C,O}$, $i=1, 2$ have the meaning from the expression (8).

We notice once again that the two differentiations can be made by term-by-term differentiation of expression (4), with the compensation of the discontinuities of the boundary, when the derived term is odd with respect to the differentiation variable. Therefore, the formula previously deduced for the functions of a single variable can be applied directly [2]:

$$\text{For } \bar{y}(x) = y_0 + \sum_{n=1}^{\infty} A_n \cos \omega_n x + \sum_{n=1}^{\infty} B_n \sin \omega_n x, \quad y_o(x) = 1/2 \cdot [y(x) - y(-x)]$$

$$\bar{y}'' = \sum_{n=1}^{\infty} \left[\left(-A_n \omega_n^2 + 2(-1)^n \frac{y'_o(L)}{L} \right) \cos \omega_n x - \left(B_n \omega_n^2 + 2(-1)^n \omega_n \frac{y_o(L)}{L} \right) \sin \omega_n x \right] + \frac{y'_o(L)}{L}$$

For the partial derivatives with respect to all the two variables:

$$\begin{aligned}
\frac{\partial^2 y}{\partial x_1 \partial x_2} &= \frac{\partial}{\partial x_1} \left[\frac{\partial y}{\partial x_2}(C, x_1) \right]^{SFS} = Y_{00}^{\prime 1} + \sum_{n=1}^{\infty} \left[\omega_{n2} Y_{0n}^{1,00} + 2(-1)^n Y_{00}^{\prime 1} \right] \cos \omega_{n2} x_2 - \sum_{n=1}^{\infty} \omega_{n2} Y_{0n}^{1,0E} \sin \omega_{n2} x_2 + \\
&+ \sum_{m=1}^{\infty} \omega_m \left\{ Y_{m0}^{2,00} + \sum_{n=1}^{\infty} \left[\omega_{n2} Y_{nm}^{OO} + 2(-1)^n Y_{m0}^{2,00} \right] \cos \omega_{n2} x_2 - \sum_{n=1}^{\infty} \omega_{n2} Y_{nm}^{OE} \sin \omega_{n2} x_2 \right\} \cos \omega_m x_1 + \\
&+ 2(-1)^m \left(Y_{00}^{\prime 1} + \sum_{n=1}^{\infty} \left[\omega_{n2} Y_{0n}^{1,00} + 2(-1)^n Y_{00}^{\prime 1} \right] \cos \omega_{n2} x_2 - \sum_{n=1}^{\infty} \omega_{n2} Y_{0n}^{1,0E} \sin \omega_{n2} x_2 \right) \cos \omega_m x_1 + \\
&- \sum_{m=1}^{\infty} \omega_m \left\{ Y_{m0}^{2,E0} + \sum_{n=1}^{\infty} \left[\omega_{n2} Y_{nm}^{EO} + 2(-1)^n Y_{m0}^{2,E0} \right] \cos \omega_{n2} x_2 - \sum_{n=1}^{\infty} \omega_{n2} Y_{nm}^{EE} \sin \omega_{n2} x_2 \right\} \sin \omega_m x_1 = \\
&= Y_{00}^{\prime 1} + \sum_{m=1}^{\infty} \left[\omega_m Y_{m0}^{2,00} + 2(-1)^m Y_{00}^{\prime 1} \right] \cos \omega_m x_1 - \sum_{m=1}^{\infty} \omega_m Y_{m0}^{2,E0} \sin \omega_m x_1 +
\end{aligned}$$

$$\begin{aligned}
& + \sum_{n=1}^{\infty} \left[\omega_{n2} Y_{0n}^{1.0O} + 2(-1)^n Y_{00}^{r1.} \right] \cos \omega_{n2} x_2 - \sum_{n=1}^{\infty} \omega_{n2} Y_{0n}^{1.0E} \sin \omega_{n2} x_2 + \\
& + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[\omega_{m1} \omega_{n2} Y_{mn}^{OO} + 2(-1)^n \omega_{m1} Y_{m0}^{2.OO} + 2(-1)^m \omega_{n2} Y_{0n}^{1.0O} + 4(-1)^{m+n} Y_{00}^{r1.} \right] \cos \omega_{m1} x_1 \cos \omega_{n2} x_2 - \\
& - \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[\omega_{m1} \omega_{n2} Y_{mn}^{EO} + 2(-1)^n \omega_{m1} Y_{m0}^{2.E0} \right] \sin \omega_{m1} x_1 \cos \omega_{n2} x_2 - \\
& - \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[\omega_{m1} \omega_{n2} Y_{mn}^{OE} + 2(-1)^m \omega_{n2} Y_{0n}^{1.0E} \right] \cos \omega_{m1} x_1 \sin \omega_{n2} x_2 + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \omega_{m1} \omega_{n2} Y_{mn}^{EE} \sin \omega_{m1} x_1 \sin \omega_{n2} x_2
\end{aligned} \tag{21}$$

$$\begin{aligned}
\left[\frac{\partial^2 y}{\partial x_2 \partial x_1} \right]^{SFS} &= \frac{\partial}{\partial x_2} \left[\frac{\partial y}{\partial x_1} (C, x_2) \right]^{SFS} = Y_{00}^{r2.} + \sum_{m=1}^{\infty} \left[\omega_{m1} Y_{m0}^{2.OO} + 2(-1)^m Y_{00}^{r2.} \right] \cos \omega_{m1} x_1 - \sum_{m=1}^{\infty} \omega_{m1} Y_{m0}^{2.E0} \sin \omega_{m1} x_1 + \\
& + \sum_{n=1}^{\infty} \left\{ \omega_{n2} Y_{0n}^{1.0O} + \omega_{n2} \sum_{m=1}^{\infty} \left[\omega_{m1} Y_{mn}^{OO} + 2(-1)^m Y_{0n}^{1.0O} \right] \cos \omega_{m1} x_1 - \omega_{n2} \sum_{m=1}^{\infty} \omega_{m1} Y_{mn}^{EO} \sin \omega_{m1} x_1 \right\} \cos \omega_{n2} x_2 + \\
& + 2(-1)^n \sum_{n=1}^{\infty} \left\{ Y_{00}^{r2.} + \sum_{m=1}^{\infty} \left[\omega_{m1} Y_{m0}^{2.OO} + 2(-1)^m Y_{00}^{r2.} \right] \cos \omega_{m1} x_1 - \sum_{m=1}^{\infty} \omega_{m1} Y_{m0}^{2.E0} \sin \omega_{m1} x_1 \right\} \cos \omega_{n2} x_2 - \\
& - \sum_{n=1}^{\infty} \left\{ \omega_{n2} Y_{0n}^{1.0E} + \omega_{n2} \sum_{m=1}^{\infty} \left[\omega_{m1} Y_{mn}^{OE} + 2(-1)^m Y_{0n}^{1.0E} \right] \cos \omega_{m1} x_1 - \omega_{n2} \sum_{m=1}^{\infty} \omega_{m1} Y_{mn}^{EE} \sin \omega_{m1} x_1 \right\} \sin \omega_{n2} x_2 = \\
& = Y_{00}^{r2.} + \sum_{m=1}^{\infty} \left[\omega_{m1} Y_{m0}^{2.OO} + 2(-1)^m Y_{00}^{r2.} \right] \cos \omega_{m1} x_1 - \sum_{m=1}^{\infty} \omega_{m1} Y_{m0}^{2.E0} \sin \omega_{m1} x_1 + \\
& + \sum_{n=1}^{\infty} \left[\omega_{n2} Y_{0n}^{1.0O} + 2(-1)^n Y_{00}^{r2.} \right] \cos \omega_{n2} x_2 - \sum_{n=1}^{\infty} \omega_{n2} Y_{0n}^{1.0E} \sin \omega_{n2} x_2 + \\
& + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[\omega_{m1} \omega_{n2} Y_{mn}^{OO} + 2(-1)^m \omega_{n2} Y_{0n}^{1.0O} + 2(-1)^n \omega_{m1} Y_{m0}^{2.OO} + 4(-1)^{m+n} Y_{00}^{r2.} \right] \cos \omega_{m1} x_1 \cos \omega_{n2} x_2 - \\
& - \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[\omega_{m1} \omega_{n2} Y_{mn}^{EO} + 2(-1)^n \omega_{m1} Y_{m0}^{2.E0} \right] \sin \omega_{m1} x_1 \cos \omega_{n2} x_2 - \\
& - \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[\omega_{m1} \omega_{n2} Y_{mn}^{OE} + 2(-1)^m \omega_{n2} Y_{0n}^{1.0E} \right] \cos \omega_{m1} x_1 \sin \omega_{n2} x_2 + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \omega_{m1} \omega_{n2} Y_{mn}^{EE} \sin \omega_{m1} x_1 \sin \omega_{n2} x_2
\end{aligned} \tag{22}$$

and because, according to (18), $Y_{00}^{r1.} = Y_{00}^{r2.}$, the two expansions are equal.

Higher-order partial integrals are calculated starting from relations (14) and (15) and applying formula (8):

$$\begin{aligned}
\left[\frac{\partial^{-2} y}{\partial x_1^2} \right]^{SFS} &= \left[\int_0^{x_1} \left(\int_0^{x_1} y(C, x_1) dx_1 \right) dx_1 \right]^{SFS} = \\
& = \sum_{m=1}^{\infty} \frac{Y_{m0}^{E0} + \sum_{n=1}^{\infty} Y_{mn}^{EE} \cos \omega_{n2} x_2 + \sum_{n=1}^{\infty} Y_{mn}^{EO} \sin \omega_{n2} x_2 - 2(-1)^m \left(Y_{00} + \sum_{n=1}^{\infty} Y_{0n}^{0E} \cos \omega_{n2} x_2 + \sum_{n=1}^{\infty} Y_{0n}^{0O} \sin \omega_{n2} x_2 \right)}{\omega_{m1}^2} - \\
& - \sum_{m=1}^{\infty} \frac{Y_{m0}^{E0} + \sum_{n=1}^{\infty} Y_{mn}^{EE} \cos \omega_{n2} x_2 + \sum_{n=1}^{\infty} Y_{mn}^{EO} \sin \omega_{n2} x_2 - 2(-1)^m \left(Y_{00} + \sum_{n=1}^{\infty} Y_{0n}^{0E} \cos \omega_{n2} x_2 + \sum_{n=1}^{\infty} Y_{0n}^{0O} \sin \omega_{n2} x_2 \right)}{\omega_{m1}^2} \cos \omega_{m1} + \\
& + \sum_{m=1}^{\infty} \frac{\left[1 - 2(-1)^m \left(Y_{m0}^{O0} + \sum_{n=1}^{\infty} Y_{mn}^{OE} \cos \omega_{n2} x_2 + \sum_{n=1}^{\infty} Y_{mn}^{OO} \sin \omega_{n2} x_2 \right) \right]}{\omega_{m1}^2} \sin \omega_{m1} x_1
\end{aligned}$$

$$\begin{aligned}
\left[\frac{\partial^{-2} y}{\partial x_1^2} \right]^{SFS} &= \sum_{m=1}^{\infty} \frac{Y_{m0}^{E0} - 2(-1)^m Y_{00}}{\omega_{m1}^2} - \sum_{m=1}^{\infty} \frac{Y_{m0}^{E0} - 2(-1)^m Y_{00}}{\omega_{m1}^2} \cos \omega_{m1} x_1 + \sum_{m=1}^{\infty} \left[1 - 2(-1)^m \right] \frac{Y_{m0}^{O0}}{\omega_{m1}^2} \sin \omega_{m1} x_1 + \\
&+ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{Y_{mn}^{EE} - 2(-1)^m Y_{0n}^{OE}}{\omega_{m1}^2} \cos \omega_{n2} x_2 + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{Y_{mn}^{EO} - 2(-1)^m Y_{0n}^{OO}}{\omega_{m1}^2} \sin \omega_{n2} x_2 - \\
&- \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{Y_{mn}^{EE} - 2(-1)^m Y_{0n}^{OE}}{\omega_{m1}^2} \cos \omega_{m1} \cos \omega_{n2} x_2 - \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{Y_{mn}^{EO} - 2(-1)^m Y_{0n}^{OO}}{\omega_{m1}^2} \cos \omega_{m1} \sin \omega_{n2} x_2 \\
&+ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[1 - 2(-1)^m \right] \frac{Y_{mn}^{OE}}{\omega_{m1}^2} \sin \omega_{m1} x_1 \cos \omega_{n2} x_2 + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[1 - 2(-1)^m \right] \frac{Y_{mn}^{OO}}{\omega_{m1}^2} \sin \omega_{m1} x_1 \sin \omega_{n2} x_2 \\
\left[\frac{\partial^{-2} y}{\partial x_1 \partial x_2} \right]^{SFS} &= \left[\int_0^{x_1} \left(\int_0^{x_2} f(C, x_2) dx_2 \right) dx_1 \right]^{SFS} \\
\left[\frac{\partial^{-1} y}{\partial x_2} (C, x_1) \right]^{SFS} &= \frac{\sum_{n=1}^{\infty} Y_{0n}^{O0} - \sum_{n=1}^{\infty} Y_{0n}^{OO} \cos \omega_{n2} x_2 + \sum_{n=1}^{\infty} \left[Y_{0n}^{OE} - 2(-1)^n Y_{00} \right] \sin \omega_{n2} x_2}{\omega_{n2}} + \\
&+ \frac{\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} Y_{mn}^{EO} - \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} Y_{mn}^{EO} \cos \omega_{n2} x_2 + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[Y_{mn}^{EE} - 2(-1)^n Y_{m0}^{E0} \right] \sin \omega_{n2} x_2}{\omega_{n2}} \cos \omega_{m1} x_1 + \\
&+ \frac{\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} Y_{mn}^{OO} - \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} Y_{mn}^{OO} \cos \omega_{n2} x_2 + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[Y_{mn}^{EO} - 2(-1)^n Y_{m0}^{O0} \right] \sin \omega_{n2} x_2}{\omega_{n2}} \sin \omega_{m1} x_1 \\
\left[\frac{\partial^{-2} y}{\partial x_1 \partial x_2} \right]^{SFS} &= \frac{\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} Y_{mn}^{OO} - \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} Y_{mn}^{OO} \cos \omega_{n2} x_2 + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[Y_{mn}^{EO} - 2(-1)^n Y_{m0}^{O0} \right] \sin \omega_{n2} x_2}{\omega_{m1} \omega_{n2}} - \\
&- \frac{\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} Y_{mn}^{OO} - \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} Y_{mn}^{OO} \cos \omega_{n2} x_2 + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[Y_{mn}^{EO} - 2(-1)^n Y_{m0}^{O0} \right] \sin \omega_{n2} x_2}{\omega_{m1} \omega_{n2}} \cos \omega_{m1} x_1 + \\
&+ \frac{\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} Y_{mn}^{EO} - \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} Y_{mn}^{EO} \cos \omega_{n2} x_2 + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[Y_{mn}^{EE} - 2(-1)^n Y_{m0}^{E0} \right] \sin \omega_{n2} x_2}{\omega_{m1} \omega_{n2}} \sin \omega_{m1} x_1 + \\
&- \sum_{m=1}^{\infty} 2(-1)^m \frac{\sum_{n=1}^{\infty} Y_{0n}^{O0} - \sum_{n=1}^{\infty} Y_{0n}^{OO} \cos \omega_{n2} x_2 + \sum_{n=1}^{\infty} \left[Y_{0n}^{OE} - 2(-1)^n Y_{00} \right] \sin \omega_{n2} x_2}{\omega_{m1} \omega_{n2}} \sin \omega_{m1} x_1 = \\
&= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{Y_{mn}^{OO}}{\omega_{m1} \omega_{n2}} - \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{Y_{mn}^{OO}}{\omega_{m1} \omega_{n2}} \cos \omega_{m1} x_1 + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{Y_{mn}^{EO} - 2(-1)^m Y_{0n}^{O0}}{\omega_{m1} \omega_{n2}} \sin \omega_{m1} x_1 + \\
&- \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{Y_{mn}^{OO}}{\omega_{m1} \omega_{n2}} \cos \omega_{n2} x_2 + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{Y_{mn}^{EO} - 2(-1)^n Y_{m0}^{O0}}{\omega_{m1} \omega_{n2}} \sin \omega_{n2} x_2 + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{Y_{mn}^{OO}}{\omega_{m1} \omega_{n2}} \cos \omega_{m1} \cos \omega_{n2} x_2 - \\
&- \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{Y_{mn}^{EO} - 2(-1)^n Y_{m0}^{O0}}{\omega_{m1} \omega_{n2}} \cos \omega_{m1} \sin \omega_{n2} x_2 - \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{Y_{mn}^{EO} - 2(-1)^m Y_{0n}^{O0}}{\omega_{m1} \omega_{n2}} \sin \omega_{m1} x_1 \cos \omega_{n2} x_2 + \\
&+ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{Y_{mn}^{EE} - 2(-1)^n Y_{m0}^{E0} - 2(-1)^m Y_{0n}^{OE} + 4(-1)^{m+n} Y_{00}}{\omega_{m1} \omega_{n2}} \sin \omega_{m1} x_1 \sin \omega_{n2} x_2
\end{aligned}$$

By repeatedly applying this procedure, the derivatives and the partial integrals of any order of two-variable functions are obtained. For functions with several variables, the

procedure is identical, but the number of coefficients to be calculated is higher, being equal to N^3 , where N is the number of the variables of the function.

5. The Fourier series expansion of the product of two periodic functions depending on two variables

As in the case of monovariate functions, in the domain $[-L_1, L_1] \times [-L_2, L_2]$, the product $p(x_1, x_2) = a(x_1, x_2) \cdot y(x_1, x_2)$ between two functions of two variables, square-integrables, it is also an integrable square-function [5]. The calculation of the coefficients of SFS expansion of the product can be done with the calculation formulas (4), by calculating the definite integrals of the type

$\frac{1}{\|f\|^2} \int_{-L_1}^{L_1} \int_{-L_2}^{L_2} p(x_1, x_2) f(\cos \omega_{m1} x_1, \cos \omega_{n2} x_2, \sin \omega_{m1} x_1, \sin \omega_{n2} x_2) dx_1 dx_2$, for each positive or zero integer numbers m and n , ($= 0, 1, 2, \dots, \infty$).

As with monovariate functions, if the Fourier coefficients of the series expansion of each of the terms of the product are known, algebraic formulas for calculating the Fourier coefficients of the series expansion of the product can be found and used. These formulas are essential if we want to transform the PDEs into algebraic equations, or if we want to calculate the dependency relations between the coefficients of the product of these two multivariable functions (in the case of the square, the two functions are identical) and the coefficients of each of its terms.

In either of the two variants, if some degree of approximation is allowed, the computation can also be performed by numerical methods.

Let be the two functions with two variables $y(x_1, x_2)$ and $a(x_1, x_2)$, defined on the domain $[-L_1, L_1] \times [-L_2, L_2]$, periodic, square-integrable, which can be expanded in SFS:

$$y(x_1, x_2) = y_{00} + y_{10}(x_1) + y_{01}(x_2) + y_{11}(x_1, x_2) = y_{00} + y_{E0}(x_1) + y_{O0}(x_1) + y_{0E}(x_2) + y_{0O}(x_2) + y_{EE}(x_1, x_2) + y_{EO}(x_1, x_2) + y_{OE}(x_1, x_2) + y_{OO}(x_1, x_2)$$

$$\begin{aligned} [y(x_1, x_2)]^{SFS} &= Y_{00} + \sum_{m=1}^{\infty} (Y_{m0}^{E0} \cos \omega_{m1} x_1 + Y_{m0}^{O0} \sin \omega_{m1} x_1) + \sum_{n=1}^{\infty} (Y_{0n}^{0E} \cos \omega_{n2} x_2 + Y_{0n}^{0O} \sin \omega_{n2} x_2) + \\ &+ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (Y_{mn}^{EE} \cos \omega_{m1} x_1 \cos \omega_{n2} x_2 + Y_{mn}^{EO} \cos \omega_{m1} x_1 \sin \omega_{n2} x_2 + Y_{mn}^{OE} \sin \omega_{m1} x_1 \cos \omega_{n2} x_2 + Y_{mn}^{OO} \sin \omega_{m1} x_1 \sin \omega_{n2} x_2) \\ a(x_1, x_2) &= a_{00} + a_{10}(x_1) + a_{01}(x_2) + a_{11}(x_1, x_2) = a_{00} + a_{E0}(x_1) + a_{O0}(x_1) + a_{0E}(x_2) + a_{0O}(x_2) + \\ &+ a_{EE}(x_1, x_2) + a_{EO}(x_1, x_2) + a_{OE}(x_1, x_2) + a_{OO}(x_1, x_2) \end{aligned}$$

$$\begin{aligned} [a(x_1, x_2)]^{SFS} &= A_{00} + \sum_{m=1}^{\infty} (A_{m0}^{E0} \cos \omega_{m1} x_1 + A_{m0}^{O0} \sin \omega_{m1} x_1) + \sum_{n=1}^{\infty} (A_{0n}^{0E} \cos \omega_{n2} x_2 + A_{0n}^{0O} \sin \omega_{n2} x_2) + \\ &+ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (A_{mn}^{EE} \cos \omega_{m1} x_1 \cos \omega_{n2} x_2 + A_{mn}^{EO} \cos \omega_{m1} x_1 \sin \omega_{n2} x_2 + A_{mn}^{OE} \sin \omega_{m1} x_1 \cos \omega_{n2} x_2 + A_{mn}^{OO} \sin \omega_{m1} x_1 \sin \omega_{n2} x_2) \end{aligned}$$

All the components of these expansions except y_{00} and a_{00} have mean values equal with zero.

Therefore, the SFS expansion of the product will take the form:

$$\begin{aligned} (ya)^{SFS} &= \bar{P}(x, y) = [(y_{00} + y_{10} + y_{01} + y_{11}) \cdot (a_{00} + a_{10} + a_{01} + a_{11})]^{SFS} = \\ &= P_{00} + \sum_{m=1}^{\infty} (P_{m0}^{E0} \cos \omega_{m1} x_1 + P_{m0}^{O0} \sin \omega_{m1} x_1) + \sum_{n=1}^{\infty} (P_{0n}^{0E} \cos \omega_{n2} x_2 + P_{0n}^{0O} \sin \omega_{n2} x_2) + \\ &+ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (P_{mn}^{EE} \cos \omega_{m1} x_1 \cos \omega_{n2} x_2 + P_{mn}^{EO} \cos \omega_{m1} x_1 \sin \omega_{n2} x_2 + P_{mn}^{OE} \sin \omega_{m1} x_1 \cos \omega_{n2} x_2 + P_{mn}^{OO} \sin \omega_{m1} x_1 \sin \omega_{n2} x_2) \end{aligned}$$

The terms of the expansion in basic functions of the product are:

$$\begin{aligned}
&= y_{00}a_{00} + (y_{00}a_{E0} + a_{00}y_{E0} + y_{00}a_{O0} + a_{00}y_{O0} + y_{00}a_{0E} + a_{00}y_{0E} + y_{00}a_{0O} + a_{00}y_{0O}) + \\
&+ (y_{E0}a_{E0} + y_{E0}a_{O0} + y_{O0}a_{E0} + y_{O0}a_{O0} + y_{0E}a_{0E} + y_{0E}a_{0O} + y_{0O}a_{0E} + y_{0O}a_{0O}) + \\
&+ (y_{00}a_{EE} + a_{00}y_{EE} + y_{E0}a_{0E} + a_{E0}y_{0E} + y_{00}a_{EO} + a_{00}y_{EO} + y_{E0}a_{0O} + a_{E0}y_{0O}) + \\
&+ (y_{00}a_{OE} + a_{00}y_{OE} + y_{O0}a_{0E} + a_{O0}y_{0E} + y_{00}a_{OO} + a_{00}y_{OO} + y_{O0}a_{0O} + a_{O0}y_{0O}) + \\
&+ (y_{EE}a_{E0} + y_{EE}a_{O0} + y_{EO}a_{E0} + y_{EO}a_{O0} + y_{OE}a_{E0} + y_{OE}a_{O0} + y_{OO}a_{E0} + y_{OO}a_{O0}) + \\
&+ (y_{EE}a_{0E} + y_{EE}a_{0O} + y_{EO}a_{0E} + y_{EO}a_{0O} + y_{OE}a_{0E} + y_{OE}a_{0O} + y_{OO}a_{0E} + y_{OO}a_{0O}) + \\
&+ (a_{EE}y_{E0} + a_{EE}y_{O0} + a_{EO}y_{E0} + a_{EO}y_{O0} + a_{OE}y_{E0} + a_{OE}y_{O0} + a_{OO}y_{E0} + a_{OO}y_{O0}) + \\
&+ (a_{EE}y_{0E} + a_{EE}y_{0O} + a_{EO}y_{0E} + a_{EO}y_{0O} + a_{OE}y_{0E} + a_{OE}y_{0O} + a_{OO}y_{0E} + a_{OO}y_{0O}) + \\
&+ (y_{EE}a_{EE} + y_{EE}a_{EO} + y_{EO}a_{EE} + y_{EO}a_{EO} + y_{OE}a_{OE} + y_{OE}a_{OO} + y_{OO}a_{OE} + y_{OO}a_{OO}) + \\
&+ (y_{EE}a_{OE} + y_{EE}a_{OO} + y_{EO}a_{OE} + y_{EO}a_{OO} + y_{OE}a_{EE} + y_{OE}a_{EO} + y_{OO}a_{EE} + y_{OO}a_{EO})
\end{aligned}$$

Therefore, the SFS expansion of the product will be the sum of SFS expansions of these 81 elementary functions:

$$\begin{aligned}
[P(x_1, x_2)]^{SFS} &= Y_{00}A_{00} + \sum_{m=1}^{\infty} [(Y_{00}A_{m0}^{E0} + A_{00}Y_{m0}^{E0})\cos \omega_{m1}x_1 + (Y_{00}A_{m0}^{O0} + A_{00}Y_{m0}^{O0})\sin \omega_{m1}x_1] + \\
&+ \sum_{n=1}^{\infty} [(Y_{00}A_{0n}^{OE} + A_{00}Y_{0n}^{OE})\cos \omega_{n2}x_2 + (Y_{00}A_{0n}^{OO} + A_{00}Y_{0n}^{OO})\sin \omega_{n2}x_2] + \\
&+ \left(\sum_{m=1}^{\infty} Y_{m0}^{E0} \cos \omega_{m1}x_1 \right) \left(\sum_{m=1}^{\infty} A_{m0}^{E0} \cos \omega_{m1}x_1 \right) + \left(\sum_{m=1}^{\infty} Y_{m0}^{E0} \cos \omega_{m1}x_1 \right) \left(\sum_{m=1}^{\infty} A_{m0}^{O0} \sin \omega_{m1}x_1 \right) + \\
&+ \left(\sum_{m=1}^{\infty} Y_{m0}^{O0} \sin \omega_{m1}x_1 \right) \left(\sum_{m=1}^{\infty} A_{m0}^{E0} \cos \omega_{m1}x_1 \right) + \left(\sum_{m=1}^{\infty} Y_{m0}^{O0} \sin \omega_{m1}x_1 \right) \left(\sum_{m=1}^{\infty} A_{m0}^{O0} \sin \omega_{m1}x_1 \right) + \left(\sum_{n=1}^{\infty} Y_{0n}^{OE} \cos \omega_{n2}x_2 \right) \left(\sum_{n=1}^{\infty} A_{0n}^{OE} \cos \omega_{n2}x_2 \right) + \\
&+ \left(\sum_{n=1}^{\infty} Y_{0n}^{OE} \cos \omega_{n2}x_2 \right) \left(\sum_{n=1}^{\infty} A_{0n}^{OO} \sin \omega_{n2}x_2 \right) + \left(\sum_{n=1}^{\infty} Y_{0n}^{OO} \sin \omega_{n2}x_2 \right) \left(\sum_{n=1}^{\infty} A_{0n}^{OE} \cos \omega_{n2}x_2 \right) + \left(\sum_{n=1}^{\infty} Y_{0n}^{OO} \sin \omega_{n2}x_2 \right) \left(\sum_{n=1}^{\infty} A_{0n}^{OO} \sin \omega_{n2}x_2 \right) + \\
&+ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (Y_{00}A_{mn}^{EE} + A_{00}Y_{mn}^{EE} + Y_{m0}^{E0}A_{0n}^{OE} + A_{m0}^{E0}Y_{0n}^{OE}) \cos \omega_{m1}x_1 \cos \omega_{n2}x_2 + \\
&+ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (Y_{00}A_{mn}^{EO} + A_{00}Y_{mn}^{EO} + Y_{m0}^{E0}A_{0n}^{OO} + A_{m0}^{E0}Y_{0n}^{OO}) \cos \omega_{m1}x_1 \sin \omega_{n2}x_2 + \\
&+ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (Y_{00}A_{mn}^{OE} + A_{00}Y_{mn}^{OE} + Y_{m0}^{O0}A_{0n}^{OE} + A_{m0}^{O0}Y_{0n}^{OE}) \sin \omega_{m1}x_1 \cos \omega_{n2}x_2 + \\
&+ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (Y_{00}A_{mn}^{OO} + A_{00}Y_{mn}^{OO} + Y_{m0}^{O0}A_{0n}^{OO} + A_{m0}^{O0}Y_{0n}^{OO}) \sin \omega_{m1}x_1 \sin \omega_{n2}x_2 + \\
&+ \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} Y_{mn}^{EE} \cos \omega_{m1}x_1 \cos \omega_{n2}x_2 \right) \left(\sum_{m=1}^{\infty} A_{m0}^{E0} \cos \omega_{m1}x_1 \right) + \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} Y_{mn}^{EE} \cos \omega_{m1}x_1 \cos \omega_{n2}x_2 \right) \left(\sum_{m=1}^{\infty} A_{m0}^{O0} \sin \omega_{m1}x_1 \right) + \\
&+ \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} Y_{mn}^{EO} \cos \omega_{m1}x_1 \sin \omega_{n2}x_2 \right) \left(\sum_{m=1}^{\infty} A_{m0}^{E0} \cos \omega_{m1}x_1 \right) + \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} Y_{mn}^{EO} \cos \omega_{m1}x_1 \sin \omega_{n2}x_2 \right) \left(\sum_{m=1}^{\infty} A_{m0}^{O0} \sin \omega_{m1}x_1 \right) + \\
&+ \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} Y_{mn}^{OE} \sin \omega_{m1}x_1 \cos \omega_{n2}x_2 \right) \left(\sum_{m=1}^{\infty} A_{m0}^{E0} \cos \omega_{m1}x_1 \right) + \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} Y_{mn}^{OE} \sin \omega_{m1}x_1 \cos \omega_{n2}x_2 \right) \left(\sum_{m=1}^{\infty} A_{m0}^{O0} \sin \omega_{m1}x_1 \right) + \\
&+ \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} Y_{mn}^{OO} \sin \omega_{m1}x_1 \sin \omega_{n2}x_2 \right) \left(\sum_{m=1}^{\infty} A_{m0}^{E0} \cos \omega_{m1}x_1 \right) + \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} Y_{mn}^{OO} \sin \omega_{m1}x_1 \sin \omega_{n2}x_2 \right) \left(\sum_{m=1}^{\infty} A_{m0}^{O0} \sin \omega_{m1}x_1 \right) + \\
&+ \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} Y_{mn}^{EE} \cos \omega_{m1}x_1 \cos \omega_{n2}x_2 \right) \left(\sum_{n=1}^{\infty} A_{0n}^{OE} \cos \omega_{n2}x_2 \right) + \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} Y_{mn}^{EE} \cos \omega_{m1}x_1 \cos \omega_{n2}x_2 \right) \left(\sum_{n=1}^{\infty} A_{0n}^{OO} \sin \omega_{n2}x_2 \right) + \\
&+ \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} Y_{mn}^{EO} \cos \omega_{m1}x_1 \sin \omega_{n2}x_2 \right) \left(\sum_{n=1}^{\infty} A_{0n}^{OE} \cos \omega_{n2}x_2 \right) + \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} Y_{mn}^{EO} \cos \omega_{m1}x_1 \sin \omega_{n2}x_2 \right) \left(\sum_{n=1}^{\infty} A_{0n}^{OO} \sin \omega_{n2}x_2 \right) +
\end{aligned}$$

$$\begin{aligned}
& + \sum_{n=1}^{\infty} \left[(Y_{00} A_{0n}^{0E} + A_{00} Y_{0n}^{0E}) \cos \omega_{n2} x_2 + (Y_{00} A_{0n}^{0O} + A_{00} Y_{0n}^{0O}) \sin \omega_{n2} x_2 \right] + \\
& + \frac{1}{2} \sum_{m=1}^{\infty} Y_{m0}^{E0} A_{m0}^{E0} + \frac{1}{2} \sum_{m=1}^{\infty} \left[\sum_{l=1}^{\infty} Y_{l,0}^{E0} (A_{l+m,0}^{E0} + A_{|l-m|,0}^{E0}) \right] \cos \omega_{m1} x_1 + \frac{1}{2} \sum_{m=1}^{\infty} \left[\sum_{l=1}^{\infty} Y_{l,0}^{E0} (A_{l+m,0}^{0O} - \text{sgn}(l-m) A_{|l-m|,0}^{0O}) \right] \sin \omega_{m1} x_1 - \\
& - \frac{1}{2} \sum_{m=1}^{\infty} \left[\sum_{l=1}^{\infty} Y_{l,0}^{0O} (A_{l+m,0}^{E0} - A_{|l-m|,0}^{E0}) \right] \sin \omega_{m1} x_1 + \frac{1}{2} \sum_{m=1}^{\infty} Y_{m0}^{0O} A_{m0}^{0O} + \frac{1}{2} \sum_{m=1}^{\infty} \left[\sum_{l=1}^{\infty} Y_{l,0}^{0O} (A_{l+m,0}^{0O} + \text{sgn}(l-m) A_{|l-m|,0}^{0O}) \right] \cos \omega_{m1} x_1 + \\
& + \frac{1}{2} \sum_{n=1}^{\infty} Y_{0n}^{0E} A_{0n}^{0E} + \frac{1}{2} \sum_{n=1}^{\infty} \left[\sum_{l=1}^{\infty} Y_{0,l}^{0E} (A_{0,l+n}^{0E} + A_{0,|l-n|}^{0E}) \right] \cos \omega_{n2} x_2 + \frac{1}{2} \sum_{n=1}^{\infty} \left[\sum_{l=1}^{\infty} Y_{0,l}^{0E} (A_{0,l+n}^{0O} - \text{sgn}(l-n) A_{0,|l-n|}^{0O}) \right] \sin \omega_{n2} x_2 - \\
& - \frac{1}{2} \sum_{n=1}^{\infty} \left[\sum_{l=1}^{\infty} Y_{0,l}^{0O} (A_{0,l+n}^{0E} - A_{0,|l-n|}^{0E}) \right] \sin \omega_{n2} x_2 + \frac{1}{2} \sum_{n=1}^{\infty} Y_{0,n}^{0O} A_{0,n}^{0O} + \frac{1}{2} \sum_{n=1}^{\infty} \left[\sum_{l=1}^{\infty} Y_{0,l}^{0O} (A_{0,l+n}^{0O} + \text{sgn}(l-n) A_{0,|l-n|}^{0O}) \right] \cos \omega_{n2} x_2 + \\
& + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (Y_{00} A_{mn}^{EO} + A_{00} Y_{mn}^{EO} + Y_{m0}^{E0} A_{0n}^{0O} + A_{m0}^{E0} Y_{0n}^{0O}) \cos \omega_{m1} x_1 \sin \omega_{n2} x_2 + \\
& + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (Y_{00} A_{mn}^{EE} + A_{00} Y_{mn}^{EE} + Y_{m0}^{E0} A_{0n}^{0E} + A_{m0}^{E0} Y_{0n}^{0E}) \cos \omega_{m1} x_1 \cos \omega_{n2} x_2 + \\
& + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (Y_{00} A_{mn}^{OO} + A_{00} Y_{mn}^{OO} + Y_{m0}^{0O} A_{0n}^{0O} + A_{m0}^{0O} Y_{0n}^{0O}) \sin \omega_{m1} x_1 \sin \omega_{n2} x_2 + \\
& + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (Y_{00} A_{mn}^{OE} + A_{00} Y_{mn}^{OE} + Y_{m0}^{0O} A_{0n}^{0E} + A_{m0}^{0O} Y_{0n}^{0E}) \sin \omega_{m1} x_1 \cos \omega_{n2} x_2 + \\
& + \frac{1}{2} \sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} Y_{nm}^{EE} A_{m0}^{E0} \right) \cos \omega_{n2} x_2 + \frac{1}{2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[\sum_{l=1}^{\infty} Y_{l,n}^{EE} (A_{l+m,0}^{E0} + A_{|l-m|,0}^{E0}) \right] \cos \omega_{m1} x_1 \cos \omega_{n2} x_2 + \\
& + \frac{1}{2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[\sum_{l=1}^{\infty} Y_{l,n}^{EE} (A_{l+m,0}^{0O} - \text{sgn}(l-m) A_{|l-m|,0}^{0O}) \right] \sin \omega_{m1} x_1 \cos \omega_{n2} x_2 + \\
& + \frac{1}{2} \sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} Y_{nm}^{EO} A_{m0}^{E0} \right) \sin \omega_{n2} x_2 + \frac{1}{2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[\sum_{l=1}^{\infty} Y_{l,n}^{EO} (A_{l+m,0}^{E0} + A_{|l-m|,0}^{E0}) \right] \cos \omega_{m1} x_1 \sin \omega_{n2} x_2 + \\
& + \frac{1}{2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[\sum_{l=1}^{\infty} Y_{l,n}^{EO} (A_{l+m,0}^{0O} - \text{sgn}(l-m) A_{|l-m|,0}^{0O}) \right] \sin \omega_{m1} x_1 \sin \omega_{n2} x_2 - \\
& - \frac{1}{2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[\sum_{l=1}^{\infty} Y_{l,n}^{OE} (A_{l+m,0}^{E0} - A_{|l-m|,0}^{E0}) \right] \sin \omega_{m1} x_1 \cos \omega_{n2} x_2 + \\
& + \frac{1}{2} \sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} Y_{mn}^{OE} A_{m0}^{0O} \right) \cos \omega_{n2} x_2 + \frac{1}{2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[\sum_{l=1}^{\infty} Y_{l,n}^{OE} (A_{l+m,0}^{0O} + \text{sgn}(l-m) A_{|l-m|,0}^{0O}) \right] \cos \omega_{m1} x_1 \cos \omega_{n2} x_2 - \\
& - \frac{1}{2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[\sum_{l=1}^{\infty} Y_{l,n}^{OO} (A_{l+m,0}^{E0} - A_{|l-m|,0}^{E0}) \right] \sin \omega_{m1} x_1 \sin \omega_{n2} x_2 + \\
& + \frac{1}{2} \sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} Y_{mn}^{OO} A_{m0}^{0O} \right) \sin \omega_{n2} x_2 + \frac{1}{2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[\sum_{l=1}^{\infty} Y_{l,n}^{OO} (A_{l+m,0}^{0O} + \text{sgn}(l-m) A_{|l-m|,0}^{0O}) \right] \cos \omega_{m1} x_1 \sin \omega_{n2} x_2 + \\
& + \frac{1}{2} \sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} Y_{mn}^{EE} A_{0n}^{0E} \right) \cos \omega_{m1} x_1 + \frac{1}{2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[\sum_{l=1}^{\infty} Y_{m,l}^{EE} (A_{0,l+n}^{0E} + A_{0,|l-n|}^{0E}) \right] \cos \omega_{m1} x_1 \cos \omega_{n2} x_2 + \\
& + \frac{1}{2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[\sum_{l=1}^{\infty} Y_{m,l}^{EE} (A_{0,l+n}^{0O} - A_{0,|l-n|}^{0O}) \right] \cos \omega_{m1} x_1 \sin \omega_{n2} x_2 - \\
& - \frac{1}{2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[\sum_{l=1}^{\infty} Y_{m,l}^{EO} (A_{0,l+n}^{0E} - A_{0,|l-n|}^{0E}) \right] \cos \omega_{m1} x_1 \sin \omega_{n2} x_2 + \\
& + \frac{1}{2} \sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} Y_{mn}^{EO} A_{0n}^{0O} \right) \cos \omega_{m1} x_1 + \frac{1}{2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[\sum_{l=1}^{\infty} Y_{m,l}^{EO} (A_{0,l+n}^{0O} + \text{sgn}(l-n) A_{0,|l-n|}^{0O}) \right] \cos \omega_{m1} x_1 \cos \omega_{n2} x_2 +
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2} \sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} Y_{mn}^{OE} A_{0n}^{OE} \right) \sin \omega_{m1} x_1 + \frac{1}{2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[\sum_{l=1}^{\infty} Y_{m,l}^{OE} \left(A_{0,l+n}^{OE} + A_{0,|l-n|}^{OE} \right) \right] \sin \omega_{m1} x_1 \cos \omega_{n2} x_2 - \\
& - \frac{1}{2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[\sum_{l=1}^{\infty} Y_{m,l}^{OE} \left(A_{0,l+n}^{OO} - A_{0,|l-n|}^{OO} \right) \right] \sin \omega_{m1} x_1 \sin \omega_{n2} x_2 - \\
& - \frac{1}{2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[\sum_{l=1}^{\infty} Y_{m,l}^{OO} \left(A_{0,l+n}^{OE} - A_{0,|l-n|}^{OE} \right) \right] \sin \omega_{m1} x_1 \sin \omega_{n2} x_2 + \\
& + \frac{1}{2} \sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} Y_{mn}^{OO} A_{0n}^{OO} \right) \sin \omega_{m1} x_1 + \frac{1}{2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[\sum_{l=1}^{\infty} Y_{m,l}^{OO} \left(A_{0,l+n}^{OO} + \operatorname{sgn}(l-n) A_{0,|l-n|}^{OO} \right) \right] \sin \omega_{m1} x_1 \cos \omega_{n2} x_2 + \\
& + \frac{1}{2} \sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} Y_{m0}^{E0} A_{mn}^{EE} \right) \cos \omega_{n2} x_2 + \frac{1}{2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[\sum_{l=1}^{\infty} Y_{l,0}^{E0} \left(A_{l+m,n}^{EE} + A_{|l-m|,n}^{EE} \right) \right] \cos \omega_{m1} x_1 \cos \omega_{n2} x_2 - \\
& - \frac{1}{2} \sum_{n=1}^{\infty} \left\{ \sum_{m=1}^{\infty} \left[\sum_{l=1}^{\infty} Y_{l,0}^{OO} \left(A_{l+m,n}^{EE} - A_{|l-m|,n}^{EE} \right) \right] \sin \omega_{m1} x_1 \right\} \cos \omega_{n2} x_2 \\
& + \frac{1}{2} \sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} Y_{m0}^{E0} A_{mn}^{EO} \right) \sin \omega_{n2} x_2 + \frac{1}{2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[\sum_{l=1}^{\infty} Y_{l,0}^{E0} \left(A_{l+m,n}^{EO} + A_{|l-m|,n}^{EO} \right) \right] \cos \omega_{m1} x_1 \sin \omega_{n2} x_2 + \\
& - \frac{1}{2} \sum_{n=1}^{\infty} \left\{ \sum_{m=1}^{\infty} \left[\sum_{l=1}^{\infty} Y_{l,0}^{OO} \left(A_{l+m,n}^{EO} - A_{|l-m|,n}^{EO} \right) \right] \sin \omega_{m1} x_1 \right\} \cos \omega_{n2} x_2 + \\
& + \frac{1}{2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[\sum_{l=1}^{\infty} Y_{l,0}^{E0} \left(A_{l+m,n}^{OE} - \operatorname{sgn}(l-m) A_{|l-m|,n}^{OE} \right) \right] \sin \omega_{m1} x_1 \cos \omega_{n2} x_2 + \\
& + \frac{1}{2} \sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} Y_{m0}^{OO} A_{mn}^{OE} \right) \cos \omega_{n2} x_2 + \frac{1}{2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[\sum_{l=1}^{\infty} Y_{l,0}^{OO} \left(A_{l+m,n}^{OE} + \operatorname{sgn}(l-m) A_{|l-m|,n}^{OE} \right) \right] \sin \omega_{m1} x_1 \cos \omega_{n2} x_2 - \\
& + \frac{1}{2} \sum_{n=1}^{\infty} \left\{ \sum_{m=1}^{\infty} \left[\sum_{l=1}^{\infty} Y_{l,0}^{E0} \left(A_{l+m,n}^{OO} - \operatorname{sgn}(l-m) A_{|l-m|,n}^{OO} \right) \right] \sin \omega_{m1} x_1 \right\} \sin \omega_{n2} x_2 + \\
& + \frac{1}{2} \sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} Y_{m0}^{OO} A_{mn}^{OO} \right) \sin \omega_{n2} x_2 + \frac{1}{2} \sum_{n=1}^{\infty} \left\{ \sum_{m=1}^{\infty} \left[\sum_{l=1}^{\infty} Y_{l,0}^{OO} \left(A_{l+m,n}^{OO} + \operatorname{sgn}(l-m) A_{|l-m|,n}^{OO} \right) \right] \cos \omega_{m1} x_1 \right\} \sin \omega_{n2} x_2 + \\
& + \frac{1}{2} \sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} Y_{0n}^{OE} A_{mn}^{EE} \right) \cos \omega_{m1} x_1 + \frac{1}{2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[\sum_{l=1}^{\infty} Y_{0,l}^{OE} \left(A_{m,l+n}^{EE} + A_{m,|l-n|}^{EE} \right) \right] \cos \omega_{m1} x_1 \cos \omega_{n2} x_2 - \\
& - \frac{1}{2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[\sum_{l=1}^{\infty} Y_{0,l}^{OO} \left(A_{m,l+n}^{EE} - A_{m,|l-n|}^{EE} \right) \right] \cos \omega_{m1} x_1 \sin \omega_{n2} x_2 \\
& + \frac{1}{2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[\sum_{l=1}^{\infty} Y_{0,l}^{OE} \left(A_{m,l+n}^{EO} - \operatorname{sgn}(l-n) A_{m,|l-n|}^{EO} \right) \right] \cos \omega_{m1} x_1 \sin \omega_{n2} x_2 + \\
& + \frac{1}{2} \sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} Y_{0n}^{OO} A_{mn}^{EO} \right) \cos \omega_{m1} x_1 + \frac{1}{2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[\sum_{l=1}^{\infty} Y_{0,l}^{OO} \left(A_{m,l+n}^{EO} + \operatorname{sgn}(l-n) A_{m,|l-n|}^{EO} \right) \right] \cos \omega_{m1} x_1 \cos \omega_{n2} x_2 + \\
& + \frac{1}{2} \sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} Y_{0n}^{OE} A_{mn}^{OE} \right) \sin \omega_{m1} x_1 + \frac{1}{2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[\sum_{l=1}^{\infty} Y_{0,l}^{OE} \left(A_{m,l+n}^{OE} + A_{m,|l-n|}^{OE} \right) \right] \sin \omega_{m1} x_1 \cos \omega_{n2} x_2 + \\
& - \frac{1}{2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[\sum_{l=1}^{\infty} Y_{0,l}^{OO} \left(A_{m,l+n}^{OE} - A_{m,|l-n|}^{OE} \right) \right] \sin \omega_{m1} x_1 \sin \omega_{n2} x_2 - \\
& + \frac{1}{2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[\sum_{l=1}^{\infty} Y_{0,l}^{OE} \left(A_{m,l+n}^{OO} - \operatorname{sgn}(l-n) A_{m,|l-n|}^{OO} \right) \right] \sin \omega_{m1} x_1 \sin \omega_{n2} x_2 + \\
& + \frac{1}{2} \sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} Y_{0n}^{OO} A_{mn}^{OO} \right) \sin \omega_{m1} x_1 + \frac{1}{2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[\sum_{l=1}^{\infty} Y_{0,l}^{OO} \left(A_{m,l+n}^{OO} + \operatorname{sgn}(l-n) A_{m,|l-n|}^{OO} \right) \right] \sin \omega_{m1} x_1 \cos \omega_{n2} x_2 +
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{4} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} Y_{m,n}^{EE} A_{m,n}^{EE} + \frac{1}{4} \sum_{m=1}^{\infty} \left[\sum_{n=1}^{\infty} \sum_{l=1}^{\infty} Y_{l,n}^{EE} (A_{l+m,n}^{EE} + A_{|l-m|,n}^{EE}) \right] \cos \omega_{m1} x_1 + \frac{1}{4} \sum_{n=1}^{\infty} \left[\sum_{m=1}^{\infty} \sum_{l=1}^{\infty} Y_{m,l}^{EE} (A_{m,l+n}^{EE} + A_{m,|l-m|}^{EE}) \right] \cos \omega_{n2} x_2 + \\
& + \frac{1}{4} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[\sum_{l=1}^{\infty} \sum_{r=1}^{\infty} (Y_{r+m,l}^{EE} + Y_{|r-m|,l}^{EE}) (A_{r,l+n}^{EE} + A_{r,|l-n|}^{EE}) \right] \cos \omega_{m1} x_1 \cos \omega_{n2} x_2 + \\
& + \frac{1}{4} \sum_{n=1}^{\infty} \left\{ \sum_{m=1}^{\infty} \sum_{l=1}^{\infty} Y_{m,l}^{EE} [A_{m,l+n}^{EO} - \text{sgn}(l-n) A_{m,|l-n|}^{EO}] \right\} \sin \omega_{n2} x + \\
& + \frac{1}{4} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[\sum_{l=1}^{\infty} \sum_{r=1}^{\infty} (Y_{r+m,l}^{EE} + Y_{|r-m|,l}^{EE}) (A_{r,l+n}^{EO} - \text{sgn}(l-n) A_{r,|l-n|}^{EO}) \right] \cos \omega_{m1} x_1 \sin \omega_{n2} x_2 + \\
& - \frac{1}{4} \sum_{n=1}^{\infty} \left[\sum_{m=1}^{\infty} \sum_{l=1}^{\infty} Y_{m,l}^{EO} (A_{m,l+n}^{EE} - A_{m,|l-n|}^{EE}) \right] \sin \omega_{n2} x - \\
& - \frac{1}{4} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[\sum_{l=1}^{\infty} \sum_{r=1}^{\infty} (Y_{r+m,l}^{EO} + Y_{|r-m|,l}^{EO}) (A_{r,l+n}^{EE} - A_{r,|l-n|}^{EE}) \right] \cos \omega_{m1} x_1 \sin \omega_{n2} x_2 + \\
& + \frac{1}{4} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} Y_{m,n}^{EO} A_{m,n}^{EO} + \frac{1}{4} \sum_{m=1}^{\infty} \left[\sum_{n=1}^{\infty} \sum_{l=1}^{\infty} Y_{l,n}^{EO} (A_{l+m,n}^{EO} + A_{|l-m|,n}^{EO}) \right] \cos \omega_{m1} x_1 + \frac{1}{4} \sum_{n=1}^{\infty} \left[\sum_{m=1}^{\infty} \sum_{l=1}^{\infty} Y_{m,l}^{EO} (A_{m,l+n}^{EO} + \text{sgn}(l-n) A_{m,|l-n|}^{EO}) \right] \cos \omega_{n2} x_2 + \\
& + \frac{1}{4} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[\sum_{l=1}^{\infty} \sum_{r=1}^{\infty} (Y_{r+m,l}^{EO} + Y_{|r-m|,l}^{EO}) (A_{r,l+n}^{EO} + \text{sgn}(l-n) A_{r,|l-n|}^{EO}) \right] \cos \omega_{m1} x_1 \cos \omega_{n2} x_2 + \frac{1}{4} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} Y_{m,n}^{OE} A_{m,n}^{OE} + \\
& + \frac{1}{4} \sum_{m=1}^{\infty} \left[\sum_{n=1}^{\infty} \sum_{l=1}^{\infty} Y_{l,n}^{OE} (A_{l+m,n}^{OE} + \text{sgn}(l-m) A_{|l-m|,n}^{OE}) \right] \cos \omega_{m1} x_1 + \frac{1}{4} \sum_{n=1}^{\infty} \left[\sum_{m=1}^{\infty} \sum_{l=1}^{\infty} Y_{m,l}^{OE} (A_{m,l+n}^{OE} + A_{m,|l-n|}^{OE}) \right] \cos \omega_{n2} x_2 + \\
& + \frac{1}{4} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[\sum_{l=1}^{\infty} \sum_{r=1}^{\infty} (Y_{r+m,l}^{OE} + \text{sgn}(r-m) Y_{|r-m|,l}^{OE}) (A_{r,l+n}^{OE} + A_{r,|l-n|}^{OE}) \right] \cos \omega_{m1} x_1 \cos \omega_{n2} x_2 + \\
& + \frac{1}{4} \sum_{n=1}^{\infty} \left[\sum_{m=1}^{\infty} \sum_{l=1}^{\infty} Y_{m,l}^{OE} (A_{m,l+n}^{OO} - \text{sgn}(l-n) A_{m,|l-n|}^{OO}) \right] \sin \omega_{n2} x_2 + \\
& + \frac{1}{4} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[\sum_{l=1}^{\infty} \sum_{r=1}^{\infty} (Y_{r+m,l}^{OE} + \text{sgn}(r-m) Y_{|r-m|,l}^{OE}) (A_{r,l+n}^{OO} - \text{sgn}(l-n) A_{r,|l-n|}^{OO}) \right] \cos \omega_{m1} x_1 \sin \omega_{n2} x_2 - \\
& - \frac{1}{4} \sum_{n=1}^{\infty} \left[\sum_{m=1}^{\infty} \sum_{l=1}^{\infty} Y_{m,l}^{OO} (A_{m,l+n}^{OE} - A_{m,|l-n|}^{OE}) \right] \sin \omega_{n2} x_2 - \\
& - \frac{1}{4} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[\sum_{l=1}^{\infty} \sum_{r=1}^{\infty} (Y_{r+m,l}^{OO} + \text{sgn}(r-m) Y_{|r-m|,l}^{OO}) (A_{r,l+n}^{OE} - A_{r,|l-n|}^{OE}) \right] \cos \omega_{m1} x_1 \sin \omega_{n2} x_2 + \frac{1}{4} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} Y_{m,n}^{OO} A_{m,n}^{OO} + \\
& + \frac{1}{4} \sum_{m=1}^{\infty} \left[\sum_{n=1}^{\infty} \sum_{l=1}^{\infty} Y_{l,n}^{OO} (A_{l+m,n}^{OO} + \text{sgn}(l-m) A_{|l-m|,n}^{OO}) \right] \cos \omega_{m1} x_1 + \frac{1}{4} \sum_{n=1}^{\infty} \left[\sum_{m=1}^{\infty} \sum_{l=1}^{\infty} Y_{m,l}^{OO} (A_{m,l+n}^{OO} + \text{sgn}(l-n) A_{m,|l-n|}^{OO}) \right] \cos \omega_{n2} x_2 + \\
& + \frac{1}{4} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[\sum_{l=1}^{\infty} \sum_{r=1}^{\infty} (Y_{r+m,l}^{OO} + \text{sgn}(r-m) Y_{|r-m|,l}^{OO}) (A_{r,l+n}^{OO} + \text{sgn}(l-n) A_{r,|l-n|}^{OO}) \right] \cos \omega_{m1} x_1 \cos \omega_{n2} x_2 = \\
& + \frac{1}{4} \sum_{n=1}^{\infty} \left[\sum_{m=1}^{\infty} \sum_{l=1}^{\infty} Y_{m,l}^{EE} (A_{m,l+n}^{OE} - \text{sgn}(l-n) A_{m,|l-n|}^{OE}) \right] \sin \omega_{m1} x_1 - \\
& - \frac{1}{4} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[\sum_{r=1}^{\infty} (Y_{r+m,l}^{EE} - Y_{|r-m|,l}^{EE}) (A_{r,l+n}^{OE} + A_{r,|l-n|}^{OE}) \right] \sin \omega_{m1} x_1 \cos \omega_{n2} x_2 - \\
& - \frac{1}{4} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[\sum_{r=1}^{\infty} (Y_{r+m,l}^{EE} - Y_{|r-m|,l}^{EE}) (A_{r,l+n}^{OO} - \text{sgn}(l-n) A_{r,|l-n|}^{OO}) \right] \sin \omega_{m1} x_1 \sin \omega_{n2} x_2 + \\
& + \frac{1}{4} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[\sum_{r=1}^{\infty} (Y_{r+m,l}^{EO} - Y_{|r-m|,l}^{EO}) (A_{r,l+n}^{OE} - A_{r,|l-n|}^{OE}) \right] \sin \omega_{m1} x_1 \sin \omega_{n2} x_2 +
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{4} \sum_{m=1}^{\infty} \left[\sum_{n=1}^{\infty} \sum_{l=1}^{\infty} Y_{l,n}^{EO} \left(A_{l+m,n}^{OO} - \text{sgn}(l-m) A_{|l-m|,n}^{OO} \right) \right] \sin \omega_{m1} x_1 - \\
& - \frac{1}{4} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_{l=1}^{\infty} \left[\sum_{r=1}^{\infty} \left(Y_{r+m,l}^{EO} - Y_{|r-m|,l}^{EO} \right) \left(A_{r,l+n}^{OO} + \text{sgn}(l-n) A_{r,|l-n|}^{OO} \right) \right] \sin \omega_{m1} x_1 \cos \omega_{n2} x_2 - \\
& - \frac{1}{4} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[\sum_{l=1}^{\infty} Y_{l,n}^{OE} \left(A_{l+m,n}^{EE} - A_{|l-m|,n}^{EE} \right) \right] \sin \omega_{m1} x_1 + \\
& + \frac{1}{4} \sum_{n=1}^{\infty} \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} \left[\sum_{r=1}^{\infty} \left(Y_{r+m,l}^{OE} - \text{sgn}(r-m) Y_{|r-m|,l}^{OE} \right) \left(A_{r,l+n}^{EE} + A_{r,|l-n|}^{EE} \right) \right] \sin \omega_{m1} x_1 \cos \omega_{n2} x_2 + \\
& + \frac{1}{4} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_{l=1}^{\infty} \left[\sum_{r=1}^{\infty} \left(Y_{r+m,l}^{OE} - \text{sgn}(r-m) Y_{|r-m|,l}^{OE} \right) \left(A_{r,l+n}^{EO} - \text{sgn}(l-n) A_{r,|l-n|}^{EO} \right) \right] \sin \omega_{m1} x_1 \sin \omega_{n2} x_2 + \\
& - \frac{1}{4} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[\sum_{l=1}^{\infty} \sum_{r=1}^{\infty} \left(Y_{r+m,l}^{OO} - \text{sgn}(r-m) Y_{|r-m|,l}^{OO} \right) \left(A_{r,l+n}^{EE} - A_{r,|l-n|}^{EE} \right) \right] \sin \omega_{m1} x_1 \sin \omega_{n2} x_2 - \\
& - \frac{1}{4} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[\sum_{l=1}^{\infty} Y_{l,n}^{OO} \left(A_{l+m,n}^{EO} - A_{|l-m|,n}^{EO} \right) \right] \sin \omega_{m1} x_1 + \\
& + \frac{1}{4} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[\sum_{l=1}^{\infty} \sum_{r=1}^{\infty} \left(Y_{r+m,l}^{OO} - \text{sgn}(r-m) Y_{|r-m|,l}^{OO} \right) \left(A_{r,l+n}^{EO} + \text{sgn}(l-n) A_{r,|l-n|}^{EO} \right) \right] \sin \omega_{m1} x_1 \cos \omega_{n2} x_2 =
\end{aligned}$$

After reorganizing and grouping the terms of the same type, we find the expressions of the coefficients of the serie expansion of the product:

$$\begin{aligned}
\mathbf{P}_{00} &= Y_{00} A_{00} + \frac{1}{2} \sum_{m=1}^{\infty} Y_{m0}^{E0} A_{m0}^{E0} + \frac{1}{2} \sum_{m=1}^{\infty} Y_{m0}^{O0} A_{m0}^{O0} + \frac{1}{2} \sum_{n=1}^{\infty} Y_{0n}^{OE} A_{0n}^{OE} + \frac{1}{2} \sum_{n=1}^{\infty} Y_{0n}^{OO} A_{0n}^{OO} + \\
& + \frac{1}{4} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} Y_{m,n}^{EE} A_{m,n}^{EE} + \frac{1}{4} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} Y_{m,n}^{OO} A_{m,n}^{OO} + \frac{1}{4} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} Y_{m,n}^{EO} A_{m,n}^{EO} + \frac{1}{4} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} Y_{m,n}^{OE} A_{m,n}^{OE} = Y_{00} A_{00} + \\
\mathbf{P}_{m,0}^{EO} &= \left(Y_{00} A_{m0}^{E0} + A_{00} Y_{m0}^{E0} \right) + \frac{1}{2} \left[\sum_{l=1}^{\infty} Y_{l,0}^{E0} \left(A_{l+m,0}^{E0} + A_{|l-m|,0}^{E0} \right) + \sum_{l=1}^{\infty} Y_{l,0}^{O0} \left(A_{l+m,0}^{O0} + \text{sgn}(l-m) A_{|l-m|,0}^{O0} \right) \right] + \\
& + \frac{1}{2} \left[\sum_{n=1}^{\infty} Y_{mn}^{EE} A_{0n}^{OE} + \sum_{n=1}^{\infty} Y_{mn}^{EO} A_{0n}^{OO} + \sum_{n=1}^{\infty} Y_{0n}^{OE} A_{mn}^{EE} + \sum_{n=1}^{\infty} Y_{0n}^{OO} A_{mn}^{EO} \right] + \frac{1}{4} \left[\sum_{n=1}^{\infty} \sum_{l=1}^{\infty} Y_{l,n}^{EE} \left(A_{l+m,n}^{EE} + A_{|l-m|,n}^{EE} \right) \right] + \\
& + \frac{1}{4} \left[\sum_{n=1}^{\infty} \sum_{l=1}^{\infty} Y_{l,n}^{EO} \left(A_{l+m,n}^{EO} + A_{|l-m|,n}^{EO} \right) + \sum_{n=1}^{\infty} \sum_{l=1}^{\infty} Y_{l,n}^{OE} \left(A_{l+m,n}^{OE} + \text{sgn}(l-m) A_{|l-m|,n}^{OE} \right) + \sum_{n=1}^{\infty} \sum_{l=1}^{\infty} Y_{l,n}^{OO} \left(A_{l+m,n}^{OO} + \text{sgn}(l-m) A_{|l-m|,n}^{OO} \right) \right] \\
\mathbf{P}_{0,n}^{OE} &= \left(Y_{00} A_{0n}^{OE} + A_{00} Y_{0n}^{OE} \right) + \frac{1}{2} \left[\sum_{l=1}^{\infty} Y_{0,l}^{OE} \left(A_{0,l+n}^{OE} + A_{0,|l-n|}^{OE} \right) + \sum_{l=1}^{\infty} Y_{0,l}^{OO} \left(A_{0,l+n}^{OO} + \text{sgn}(l-n) A_{0,|l-n|}^{OO} \right) \right] + \\
& \frac{1}{2} \left[\sum_{m=1}^{\infty} Y_{mn}^{EE} A_{m0}^{E0} + \sum_{m=1}^{\infty} Y_{mn}^{OE} A_{m0}^{O0} + \sum_{m=1}^{\infty} Y_{m0}^{EO} A_{mn}^{EE} + \sum_{m=1}^{\infty} Y_{m0}^{OO} A_{mn}^{EO} \right] + \frac{1}{4} \left[\sum_{m=1}^{\infty} \sum_{l=1}^{\infty} Y_{m,l}^{EE} \left(A_{m,l+n}^{EE} + A_{m,|l-m|}^{EE} \right) \right] + \\
& + \frac{1}{4} \left[\sum_{m=1}^{\infty} \sum_{l=1}^{\infty} Y_{m,l}^{EO} \left(A_{m,l+n}^{EO} + \text{sgn}(l-n) A_{m,|l-n|}^{EO} \right) + \sum_{m=1}^{\infty} \sum_{l=1}^{\infty} Y_{m,l}^{OE} \left(A_{m,l+n}^{OE} + A_{m,|l-n|}^{OE} \right) + \sum_{m=1}^{\infty} \sum_{l=1}^{\infty} Y_{m,l}^{OO} \left(A_{m,l+n}^{OO} + \text{sgn}(l-n) A_{m,|l-n|}^{OO} \right) \right] \\
\mathbf{P}_{m,0}^{OO} &= \left(Y_{00} A_{m0}^{O0} + A_{00} Y_{m0}^{O0} \right) + \frac{1}{2} \left[\sum_{l=1}^{\infty} Y_{l,0}^{EO} \left(A_{l+m,0}^{OO} - \text{sgn}(l-m) A_{|l-m|,0}^{OO} \right) - \sum_{l=1}^{\infty} Y_{l,0}^{OO} \left(A_{l+m,0}^{EO} - A_{|l-m|,0}^{EO} \right) \right] + \\
& + \frac{1}{2} \left[\sum_{n=1}^{\infty} Y_{mn}^{OE} A_{0n}^{OE} + \sum_{n=1}^{\infty} Y_{mn}^{OO} A_{0n}^{OO} + \sum_{n=1}^{\infty} Y_{0n}^{OE} A_{mn}^{EE} + \sum_{n=1}^{\infty} Y_{0n}^{OO} A_{mn}^{EO} \right] + \frac{1}{4} \left[\sum_{n=1}^{\infty} \sum_{l=1}^{\infty} Y_{m,l}^{EE} \left(A_{m,l+n}^{OE} - \text{sgn}(l-n) A_{m,|l-n|}^{OE} \right) \right] + \\
& + \frac{1}{4} \left[\sum_{n=1}^{\infty} \sum_{l=1}^{\infty} Y_{l,n}^{EO} \left(A_{l+m,n}^{OO} - \text{sgn}(l-m) A_{|l-m|,n}^{OO} \right) - \sum_{n=1}^{\infty} \sum_{l=1}^{\infty} Y_{l,n}^{OE} \left(A_{l+m,n}^{EE} - A_{|l-m|,n}^{EE} \right) - \sum_{n=1}^{\infty} \sum_{l=1}^{\infty} Y_{l,n}^{OO} \left(A_{l+m,n}^{EO} - A_{|l-m|,n}^{EO} \right) \right] \\
\mathbf{P}_{0,n}^{OO} &= \left(Y_{00} A_{0n}^{O0} + A_{00} Y_{0n}^{O0} \right) + \frac{1}{2} \left[\sum_{l=1}^{\infty} Y_{0,l}^{OE} \left(A_{0,l+n}^{OO} - \text{sgn}(l-n) A_{0,|l-n|}^{OO} \right) - \sum_{l=1}^{\infty} Y_{0,l}^{OO} \left(A_{0,l+n}^{OE} - A_{0,|l-n|}^{OE} \right) \right] +
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2} \left[\sum_{m=1}^{\infty} Y_{mn}^{EO} A_{m0}^{E0} + \sum_{m=1}^{\infty} Y_{mn}^{OO} A_{m0}^{O0} + \sum_{m=1}^{\infty} Y_{m0}^{OO} A_{mn}^{OO} + \sum_{m=1}^{\infty} Y_{m0}^{EO} A_{mn}^{EO} \right] - \frac{1}{4} \left[\sum_{m=1}^{\infty} \sum_{l=1}^{\infty} Y_{m,l}^{EO} (A_{m,l+n}^{EE} - A_{m,|l-n|}^{EE}) \right] + \\
& + \frac{1}{4} \left[\sum_{m=1}^{\infty} \sum_{l=1}^{\infty} Y_{m,l}^{EE} [A_{m,l+n}^{EO} - \text{sgn}(l-n)A_{m,|l-n|}^{EO}] + \sum_{m=1}^{\infty} \sum_{l=1}^{\infty} Y_{m,l}^{OE} (A_{m,l+n}^{OO} - \text{sgn}(l-n)A_{m,|l-n|}^{OO}) - \sum_{m=1}^{\infty} \sum_{l=1}^{\infty} Y_{m,l}^{OO} (A_{m,l+n}^{OE} - A_{m,|l-n|}^{OE}) \right] \\
\mathbf{P}_{mn}^{EE} & = (Y_{00}A_{mn}^{EE} + A_{00}Y_{mn}^{EE} + Y_{m0}A_{0n}^{EO} + A_{m0}Y_{0n}^{OE}) + \frac{1}{2} \left[\sum_{l=1}^{\infty} Y_{l,n}^{EE} (A_{l+m,0}^{E0} + A_{|l-m|,0}^{E0}) + \sum_{l=1}^{\infty} Y_{l,n}^{OE} (A_{l+m,0}^{OO} + \text{sgn}(l-m)A_{|l-m|,0}^{OO}) \right] + \\
& + \frac{1}{2} \left[\sum_{l=1}^{\infty} Y_{m,l}^{EE} (A_{0,l+n}^{OE} + A_{0,|l-n|}^{OE}) + \sum_{l=1}^{\infty} Y_{m,l}^{EO} (A_{0,l+n}^{OO} + \text{sgn}(l-n)A_{0,|l-n|}^{OO}) + \sum_{l=1}^{\infty} Y_{l,0}^{EO} (A_{l+m,n}^{EE} + A_{|l-m|,n}^{EE}) \right] + \\
& + \frac{1}{2} \left[\sum_{l=1}^{\infty} Y_{l,0}^{OO} (A_{l+m,n}^{OE} + \text{sgn}(l-m)A_{|l-m|,n}^{OE}) + \sum_{l=1}^{\infty} Y_{0,l}^{OE} (A_{m,l+n}^{EE} + A_{m,|l-n|}^{EE}) + \sum_{l=1}^{\infty} Y_{m,l}^{OO} (A_{0,l+n}^{EO} + \text{sgn}(l-n)A_{0,|l-n|}^{EO}) \right] + \\
& + \frac{1}{4} \left[\sum_{l=1}^{\infty} \sum_{r=1}^{\infty} (Y_{r+m,l}^{EE} + Y_{|r-m|,l}^{EE}) (A_{r,l+n}^{EE} + A_{r,|l-n|}^{EE}) + \sum_{l=1}^{\infty} \sum_{r=1}^{\infty} (Y_{r+m,l}^{EO} + Y_{|r-m|,l}^{EO}) (A_{r,l+n}^{EO} + \text{sgn}(l-n)A_{r,|l-n|}^{EO}) \right] + \\
& + \frac{1}{4} \left[\sum_{l=1}^{\infty} \sum_{r=1}^{\infty} (Y_{r+m,l}^{OE} + \text{sgn}(r-m)Y_{|r-m|,l}^{OE}) (A_{r,l+n}^{OE} + A_{r,|l-n|}^{OE}) + \sum_{l=1}^{\infty} \sum_{r=1}^{\infty} (Y_{r+m,l}^{OO} + \text{sgn}(r-m)Y_{|r-m|,l}^{OO}) (A_{r,l+n}^{OO} + \text{sgn}(l-n)A_{r,|l-n|}^{OO}) \right] \\
\mathbf{P}_{mn}^{EO} & = (Y_{00}A_{mn}^{EO} + A_{00}Y_{mn}^{EO} + Y_{m0}A_{0n}^{EO} + A_{m0}Y_{0n}^{OO}) + \frac{1}{2} \left[\sum_{l=1}^{\infty} Y_{l,n}^{EO} (A_{l+m,0}^{E0} + A_{|l-m|,0}^{E0}) + \sum_{l=1}^{\infty} Y_{l,n}^{OO} (A_{l+m,0}^{OO} + \text{sgn}(l-m)A_{|l-m|,0}^{OO}) \right] + \\
& + \frac{1}{2} \left[\sum_{l=1}^{\infty} Y_{m,l}^{EE} (A_{0,l+n}^{OO} - A_{0,|l-n|}^{OO}) - \sum_{l=1}^{\infty} Y_{m,l}^{EO} (A_{0,l+n}^{OE} - A_{0,|l-n|}^{OE}) + \sum_{l=1}^{\infty} Y_{l,0}^{EO} (A_{l+m,n}^{EO} + A_{|l-m|,n}^{EO}) \right] + \\
& + \frac{1}{2} \left[\sum_{l=1}^{\infty} Y_{l,0}^{OO} (A_{l+m,n}^{OO} + \text{sgn}(l-m)A_{|l-m|,n}^{OO}) + \sum_{l=1}^{\infty} Y_{0,l}^{OO} (A_{m,l+n}^{EE} - A_{m,|l-n|}^{EE}) + \sum_{l=1}^{\infty} Y_{0,l}^{OE} (A_{m,l+n}^{EO} - \text{sgn}(l-n)A_{m,|l-n|}^{EO}) \right] + \\
& + \frac{1}{4} \left[\sum_{l=1}^{\infty} \sum_{r=1}^{\infty} (Y_{r+m,l}^{EE} + Y_{|r-m|,l}^{EE}) (A_{r,l+n}^{EO} - \text{sgn}(l-n)A_{r,|l-n|}^{EO}) + \sum_{l=1}^{\infty} \sum_{r=1}^{\infty} (Y_{r+m,l}^{EO} + Y_{|r-m|,l}^{EO}) (A_{r,l+n}^{EE} - A_{r,|l-n|}^{EE}) \right] + \\
& + \frac{1}{4} \left[\sum_{l=1}^{\infty} \sum_{r=1}^{\infty} (Y_{r+m,l}^{OE} + \text{sgn}(r-m)Y_{|r-m|,l}^{OE}) (A_{r,l+n}^{OO} - \text{sgn}(l-n)A_{r,|l-n|}^{OO}) + \sum_{l=1}^{\infty} \sum_{r=1}^{\infty} (Y_{r+m,l}^{OO} + \text{sgn}(r-m)Y_{|r-m|,l}^{OO}) (A_{r,l+n}^{OE} - A_{r,|l-n|}^{OE}) \right] \\
\mathbf{P}_{mn}^{OE} & = (Y_{00}A_{mn}^{OE} + A_{00}Y_{mn}^{OE} + Y_{m0}A_{0n}^{OO} + A_{m0}Y_{0n}^{OE}) + \frac{1}{2} \left[\sum_{l=1}^{\infty} Y_{l,n}^{EE} (A_{l+m,0}^{OO} - \text{sgn}(l-m)A_{|l-m|,0}^{OO}) - \sum_{l=1}^{\infty} Y_{l,n}^{OE} (A_{l+m,0}^{E0} - A_{|l-m|,0}^{E0}) \right] + \\
& + \frac{1}{2} \left[\sum_{l=1}^{\infty} Y_{m,l}^{OE} (A_{0,l+n}^{OE} + A_{0,|l-n|}^{OE}) + \sum_{l=1}^{\infty} Y_{m,l}^{OO} (A_{0,l+n}^{OO} + \text{sgn}(l-n)A_{0,|l-n|}^{OO}) - \sum_{l=1}^{\infty} Y_{l,0}^{OO} (A_{l+m,n}^{EE} - A_{|l-m|,n}^{EE}) \right] + \\
& + \frac{1}{2} \left[\sum_{l=1}^{\infty} Y_{l,0}^{EO} (A_{l+m,n}^{OE} - \text{sgn}(l-m)A_{|l-m|,n}^{OE}) + \sum_{l=1}^{\infty} Y_{m,l}^{OO} (A_{0,l+n}^{OO} + \text{sgn}(l-n)A_{0,|l-n|}^{OO}) + \sum_{l=1}^{\infty} Y_{m,l}^{OE} (A_{0,l+n}^{EO} + A_{0,|l-n|}^{EO}) \right] + \\
& + \frac{1}{4} \left[- \sum_{l=1}^{\infty} \sum_{r=1}^{\infty} (Y_{r+m,l}^{EE} - Y_{|r-m|,l}^{EE}) (A_{r,l+n}^{OE} + A_{r,|l-n|}^{OE}) - \sum_{l=1}^{\infty} \sum_{r=1}^{\infty} (Y_{r+m,l}^{EO} - Y_{|r-m|,l}^{EO}) (A_{r,l+n}^{OO} + \text{sgn}(l-n)A_{r,|l-n|}^{OO}) \right] + \\
& + \frac{1}{4} \left[\sum_{l=1}^{\infty} \sum_{r=1}^{\infty} (Y_{r+m,l}^{OE} - \text{sgn}(r-m)Y_{|r-m|,l}^{OE}) (A_{r,l+n}^{EE} + A_{r,|l-n|}^{EE}) + \sum_{l=1}^{\infty} \sum_{r=1}^{\infty} (Y_{r+m,l}^{OO} - \text{sgn}(r-m)Y_{|r-m|,l}^{OO}) (A_{r,l+n}^{EO} + \text{sgn}(l-n)A_{r,|l-n|}^{EO}) \right] \\
\mathbf{P}_{mn}^{OO} & = (Y_{00}A_{mn}^{OO} + A_{00}Y_{mn}^{OO} + Y_{m0}A_{0n}^{OO} + A_{m0}Y_{0n}^{OO}) + \frac{1}{2} \left[\sum_{l=1}^{\infty} Y_{l,n}^{EO} (A_{l+m,0}^{OO} - \text{sgn}(l-m)A_{|l-m|,0}^{OO}) - \sum_{l=1}^{\infty} Y_{l,n}^{OO} (A_{l+m,0}^{E0} - A_{|l-m|,0}^{E0}) \right] + \\
& + \frac{1}{2} \left[\sum_{l=1}^{\infty} Y_{m,l}^{OE} (A_{0,l+n}^{OO} - A_{0,|l-n|}^{OO}) - \sum_{l=1}^{\infty} Y_{m,l}^{OO} (A_{0,l+n}^{OE} - A_{0,|l-n|}^{OE}) - \sum_{l=1}^{\infty} Y_{l,0}^{OO} (A_{l+m,n}^{EO} - A_{|l-m|,n}^{EO}) \right] + \\
& + \frac{1}{2} \left[\sum_{l=1}^{\infty} Y_{l,0}^{EO} (A_{l+m,n}^{OO} - \text{sgn}(l-m)A_{|l-m|,n}^{OO}) - \sum_{l=1}^{\infty} Y_{m,l}^{OO} (A_{0,l+n}^{OE} - A_{0,|l-n|}^{OE}) + \sum_{l=1}^{\infty} Y_{m,l}^{OE} (A_{0,l+n}^{OO} - \text{sgn}(l-n)A_{0,|l-n|}^{OO}) \right] + \\
& + \frac{1}{4} \left[- \sum_{l=1}^{\infty} \sum_{r=1}^{\infty} (Y_{r+m,l}^{EE} - Y_{|r-m|,l}^{EE}) (A_{r,l+n}^{OO} - \text{sgn}(l-n)A_{r,|l-n|}^{OO}) - \sum_{l=1}^{\infty} \sum_{r=1}^{\infty} (Y_{r+m,l}^{EO} - Y_{|r-m|,l}^{EO}) (A_{r,l+n}^{OE} - A_{r,|l-n|}^{OE}) \right] + \\
& + \frac{1}{4} \left[\sum_{l=1}^{\infty} \sum_{r=1}^{\infty} (Y_{r+m,l}^{OE} - \text{sgn}(r-m)Y_{|r-m|,l}^{OE}) (A_{r,l+n}^{EE} + A_{r,|l-n|}^{EE}) + \sum_{l=1}^{\infty} \sum_{r=1}^{\infty} (Y_{r+m,l}^{OO} - \text{sgn}(r-m)Y_{|r-m|,l}^{OO}) (A_{r,l+n}^{EO} + \text{sgn}(l-n)A_{r,|l-n|}^{EO}) \right]
\end{aligned}$$

The coefficients P_{mn} (where $m, n=0, 1, 2, \dots, \infty$) of the serie expansion of the product of the two functions, each contain expressions of one or more infinite numerical series. If the coefficients of the expansion of each of the terms of the product are known, in some particular cases the limit of these series can be determined exactly. In other cases, the limit is calculated with an approximation based on the number of terms calculated for the partial sums. Given the fairly large number of coefficients and series that appear in their expressions (especially in the case of functions with a large number of variables), a large volume of calculations is required, which requires the use of modern computing technology. .

In addition to their usefulness in solving PDEs, the previously determined formulas may be useful in the study of numerical series and function series. If the expression of a coefficient P_{mn} of the series expansion of certain functions, calculated by applying the formulas (4.a) (or those corresponding to a greater number of variables) is compared with the expression of the same coefficient determined based on the formulas deduced in this section, a relationship of equality can be established between the integral provided by (4.a) and the corresponding serie. In addition, we found that the same coefficient can be expressed in several ways, depending on the order in which its components are added together. These equations are also valid when all of these variables acquire certain particular values.

In the case of a function of $N=3$ variables, the number of types of coefficients of the series expansion SFS is 27 (N^3), and the product of their series expansions is composed of 729 (N^6) elementary functions, therefore the mathematical expressions of the coefficients can contain a considerable number of the numerical series. However, in the presence of a properly designed calculation algorithm, these problems are easily overcome by a computer with usual performance.

6. Conclusions

A PDEs equation consists of a series of terms of the form $y(\mathbf{x})$, $y'(\mathbf{x})$, $\partial y(\mathbf{x})/\partial x_i$, $\partial^2 y(\mathbf{x})/\partial x_i \partial x_j$, $\partial^n y(\mathbf{x})/\partial x_i \dots \partial x_j$, $\partial^{-n} y(\mathbf{x})/\partial x_i \dots \partial x_j$, $y^n(\mathbf{x}) \partial^n y(\mathbf{x})/\partial x_i \dots \partial x_j$, and their products with other multivariable functions $p(\mathbf{x})$, $q(\mathbf{x})$, ..., where \mathbf{x} is a d -tuple, and x_i , x_j , are its variables. The replacement of these terms, when they appear in a differential equation, by their SFS expansion, or by their Fourier sums S_N , which approximate them (the expansion containing unknown Fourier coefficients of the function $y(\mathbf{x})$), leads to a system with $2N+1$ ($N \rightarrow \infty$) algebraic equations. By solving it, we can find the coefficients P_{mn} , and in this way, the unknown function (or functions, if it is a system of equations).

As for the integer series expansion method, the differentiation and the integration operations performed on the series expansions, as well as the product between the multivariable functions are transformed into algebraic operations performed on the expansion coefficients. Therefore, solving PDEs turns into solving algebraic equations. From this point of view, the method of solving by determining the coefficients of development in sinusoidal series is more advantageous, due to the ease with which the coefficients of the development can be calculated and because of the high speed of convergence (which makes them ideal for numerical approximation methods).

This method of solving ordinary differential equations and equations with partial derivatives, by determining the coefficients of the development in sinusoidal series of the unknown functions can become an extremely efficient method, applicable to all types of differential and integro-differential, linear equations. and nonlinear, to linear and nonlinear partial differential equations, to systems of such equations, whatever their order and the complexity of their coefficients, in particular for situations encountered in physics and engineering. It follows that this method will be studied continuously, in terms of application

conditions and conditionalities that must satisfy the functions involved at the limits of the domain. In addition, the implications that emerge by comparing it with other PDE resolution methods need to be investigated.

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PROPRIÉTÉS APPLICATIVES DES DÉVELOPPEMENTS EN SÉRIES DE FOURIER SINUSOÏDALES DES FONCTIONS À PLUSIEURS VARIABLES

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Résumé: Dans nos recherches précédentes [1, 2], nous avons développé une méthode de résolution d'équations différentielles en déterminant les coefficients du développement en séries sinusoïdales (ou non sinusoïdales) des fonctions inconnues. Cela s'est avéré être une méthode extrêmement puissante, applicable à tous les types d'équations différentielles (ODEs) et intégró-différentielles, linéaires et non linéaires, des systèmes de telles équations, valables sur un intervalle fermé, quels que soient leur ordre et la complexité des leurs coefficients.

Les conditions requises pour l'application de la méthode ne sont pas très contraignantes et sont faciles à remplir, notamment pour les situations rencontrées en physique et en ingénierie. Pour appliquer cette méthode aussi dans le cas des équations et des systèmes d'équations aux dérivées partielles (pour des fonctions de plusieurs variables), il est nécessaire de généraliser les formules applicatives lesquels décrivent les propriétés des fonctions monovariabiles, utilisées dans le cas des ODEs, pour les fonctions dépendant de plusieurs variables, sujet d'équations aux dérivées partielles (PDEs), valables dans un domaine fermé. Comme dans le cas des ODEs, les opérations de dérivation et d'intégration effectuées sur les développements en séries sinusoïdales de Fourier (SFS) se transformeront en opérations algébriques effectuées sur les coefficients du développement. Par conséquent, la résolution des PDEs se transformera en la résolution de certaines équations algébriques.

Mots clés: séries de Fourier sinusoïdales périodiques, séries non sinusoïdales, équations différentielles linéaires, équations différentielles avec dérivées partielles

1. Introduction

Des recherches initiées par Fourier à partir de l'étude de la diffusion de la chaleur [3] et poursuivies par un grand nombre de mathématiciens, ont montré que toute fonction réelle, d'argument réel, $f(x) : R \rightarrow R$, T -périodique sur l'intervalle (x_1, x_2) , où $T = x_2 - x_1$, continue et dérivable par morceaux, ou de carré intégrable sur l'intervalle T , c'est à dire: $\int_{x_1}^{x_2} |f(x)|^2 dx < +\infty$, peut être représenté par une somme pondérée de fonctions trigonométriques simples:

$$\bar{f}(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} [a_n \cos(\omega_n x) + b_n \sin(\omega_n x)], \text{ où pour } \forall n \in \mathbf{N}, \text{ nous définirons } \omega_n = n \frac{2\pi}{T} \quad (1)$$

Les coefficients de ce développement peuvent être calculés à l'aide des formules d'Euler:

$$a_n = \frac{2}{T} \int_{x_1}^{x_2} f(x) \cos(\omega_n x) dx \quad n=0, 1, 2, 3, \dots \quad (2)$$

$$b_n = \frac{2}{T} \int_{x_1}^{x_2} f(x) \sin(\omega_n x) dx \quad n=1, 2, 3, \dots$$

Cette assertion a été étendue aux fonctions réelles avec plusieurs variables [4-7]. En utilisant les notations vectorielles, la fonction $f(\mathbf{x}) : R^d \rightarrow R$, est périodique, de période

$\mathbf{T} = (T_1, T_2, \dots, T_d)$ si, pour tous d -uplet d'entiers $\mathbf{n} \in \mathbf{Z}^d$, $\mathbf{n} = (n_1, n_2, \dots, n_d)$, nous avons: $f(\mathbf{x} + \mathbf{n} \cdot \mathbf{T}) = f(\mathbf{x})$. Ici, $\mathbf{n} \cdot \mathbf{T}$ est un produit scalaire, et l'ensemble des points \mathbf{n} de \mathbf{R}^d lequel contient seulement des entiers est appelé *le trellis entier d -dimensionnel \mathbf{Z}^d* .

Si nous notons $L_i = T_i/2$, $i = 1, 2, \dots, d$, par des changements de variables appropriés, on peut arriver à une fonction périodique $f(\mathbf{x})$ sur le prisme $\mathbf{L}^d = \prod_{i=1}^d [-L_i, L_i]$. Si cette fonction est continue et dérivable par portions par rapport à l'une quelconque des variables x_i et si elle est de carré intégrable sur le prisme \mathbf{L}^d , c'est-à-dire: $\int_{\mathbf{L}^d} |f(\mathbf{x})|^2 d\mathbf{x} < +\infty$ (elle appartient à l'espace $L^2(\mathbf{L}^d)$), les formules et les théorèmes valides dans l'espace $L^2([-L_1, L_1])$ pour les fonctions monovariées, sont pratiquement identiques. Les «briques» pour le développement en série de Fourier de fonctions périodiques multivariées seront (comme dans le cas de celles à une seule variable) les exponentielles complexes unidimensionnelles:

$$(f(x_1, x_2, \dots, x_d))^{SFS} = \sum_{n_1, n_2, \dots, n_d} c_{n_1 n_2 \dots n_d} e^{i\omega_{n_1} x_1} e^{i\omega_{n_2} x_2} \dots e^{i\omega_{n_d} x_d}, \text{ où pour } \forall n_i \in \mathbf{Z}^d, \text{ on définit } \omega_{n_i} = n_i \frac{\pi}{L_i}$$

Les coefficients de ce développement sont calculés à l'aide des formules d'Euler:

$$c_{n_1 n_2 \dots n_d} = \frac{1}{\prod_{i=1}^d T_i} \int_{-L_1}^{L_1} \int_{-L_2}^{L_2} \dots \int_{-L_d}^{L_d} e^{-i\omega_{n_1} x_1} e^{-i\omega_{n_2} x_2} \dots e^{-i\omega_{n_d} x_d} f(x_1, x_2, \dots, x_d) dx_1 dx_2 \dots dx_d$$

Avec les notations vectorielles, on peut écrire:

$$\bar{f}(\mathbf{x}) = \sum_{\mathbf{Z}^d} c_{\mathbf{n}} \exp(\boldsymbol{\omega}_{\mathbf{n}} \cdot \mathbf{x}_{\mathbf{n}}), \text{ où } \boldsymbol{\omega}_{\mathbf{n}} = (\omega_{n_1}, \omega_{n_2}, \dots, \omega_{n_d}), \text{ et } c_{\mathbf{n}} = \left(2^d \prod_{i=1}^d L_i \right)^{-1} \int_{\mathbf{L}^d} e^{-i\boldsymbol{\omega}_{\mathbf{n}} \cdot \mathbf{x}} f(\mathbf{x}) d\mathbf{x} \quad (3)$$

Comme dans le cas unidimensionnel [2,5], une fois que l'on connaît les coefficients de développement en SFS d'une fonction multivariée, on peut trouver des formules de calcul pour les coefficients de développement en SFS du produit $f(\mathbf{x}) \cdot g(\mathbf{x})$ de deux fonctions de plusieurs variables, et pour $f(\mathbf{x}) = g(\mathbf{x})$, ces du carré d'une telle fonction. En appliquant successivement, à plusieurs reprises, ces formules, les coefficients de Fourier d'expressions analytiques plus compliqués peuvent être calculés.

De plus, dans certaines conditions, la relation (3) peut être intégrée, ou dérivée de façon répétée, terme par terme. Tant la dérivation que l'intégration par rapport à une seule variable (dérivation partielle et, par extension, "**intégration partielle**" de premier ordre) sont identiques à la dérivation, respectivement l'intégration d'une fonction monovariée (en gardant les autres variables à une valeur constante). En conséquence, les résultats obtenus dans le cas du développement en SFS pour les fonctions monovariées [2] restent valables aussi dans le cas des dérivées partielles et des intégrales partielles de premier ordre, et par répétition, également sur ceux d'un ordre supérieur.

Soit la fonction $z = f(\mathbf{x})$, définie sur le domaine $(d+1)$ -dimensionnel $\mathbf{R} \times \mathbf{L}^d$. Dans ce domaine, $f(\mathbf{x})$ sont des hypersurfaces. Pour la dérivation par rapport à l'une des variables x_i (la dérivée partielle $\partial f / \partial x_i$, ou f_{x_i}), nous définirons d'abord la fonction x_i -monovariée $f(C, x_i) = f(C_1, \dots, x_i, \dots, C_d)$, ou $C_j, j = 1, 2, \dots, d, j \neq i$, sont des constantes réelles que ne changent pas lors de la dérivation. Pour chaque combinaison C_0 de variables autres que x_i , on obtient une courbe x_i -monovariée sur l'hypersurface $f(\mathbf{x})$, et pour chaque valeur $x_i = x_0$, on obtient l'intersection de cette hypersurface avec le plan $x_i = x_0$.

Pour la dérivation de la fonction x_i -monovariée $f(C, x_i)$, nous envisagerons que l'existence des discontinuités finies de cette fonction (fréquentes notamment sur les surfaces latérales du prisme \mathbf{L}^d , d -dimensionnelle) introduites par cette variable, conduit à la modification des coefficients des composantes impaires de son développement en série, ainsi qu'à un changement de la valeur moyenne du dérivé [1, 2]. Pour toutes les fonctions x_j -

monovariabes, les valeurs de ces changements dépendent de l'empreinte de l'intersection de la fonction $f(\mathbf{x})$ avec l'hypersurface bordant le domaine de définition. Pour chaque dérivé d'ordre supérieur, il est nécessaire de connaître cette empreinte de la fonction dérivé d'ordre inférieur. Dans le cas des PDEs périodiques, ces empreintes sont les conditions aux limites de l'équation (à la limite du domaine de définition \mathbf{L}^d).

Pour l'intégration par rapport à l'une des variables x_i (l'intégrale partielle $\partial^{-1}f/\partial x_i$, ou $f^{-1}_{x_i}$, est la primitive de la fonction x_i -monovariabes $f(C, x_i)$) on procédera de la même manière que dans le cas des fonctions d'une seule variable. Dans la relation du développement en SFS d'une fonction multivariabes à variables réelles (résultant de la relation (3)), «l'intégration partielle» (par rapport à x_i) de la composante paire (par rapport à x_i) de la fonction x_i -monovariabes $f(C, x_i)$ génère une composante impaire (relative à cette variable) de l'intégrale, et «l'intégration partielle» d'une composante impaire (par rapport à x_i) génère une composante paire (par rapport à cette variable) et une composante constante (dont la valeur devient la valeur moyenne du «primitif» de la fonction x_i -monovariabes). L'intégration partielle (par rapport à l'une des variables) de la valeur moyenne de la fonction x_i -monovariabes génère (par rapport à x_i) une fonction rampe, dont le développement en série de Fourier conduit au changement des coefficients de la composante impaire du développement de la primitive.

Avec les formules obtenues et celles résultant de leur combinaison, nous avons les outils nécessaires pour transformer, pareil avec le cas des ODEs [2], une équation aux dérivées partielles (PDE), pour laquelle nous connaissons assez des conditions aux limites, en un système d'équations algébriques, dans lesquelles les valeurs inconnues sont les coefficients du développement en série sinusoïdale de la fonction multivariabes périodique inconnue $f(\mathbf{x})$. Dans les sections suivantes, nous analyserons leur formation et déterminerons des formules de calcul pour les expressions des coefficients du développement en SFS des intégrales, des dérivées des fonctions multivariabes et du produit de deux tels fonctions, avec exemplification dans le cas des fonctions $y(x_1, x_2)$ de deux variables, définies sur l'espace $\mathbf{L}^2 = [-L_1, L_1] \times [-L_2, L_2]$.

Le développement en SFS de cette fonction est [5]:

$$\begin{aligned} [y(x_1, x_2)]^{\text{SFS}} &= [y_{00} + y_{E0}(x_1) + y_{O0}(x_1) + y_{0E}(x_2) + y_{O0}(x_2) + y_{EE}(x_1, x_2) + y_{EO}(x_1, x_2) + y_{OE}(x_1, x_2) + y_{OO}(x_1, x_2)]^{\text{SFS}} = \\ &= Y_{00} + \left(\sum_{m=1}^{\infty} Y_{m0}^{E0} \cos \omega_{m1} x_1 \right) + \left(\sum_{m=1}^{\infty} Y_{m0}^{O0} \sin \omega_{m1} x_1 \right) + \left(\sum_{n=1}^{\infty} Y_{0n}^{0E} \cos \omega_{n2} x_2 \right) + \left(\sum_{n=1}^{\infty} Y_{0n}^{OO} \sin \omega_{n2} x_2 \right) + \\ &+ \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} Y_{mn}^{EE} \cos \omega_{m1} x_1 \cos \omega_{n2} x_2 \right) + \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} Y_{mn}^{EO} \cos \omega_{m1} x_1 \sin \omega_{n2} x_2 \right) + \\ &+ \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} Y_{mn}^{OE} \sin \omega_{m1} x_1 \cos \omega_{n2} x_2 \right) + \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} Y_{mn}^{OO} \sin \omega_{m1} x_1 \sin \omega_{n2} x_2 \right), \text{ où:} \end{aligned} \quad (4)$$

$$Y_{00} = \frac{1}{4L_1L_2} \int_{-L_1}^{L_1} \int_{-L_2}^{L_2} y(x_1, x_2) dx_1 dx_2$$

$$Y_{m0}^{E0} = \frac{1}{2L_1L_2} \int_{-L_1}^{L_1} \int_{-L_2}^{L_2} y(x_1, x_2) \cos \omega_{m1} x_1 dx_1 dx_2$$

$$Y_{m0}^{O0} = \frac{1}{2L_1L_2} \int_{-L_1}^{L_1} \int_{-L_2}^{L_2} y(x_1, x_2) \sin \omega_{m1} x_1 dx_1 dx_2$$

$$Y_{0n}^{0E} = \frac{1}{2L_1L_2} \int_{-L_1}^{L_1} \int_{-L_2}^{L_2} y(x_1, x_2) \cos \omega_{n2} x_2 dx_1 dx_2$$

$$Y_{0n}^{OO} = \frac{1}{2L_1L_2} \int_{-L_1}^{L_1} \int_{-L_2}^{L_2} y(x_1, x_2) \sin \omega_{n2} x_2 dx_1 dx_2$$

$$Y_{mn}^{EE} = \frac{1}{L_1 L_2} \int_{-L_1}^{L_1} \int_{-L_2}^{L_2} y(x_1, x_2) \cos \omega_{m1} x_1 \cos \omega_{n2} x_2 dx_1 dx_2 \quad (4a)$$

$$Y_{mn}^{EO} = \frac{1}{L_1 L_2} \int_{-L_1}^{L_1} \int_{-L_2}^{L_2} y(x_1, x_2) \cos \omega_{m1} x_1 \sin \omega_{n2} x_2 dx_1 dx_2$$

$$Y_{mn}^{OE} = \frac{1}{L_1 L_2} \int_{-L_1}^{L_1} \int_{-L_2}^{L_2} y(x_1, x_2) \sin \omega_{m1} x_1 \cos \omega_{n2} x_2 dx_1 dx_2$$

$$Y_{mn}^{OO} = \frac{1}{L_1 L_2} \int_{-L_1}^{L_1} \int_{-L_2}^{L_2} y(x_1, x_2) \sin \omega_{m1} x_1 \sin \omega_{n2} x_2 dx_1 dx_2$$

$$\omega_{m_1} = m\omega_1 = m \frac{\pi}{L_1}, \quad \omega_{n_2} = n\omega_2 = n \frac{\pi}{L_2}, \quad \text{ou } m, n \in \mathbf{N}^+$$

Notons que, dans ce que suit, nous considérerons toujours que les composantes impaires y_O , et les composantes impaires y_E aussi, ont la valeur moyenne égale à zéro.

2. Le développement en SFS des dérivées et des intégrales partielles des fonctions multivariées

Si dans le développement en SFS de la fonction $f(\mathbf{x}) \rightarrow \mathbf{L}^d = \prod_{i=1}^d [-L_i, L_i]$, on sépare les

termes lesquels contiennent les fonctions $\cos \omega_{mi} x_i$, respectivement $\sin \omega_{mi} x_i$, on obtient le développement en SFS de la fonction x_i -monovariée $f(C, x_i)$: (5)

$$[f(C, x_i)]^{SFS} = F_{0,i}^{C,0} + \sum_{m=1}^{\infty} F_{m,i}^{C,E} \cos \omega_{mi} x_i + \sum_{m=1}^{\infty} F_{m,i}^{C,O} \sin \omega_{mi} x_i = F_{0,i}^C + F_{E,i}^C + F_{O,i}^C, \quad m = 1, 2, \dots, \infty$$

ou $F_{0,i}^C, F_{E,i}^C, F_{O,i}^C$ sont les 3 composantes de la fonction (la valeur moyenne, la composante paire et celle impaire, par rapport à la variable x_i) et les coefficients $F_{0,i}^{C,0}, F_{m,i}^{C,E}$ et $F_{m,i}^{C,O}$, sont des fonctions dépendant des autres $d-1$ variables. Évidemment:

$$F_{E,i}^C(L_i) = F_{E,i}^C(-L_i) \quad \text{et} \quad F_{O,i}^C(L_i) = -F_{O,i}^C(-L_i) \quad (6)$$

Toute fonction $f(\mathbf{x})$, continue par morceaux, de carré intégrable, définie sur un prisme d -dimensionnel $f(\mathbf{x}) \rightarrow \mathbf{L}^d = \prod_{i=1}^d [-L_i, L_i]$, peut être prolongé sur tout l'espace \mathbf{R}^d , en

prolongeant chaque fonction monovariée $f(C, x_i)$ sur tout l'axe \mathbf{R} , le résultat étant une fonction $2L_i$ -périodique par rapport à chaque variable x_i [1]. Si la fonction $f(\mathbf{x})$ est continue à l'intérieur du prisme par rapport à chacune des variables x_i et si $f(C, -L_i) = f(C, L_i)$, $i=1, 2, \dots, d$, la fonction $f(\mathbf{x})$ est *continue* sur tout l'espace \mathbf{R}^d .

Considérant que la dérivée partielle du premier ordre, par rapport à la variable x_i , d'une fonction multivariée a les mêmes propriétés que toute dérivée du premier ordre d'une fonction monovariée $f(C, x_i)$, on peut prolongé sur ce type des fonctions les résultats obtenus pour les fonctions monovariées et pour leurs développements en SFS. Par conséquent, nous pouvons affirmer [5, ch.5]:

Théorème 1. Pour toute fonction *continue* $f(\mathbf{x}) \rightarrow \mathbf{R}^d$, $2L_i$ -périodique par rapport à chacune des variables x_i , ayant toutes les dérivées partielles des fonctions absolument intégrables (mais lesquels peuvent être indéfinies en un nombre fini de points), la série de Fourier de ses dérivées partielles $\partial f / \partial x_i$ est obtenue à partir de la série de Fourier de la fonction $f(\mathbf{x})$ par dérivation terme par terme, par rapport à la variable x_i .

Pour $[f(C, x_i)]^{SFS} = F_{0,i}^{C,0} + \sum_{m=1}^{\infty} F_{m,i}^{C,E} \cos \omega_{mi} x_i + \sum_{m=1}^{\infty} F_{m,i}^{C,O} \sin \omega_{mi} x_i$, on obtient:

$$\left[\frac{\partial f(\mathbf{x})}{\partial x_i}\right]^{SFS} = [f'(C, x_i)]^{SFS} = \omega_{mi} \sum_{m=1}^{\infty} (F_{m,i}^{C.O} \cos \omega_{mi} x_i - F_{m,i}^{C.E} \sin \omega_{mi} x_i) \quad (6)$$

Puisque la fonction $f(C, x_i)$ est absolument intégrable, on peut écrire:

$$\lim_{m \rightarrow \infty} m F_{m,i}^{C.E} = \lim_{m \rightarrow \infty} m F_{m,i}^{C.O} = 0,$$

ce que signifie que pour $m \rightarrow \infty$, les suites de fonctions $F_{m,i}^{C.E}$ et $F_{m,i}^{C.O}$ convergent vers 0 plus rapidement que la suite $1/m$.

Puisque la fonction $\partial f/\partial x_i$ satisfait toutes les conditions imposées par le théorème 1, elle peut être dérivée à nouveau, de la même manière, par rapport à l'un des autres variables. En ce que concerne la dérivation répétée par rapport à la même variable, peut être appliqué le

Théorème 2. Pour toute fonction continue $f(\mathbf{x}) \rightarrow \mathbf{R}^d$, $2L_i$ -périodique par rapport à chacune des variables x_i , laquelle par rapport à la variable x_i a $n-1$ dérivées continues et la dérivée d'ordre n absolument intégrable (cette dérivée peut être indéfinie en un nombre fini de points), les séries de Fourier des dérivées partielles d'ordre r , $\partial^r f/\partial x_i^r$, $r < n$ sont obtenues à partir de la série de Fourier de la fonction $f(\mathbf{x})$ en la dérivant r fois, terme par terme, par rapport à la variable x_i . Toutes les séries obtenues par ces dérivations, à l'exception possible de la dérivée d'ordre n , convergent uniformément vers la dérivée correspondante.

Les coefficients de Fourier de la fonction $f(\mathbf{x})$ satisfont la relation:

$$\lim_{m \rightarrow \infty} m^n F_{m,i}^{C.E} = \lim_{m \rightarrow \infty} m^n F_{m,i}^{C.O} = 0$$

Et la dérivée partielle d'ordre r , $\partial^r f/\partial x_i^r$ peut être dérivée une fois par rapport aux autres variables, si $f(\mathbf{x})$ satisfait, par rapport à cette variable, les conditions du théorème 1, ou d'un nombre répété de fois, si elle satisfait les conditions du théorème 2.

L'extension de la fonction discontinue $f_d(\mathbf{x})$, pour laquelle $f(C, -L_i) \neq f(C, L_i)$, pour au moins une des variables x_i , pose le problème de la valeur de la fonction $f(C, x_i)$ en les points $x_i = -L_i$ et $x_i = L_i$. Il y a une classe des fonctions $f_i(x_i)$, "égales presque partout" [1], lesquelles remplissent les conditions pour avoir un développement en SFS convergent, et lesquelles dans l'intervalle $[-L_i, L_i]$ sont égales à $f(C, x_i)$, mais à ses extrémités (ou/et à en autres points de discontinuité $x_{i,s}$), peuvent chacun prendre des valeurs finies différentes. Toutes ces fonctions ont le même développement en série de Fourier, et satisfont les égalités:

$$\bar{f}_r(-L_i) = \bar{f}_r(L_i) = \frac{1}{2} \left[\lim_{x_i \rightarrow (-L_i)^+} f(x_i) + \lim_{x_i \rightarrow (L_i)^-} f(x_i) \right] \text{ et } \bar{f}_r(x_{i,s}) = \frac{1}{2} \left[\lim_{x_i \rightarrow x_{i,s}^+} f(x) + \lim_{x_i \rightarrow x_{i,s}^-} f(x) \right].$$

Le comportement de toute cette classe de fonctions $f_i(x_i)$, peut être décrit par une *fonction de Fourier* $f_F(x_i)$, laquelle appartient à cette classe et laquelle satisfait la condition $f_F(x_i) = \bar{f}_F(x_i)$. Dans ce que suit, nous considérerons toutes les fonctions $f(C, x_i)$ lesquelles satisfont aux conditions d'avoir le développement en SFS convergent, comme fonctions de Fourier, et pour lesquelles à l'intérieur de l'intervalle $[-L_i, L_i]$:

$$f(-L_i) = \lim_{h \rightarrow 0} f(-L_i + h), \text{ et } f(L_i) = \lim_{h \rightarrow 0} f(L_i - h).$$

Théorème 3. Si la fonction $f(\mathbf{x}) \rightarrow \mathbf{L}^d = \prod_{i=1}^d [-L_i, L_i]$, pour laquelle $f(C, -L_i) \neq f(C, L_i)$,

à la dérivée partielle par rapport à la variable x_i absolument intégrable (mais laquelle peut être indéfinie en un nombre fini de points), elle admet le développement en SFS:

$$\left[\frac{\partial f(\mathbf{x})}{\partial x_i}\right]^{SFS} = [f'(C, x_i)]^{SFS} = f_{0,i}(C, L_i) + \sum_{m=1}^{\infty} \left[(\omega_{mi} F_{m,i}^{C.O} + 2(-1)^n f_{0,i}(C, L_i)) \cos \omega_{mi} x_i - \omega_{mi} F_{m,i}^{C.E} \sin \omega_{mi} x_i \right] \quad (7)$$

où: $[f(C, x_i)]^{SFS} = F_{0,i}^{C.O} + \sum_{m=1}^{\infty} (F_{m,i}^{C.E} \cos \omega_{mi} x_i + F_{m,i}^{C.O} \sin \omega_{mi} x_i)$, et $f_{0,i}(C, L_i) = \frac{1}{2L_i} [f(C, L_i) - f(C, -L_i)]$

Le théorème 3 est une conséquence du théorème similaire appliqué aux fonctions monovariées: avant la dérivation, dans le domaine \mathbf{L}^d , nous avons décomposé la fonction $F_{O,i}^C$ (la composante impaire par rapport à la variable x_i de la fonction $f(\mathbf{x})$), en la somme de la fonction $f_{O,i.c}(\mathbf{x})=F_{O,i}^C - f_o(C, L_i) \cdot x_i$, continue sur \mathbf{R}^d , et la fonction $f_{O,i}(\mathbf{x})=f_o(C, L_i) \cdot x_i$, laquelle contient un „hyper-saut Heaviside” dans le plan $x_i=L_i$, entre les courbes-„crête” $f_o(C, L_i)$ et $f_o(C, -L_i)$ (ces sauts sont la seule cause de la génération de phénomènes de type Gibbs).

Remarque 1: Si $f(\mathbf{x})$ est une fonction Fourier sur tout le domaine de définition \mathbf{R}^d , cette propriété appartient également à la fonction $f_{O,i.c}(\mathbf{x})$ et à son développement en SFS aussi. De plus, le développement dans le SFS de la fonction $f_o(C, L_i) \cdot x_i$ est une fonction de Fourier. Par conséquent, leur somme $\bar{f}(\mathbf{x})=f_F(\mathbf{x})$ bénéficie des mêmes propriétés. Comme dans le cas des fonctions monovariées, dans les phénomènes naturels, à énergie limitée, ce type de fonctions est plus plausible que celui des fonctions «hyper-rampe».

Remarque 2: pour $f(C, L_i)=f(C, -L_i) \Rightarrow f_{O,i}(C, L_i)=0$ et la relation (7) devient (6).

Remarque 3: Comme pour les fonctions monovariées:

$f_{O,i}(C, L_i)=\lim_{m \rightarrow \infty} 2(-1)^{m+1} m F_{m,i}^{C,O}$. Ici $f_{O,i}(C, L_i)$ (et aussi $F_{0,i}^{C,O}$, $F_{m,i}^{C,O}$ și $F_{m,i}^{C,E}$, $m \in \mathbf{N}^+$), est une fonction de $d - 1$ variables.

Remarque 4: Ce théorème s'applique de la même manière aux dérivées partielles de la fonction $f(\mathbf{x})$ par rapport à la même, ou à une autre variable. Dans ce cas, les courbes-crête $f_{O,i}(C, L_i)$ sont dérivées en conséquence.

La dérivation est effectuée de la même manière si les discontinuités par rapport à l'une des variables se produisent dans l'intervalle de définition. Si la fonction $f_d(\mathbf{x}) \rightarrow \mathbf{L}^d$ est discontinue par rapport à la variable x_i , en un nombre fini de points, la fonction $f_{O,i}(\mathbf{x})$, impaire par rapport à la variable x_i , peut s'écrire comme la somme d'une fonction continue $f_{O,i.c}(\mathbf{x})$, obtenu en translatant le long de l'axe Oz les portions continues des hypersurfaces respectives, de sorte que leurs extrémités coïncident (et pour la dérivation de laquelle s'applique le théorème 3) et une fonction discontinue $f_{O,i.H}(\mathbf{x})$, laquelle contient de sauts plans entre les courbes-crête intérieurs (et dont la dérivée est une somme des „hyper-peigne” de Dirac).

Théorème 4. Pour toute fonction absolument intégrable $f(\mathbf{x}) \rightarrow \mathbf{R}^d$, $2L_i$ -périodique par rapport à chacune des variables x_i , l'intégrale partielle définie, par rapport à la variable x_i , est obtenue à partir de la série de Fourier de la fonction $f(\mathbf{x})$ en l'intégrant terme par terme, en relation à la variable x_i , entre toutes limites d'intégration, indifférent si la série est convergente ou non.

Pour $[f(C, x_i)]^{SFS} = F_{C,i}^{C,0} + \sum_{m=1}^{\infty} F_{C,i}^{C,E} \cos \omega_m x_i + \sum_{m=1}^{\infty} F_{C,i}^{C,O} \sin \omega_m x_i$, on obtient:

$$\frac{\partial^{-1}}{\partial x_i} f(\mathbf{x}) \Big|_a^b = \int_a^b f(C, x_i) dx_i = F_{C,i}^{C,0}(b-a) + \sum_{m=1}^{\infty} \frac{-F_{C,i}^{C,O}(\cos \omega_m b - \cos \omega_m a) + F_{C,i}^{C,E}(\sin \omega_m b - \sin \omega_m a)}{\omega_m}$$

Théorème 5. Pour toute fonction $f(\mathbf{x}) \rightarrow \mathbf{R}^d$, $2L_i$ -périodique par rapport à chacune des variables x_i , ayant toutes les dérivées partielles des fonctions absolument intégrables, la série de Fourier de l'intégrale partielle $\partial f^{-1} / \partial x_i$ s'obtient à partir de la série de Fourier de la fonction $f(\mathbf{x})$ en intégrant terme par terme par rapport à la variable x_i , indifférent si la série est convergente ou non.

Pour $[f(C, x_i)]^{SFS} = F_{C,i}^{C.0} + \sum_{m=1}^{\infty} F_{C,i}^{C.E} \cos \omega_m x_i + \sum_{m=1}^{\infty} F_{C,i}^{C.O} \sin \omega_m x_i$, on obtient: (8)

$$\left[\frac{\partial^{-1} f(\mathbf{x})}{\partial x_i} \right]^{SFS} = \left[\int_0^x f(C, x_i) dx_i \right]^{SFS} = \sum_{m=1}^{\infty} \frac{F_{C,i}^{C.O}}{\omega_m} + \sum_{m=1}^{\infty} \frac{-F_{C,i}^{C.O} \cos \omega_m x_i + [F_{C,i}^{C.E} + 2(-1)^{m+1} F_{C,i}^{C.O}] \sin \omega_m x_i}{\omega_m}$$

Puisque la fonction $\partial^{-1} f / \partial x_i$ satisfait toutes les conditions imposées par le théorème 4, elle peut être partiellement intégrée, de la même manière, en relation avec l'une quelconque des autres variables.

3. Les coefficients des développements en SFS des dérivées et des intégrales partielles du premier ordre des fonctions à deux variables

Avec les significations de (4a), le développement en SFS de la fonction de deux variables $y(x_1, x_2)$ est:

$$\begin{aligned} [y(x_1, x_2)]^{SFS} &= Y_{00} + \left(\sum_{m=1}^{\infty} Y_{m0}^{E0} \cos \omega_m x_1 \right) + \left(\sum_{m=1}^{\infty} Y_{m0}^{O0} \sin \omega_m x_1 \right) + \left(\sum_{n=1}^{\infty} Y_{0n}^{OE} \cos \omega_n x_2 \right) + \left(\sum_{n=1}^{\infty} Y_{0n}^{OO} \sin \omega_n x_2 \right) + \\ &+ \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} Y_{mn}^{EE} \cos \omega_m x_1 \cos \omega_n x_2 \right) + \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} Y_{mn}^{EO} \cos \omega_m x_1 \sin \omega_n x_2 \right) + \\ &+ \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} Y_{mn}^{OE} \sin \omega_m x_1 \cos \omega_n x_2 \right) + \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} Y_{mn}^{OO} \sin \omega_m x_1 \sin \omega_n x_2 \right) \end{aligned} \quad (4)$$

Les développements en SFS des deux fonctions x_1 -monovariante et x_2 -monovariante sont:

$$\begin{aligned} [y(C, x_1)]^{SFS} &= Y_{C,1}^{C.0}(x_2) + \sum_{m=1}^{\infty} Y_{m,1}^{C.E}(x_2) \cos \omega_m x_1 + \sum_{m=1}^{\infty} Y_{m,1}^{C.O}(x_2) \sin \omega_m x_1 = \\ &= \left(Y_{00} + \sum_{n=1}^{\infty} Y_{0n}^{OE} \cos \omega_n x_2 + \sum_{n=1}^{\infty} Y_{0n}^{OO} \sin \omega_n x_2 \right) + \sum_{m=1}^{\infty} \left(Y_{m0}^{E0} + \sum_{n=1}^{\infty} Y_{mn}^{EE} \cos \omega_n x_2 + \sum_{n=1}^{\infty} Y_{mn}^{EO} \sin \omega_n x_2 \right) \cos \omega_m x_1 + \\ &+ \sum_{m=1}^{\infty} \left(Y_{m0}^{O0} + \sum_{n=1}^{\infty} Y_{mn}^{OE} \cos \omega_n x_2 + \sum_{n=1}^{\infty} Y_{mn}^{OO} \sin \omega_n x_2 \right) \sin \omega_m x_1 \end{aligned} \quad (4.1)$$

$$\begin{aligned} [y(C, x_2)]^{SFS} &= Y_{C,2}^{C.0}(x_1) + \sum_{n=1}^{\infty} Y_{n,2}^{C.E}(x_1) \cos \omega_n x_2 + \sum_{n=1}^{\infty} Y_{n,2}^{C.O}(x_1) \sin \omega_n x_2 = \\ &= \left(Y_{00} + \sum_{m=1}^{\infty} Y_{m0}^{E0} \cos \omega_m x_1 + \sum_{m=1}^{\infty} Y_{m0}^{O0} \sin \omega_m x_1 \right) + \sum_{n=1}^{\infty} \left(Y_{0n}^{OE} + \sum_{m=1}^{\infty} Y_{mn}^{EE} \cos \omega_m x_1 + \sum_{m=1}^{\infty} Y_{mn}^{EO} \sin \omega_m x_1 \right) \cos \omega_n x_2 + \\ &+ \sum_{n=1}^{\infty} \left(Y_{0n}^{OO} + \sum_{m=1}^{\infty} Y_{mn}^{OE} \cos \omega_m x_1 + \sum_{m=1}^{\infty} Y_{mn}^{OO} \sin \omega_m x_1 \right) \sin \omega_n x_2 \end{aligned} \quad (4.2)$$

Si, sur tout l'intervalle de définition $\mathbf{L}^2 = [-L_1, L_1] \times [-L_2, L_2]$, la fonction réelle, périodique de deux variables $y(x_1, x_2)$ est continue par rapport à tout les deux variables, si elle a les deux dérivées partielles de premier ordre lisses par morceaux et si ont lieu les égalités $y(-L_1, x_2) = y(L_1, x_2)$, pour $\forall x_2 \in [-L_2, L_2]$ et $y(x_1, -L_2) = y(x_1, L_2)$, pour $\forall x_1 \in [-L_1, L_1]$, le développement en série de Fourier sinusoïdale de tous les deux dérivées partielles peut être obtenu en dérivant terme par terme, par rapport à la variable respective, le développement de Fourier de la fonction $y(x_1, x_2)$. La série obtenue converge uniformément vers $\partial f / \partial x_i$, pour $i=1, 2, \dots$, en tous les points de l'intervalle de définition.

Avec les notations de (4), on peut écrire:

$$\begin{aligned} \left(\frac{\partial y}{\partial x_1} \right)^{SFS} &= \sum_{m=1}^{\infty} \omega_m Y_{m0}^{O0} \cos \omega_m x_1 - \sum_{m=1}^{\infty} \omega_m Y_{m0}^{E0} \sin \omega_m x_1 + \\ &+ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \omega_m Y_{mn}^{OE} \cos \omega_m x_1 \cos \omega_n x_2 + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \omega_m Y_{mn}^{OO} \cos \omega_m x_1 \sin \omega_n x_2 + \end{aligned} \quad (9)$$

$$\begin{aligned}
& - \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \omega_{m1} Y_{mn}^{EE} \sin \omega_{m1} x_1 \cos \omega_{n2} x_2 - \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \omega_{m1} Y_{mn}^{EO} \sin \omega_{m1} x_1 \sin \omega_{n2} x_2 \\
& \left(\frac{\partial y}{\partial x_2} \right)^{SFS} = \sum_{n=1}^{\infty} \omega_{n2} Y_{0n}^{OO} \cos \omega_{n2} x_2 - \sum_{n=1}^{\infty} \omega_{n2} Y_{0n}^{OE} \sin \omega_{n2} x_2 + \\
& + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \omega_{n2} Y_{mn}^{EO} \cos \omega_{m1} x_1 \cos \omega_{n2} x_2 - \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \omega_{n2} Y_{mn}^{EE} \cos \omega_{m1} x_1 \sin \omega_{n2} x_2 + \\
& + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \omega_{n2} Y_{mn}^{OO} \sin \omega_{m1} x_1 \cos \omega_{n2} x_2 - \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \omega_{n2} Y_{mn}^{OE} \sin \omega_{m1} x_1 \sin \omega_{n2} x_2
\end{aligned} \tag{10}$$

Le cas où $y(-L_1, x_2) \neq y(L_1, x_2)$ et/ou $y(x_1, -L_2) \neq y(x_1, L_2)$ est un cas particulier des fonctions multivariées décrit dans la section précédente, celle des courbes monovariées $y(C, x_2)$, respectivement $y(x_1, C)$, pour lesquelles C prend les valeurs particulières $C=x_1$ et $C_0=-L_1, C_0=L_1$, respectivement $C=x_2$ et $C_0=-L_2, C_0=L_2$. Dans le cas de la fonction $y(C, x_2)$, pour chaque valeur $x_1=C_k$, la fonction monovariée $y(C_k, x_2)$ situé dans le plan $x_1=C_k$ est associée au segment de droite lequel unit les points $y(C_k, -L_2)$ et $y(C_k, L_2)$, et tous ces segments (pour tous les valeurs de k) constituent la surface $y_{O.i}.x_1$. Pour le calcul des dérivées partielles de premier ordre, nous allons d'abord déterminer l'expression de ces courbes-crête: $y(-L_1, x_2)$, $y(L_1, x_2)$, $y(x_1, -L_2)$ et $y(x_1, L_2)$ résultant de l'intersection de la surface $y(x_1, x_2)$ avec chacun des plans $x_1=-L_1, x_1=L_1, x_2=-L_2$ et $x_2=L_2$ lesquelles bordent le prisme de définition.

Les différences entre les valeurs prises, à la limite, par la fonction $y_F(x_1, x_2)$ au voisinage des frontières (avant et après le plan de séparation) génèrent entre les courbes de la frontières („courbes-crête”), des „surfaces-contremarche” pour chaque discontinuité de la fonction. Le saut infini qui se produit en raison de la dérivation de ce discontinuité peut être évité par l'introduction d'une «surface contremarche» d'effet inverse, générée par une surface-rampe $y_{O.i}.x_i$.

Les coefficients du développement en SFS des courbes-crête interviennent dans le calcul des coefficients du développement en série des dérivées partielles (nous attribuerons, par extension, l'indice O , par analogie avec les constantes $f_O(L)$ issues de l'analyse des fonctions monovariées, bien que la fonction $y_{O.i}$ peut avoir tous les trois composants):

$$y_{O.1}(L_1, x_2) = \frac{y(L_1, x_2) - y(-L_1, x_2)}{2L_1} = Y_{00}^1 + \sum_{n=1}^{\infty} Y_{0n}^{1.OE} \cos \omega_{n2} x_2 + \sum_{n=1}^{\infty} Y_{0n}^{1.OO} \sin \omega_{n2} x_2 \tag{11}$$

$$y_{O.2}(x_1, L_2) = \frac{y(x_1, L_2) - y(x_1, -L_2)}{2L_2} = Y_{00}^2 + \sum_{m=1}^{\infty} Y_{m0}^{2.EO} \cos \omega_{m1} x_1 + \sum_{m=1}^{\infty} Y_{m0}^{2.OO} \sin \omega_{m1} x_1$$

En appliquant la formule (7) sur les relations (4.1), respectivement (4.2), on obtient:

$$\begin{aligned}
& \left(\frac{\partial y}{\partial x_1} \right)^{SFS} = \sum_{m=1}^{\infty} \left[\omega_{m1} \left(Y_{m0}^{OO} + \sum_{n=1}^{\infty} Y_{mn}^{OE} \cos \omega_{n2} x_2 + \sum_{n=1}^{\infty} Y_{mn}^{EO} \sin \omega_{n2} x_2 \right) + 2(-1)^m y_{O.1}(L_1, x_2) \right] \cos \omega_{m1} x_1 - \\
& - \sum_{m=1}^{\infty} \omega_{m1} \left(Y_{m0}^{EO} + \sum_{n=1}^{\infty} Y_{mn}^{EE} \cos \omega_{n2} x_2 + \sum_{n=1}^{\infty} Y_{mn}^{EO} \sin \omega_{n2} x_2 \right) \sin \omega_{m1} x_1 + y_{O.1}(L_1, x_2) = \\
& = Y_{00}^1 + \sum_{m=1}^{\infty} \left[\omega_{m1} Y_{m0}^{OO} + 2(-1)^m Y_{00}^1 \right] \cos \omega_{m1} x_1 - \sum_{m=1}^{\infty} \omega_{m1} Y_{m0}^{EO} \sin \omega_{m1} x_1 + \sum_{n=1}^{\infty} Y_{0n}^{1.OE} \cos \omega_{n2} x_2 + \sum_{n=1}^{\infty} Y_{0n}^{1.OO} \sin \omega_{n2} x_2 + \\
& + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[\omega_{m1} Y_{mn}^{OE} + 2(-1)^m Y_{0n}^{1.OE} \right] \cos \omega_{m1} x_1 \cos \omega_{n2} x_2 + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[\omega_{m1} Y_{mn}^{EO} + 2(-1)^m Y_{0n}^{1.OO} \right] \cos \omega_{m1} x_1 \sin \omega_{n2} x_2 - \\
& - \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \omega_{m1} Y_{mn}^{EE} \sin \omega_{m1} x_1 \cos \omega_{n2} x_2 - \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \omega_{m1} Y_{mn}^{EO} \sin \omega_{m1} x_1 \sin \omega_{n2} x_2
\end{aligned} \tag{12}$$

$$\begin{aligned}
\left(\frac{\partial y}{\partial x_2}\right)^{SFS} &= \sum_{n=1}^{\infty} \left[\omega_{n2} \left(Y_{0n}^{00} + \sum_{m=1}^{\infty} Y_{mn}^{EO} \cos \omega_{m1} x_1 + \sum_{m=1}^{\infty} Y_{mn}^{OO} \sin \omega_{m1} x_1 \right) + 2(-1)^n y_{O.2}(L_2, x_1) \right] \cos \omega_{n2} x_2 - \\
&- \sum_{n=1}^{\infty} \omega_{n2} \left(Y_{0n}^{E0} + \sum_{m=1}^{\infty} Y_{mn}^{EE} \cos \omega_{m1} x_1 + \sum_{m=1}^{\infty} Y_{mn}^{OE} \sin \omega_{m1} x_1 \right) \sin \omega_{n2} x_2 + y_{O.2}(L_2, x_1) = \\
&= Y_{00}^2 + \sum_{m=1}^{\infty} Y_{m0}^{2.E0} \cos \omega_{m1} x_1 + \sum_{m=1}^{\infty} Y_{m0}^{2.O0} \sin \omega_{m1} x_1 + \sum_{n=1}^{\infty} \left[\omega_{n2} Y_{0n}^{00} + 2(-1)^n Y_{00}^2 \right] \cos \omega_{n2} x_2 - \sum_{n=1}^{\infty} \omega_{n2} Y_{0n}^{E0} \sin \omega_{n2} x_2 + \\
&+ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[\omega_{n2} Y_{mn}^{EO} + 2(-1)^n Y_{m0}^{2.E0} \right] \cos \omega_{m1} x_1 \cos \omega_{n2} x_2 - \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \omega_{n2} Y_{mn}^{EE} \cos \omega_{m1} x_1 \sin \omega_{n2} x_2 + \\
&+ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[\omega_{n2} Y_{mn}^{OO} + 2(-1)^n Y_{m0}^{2.O0} \right] \sin \omega_{m1} x_1 \cos \omega_{n2} x_2 - \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \omega_{n2} Y_{mn}^{OE} \sin \omega_{m1} x_1 \sin \omega_{n2} x_2 \quad (13)
\end{aligned}$$

Remarque 1: pour $y(+L_1, x_2) = y(-L_1, x_2) \Rightarrow y_{O.1}(x_2) = 0$ et la relation (12) devienne (9).

Remarque 2: Comme pour les fonctions monovariables:

$$y_{O.1}(x_2) = \lim_{m \rightarrow \infty} 2(-1)^{m+1} \omega_{m1} \left(Y_{m0}^{00} + \sum_{n=1}^{\infty} Y_{mn}^{OE} \cos \omega_{n2} x_2 + \sum_{n=1}^{\infty} Y_{mn}^{OO} \sin \omega_{n2} x_2 \right) \quad (11.1)$$

Remarque 3: Les courbes-crête de la fonction $y(x_1, x_2)$ sont des cas particuliers des fonctions monovariables $y(C, x_i)$, qui sont obtenues lorsque C prend les valeurs particulières $C = \pm L_i$. Les extrémités de chacune de ces courbes-crête coïncident avec une extrémité des courbes-crête correspondant à l'autre variable:

$$y(C, x_1) \Big|_{C=-L_2, x_1=-L_1} = y(C, x_2) \Big|_{C=-L_1, x_2=-L_2}, \quad y(C, x_1) \Big|_{C=L_2, x_1=-L_1} = y(C, x_2) \Big|_{C=-L_1, x_2=L_2}, \quad \text{etc}$$

Cette observation est également valable pour les courbes issues de la dérivation, quel que soit le nombre de dérivations et la variable par rapport à laquelle la dérivation est faite.

Il est facile d'observer que tous les deux dérivations peuvent être faites par dérivation terme par terme de l'expression (4), mais en compensant le saut de la périphérie du prisme, chaque fois qu'on dérive un terme impair par rapport à la variable de dérivation.

Avec les mêmes données initiales, en appliquant la formule (8) vers les mêmes fonctions x_i -monovariables, on obtient les développements en série des intégrales partielles du premier ordre de la fonction y :

$$\begin{aligned}
\left[\frac{\partial^{-1} y}{\partial x_1} \right]^{SFS} &= \left[\int_0^{x_1} y(C, x_1) dx_1 \right]^{SFS} = \sum_{m=1}^{\infty} \frac{Y_{C.1}^{C.O}}{\omega_{m1}} + \sum_{m=1}^{\infty} \frac{-Y_{C.1}^{C.O} \cos \omega_{m1} x_1 + \left[Y_{C.1}^{C.E} + 2(-1)^{m+1} Y_{C.1}^{C.0} \right] \sin \omega_{m1} x_1}{\omega_{m1}} \\
&= \sum_{m=1}^{\infty} \frac{Y_{m0}^{00} + \sum_{n=1}^{\infty} Y_{mn}^{OE} \cos \omega_{n2} x_2 + \sum_{n=1}^{\infty} Y_{mn}^{OO} \sin \omega_{n2} x_2}{\omega_{m1}} - \sum_{m=1}^{\infty} \frac{Y_{m0}^{00} + \sum_{n=1}^{\infty} Y_{mn}^{OE} \cos \omega_{n2} x_2 + \sum_{n=1}^{\infty} Y_{mn}^{OO} \sin \omega_{n2} x_2}{\omega_{m1}} \cos \omega_{m1} x_1 + \\
&+ \sum_{m=1}^{\infty} \frac{Y_{m0}^{E0} + \sum_{n=1}^{\infty} Y_{mn}^{EE} \cos \omega_{n2} x_2 + \sum_{n=1}^{\infty} Y_{mn}^{EO} \sin \omega_{n2} x_2 - 2(-1)^m \left(Y_{00} + \sum_{n=1}^{\infty} Y_{0n}^{0E} \cos \omega_{n2} x_2 + \sum_{n=1}^{\infty} Y_{0n}^{0O} \sin \omega_{n2} x_2 \right)}{\omega_{m1}} \sin \omega_{m1} x_1 \\
\left[\frac{\partial^{-1} y}{\partial x_1} \right]^{SFS} &= \sum_{m=1}^{\infty} \frac{Y_{m0}^{00}}{\omega_{m1}} - \sum_{m=1}^{\infty} \frac{Y_{m0}^{00}}{\omega_{m1}} \cos \omega_{m1} x_1 + \sum_{m=1}^{\infty} \frac{Y_{m0}^{E0} - 2(-1)^m Y_{00}}{\omega_{m1}} \sin \omega_{m1} x_1 + \sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} \frac{Y_{mn}^{OE}}{\omega_{m1}} \right) \cos \omega_{n2} x_2 + \sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} \frac{Y_{mn}^{OO}}{\omega_{m1}} \right) \sin \omega_{n2} x_2 - \\
&- \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{Y_{mn}^{OE}}{\omega_{m1}} \cos \omega_{m1} x_1 \cos \omega_{n2} x_2 - \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{Y_{mn}^{OO}}{\omega_{m1}} \cos \omega_{m1} x_1 \sin \omega_{n2} x_2 + \\
&+ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{Y_{mn}^{EE} - 2(-1)^m Y_{0n}^{0E}}{\omega_{m1}} \sin \omega_{m1} x_1 \cos \omega_{n2} x_2 + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{Y_{mn}^{EO} - 2(-1)^m Y_{0n}^{0O}}{\omega_{m1}} \sin \omega_{m1} x_1 \sin \omega_{n2} x_2 \\
\left[\frac{\partial^{-1} y}{\partial x_2} \right]^{SFS} &= \left[\int_0^{x_2} y(C, x_2) dx_2 \right]^{SFS} = \sum_{n=1}^{\infty} \frac{Y_{C.2}^{C.O}}{\omega_{n2}} + \sum_{n=1}^{\infty} \frac{-Y_{C.2}^{C.O} \cos \omega_{n2} x_2 + \left[Y_{C.2}^{C.E} + 2(-1)^{n+1} Y_{C.2}^{C.0} \right] \sin \omega_{n2} x_2}{\omega_{n2}}
\end{aligned} \quad (14)$$

$$\begin{aligned}
&= \sum_{n=1}^{\infty} \frac{Y_{0n}^{00} + \sum_{m=1}^{\infty} Y_{mn}^{EO} \cos \omega_{m1} x_1 + \sum_{m=1}^{\infty} Y_{mn}^{OO} \sin \omega_{m1} x_1}{\omega_{n2}} - \sum_{n=1}^{\infty} \frac{Y_{0n}^{00} + \sum_{m=1}^{\infty} Y_{mn}^{EO} \cos \omega_{m1} x_1 + \sum_{m=1}^{\infty} Y_{mn}^{OO} \sin \omega_{m1} x_1}{\omega_{n2}} \cos \omega_{n2} x_2 + \\
&+ \sum_{n=1}^{\infty} \frac{Y_{0n}^{0E} + \sum_{m=1}^{\infty} Y_{mn}^{EE} \cos \omega_{m1} x_1 + \sum_{m=1}^{\infty} Y_{mn}^{EO} \sin \omega_{m1} x_1 - 2(-1)^n \left(Y_{00} + \sum_{m=1}^{\infty} Y_{m0}^{E0} \cos \omega_{m1} x_1 + \sum_{m=1}^{\infty} Y_{m0}^{O0} \sin \omega_{m1} x_1 \right)}{\omega_{n2}} \sin \omega_{n2} x_2 \\
\left[\frac{\partial^{-1} y}{\partial x_2} \right]^{SFS} &= \sum_{n=1}^{\infty} \frac{Y_{0n}^{00}}{\omega_{n2}} + \sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} \frac{Y_{mn}^{EO}}{\omega_{n2}} \right) \cos \omega_{m1} x_1 + \sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} \frac{Y_{mn}^{OO}}{\omega_{n2}} \right) \sin \omega_{m1} x_1 - \sum_{n=1}^{\infty} \frac{Y_{0n}^{00}}{\omega_{n2}} \cos \omega_{n2} x_2 + \sum_{n=1}^{\infty} \frac{Y_{0n}^{0E} - 2(-1)^n Y_{00}}{\omega_{n2}} \sin \omega_{n2} x_2 - \\
&- \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{Y_{mn}^{EO}}{\omega_{n2}} \cos \omega_{m1} x_1 \cos \omega_{n2} x_2 + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{Y_{mn}^{EE} - 2(-1)^n Y_{0n}^{EO}}{\omega_{n2}} \cos \omega_{m1} x_1 \sin \omega_{n2} x_2 + \\
&+ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{Y_{mn}^{OO}}{\omega_{n2}} \sin \omega_{m1} x_1 \cos \omega_{n2} x_2 + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{Y_{mn}^{EO} - 2(-1)^n Y_{m0}^{O0}}{\omega_{n2}} \sin \omega_{m1} x_1 \sin \omega_{n2} x_2
\end{aligned} \tag{15}$$

4. Les coefficients des développements en SFS des dérivées et des intégrales partielles d'un ordre supérieur pour les fonctions à deux variables

Les développements en SFS des dérivées et des intégrales partielles d'ordre supérieur de la fonction multivariable $y(\mathbf{x})$, par rapport à la variable x_i , sont obtenus en appliquant les formules (7), respectivement (8) (la dérivation, respectivement l'intégration, par rapport à la variable x_i), aux fonctions x_i -monovariées résultant de la dérivation, respectivement de l'intégration partielle d'un ordre inférieur de la fonction $y(\mathbf{x})$.

Par exemple, des dérivées du second ordre et des intégrales partielles du second ordre sont obtenues à partir des fonctions x_i -monovariées à partir des expressions (12) et (13), respectivement (14) et (15).

Le calcul des dérivées partielles (par rapport à la variable x_i) nécessite la détermination des nouvelles conditions aux frontières du domaine de définition: si la fonction $y(\mathbf{x})$ a été dérivée par rapport à la variable x_i , les expressions des courbes-crête (générées par l'intersection des hypersurfaces $y(C, x_i)$ avec chacun des plans $x_i = -L_i$ et $x_i = L_i$), sont également dérivées par rapport à cette variable; les nouvelles courbes-crête sont générées par la fonction $\partial Y / \partial x_i$ (où Y peut être l'une des dérivées calculées précédemment, par rapport à d'autres variables et a des conditions aux limites connues). Pour la fonction $y(x_1, x_2)$, les nouvelles courbes-crête, qui bordent les „surfaces-contremarche” de la discontinuité de la fonction $\partial y / \partial x_i$ et pour les coefficients de leurs développements en SFS nous allons obtenir:

$$\begin{aligned}
y'_{0.1} &= \frac{1}{2L_1} \left[\frac{\partial y(x_1, x_2)}{\partial x_2} \Big|_{x_1=L_1} - \frac{\partial y(x_1, x_2)}{\partial x_2} \Big|_{x_1=-L_1} \right] = \frac{\partial}{\partial x_2} \left(Y_{00}^1 + \sum_{n=1}^{\infty} Y_{0n}^{1.0E} \cos \omega_{n2} x_2 + \sum_{n=1}^{\infty} Y_{0n}^{1.0O} \sin \omega_{n2} x_2 \right) = \\
&= Y_{00}^{1.1} + \sum_{n=1}^{\infty} \left[\omega_{n2} Y_{0n}^{1.0O} + 2(-1)^n Y_{00}^{1.1} \right] \cos \omega_{n2} x_2 - \sum_{n=1}^{\infty} \omega_{n2} Y_{0n}^{1.0E} \sin \omega_{n2} x_2
\end{aligned} \tag{16}$$

$$\begin{aligned}
y'_{0.2}(x_1, L_2) &= \frac{1}{2L_2} \left[\frac{\partial y(x_1, x_2)}{\partial x_1} \Big|_{x_2=L_2} - \frac{\partial y(x_1, x_2)}{\partial x_1} \Big|_{x_2=-L_2} \right] = \frac{\partial}{\partial x_1} \left(Y_{00}^2 + \sum_{m=1}^{\infty} Y_{m0}^{2.E0} \cos \omega_{m1} x_1 + \sum_{m=1}^{\infty} Y_{m0}^{2.O0} \sin \omega_{m1} x_1 \right) = \\
&= Y_{00}^{2.2} + \sum_{m=1}^{\infty} \left[\omega_{m1} Y_{m0}^{2.O0} + 2(-1)^m Y_{00}^{2.2} \right] \cos \omega_{m1} x_1 - \sum_{m=1}^{\infty} \omega_{m1} Y_{m0}^{2.E0} \sin \omega_{m1} x_1
\end{aligned} \tag{17}$$

On observe que:

$$Y_{00}^{1.1} = \frac{1}{2L_2} \int_{-L_2}^{L_2} \frac{\partial}{\partial x_2} y_{0.1}(L_1, x_2) dx_2 = \frac{y(L_1, L_2) - y(L_1, -L_2) - y(-L_1, L_2) + y(-L_1, -L_2)}{4L_1 L_2} \text{ et}$$

$$Y_{00}^{r2} = \frac{1}{2L_1} \int_{-L_1}^{L_1} \frac{\partial}{\partial x_1} y_{0,2}(x_1, L_2) dx_1 = \frac{y(L_1, L_2) - y(L_1, -L_2) - y(-L_1, L_2) + y(-L_1, -L_2)}{4LL_2}$$

$$\text{Donc, } Y_{00}^{r1} = Y_{00}^{r2} \quad (18)$$

Chacune des ces deux dérivées partielles de premier ordre peut être écrite de deux manières sous la forme d'une fonction x_i -monovariable:

$$\begin{aligned} \left(\frac{\partial y}{\partial x_1} (C, x_1) \right)^{SFS} &= \left(Y_{00}^1 + \sum_{n=1}^{\infty} Y_{0n}^{1,0E} \cos \omega_{n2} x_2 + \sum_{n=1}^{\infty} Y_{0n}^{1,0O} \sin \omega_{n2} x_2 \right) + \\ &+ \sum_{m=1}^{\infty} \left\{ \left[\omega_{m1} Y_{m0}^{O0} + 2(-1)^m Y_{00}^1 \right] + \sum_{n=1}^{\infty} \left[\omega_{m1} Y_{mn}^{OE} + 2(-1)^m Y_{0n}^{1,0E} \right] \cos \omega_{n2} x_2 + \sum_{n=1}^{\infty} \left[\omega_{m1} Y_{mn}^{OO} + 2(-1)^m Y_{0n}^{1,0O} \right] \sin \omega_{n2} x_2 \right\} \cos \omega_{m1} x_1 + \\ &- \sum_{m=1}^{\infty} \left(\omega_{m1} Y_{m0}^{E0} + \sum_{n=1}^{\infty} \omega_{m1} Y_{mn}^{EE} \cos \omega_{n2} x_2 + \sum_{n=1}^{\infty} \omega_{m1} Y_{mn}^{EO} \sin \omega_{n2} x_2 \right) \sin \omega_{m1} x_1 \end{aligned} \quad (12.1)$$

$$\left(\frac{\partial y}{\partial x_1} (C, x_2) \right)^{SFS} = \left\{ Y_{00}^1 + \sum_{m=1}^{\infty} \left[\omega_{m1} Y_{m0}^{O0} + 2(-1)^m Y_{00}^1 \right] \cos \omega_{m1} x_1 - \sum_{m=1}^{\infty} \omega_{m1} Y_{m0}^{E0} \sin \omega_{m1} x_1 \right\} + \quad (12.2)$$

$$\begin{aligned} &+ \sum_{n=1}^{\infty} \left\{ Y_{0n}^{1,0E} + \sum_{m=1}^{\infty} \left[\omega_{m1} Y_{mn}^{OE} + 2(-1)^m Y_{0n}^{1,0E} \right] \cos \omega_{m1} x_1 - \sum_{m=1}^{\infty} \omega_{m1} Y_{mn}^{EE} \sin \omega_{m1} x_1 \right\} \cos \omega_{n2} x_2 \\ &+ \sum_{n=1}^{\infty} \left\{ Y_{0n}^{1,0O} + \sum_{m=1}^{\infty} \left[\omega_{m1} Y_{mn}^{OO} + 2(-1)^m Y_{0n}^{1,0O} \right] \cos \omega_{m1} x_1 - \sum_{m=1}^{\infty} \omega_{m1} Y_{mn}^{EO} \sin \omega_{m1} x_1 \right\} \sin \omega_{n2} x_2 \end{aligned}$$

$$\begin{aligned} \left(\frac{\partial y}{\partial x_2} (C, x_1) \right)^{SFS} &= \left\{ Y_{00}^2 + \sum_{n=1}^{\infty} \left[\omega_{n2} Y_{0n}^{O0} + 2(-1)^n Y_{00}^2 \right] \cos \omega_{n2} x_2 - \sum_{n=1}^{\infty} \omega_{n2} Y_{0n}^{E0} \sin \omega_{n2} x_2 \right\} + \\ &+ \sum_{m=1}^{\infty} \left\{ Y_{m0}^{2,E0} + \sum_{n=1}^{\infty} \left[\omega_{n2} Y_{mn}^{EO} + 2(-1)^n Y_{m0}^{2,E0} \right] \cos \omega_{n2} x_2 - \sum_{n=1}^{\infty} \omega_{n2} Y_{mn}^{EE} \sin \omega_{n2} x_2 \right\} \cos \omega_{m1} x_1 \\ &+ \sum_{m=1}^{\infty} \left\{ Y_{m0}^{2,O0} + \sum_{n=1}^{\infty} \left[\omega_{n2} Y_{mn}^{OO} + 2(-1)^n Y_{m0}^{2,O0} \right] \cos \omega_{n2} x_2 - \sum_{n=1}^{\infty} \omega_{n2} Y_{mn}^{OE} \sin \omega_{n2} x_2 \right\} \sin \omega_{m1} x_1 \end{aligned} \quad (13.1)$$

$$\begin{aligned} \left(\frac{\partial y}{\partial x_2} (C, x_2) \right)^{SFS} &= \left(Y_{00}^2 + \sum_{m=1}^{\infty} Y_{m0}^{2,E0} \cos \omega_{m1} x_1 + \sum_{m=1}^{\infty} Y_{m0}^{2,O0} \sin \omega_{m1} x_1 \right) + \\ &+ \sum_{n=1}^{\infty} \left\{ \left[\omega_{n2} Y_{0n}^{O0} + 2(-1)^n Y_{00}^2 \right] + \sum_{m=1}^{\infty} \left[\omega_{n2} Y_{mn}^{OO} + 2(-1)^n Y_{m0}^{2,O0} \right] \sin \omega_{m1} x_1 + \sum_{m=1}^{\infty} \left[\omega_{n2} Y_{mn}^{EO} + 2(-1)^n Y_{m0}^{2,E0} \right] \cos \omega_{m1} x_1 \right\} \cos \omega_{n2} x_2 \\ &- \sum_{n=1}^{\infty} \left(\omega_{n2} Y_{0n}^{E0} + \sum_{m=1}^{\infty} \omega_{n2} Y_{mn}^{EE} \cos \omega_{m1} x_1 + \sum_{m=1}^{\infty} \omega_{n2} Y_{mn}^{EO} \sin \omega_{m1} x_1 \right) \sin \omega_{n2} x_2 \end{aligned} \quad (13.2)$$

Avec les conditions aux limites (16), en appliquant la formule (7) sur ces fonctions x_i -monovariables, on obtient les développements en série pour les dérivées partielles du second ordre:

$$\begin{aligned} \left(\frac{\partial^2 y}{\partial x_1^2} \right)^{SFS} &= \frac{\partial}{\partial x_1} \left[\frac{\partial y}{\partial x_1} (C, x_1) \right]^{SFS} = Y_{00}^{r1} + \sum_{n=1}^{\infty} \left[\omega_{n2} Y_{0n}^{1,0O} + 2(-1)^n Y_{00}^{r1} \right] \cos \omega_{n2} x_2 - \sum_{n=1}^{\infty} \omega_{n2} Y_{0n}^{1,0E} \sin \omega_{n2} x_2 - \\ &- \sum_{m=1}^{\infty} \left[\omega_{m1}^2 \left(Y_{m0}^{E0} + \sum_{n=1}^{\infty} Y_{mn}^{EE} \cos \omega_{n2} x_2 + \sum_{n=1}^{\infty} Y_{mn}^{EO} \sin \omega_{n2} x_2 \right) \right] \cos \omega_{m1} x_1 - \\ &- \sum_{m=1}^{\infty} 2(-1)^m \left\{ Y_{00}^{r1} + \sum_{n=1}^{\infty} \left[\omega_{n2} Y_{0n}^{1,0O} + 2(-1)^n Y_{00}^{r1} \right] \cos \omega_{n2} x_2 - \sum_{n=1}^{\infty} \omega_{n2} Y_{0n}^{1,0E} \sin \omega_{n2} x_2 \right\} \cos \omega_{m1} x_1 - \\ &- \sum_{m=1}^{\infty} \omega_{m1}^2 \left(Y_{m0}^{O0} + \sum_{n=1}^{\infty} Y_{mn}^{OE} \cos \omega_{n2} x_2 + \sum_{n=1}^{\infty} Y_{mn}^{OO} \sin \omega_{n2} x_2 \right) \sin \omega_{m1} x_1 - \end{aligned}$$

$$\begin{aligned}
& - \sum_{m=1}^{\infty} 2(-1)^m \omega_{m1} \left(Y_{00}^1 + \sum_{n=1}^{\infty} Y_{0n}^{1.0E} \cos \omega_{n2} x_2 + \sum_{n=1}^{\infty} Y_{0n}^{1.0O} \sin \omega_{n2} x_2 \right) \sin \omega_{m1} x_1 = \\
& = Y_{00}^{1.1} - \sum_{m=1}^{\infty} \left[\omega_{m1}^2 Y_{m0}^{E0} + 2(-1)^m Y_{00}^{1.1} \right] \cos \omega_{m1} x_1 - \sum_{m=1}^{\infty} \left[\omega_{m1}^2 Y_{m0}^{O0} + 2(-1)^m \omega_{m1} Y_{00}^{1.1} \right] \sin \omega_{m1} x_1 + \\
& + \sum_{n=1}^{\infty} \left[\omega_{n2} Y_{0n}^{1.0O} + 2(-1)^n Y_{00}^{1.1} \right] \cos \omega_{n2} x_2 - \sum_{n=1}^{\infty} \omega_{n2} Y_{0n}^{1.0E} \sin \omega_{n2} x_2 - \\
& - \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[\omega_{m1}^2 Y_{mn}^{EE} + 2(-1)^m \omega_{n2} Y_{0n}^{1.0O} + 4(-1)^{m+n} Y_{00}^{1.1} \right] \cos \omega_{m1} x_1 \cos \omega_{n2} x_2 - \\
& - \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[\omega_{m1}^2 Y_{mn}^{EO} + 2(-1)^m \omega_{n2} Y_{0n}^{1.0E} \right] \cos \omega_{m1} x_1 \sin \omega_{n2} x_2 - \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[\omega_{m1}^2 Y_{mn}^{OE} + 2(-1)^m \omega_{m1} Y_{0n}^{1.0E} \right] \sin \omega_{m1} x_1 \cos \omega_{n2} x_2 - \\
& - \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[\omega_{m1}^2 Y_{mn}^{OO} + \sum_{n=1}^{\infty} 2(-1)^m \omega_{m1} Y_{0n}^{1.0O} \right] \sin \omega_{m1} x_1 \sin \omega_{n2} x_2 = \tag{19}
\end{aligned}$$

$$\begin{aligned}
& = y'_{0.1}(L_1, x_2) + \sum_{m=1}^{\infty} \left[-\omega_{m1}^2 Y_{m.1}^{C.E}(x_2) + 2(-1)^m y'_{0.1}(L_1, x_2) \right] \cos \omega_{m1} x_1 - \sum_{m=1}^{\infty} \left[\omega_{m1}^2 Y_{m.1}^{C.O}(x_2) + 2(-1)^m \omega_{m1} y_{0.1} \right] \sin \omega_{m1} x_1 \\
& \left[\frac{\partial^2 y}{\partial x_2^2} \right]^{SFS} = \frac{\partial}{\partial x_2} \left[\frac{\partial y}{\partial x_2} (C, x_2) \right]^{SFS} = Y_{00}^{1.2} + \sum_{m=1}^{\infty} \left[\omega_{m1} Y_{m0}^{2.O0} + 2(-1)^m Y_{00}^{1.2} \right] \cos \omega_{m1} x_1 - \sum_{m=1}^{\infty} \omega_{m1} Y_{m0}^{2.E0} \sin \omega_{m1} x_1 - \\
& - \sum_{n=1}^{\infty} \omega_{n2}^2 \left(Y_{0n}^{0E} + \sum_{m=1}^{\infty} Y_{mn}^{EE} \cos \omega_{m1} x_1 + \sum_{m=1}^{\infty} Y_{mn}^{OE} \sin \omega_{m1} x_1 \right) \cos \omega_{n2} x_2 - \\
& - \sum_{n=1}^{\infty} 2(-1)^n \left\{ Y_{00}^{1.2} + \sum_{m=1}^{\infty} \left[\omega_{m1} Y_{m0}^{2.O0} + 2(-1)^m Y_{00}^{1.2} \right] \cos \omega_{m1} x_1 - \sum_{m=1}^{\infty} \omega_{m1} Y_{m0}^{2.E0} \sin \omega_{m1} x_1 \right\} \cos \omega_{n2} x_2 - \\
& - \sum_{n=1}^{\infty} \omega_{n2}^2 \left[Y_{0n}^{0O} + \sum_{m=1}^{\infty} \left(Y_{mn}^{OO} \sin \omega_{m1} x_1 + Y_{mn}^{EO} \cos \omega_{m1} x_1 \right) \right] \sin \omega_{n2} x_2 - \\
& - \sum_{n=1}^{\infty} 2(-1)^n \omega_{n2} \left[Y_{00}^{1.2} + \sum_{m=1}^{\infty} \left(Y_{m0}^{2.O0} \sin \omega_{m1} x_1 + Y_{m0}^{2.E0} \cos \omega_{m1} x_1 \right) \right] \sin \omega_{n2} x_2 = \\
& = Y_{00}^{1.2} + \sum_{m=1}^{\infty} \left[\omega_{m1} Y_{m0}^{2.O0} + 2(-1)^m Y_{00}^{1.2} \right] \cos \omega_{m1} x_1 - \sum_{m=1}^{\infty} \omega_{m1} Y_{m0}^{2.E0} \sin \omega_{m1} x_1 - \sum_{n=1}^{\infty} \left[\omega_{n2}^2 Y_{0n}^{0E} + 2(-1)^n Y_{00}^{1.2} \right] \cos \omega_{n2} x_2 - \\
& - \sum_{n=1}^{\infty} \left[\omega_{n2}^2 Y_{0n}^{0O} + 2(-1)^n \omega_{n2} Y_{00}^{1.2} \right] \sin \omega_{n2} x_2 - \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[\omega_{n2}^2 Y_{mn}^{EE} + 2(-1)^n \omega_{m1} Y_{m0}^{2.O0} + 4(-1)^{m+n} Y_{00}^{1.2} \right] \cos \omega_{m1} x_1 \cos \omega_{n2} x_2 - \\
& - \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[\omega_{n2}^2 Y_{mn}^{OE} + 2(-1)^n \omega_{m1} Y_{m0}^{2.E0} \right] \sin \omega_{m1} x_1 \cos \omega_{n2} x_2 - \tag{20} \\
& - \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[\omega_{n2}^2 Y_{mn}^{EO} + 2(-1)^n \omega_{n2} Y_{m0}^{2.E0} \right] \cos \omega_{m1} x_1 \sin \omega_{n2} x_2 - \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[\omega_{n2}^2 Y_{mn}^{OO} + 2(-1)^n \omega_{n2} Y_{m0}^{2.O0} \right] \sin \omega_{m1} x_1 \sin \omega_{n2} x_2 = \\
& = y'_{0.2}(x_1, L_2) + \sum_{n=1}^{\infty} \left[-\omega_{n2}^2 Y_{n.2}^{C.E}(x_1) + 2(-1)^n y'_{0.2}(x_1, L_2) \right] \cos \omega_{n2} x_2 - \sum_{n=1}^{\infty} \left[\omega_{n2}^2 Y_{n.2}^{C.O}(x_1) + 2(-1)^n \omega_{n2} y_{0.2} \right] \sin \omega_{n2} x_2
\end{aligned}$$

Dans tout les deux relations, $F_{C.i}^{C.O}, F_{C.i}^{C.E}, F_{C.i}^{C.O}$, $i=1, 2$ ont le sens de l'expression (8).

Nous remarquons encore une fois que les deux dérivations peuvent être faites par dérivation terme par terme de l'expression (4), avec la compensation de la discontinuité de la frontière, lorsque le terme dérivé est impair par rapport à la variable de dérivation. Par conséquent, la formule précédemment déduite pour les fonctions d'une seule variable peut être appliquée directement [2]:

$$\text{Pour } \bar{y}(x) = y_0 + \sum_{n=1}^{\infty} A_n \cos \omega_n x + \sum_{n=1}^{\infty} B_n \sin \omega_n x, \quad y_0(x) = 1/2 \cdot [y(x) - y(-x)]$$

$$\bar{y}'' = \sum_{n=1}^{\infty} \left[\left(-A_n \omega_n^2 + 2(-1)^n \frac{y'_0(L)}{L} \right) \cos \omega_n x - \left(B_n \omega_n^2 + 2(-1)^n \omega_n \frac{y_0(L)}{L} \right) \sin \omega_n x \right] + \frac{y'_0(L)}{L}$$

Pour les dérivées partielles par rapport à tout les deux variables:

$$\begin{aligned}
\frac{\partial^2 y}{\partial x_1 \partial x_2} &= \frac{\partial}{\partial x_1} \left[\frac{\partial y}{\partial x_2} (C, x_1) \right]^{SFS} = Y_{00}^{r1.} + \sum_{n=1}^{\infty} [\omega_{n2} Y_{0n}^{1.0O} + 2(-1)^n Y_{00}^{r1.}] \cos \omega_{n2} x_2 - \sum_{n=1}^{\infty} \omega_{n2} Y_{0n}^{1.0E} \sin \omega_{n2} x_2 + \\
&+ \sum_{m=1}^{\infty} \omega_{m1} \left\{ Y_{m0}^{2.OO} + \sum_{n=1}^{\infty} [\omega_{n2} Y_{mn}^{OO} + 2(-1)^n Y_{m0}^{2.OO}] \cos \omega_{n2} x_2 - \sum_{n=1}^{\infty} \omega_{n2} Y_{mn}^{OE} \sin \omega_{n2} x_2 \right\} \cos \omega_{m1} x_1 + \\
&+ 2(-1)^m \left(Y_{00}^{r1.} + \sum_{n=1}^{\infty} [\omega_{n2} Y_{0n}^{1.0O} + 2(-1)^n Y_{00}^{r1.}] \cos \omega_{n2} x_2 - \sum_{n=1}^{\infty} \omega_{n2} Y_{0n}^{1.0E} \sin \omega_{n2} x_2 \right) \cos \omega_{m1} x_1 + \\
&- \sum_{m=1}^{\infty} \omega_{m1} \left\{ Y_{m0}^{2.EO} + \sum_{n=1}^{\infty} [\omega_{n2} Y_{mn}^{EO} + 2(-1)^n Y_{m0}^{2.EO}] \cos \omega_{n2} x_2 - \sum_{n=1}^{\infty} \omega_{n2} Y_{mn}^{EE} \sin \omega_{n2} x_2 \right\} \sin \omega_{m1} x_1 = \\
&= Y_{00}^{r1.} + \sum_{m=1}^{\infty} [\omega_{m1} Y_{m0}^{2.OO} + 2(-1)^m Y_{00}^{r1.}] \cos \omega_{m1} x_1 - \sum_{m=1}^{\infty} \omega_{m1} Y_{m0}^{2.EO} \sin \omega_{m1} x_1 + \\
&+ \sum_{n=1}^{\infty} [\omega_{n2} Y_{0n}^{1.0O} + 2(-1)^n Y_{00}^{r1.}] \cos \omega_{n2} x_2 - \sum_{n=1}^{\infty} \omega_{n2} Y_{0n}^{1.0E} \sin \omega_{n2} x_2 + \\
&+ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} [\omega_{m1} \omega_{n2} Y_{mn}^{OO} + 2(-1)^n \omega_{m1} Y_{m0}^{2.OO} + 2(-1)^m \omega_{n2} Y_{0n}^{1.0O} + 4(-1)^{m+n} Y_{00}^{r1.}] \cos \omega_{m1} x_1 \cos \omega_{n2} x_2 - \\
&- \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} [\omega_{m1} \omega_{n2} Y_{mn}^{EO} + 2(-1)^n \omega_{m1} Y_{m0}^{2.EO}] \sin \omega_{m1} x_1 \cos \omega_{n2} x_2 - \\
&- \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} [\omega_{m1} \omega_{n2} Y_{mn}^{OE} + 2(-1)^m \omega_{n2} Y_{0n}^{1.0E}] \cos \omega_{m1} x_1 \sin \omega_{n2} x_2 + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \omega_{m1} \omega_{n2} Y_{mn}^{EE} \sin \omega_{m1} x_1 \sin \omega_{n2} x_2
\end{aligned} \tag{21}$$

$$\begin{aligned}
\left[\frac{\partial^2 y}{\partial x_2 \partial x_1} \right]^{SFS} &= \frac{\partial}{\partial x_2} \left[\frac{\partial y}{\partial x_1} (C, x_2) \right]^{SFS} = Y_{00}^{r2.} + \sum_{m=1}^{\infty} [\omega_{m1} Y_{m0}^{2.OO} + 2(-1)^m Y_{00}^{r2.}] \cos \omega_{m1} x_1 - \sum_{m=1}^{\infty} \omega_{m1} Y_{m0}^{2.EO} \sin \omega_{m1} x_1 + \\
&+ \sum_{n=1}^{\infty} \left\{ \omega_{n2} Y_{0n}^{1.0O} + \omega_{n2} \sum_{m=1}^{\infty} [\omega_{m1} Y_{mn}^{OO} + 2(-1)^m Y_{0n}^{1.0O}] \cos \omega_{m1} x_1 - \omega_{n2} \sum_{m=1}^{\infty} \omega_{m1} Y_{mn}^{EO} \sin \omega_{m1} x_1 \right\} \cos \omega_{n2} x_2 + \\
&+ 2(-1)^n \sum_{n=1}^{\infty} \left\{ Y_{00}^{r2.} + \sum_{m=1}^{\infty} [\omega_{m1} Y_{m0}^{2.OO} + 2(-1)^m Y_{00}^{r2.}] \cos \omega_{m1} x_1 - \sum_{m=1}^{\infty} \omega_{m1} Y_{m0}^{2.EO} \sin \omega_{m1} x_1 \right\} \cos \omega_{n2} x_2 - \\
&- \sum_{n=1}^{\infty} \left\{ \omega_{n2} Y_{0n}^{1.0E} + \omega_{n2} \sum_{m=1}^{\infty} [\omega_{m1} Y_{mn}^{EO} + 2(-1)^m Y_{0n}^{1.0E}] \cos \omega_{m1} x_1 - \omega_{n2} \sum_{m=1}^{\infty} \omega_{m1} Y_{mn}^{EE} \sin \omega_{m1} x_1 \right\} \sin \omega_{n2} x_2 = \\
&= Y_{00}^{r2.} + \sum_{m=1}^{\infty} [\omega_{m1} Y_{m0}^{2.OO} + 2(-1)^m Y_{00}^{r2.}] \cos \omega_{m1} x_1 - \sum_{m=1}^{\infty} \omega_{m1} Y_{m0}^{2.EO} \sin \omega_{m1} x_1 + \\
&+ \sum_{n=1}^{\infty} [\omega_{n2} Y_{0n}^{1.0O} + 2(-1)^n Y_{00}^{r2.}] \cos \omega_{n2} x_2 - \sum_{n=1}^{\infty} \omega_{n2} Y_{0n}^{1.0E} \sin \omega_{n2} x_2 + \\
&+ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} [\omega_{m1} \omega_{n2} Y_{mn}^{OO} + 2(-1)^m \omega_{n2} Y_{0n}^{1.0O} + 2(-1)^n \omega_{m1} Y_{m0}^{2.OO} + 4(-1)^{m+n} Y_{00}^{r2.}] \cos \omega_{m1} x_1 \cos \omega_{n2} x_2 - \\
&- \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} [\omega_{m1} \omega_{n2} Y_{mn}^{EO} + 2(-1)^n \omega_{m1} Y_{m0}^{2.EO}] \sin \omega_{m1} x_1 \cos \omega_{n2} x_2 - \\
&- \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} [\omega_{m1} \omega_{n2} Y_{mn}^{OE} + 2(-1)^m \omega_{n2} Y_{0n}^{1.0E}] \cos \omega_{m1} x_1 \sin \omega_{n2} x_2 + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \omega_{m1} \omega_{n2} Y_{mn}^{EE} \sin \omega_{m1} x_1 \sin \omega_{n2} x_2
\end{aligned} \tag{22}$$

et parce que, selon (18), $Y_{00}^{r1.} = Y_{00}^{r2.}$, les deux développements sont égaux.

Les intégrales partielles d'ordre supérieur sont calculées à partir des relations (14) et (15) et en appliquant la formule (8):

$$\begin{aligned}
\left[\frac{\partial^{-2} y}{\partial x_1^2} \right]^{SFS} &= \left[\int_0^{x_1} \left(\int_0^{x_1} y(C, x_1) dx_1 \right) dx_1 \right]^{SFS} = \\
&= \sum_{m=1}^{\infty} \frac{Y_{m0}^{E0} + \sum_{n=1}^{\infty} Y_{mn}^{EE} \cos \omega_{n2} x_2 + \sum_{n=1}^{\infty} Y_{mn}^{EO} \sin \omega_{n2} x_2 - 2(-1)^m \left(Y_{00} + \sum_{n=1}^{\infty} Y_{0n}^{0E} \cos \omega_{n2} x_2 + \sum_{n=1}^{\infty} Y_{0n}^{0O} \sin \omega_{n2} x_2 \right)}{\omega_{m1}^2} \\
&- \sum_{m=1}^{\infty} \frac{Y_{m0}^{E0} + \sum_{n=1}^{\infty} Y_{mn}^{EE} \cos \omega_{n2} x_2 + \sum_{n=1}^{\infty} Y_{mn}^{EO} \sin \omega_{n2} x_2 - 2(-1)^m \left(Y_{00} + \sum_{n=1}^{\infty} Y_{0n}^{0E} \cos \omega_{n2} x_2 + \sum_{n=1}^{\infty} Y_{0n}^{0O} \sin \omega_{n2} x_2 \right)}{\omega_{m1}^2} \cos \omega_{m1} + \\
&+ \sum_{m=1}^{\infty} \frac{\left[1 - 2(-1)^m \left(Y_{m0}^{0O} + \sum_{n=1}^{\infty} Y_{mn}^{OE} \cos \omega_{n2} x_2 + \sum_{n=1}^{\infty} Y_{mn}^{OO} \sin \omega_{n2} x_2 \right) \right]}{\omega_{m1}^2} \sin \omega_{m1} x_1 \\
\left[\frac{\partial^{-2} y}{\partial x_1^2} \right]^{SFS} &= \sum_{m=1}^{\infty} \frac{Y_{m0}^{E0} - 2(-1)^m Y_{00}}{\omega_{m1}^2} - \sum_{m=1}^{\infty} \frac{Y_{m0}^{E0} - 2(-1)^m Y_{00}}{\omega_{m1}^2} \cos \omega_{m1} x_1 + \sum_{m=1}^{\infty} \left[1 - 2(-1)^m \right] \frac{Y_{m0}^{0O}}{\omega_{m1}^2} \sin \omega_{m1} x_1 + \\
&+ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\left[Y_{mn}^{EE} - 2(-1)^m Y_{0n}^{0E} \right]}{\omega_{m1}^2} \cos \omega_{n2} x_2 + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\left[Y_{mn}^{EO} - 2(-1)^m Y_{0n}^{0O} \right]}{\omega_{m1}^2} \sin \omega_{n2} x_2 - \\
&- \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{Y_{mn}^{EE} - 2(-1)^m Y_{0n}^{0E}}{\omega_{m1}^2} \cos \omega_{m1} \cos \omega_{n2} x_2 - \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{Y_{mn}^{EO} - 2(-1)^m Y_{0n}^{0O}}{\omega_{m1}^2} \cos \omega_{m1} \sin \omega_{n2} x_2 \\
&+ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[1 - 2(-1)^m \right] \frac{Y_{mn}^{OE}}{\omega_{m1}^2} \sin \omega_{m1} x_1 \cos \omega_{n2} x_2 + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[1 - 2(-1)^m \right] \frac{Y_{mn}^{OO}}{\omega_{m1}^2} \sin \omega_{m1} x_1 \sin \omega_{n2} x_2 \\
\left[\frac{\partial^{-2} y}{\partial x_1 \partial x_2} \right]^{SFS} &= \left[\int_0^{x_1} \left(\int_0^{x_2} f(C, x_2) dx_2 \right) dx_1 \right]^{SFS} \\
\left[\frac{\partial^{-1} y}{\partial x_2} (C, x_1) \right]^{SFS} &= \frac{\sum_{n=1}^{\infty} Y_{0n}^{0O} - \sum_{n=1}^{\infty} Y_{0n}^{0O} \cos \omega_{n2} x_2 + \sum_{n=1}^{\infty} \left[Y_{0n}^{0E} - 2(-1)^n Y_{00} \right] \sin \omega_{n2} x_2}{\omega_{n2}} + \\
&+ \frac{\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} Y_{mn}^{EO} - \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} Y_{mn}^{EO} \cos \omega_{n2} x_2 + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[Y_{mn}^{EE} - 2(-1)^n Y_{m0}^{E0} \right] \sin \omega_{n2} x_2}{\omega_{n2}} \cos \omega_{m1} x_1 + \\
&+ \frac{\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} Y_{mn}^{OO} - \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} Y_{mn}^{OO} \cos \omega_{n2} x_2 + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[Y_{mn}^{EO} - 2(-1)^n Y_{m0}^{0O} \right] \sin \omega_{n2} x_2}{\omega_{n2}} \sin \omega_{m1} x_1 \\
\left[\frac{\partial^{-2} y}{\partial x_1 \partial x_2} \right]^{SFS} &= \frac{\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} Y_{mn}^{OO} - \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} Y_{mn}^{OO} \cos \omega_{n2} x_2 + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[Y_{mn}^{EO} - 2(-1)^n Y_{m0}^{0O} \right] \sin \omega_{n2} x_2}{\omega_{m1} \omega_{n2}} - \\
&- \frac{\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} Y_{mn}^{OO} - \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} Y_{mn}^{OO} \cos \omega_{n2} x_2 + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[Y_{mn}^{EO} - 2(-1)^n Y_{m0}^{0O} \right] \sin \omega_{n2} x_2}{\omega_{m1} \omega_{n2}} \cos \omega_{m1} x_1 + \\
&+ \frac{\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} Y_{mn}^{EO} - \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} Y_{mn}^{EO} \cos \omega_{n2} x_2 + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[Y_{mn}^{EE} - 2(-1)^n Y_{m0}^{E0} \right] \sin \omega_{n2} x_2}{\omega_{m1} \omega_{n2}} \sin \omega_{m1} x_1 + \\
&- \sum_{m=1}^{\infty} 2(-1)^m \frac{\sum_{n=1}^{\infty} Y_{0n}^{0O} - \sum_{n=1}^{\infty} Y_{0n}^{0O} \cos \omega_{n2} x_2 + \sum_{n=1}^{\infty} \left[Y_{0n}^{0E} - 2(-1)^n Y_{00} \right] \sin \omega_{n2} x_2}{\omega_{m1} \omega_{n2}} \sin \omega_{m1} x_1 =
\end{aligned}$$

$$\begin{aligned}
&= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{Y_{mn}^{OO}}{\omega_m \omega_n} - \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{Y_{mn}^{OO}}{\omega_m \omega_n} \cos \omega_m x_1 + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{Y_{mn}^{EO} - 2(-1)^m Y_{0n}^{OO}}{\omega_m \omega_n} \sin \omega_m x_1 + \\
&- \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{Y_{mn}^{OO}}{\omega_m \omega_n} \cos \omega_n x_2 + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{Y_{mn}^{EO} - 2(-1)^n Y_{m0}^{OO}}{\omega_m \omega_n} \sin \omega_n x_2 + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{Y_{mn}^{OO}}{\omega_m \omega_n} \cos \omega_m \cos \omega_n x_2 - \\
&- \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{Y_{mn}^{EO} - 2(-1)^n Y_{m0}^{OO}}{\omega_m \omega_n} \cos \omega_m \sin \omega_n x_2 - \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{Y_{mn}^{EO} - 2(-1)^m Y_{0n}^{OO}}{\omega_m \omega_n} \sin \omega_m x_1 \cos \omega_n x_2 + \\
&+ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{[Y_{mn}^{EE} - 2(-1)^n Y_{m0}^{EO} - 2(-1)^m Y_{0n}^{OE} + 4(-1)^{m+n} Y_{00}]}{\omega_m \omega_n} \sin \omega_m x_1 \sin \omega_n x_2
\end{aligned}$$

En appliquant à plusieurs reprises cette procédure, sont obtenues les dérivées et les intégrales partielles de tout ordre de fonctions à deux variables. Pour les fonctions à plusieurs variables, la procédure est identique, mais le nombre de coefficients à calculer est plus élevé, étant égal à N^3 , où N est le nombre de variables de la fonction.

5. Le développement en série de Fourier du produit de deux fonctions périodiques dépendant de deux variables

Comme dans le cas des fonctions monovariées, dans le domaine $[-L_1, L_1] \times [-L_2, L_2]$, le produit $p(x_1, x_2) = a(x_1, x_2) \cdot y(x_1, x_2)$ entre deux fonctions de deux variables, de carré intégrables, c'est aussi une fonction de carré intégrables [5]. Le calcul des coefficients du développement en SFS du produit peut être fait avec les formules de calcul (4), en calculant les intégrales définies de type

$$\frac{1}{\|f\|^2} \int_{-L_1}^{L_1} \int_{-L_2}^{L_2} p(x_1, x_2) f(\cos \omega_m x_1, \cos \omega_n x_2, \sin \omega_m x_1, \sin \omega_n x_2) dx_1 dx_2, \text{ pour chaque nombre}$$

entier positif ou nul m et n , ($=0, 1, 2, \dots, \infty$).

Comme pour les fonctions monovariées, si les coefficients de Fourier du développement en série de chacun des termes du produit sont connus, des formules algébriques pour calculer les coefficients de Fourier du développement en série du produit peuvent être trouvées et utilisées. Ces formules sont indispensables si l'on veut transformer les PDEs en équations algébriques, ou si l'on veut calculer les relations de dépendance entre les coefficients du produit de ces deux fonctions multivariées (dans le cas du carré, les deux fonctions sont identiques) et les coefficients de chacun de ses termes.

Dans tout les deux variantes, si un certain degré d'approximation est permis, le calcul peut également être effectué par des méthodes numériques.

Soit les deux fonctions à deux variables $y(x_1, x_2)$ et $a(x_1, x_2)$, définis sur le domaine $[-L_1, L_1] \times [-L_2, L_2]$, périodiques, de carrés intégrables, qui peuvent être développés en SFS $y(x_1, x_2) = y_{00} + y_{10}(x_1) + y_{01}(x_2) + y_{11}(x_1, x_2) = y_{00} + y_{E0}(x_1) + y_{O0}(x_1) + y_{OE}(x_2) + y_{OO}(x_2) + y_{EE}(x_1, x_2) + y_{EO}(x_1, x_2) + y_{OE}(x_1, x_2) + y_{OO}(x_1, x_2)$

$$\begin{aligned}
[y(x_1, x_2)]^{SFS} &= Y_{00} + \sum_{m=1}^{\infty} (Y_{m0}^{E0} \cos \omega_m x_1 + Y_{m0}^{O0} \sin \omega_m x_1) + \sum_{n=1}^{\infty} (Y_{0n}^{OE} \cos \omega_n x_2 + Y_{0n}^{OO} \sin \omega_n x_2) + \\
&+ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (Y_{mn}^{EE} \cos \omega_m x_1 \cos \omega_n x_2 + Y_{mn}^{EO} \cos \omega_m x_1 \sin \omega_n x_2 + Y_{mn}^{OE} \sin \omega_m x_1 \cos \omega_n x_2 + Y_{mn}^{OO} \sin \omega_m x_1 \sin \omega_n x_2)
\end{aligned}$$

$$\begin{aligned}
a(x_1, x_2) &= a_{00} + a_{10}(x_1) + a_{01}(x_2) + a_{11}(x_1, x_2) = a_{00} + a_{E0}(x_1) + a_{O0}(x_1) + a_{OE}(x_2) + a_{OO}(x_2) + \\
&+ a_{EE}(x_1, x_2) + a_{EO}(x_1, x_2) + a_{OE}(x_1, x_2) + a_{OO}(x_1, x_2)
\end{aligned}$$

$$[a(x_1, x_2)]^{SFS} = A_{00} + \sum_{m=1}^{\infty} (A_{m0}^{E0} \cos \omega_m x_1 + A_{m0}^{O0} \sin \omega_m x_1) + \sum_{n=1}^{\infty} (A_{0n}^{OE} \cos \omega_n x_2 + A_{0n}^{OO} \sin \omega_n x_2) +$$

$$+ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (A_{mn}^{EE} \cos \omega_{m1} x_1 \cos \omega_{n2} x_2 + A_{mn}^{EO} \cos \omega_{m1} x_1 \sin \omega_{n2} x_2 + A_{mn}^{OE} \sin \omega_{m1} x_1 \cos \omega_{n2} x_2 + A_{mn}^{OO} \sin \omega_{m1} x_1 \sin \omega_{n2} x_2)$$

Tous les composants de ces développements, sauf y_{00} et a_{00} ont des valeurs moyennes nulles.

Par conséquent, le développement en SFS du produit prendra la forme:

$$\begin{aligned} (ya)^{SFS} &= \bar{P}(x, y) = [(y_{00} + y_{10} + y_{01} + y_{11}) \cdot (a_{00} + a_{10} + a_{01} + a_{11})]^{SFS} = \\ &= P_{00} + \sum_{m=1}^{\infty} (P_{m0}^{E0} \cos \omega_{m1} x_1 + P_{m0}^{O0} \sin \omega_{m1} x_1) + \sum_{n=1}^{\infty} (P_{0n}^{OE} \cos \omega_{n2} x_2 + P_{0n}^{OO} \sin \omega_{n2} x_2) + \\ &+ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (P_{mn}^{EE} \cos \omega_{m1} x_1 \cos \omega_{n2} x_2 + P_{mn}^{EO} \cos \omega_{m1} x_1 \sin \omega_{n2} x_2 + P_{mn}^{OE} \sin \omega_{m1} x_1 \cos \omega_{n2} x_2 + P_{mn}^{OO} \sin \omega_{m1} x_1 \sin \omega_{n2} x_2) \end{aligned}$$

Les termes du développement en des fonctions élémentaires du produit sont:

$$\begin{aligned} y(x_1, x_2) \cdot a(x_1, x_2) &= (y_{00} + y_{E0} + y_{O0} + y_{0E} + y_{0O} + y_{EE} + y_{EO} + y_{OE} + y_{OO}) \cdot \\ &\cdot (a_{00} + a_{E0} + a_{O0} + a_{0E} + a_{0O} + a_{EE} + a_{EO} + a_{OE} + a_{OO}) = \\ &= y_{00} a_{00} + (y_{00} a_{E0} + a_{00} y_{E0} + y_{00} a_{O0} + a_{00} y_{O0} + y_{00} a_{0E} + a_{00} y_{0E} + y_{00} a_{0O} + a_{00} y_{0O}) + \\ &+ (y_{E0} a_{E0} + y_{E0} a_{O0} + y_{O0} a_{E0} + y_{O0} a_{O0} + y_{0E} a_{0E} + y_{0E} a_{0O} + y_{0O} a_{0E} + y_{0O} a_{0O}) + \\ &+ (y_{00} a_{EE} + a_{00} y_{EE} + y_{E0} a_{0E} + a_{E0} y_{0E} + y_{00} a_{EO} + a_{00} y_{EO} + y_{E0} a_{0O} + a_{E0} y_{0O}) + \\ &+ (y_{00} a_{OE} + a_{00} y_{OE} + y_{O0} a_{0E} + a_{O0} y_{0E} + y_{00} a_{OO} + a_{00} y_{OO} + y_{O0} a_{0O} + a_{O0} y_{0O}) + \\ &+ (y_{EE} a_{E0} + y_{EE} a_{O0} + y_{EO} a_{E0} + y_{EO} a_{O0} + y_{OE} a_{E0} + y_{OE} a_{O0} + y_{OO} a_{E0} + y_{OO} a_{O0}) + \\ &+ (y_{EE} a_{0E} + y_{EE} a_{0O} + y_{EO} a_{0E} + y_{EO} a_{0O} + y_{OE} a_{0E} + y_{OE} a_{0O} + y_{OO} a_{0E} + y_{OO} a_{0O}) + \\ &+ (a_{EE} y_{E0} + a_{EE} y_{O0} + a_{EO} y_{E0} + a_{EO} y_{O0} + a_{OE} y_{E0} + a_{OE} y_{O0} + a_{OO} y_{E0} + a_{OO} y_{O0}) + \\ &+ (a_{EE} y_{0E} + a_{EE} y_{0O} + a_{EO} y_{0E} + a_{EO} y_{0O} + a_{OE} y_{0E} + a_{OE} y_{0O} + a_{OO} y_{0E} + a_{OO} y_{0O}) + \\ &+ (y_{EE} a_{EE} + y_{EE} a_{EO} + y_{EO} a_{EE} + y_{EO} a_{EO} + y_{OE} a_{OE} + y_{OE} a_{OO} + y_{OO} a_{OE} + y_{OO} a_{OO}) + \\ &+ (y_{EE} a_{OE} + y_{EE} a_{OO} + y_{EO} a_{OE} + y_{EO} a_{OO} + y_{OE} a_{EE} + y_{OE} a_{EO} + y_{OO} a_{EE} + y_{OO} a_{EO}) \end{aligned}$$

Par conséquent, le développement en SFS du produit sera la somme des développements de ces 81 fonctions élémentaires:

$$\begin{aligned} [P(x_1, x_2)]^{SFS} &= Y_{00} A_{00} + \sum_{m=1}^{\infty} [(Y_{00} A_{m0}^{E0} + A_{00} Y_{m0}^{E0}) \cos \omega_{m1} x_1 + (Y_{00} A_{m0}^{O0} + A_{00} Y_{m0}^{O0}) \sin \omega_{m1} x_1] + \\ &+ \sum_{n=1}^{\infty} [(Y_{00} A_{0n}^{OE} + A_{00} Y_{0n}^{OE}) \cos \omega_{n2} x_2 + (Y_{00} A_{0n}^{OO} + A_{00} Y_{0n}^{OO}) \sin \omega_{n2} x_2] + \\ &+ \left(\sum_{m=1}^{\infty} Y_{m0}^{E0} \cos \omega_{m1} x_1 \right) \left(\sum_{m=1}^{\infty} A_{m0}^{E0} \cos \omega_{m1} x_1 \right) + \left(\sum_{m=1}^{\infty} Y_{m0}^{O0} \cos \omega_{m1} x_1 \right) \left(\sum_{m=1}^{\infty} A_{m0}^{O0} \sin \omega_{m1} x_1 \right) + \\ &+ \left(\sum_{m=1}^{\infty} Y_{m0}^{OE} \sin \omega_{m1} x_1 \right) \left(\sum_{m=1}^{\infty} A_{m0}^{OE} \cos \omega_{m1} x_1 \right) + \left(\sum_{m=1}^{\infty} Y_{m0}^{OO} \sin \omega_{m1} x_1 \right) \left(\sum_{m=1}^{\infty} A_{m0}^{OO} \sin \omega_{m1} x_1 \right) + \left(\sum_{n=1}^{\infty} Y_{0n}^{OE} \cos \omega_{n2} x_2 \right) \left(\sum_{n=1}^{\infty} A_{0n}^{OE} \cos \omega_{n2} x_2 \right) + \\ &+ \left(\sum_{n=1}^{\infty} Y_{0n}^{OO} \cos \omega_{n2} x_2 \right) \left(\sum_{n=1}^{\infty} A_{0n}^{OO} \sin \omega_{n2} x_2 \right) + \left(\sum_{n=1}^{\infty} Y_{0n}^{OE} \cos \omega_{n2} x_2 \right) \left(\sum_{n=1}^{\infty} A_{0n}^{OE} \cos \omega_{n2} x_2 \right) + \left(\sum_{n=1}^{\infty} Y_{0n}^{OO} \cos \omega_{n2} x_2 \right) \left(\sum_{n=1}^{\infty} A_{0n}^{OO} \sin \omega_{n2} x_2 \right) + \\ &+ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (Y_{00} A_{mn}^{EE} + A_{00} Y_{mn}^{EE} + Y_{m0}^{E0} A_{0n}^{OE} + A_{m0}^{E0} Y_{0n}^{OE}) \cos \omega_{m1} x_1 \cos \omega_{n2} x_2 + \\ &+ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (Y_{00} A_{mn}^{EO} + A_{00} Y_{mn}^{EO} + Y_{m0}^{E0} A_{0n}^{OO} + A_{m0}^{E0} Y_{0n}^{OO}) \cos \omega_{m1} x_1 \sin \omega_{n2} x_2 + \\ &+ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (Y_{00} A_{mn}^{OE} + A_{00} Y_{mn}^{OE} + Y_{m0}^{O0} A_{0n}^{OE} + A_{m0}^{O0} Y_{0n}^{OE}) \sin \omega_{m1} x_1 \cos \omega_{n2} x_2 + \\ &+ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (Y_{00} A_{mn}^{OO} + A_{00} Y_{mn}^{OO} + Y_{m0}^{O0} A_{0n}^{OO} + A_{m0}^{O0} Y_{0n}^{OO}) \sin \omega_{m1} x_1 \sin \omega_{n2} x_2 + \end{aligned}$$

$$\begin{aligned}
& + \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} Y_{mn}^{OE} \sin \omega_{m1} x_1 \cos \omega_{n2} x_2 \right) \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn}^{EE} \cos \omega_{m1} x_1 \cos \omega_{n2} x_2 \right) + \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} Y_{mn}^{OE} \sin \omega_{m1} x_1 \cos \omega_{n2} x_2 \right) \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn}^{EO} \cos \omega_{m1} x_1 \sin \omega_{n2} x_2 \right) + \\
& + \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} Y_{mn}^{OO} \sin \omega_{m1} x_1 \sin \omega_{n2} x_2 \right) \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn}^{EE} \cos \omega_{m1} x_1 \cos \omega_{n2} x_2 \right) + \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} Y_{mn}^{OO} \sin \omega_{m1} x_1 \sin \omega_{n2} x_2 \right) \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn}^{EO} \cos \omega_{m1} x_1 \sin \omega_{n2} x_2 \right)
\end{aligned}$$

Dans ces développements apparaissent plusieurs types de composants: fonctions simples, produits entre deux fonctions monovariables d'une même variable, produits entre deux fonctions monovariables à différentes variables, produits entre une fonction monovariabile et une autre à deux variables, produits entre deux fonctions à deux variables. Les produits entre les fonctions monovariables d'une même variable ont été traités dans un article précédent [2]. En utilisant cette méthodologie, appliquée à certaines fonctions élémentaires initialement considérées comme monovariables, peuvent être déduites des formules pour tous les types de produits de fonctions. Les calculs détaillés peuvent être suivis dans l'annexe (dernière section). Adequate à ces calculs, on obtient:

$$\begin{aligned}
[P(x_1, x_2)]^{SFS} &= Y_{00} A_{00} + \sum_{m=1}^{\infty} \left[(Y_{00} A_{m0}^{E0} + A_{00} Y_{m0}^{E0}) \cos \omega_{m1} x_1 + (Y_{00} A_{m0}^{O0} + A_{00} Y_{m0}^{O0}) \sin \omega_{m1} x_1 \right] + \\
& + \sum_{n=1}^{\infty} \left[(Y_{00} A_{0n}^{OE} + A_{00} Y_{0n}^{OE}) \cos \omega_{n2} x_2 + (Y_{00} A_{0n}^{OO} + A_{00} Y_{0n}^{OO}) \sin \omega_{n2} x_2 \right] + \\
& + \frac{1}{2} \sum_{m=1}^{\infty} Y_{m0}^{E0} A_{m0}^{E0} + \frac{1}{2} \sum_{m=1}^{\infty} \left[\sum_{l=1}^{\infty} Y_{l,0}^{E0} (A_{l+m,0}^{E0} + A_{|l-m|,0}^{E0}) \right] \cos \omega_{m1} x_1 + \frac{1}{2} \sum_{m=1}^{\infty} \left[\sum_{l=1}^{\infty} Y_{l,0}^{E0} (A_{l+m,0}^{O0} - \text{sgn}(l-m) A_{|l-m|,0}^{O0}) \right] \sin \omega_{m1} x_1 - \\
& - \frac{1}{2} \sum_{m=1}^{\infty} \left[\sum_{l=1}^{\infty} Y_{l,0}^{O0} (A_{l+m,0}^{E0} - A_{|l-m|,0}^{E0}) \right] \sin \omega_{m1} x_1 + \frac{1}{2} \sum_{m=1}^{\infty} Y_{m0}^{O0} A_{m0}^{O0} + \frac{1}{2} \sum_{m=1}^{\infty} \left[\sum_{l=1}^{\infty} Y_{l,0}^{O0} (A_{l+m,0}^{O0} + \text{sgn}(l-m) A_{|l-m|,0}^{O0}) \right] \cos \omega_{m1} x_1 + \\
& + \frac{1}{2} \sum_{n=1}^{\infty} Y_{0n}^{OE} A_{0n}^{OE} + \frac{1}{2} \sum_{n=1}^{\infty} \left[\sum_{l=1}^{\infty} Y_{0,l}^{OE} (A_{0,l+n}^{OE} + A_{0,|l-n|}^{OE}) \right] \cos \omega_{n2} x_2 + \frac{1}{2} \sum_{n=1}^{\infty} \left[\sum_{l=1}^{\infty} Y_{0,l}^{OE} (A_{0,l+n}^{OO} - \text{sgn}(l-n) A_{0,|l-n|}^{OO}) \right] \sin \omega_{n2} x_2 - \\
& - \frac{1}{2} \sum_{n=1}^{\infty} \left[\sum_{l=1}^{\infty} Y_{0,l}^{OO} (A_{0,l+n}^{OE} - A_{0,|l-n|}^{OE}) \right] \sin \omega_{n2} x_2 + \frac{1}{2} \sum_{n=1}^{\infty} Y_{0n}^{OO} A_{0n}^{OO} + \frac{1}{2} \sum_{n=1}^{\infty} \left[\sum_{l=1}^{\infty} Y_{0,l}^{OO} (A_{0,l+n}^{OO} + \text{sgn}(l-n) A_{0,|l-n|}^{OO}) \right] \cos \omega_{n2} x_2 + \\
& + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (Y_{00} A_{mn}^{EO} + A_{00} Y_{mn}^{EO} + Y_{m0}^{E0} A_{0n}^{O0} + A_{m0}^{E0} Y_{0n}^{O0}) \cos \omega_{m1} x_1 \sin \omega_{n2} x_2 + \\
& + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (Y_{00} A_{mn}^{EE} + A_{00} Y_{mn}^{EE} + Y_{m0}^{E0} A_{0n}^{OE} + A_{m0}^{E0} Y_{0n}^{OE}) \cos \omega_{m1} x_1 \cos \omega_{n2} x_2 + \\
& + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (Y_{00} A_{mn}^{OO} + A_{00} Y_{mn}^{OO} + Y_{m0}^{O0} A_{0n}^{O0} + A_{m0}^{O0} Y_{0n}^{O0}) \sin \omega_{m1} x_1 \sin \omega_{n2} x_2 + \\
& + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (Y_{00} A_{mn}^{OE} + A_{00} Y_{mn}^{OE} + Y_{m0}^{O0} A_{0n}^{OE} + A_{m0}^{O0} Y_{0n}^{OE}) \sin \omega_{m1} x_1 \cos \omega_{n2} x_2 + \\
& + \frac{1}{2} \sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} Y_{mn}^{EE} A_{m0}^{E0} \right) \cos \omega_{n2} x_2 + \frac{1}{2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[\sum_{l=1}^{\infty} Y_{l,n}^{EE} (A_{l+m,0}^{E0} + A_{|l-m|,0}^{E0}) \right] \cos \omega_{m1} x_1 \cos \omega_{n2} x_2 + \\
& + \frac{1}{2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[\sum_{l=1}^{\infty} Y_{l,n}^{EE} (A_{l+m,0}^{O0} - \text{sgn}(l-m) A_{|l-m|,0}^{O0}) \right] \sin \omega_{m1} x_1 \cos \omega_{n2} x_2 + \\
& + \frac{1}{2} \sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} Y_{mn}^{EO} A_{m0}^{E0} \right) \sin \omega_{n2} x_2 + \frac{1}{2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[\sum_{l=1}^{\infty} Y_{l,n}^{EO} (A_{l+m,0}^{E0} + A_{|l-m|,0}^{E0}) \right] \cos \omega_{m1} x_1 \sin \omega_{n2} x_2 + \\
& + \frac{1}{2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[\sum_{l=1}^{\infty} Y_{l,n}^{EO} (A_{l+m,0}^{O0} - \text{sgn}(l-m) A_{|l-m|,0}^{O0}) \right] \sin \omega_{m1} x_1 \sin \omega_{n2} x_2 - \\
& - \frac{1}{2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[\sum_{l=1}^{\infty} Y_{l,n}^{OE} (A_{l+m,0}^{E0} - A_{|l-m|,0}^{E0}) \right] \sin \omega_{m1} x_1 \cos \omega_{n2} x_2 +
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[\sum_{l=1}^{\infty} Y_{0,l}^{OE} \left(A_{m,l+n}^{EO} - \operatorname{sgn}(l-n) A_{m,|l-n|}^{EO} \right) \right] \cos \omega_{m1} x_1 \sin \omega_{n2} x_2 + \\
& + \frac{1}{2} \sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} Y_{0n}^{OO} A_{mm}^{EO} \right) \cos \omega_{m1} x_1 + \frac{1}{2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[\sum_{l=1}^{\infty} Y_{0,l}^{OO} \left(A_{m,l+n}^{EO} + \operatorname{sgn}(l-n) A_{m,|l-n|}^{EO} \right) \right] \cos \omega_{m1} x_1 \cos \omega_{n2} x_2 + \\
& + \frac{1}{2} \sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} Y_{0n}^{OE} A_{mm}^{OE} \right) \sin \omega_{m1} x_1 + \frac{1}{2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[\sum_{l=1}^{\infty} Y_{0,l}^{OE} \left(A_{m,l+n}^{OE} + A_{m,|l-n|}^{OE} \right) \right] \sin \omega_{m1} x_1 \cos \omega_{n2} x_2 + \\
& - \frac{1}{2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[\sum_{l=1}^{\infty} Y_{0,l}^{OO} \left(A_{m,l+n}^{OE} - A_{m,|l-n|}^{OE} \right) \right] \sin \omega_{m1} x_1 \sin \omega_{n2} x_2 - \\
& + \frac{1}{2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[\sum_{l=1}^{\infty} Y_{0,l}^{OE} \left(A_{m,l+n}^{OO} - \operatorname{sgn}(l-n) A_{m,|l-n|}^{OO} \right) \right] \sin \omega_{m1} x_1 \sin \omega_{n2} x_2 + \\
& + \frac{1}{2} \sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} Y_{0n}^{OO} A_{mm}^{OO} \right) \sin \omega_{m1} x_1 + \frac{1}{2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[\sum_{l=1}^{\infty} Y_{0,l}^{OO} \left(A_{m,l+n}^{OO} + \operatorname{sgn}(l-n) A_{m,|l-n|}^{OO} \right) \right] \sin \omega_{m1} x_1 \cos \omega_{n2} x_2 + \\
& + \frac{1}{4} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} Y_{m,n}^{EE} A_{m,n}^{EE} + \frac{1}{4} \sum_{m=1}^{\infty} \left[\sum_{n=1}^{\infty} \sum_{l=1}^{\infty} Y_{l,n}^{EE} \left(A_{l+m,n}^{EE} + A_{|l-m|,n}^{EE} \right) \right] \cos \omega_{m1} x_1 + \frac{1}{4} \sum_{n=1}^{\infty} \left[\sum_{m=1}^{\infty} \sum_{l=1}^{\infty} Y_{m,l}^{EE} \left(A_{m,l+n}^{EE} + A_{m,|l-m|}^{EE} \right) \right] \cos \omega_{n2} x_2 + \\
& + \frac{1}{4} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[\sum_{l=1}^{\infty} \sum_{r=1}^{\infty} \left(Y_{r+m,l}^{EE} + Y_{|r-m|,l}^{EE} \right) \left(A_{r,l+n}^{EE} + A_{r,|l-n|}^{EE} \right) \right] \cos \omega_{m1} x_1 \cos \omega_{n2} x_2 + \\
& + \frac{1}{4} \sum_{n=1}^{\infty} \left\{ \sum_{m=1}^{\infty} \sum_{l=1}^{\infty} Y_{m,l}^{EE} \left[A_{m,l+n}^{EO} - \operatorname{sgn}(l-n) A_{m,|l-n|}^{EO} \right] \right\} \sin \omega_{n2} x_2 + \\
& + \frac{1}{4} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[\sum_{l=1}^{\infty} \sum_{r=1}^{\infty} \left(Y_{r+m,l}^{EE} + Y_{|r-m|,l}^{EE} \right) \left(A_{r,l+n}^{EO} - \operatorname{sgn}(l-n) A_{r,|l-n|}^{EO} \right) \right] \cos \omega_{m1} x_1 \sin \omega_{n2} x_2 + \\
& - \frac{1}{4} \sum_{n=1}^{\infty} \left[\sum_{m=1}^{\infty} \sum_{l=1}^{\infty} Y_{m,l}^{EO} \left(A_{m,l+n}^{EE} - A_{m,|l-n|}^{EE} \right) \right] \sin \omega_{n2} x_2 - \\
& - \frac{1}{4} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[\sum_{l=1}^{\infty} \sum_{r=1}^{\infty} \left(Y_{r+m,l}^{EO} + Y_{|r-m|,l}^{EO} \right) \left(A_{r,l+n}^{EE} - A_{r,|l-n|}^{EE} \right) \right] \cos \omega_{m1} x_1 \sin \omega_{n2} x_2 + \\
& + \frac{1}{4} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} Y_{m,n}^{EO} A_{m,n}^{EO} + \frac{1}{4} \sum_{m=1}^{\infty} \left[\sum_{n=1}^{\infty} \sum_{l=1}^{\infty} Y_{l,n}^{EO} \left(A_{l+m,n}^{EO} + A_{|l-m|,n}^{EO} \right) \right] \cos \omega_{m1} x_1 + \frac{1}{4} \sum_{n=1}^{\infty} \left[\sum_{m=1}^{\infty} \sum_{l=1}^{\infty} Y_{m,l}^{EO} \left(A_{m,l+n}^{EO} + \operatorname{sgn}(l-n) A_{m,|l-n|}^{EO} \right) \right] \cos \omega_{n2} x_2 + \\
& + \frac{1}{4} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[\sum_{l=1}^{\infty} \sum_{r=1}^{\infty} \left(Y_{r+m,l}^{EO} + Y_{|r-m|,l}^{EO} \right) \left(A_{r,l+n}^{EO} + \operatorname{sgn}(l-n) A_{r,|l-n|}^{EO} \right) \right] \cos \omega_{m1} x_1 \cos \omega_{n2} x_2 + \frac{1}{4} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} Y_{m,n}^{EO} A_{m,n}^{EO} + \\
& + \frac{1}{4} \sum_{m=1}^{\infty} \left[\sum_{n=1}^{\infty} \sum_{l=1}^{\infty} Y_{l,n}^{OE} \left(A_{l+m,n}^{OE} + \operatorname{sgn}(l-m) A_{|l-m|,n}^{OE} \right) \right] \cos \omega_{m1} x_1 + \frac{1}{4} \sum_{n=1}^{\infty} \left[\sum_{m=1}^{\infty} \sum_{l=1}^{\infty} Y_{m,l}^{OE} \left(A_{m,l+n}^{OE} + A_{m,|l-n|}^{OE} \right) \right] \cos \omega_{n2} x_2 + \\
& + \frac{1}{4} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[\sum_{l=1}^{\infty} \sum_{r=1}^{\infty} \left(Y_{r+m,l}^{OE} + \operatorname{sgn}(r-m) Y_{|r-m|,l}^{OE} \right) \left(A_{r,l+n}^{OE} + A_{r,|l-n|}^{OE} \right) \right] \cos \omega_{m1} x_1 \cos \omega_{n2} x_2 + \\
& + \frac{1}{4} \sum_{n=1}^{\infty} \left[\sum_{m=1}^{\infty} \sum_{l=1}^{\infty} Y_{m,l}^{OE} \left(A_{m,l+n}^{OO} - \operatorname{sgn}(l-n) A_{m,|l-n|}^{OO} \right) \right] \sin \omega_{n2} x_2 + \\
& + \frac{1}{4} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[\sum_{l=1}^{\infty} \sum_{r=1}^{\infty} \left(Y_{r+m,l}^{OE} + \operatorname{sgn}(r-m) Y_{|r-m|,l}^{OE} \right) \left(A_{r,l+n}^{OO} - \operatorname{sgn}(l-n) A_{r,|l-n|}^{OO} \right) \right] \cos \omega_{m1} x_1 \sin \omega_{n2} x_2 - \\
& - \frac{1}{4} \sum_{n=1}^{\infty} \left[\sum_{m=1}^{\infty} \sum_{l=1}^{\infty} Y_{m,l}^{OO} \left(A_{m,l+n}^{OE} - A_{m,|l-n|}^{OE} \right) \right] \sin \omega_{n2} x_2 - \\
& - \frac{1}{4} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[\sum_{l=1}^{\infty} \sum_{r=1}^{\infty} \left(Y_{r+m,l}^{OO} + \operatorname{sgn}(r-m) Y_{|r-m|,l}^{OO} \right) \left(A_{r,l+n}^{OE} - A_{r,|l-n|}^{OE} \right) \right] \cos \omega_{m1} x_1 \sin \omega_{n2} x_2 + \frac{1}{4} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} Y_{m,n}^{OO} A_{m,n}^{OO} +
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{4} \sum_{m=1}^{\infty} \left[\sum_{n=1}^{\infty} \sum_{l=1}^{\infty} Y_{l,n}^{OO} \left(A_{l+m,n}^{OO} + \operatorname{sgn}(l-m) A_{|l-m|,n}^{OO} \right) \right] \cos \omega_{m1} x_1 + \frac{1}{4} \sum_{n=1}^{\infty} \left[\sum_{m=1}^{\infty} \sum_{l=1}^{\infty} Y_{m,l}^{OO} \left(A_{m,l+n}^{OO} + \operatorname{sgn}(l-n) A_{m,|l-n|}^{OO} \right) \right] \cos \omega_{n2} x_2 + \\
& + \frac{1}{4} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[\sum_{l=1}^{\infty} \sum_{r=1}^{\infty} \left(Y_{r+m,l}^{OO} + \operatorname{sgn}(r-m) Y_{|r-m|,l}^{OO} \right) \left(A_{r,l+n}^{OO} + \operatorname{sgn}(l-n) A_{r,|l-n|}^{OO} \right) \right] \cos \omega_{m1} x_1 \cos \omega_{n2} x_2 = \\
& + \frac{1}{4} \sum_{n=1}^{\infty} \left[\sum_{m=1}^{\infty} \sum_{l=1}^{\infty} Y_{m,l}^{EE} \left(A_{m,l+n}^{OE} - \operatorname{sgn}(l-n) A_{m,|l-n|}^{OE} \right) \right] \sin \omega_{m1} x_1 - \\
& - \frac{1}{4} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_{l=1}^{\infty} \left[\sum_{r=1}^{\infty} \left(Y_{r+m,l}^{EE} - Y_{|r-m|,l}^{EE} \right) \left(A_{r,l+n}^{OE} + A_{r,|l-n|}^{OE} \right) \right] \sin \omega_{m1} x_1 \cos \omega_{n2} x_2 - \\
& - \frac{1}{4} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_{l=1}^{\infty} \left[\sum_{r=1}^{\infty} \left(Y_{r+m,l}^{EE} - Y_{|r-m|,l}^{EE} \right) \left(A_{r,l+n}^{OO} - \operatorname{sgn}(l-n) A_{r,|l-n|}^{OO} \right) \right] \sin \omega_{m1} x_1 \sin \omega_{n2} x_2 + \\
& + \frac{1}{4} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_{l=1}^{\infty} \left[\sum_{r=1}^{\infty} \left(Y_{r+m,l}^{EO} - Y_{|r-m|,l}^{EO} \right) \left(A_{r,l+n}^{OE} - A_{r,|l-n|}^{OE} \right) \right] \sin \omega_{m1} x_1 \sin \omega_{n2} x_2 + \\
& + \frac{1}{4} \sum_{m=1}^{\infty} \left[\sum_{n=1}^{\infty} \sum_{l=1}^{\infty} Y_{l,n}^{EO} \left(A_{l+m,n}^{OO} - \operatorname{sgn}(l-m) A_{|l-m|,n}^{OO} \right) \right] \sin \omega_{m1} x_1 - \\
& - \frac{1}{4} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_{l=1}^{\infty} \left[\sum_{r=1}^{\infty} \left(Y_{r+m,l}^{EO} - Y_{|r-m|,l}^{EO} \right) \left(A_{r,l+n}^{OO} + \operatorname{sgn}(l-n) A_{r,|l-n|}^{OO} \right) \right] \sin \omega_{m1} x_1 \cos \omega_{n2} x_2 - \\
& - \frac{1}{4} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[\sum_{l=1}^{\infty} Y_{l,n}^{OE} \left(A_{l+m,n}^{EE} - A_{|l-m|,n}^{EE} \right) \right] \sin \omega_{m1} x_1 + \\
& + \frac{1}{4} \sum_{n=1}^{\infty} \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} \left[\sum_{r=1}^{\infty} \left(Y_{r+m,l}^{OE} - \operatorname{sgn}(r-m) Y_{|r-m|,l}^{OE} \right) \left(A_{r,l+n}^{EE} + A_{r,|l-n|}^{EE} \right) \right] \sin \omega_{m1} x_1 \cos \omega_{n2} x_2 + \\
& + \frac{1}{4} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_{l=1}^{\infty} \left[\sum_{r=1}^{\infty} \left(Y_{r+m,l}^{OE} - \operatorname{sgn}(r-m) Y_{|r-m|,l}^{OE} \right) \left(A_{r,l+n}^{EO} - \operatorname{sgn}(l-n) A_{r,|l-n|}^{EO} \right) \right] \sin \omega_{m1} x_1 \sin \omega_{n2} x_2 + \\
& - \frac{1}{4} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[\sum_{l=1}^{\infty} \sum_{r=1}^{\infty} \left(Y_{r+m,l}^{OO} - \operatorname{sgn}(r-m) Y_{|r-m|,l}^{OO} \right) \left(A_{r,l+n}^{EE} - A_{r,|l-n|}^{EE} \right) \right] \sin \omega_{m1} x_1 \sin \omega_{n2} x_2 - \\
& - \frac{1}{4} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[\sum_{l=1}^{\infty} Y_{l,n}^{OO} \left(A_{l+m,n}^{EO} - A_{|l-m|,n}^{EO} \right) \right] \sin \omega_{m1} x_1 + \\
& + \frac{1}{4} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[\sum_{l=1}^{\infty} \sum_{r=1}^{\infty} \left(Y_{r+m,l}^{OO} - \operatorname{sgn}(r-m) Y_{|r-m|,l}^{OO} \right) \left(A_{r,l+n}^{EO} + \operatorname{sgn}(l-n) A_{r,|l-n|}^{EO} \right) \right] \sin \omega_{m1} x_1 \cos \omega_{n2} x_2 =
\end{aligned}$$

Après avoir réorganisé et regroupé les termes d'un même type, on retrouve les expressions des coefficients du développement en série du produit:

$$\begin{aligned}
\mathbf{P}_{00} &= Y_{00} A_{00} + \frac{1}{2} \sum_{m=1}^{\infty} Y_{m0}^{E0} A_{m0}^{E0} + \frac{1}{2} \sum_{m=1}^{\infty} Y_{m0}^{OO} A_{m0}^{OO} + \frac{1}{2} \sum_{n=1}^{\infty} Y_{0n}^{OE} A_{0n}^{OE} + \frac{1}{2} \sum_{n=1}^{\infty} Y_{0n}^{OO} A_{0n}^{OO} + \\
& + \frac{1}{4} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} Y_{m,n}^{EE} A_{m,n}^{EE} + \frac{1}{4} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} Y_{m,n}^{OO} A_{m,n}^{OO} + \frac{1}{4} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} Y_{m,n}^{EO} A_{m,n}^{EO} + \frac{1}{4} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} Y_{m,n}^{OE} A_{m,n}^{OE} = Y_{00} A_{00} + \\
\mathbf{P}_{m,0}^{EO} &= \left(Y_{00} A_{m0}^{E0} + A_{00} Y_{m0}^{E0} \right) + \frac{1}{2} \left[\sum_{l=1}^{\infty} Y_{l,0}^{E0} \left(A_{l+m,0}^{E0} + A_{|l-m|,0}^{E0} \right) + \sum_{l=1}^{\infty} Y_{l,0}^{OO} \left(A_{l+m,0}^{OO} + \operatorname{sgn}(l-m) A_{|l-m|,0}^{OO} \right) \right] + \\
& + \frac{1}{2} \left[\sum_{n=1}^{\infty} Y_{mn}^{EE} A_{0n}^{OE} + \sum_{n=1}^{\infty} Y_{mn}^{EO} A_{0n}^{OO} + \sum_{n=1}^{\infty} Y_{0n}^{OE} A_{mn}^{EE} + \sum_{n=1}^{\infty} Y_{0n}^{OO} A_{mn}^{EO} \right] + \frac{1}{4} \left[\sum_{n=1}^{\infty} \sum_{l=1}^{\infty} Y_{l,n}^{EE} \left(A_{l+m,n}^{EE} + A_{|l-m|,n}^{EE} \right) \right] + \\
& + \frac{1}{4} \left[\sum_{n=1}^{\infty} \sum_{l=1}^{\infty} Y_{l,n}^{EO} \left(A_{l+m,n}^{EO} + A_{|l-m|,n}^{EO} \right) + \sum_{n=1}^{\infty} \sum_{l=1}^{\infty} Y_{l,n}^{OE} \left(A_{l+m,n}^{OE} + \operatorname{sgn}(l-m) A_{|l-m|,n}^{OE} \right) + \sum_{n=1}^{\infty} \sum_{l=1}^{\infty} Y_{l,n}^{OO} \left(A_{l+m,n}^{OO} + \operatorname{sgn}(l-m) A_{|l-m|,n}^{OO} \right) \right]
\end{aligned}$$

$$\begin{aligned}
\mathbf{P}_{0,n}^{0E} &= (Y_{00}A_{0n}^{0E} + A_{00}Y_{0n}^{0E}) + \frac{1}{2} \left[\sum_{l=1}^{\infty} Y_{0,l}^{0E} (A_{0,l+n}^{0E} + A_{0,|l-n|}^{0E}) + \sum_{l=1}^{\infty} Y_{0,l}^{0O} (A_{0,l+n}^{0O} + \text{sgn}(l-n)A_{0,|l-n|}^{0O}) \right] + \\
&\frac{1}{2} \left[\sum_{m=1}^{\infty} Y_{mn}^{EE} A_{m0}^{E0} + \sum_{m=1}^{\infty} Y_{mn}^{OE} A_{m0}^{O0} + \sum_{m=1}^{\infty} Y_{m0}^{E0} A_{mn}^{EE} + \sum_{m=1}^{\infty} Y_{m0}^{O0} A_{mn}^{OE} \right] + \frac{1}{4} \left[\sum_{m=1}^{\infty} \sum_{l=1}^{\infty} Y_{m,l}^{EE} (A_{m,l+n}^{EE} + A_{m,|l-m|}^{EE}) \right] + \\
&+ \frac{1}{4} \left[\sum_{m=1}^{\infty} \sum_{l=1}^{\infty} Y_{m,l}^{EO} (A_{m,l+n}^{EO} + \text{sgn}(l-n)A_{m,|l-n|}^{EO}) + \sum_{m=1}^{\infty} \sum_{l=1}^{\infty} Y_{m,l}^{OE} (A_{m,l+n}^{OE} + A_{m,|l-n|}^{OE}) + \sum_{m=1}^{\infty} \sum_{l=1}^{\infty} Y_{m,l}^{OO} (A_{m,l+n}^{OO} + \text{sgn}(l-n)A_{m,|l-n|}^{OO}) \right] \\
\mathbf{P}_{m,0}^{0O} &= (Y_{00}A_{m0}^{0O} + A_{00}Y_{m0}^{0O}) + \frac{1}{2} \left[\sum_{l=1}^{\infty} Y_{l,0}^{EO} (A_{l+m,0}^{OO} - \text{sgn}(l-m)A_{|l-m|,0}^{OO}) - \sum_{l=1}^{\infty} Y_{l,0}^{OO} (A_{l+m,0}^{EO} - A_{|l-m|,0}^{EO}) \right] + \\
&+ \frac{1}{2} \left[\sum_{n=1}^{\infty} Y_{mn}^{OE} A_{0n}^{OE} + \sum_{n=1}^{\infty} Y_{mn}^{OO} A_{0n}^{OO} + \sum_{n=1}^{\infty} Y_{0n}^{OE} A_{mn}^{OE} + \sum_{n=1}^{\infty} Y_{0n}^{OO} A_{mn}^{OO} \right] + \frac{1}{4} \left[\sum_{n=1}^{\infty} \sum_{l=1}^{\infty} Y_{m,l}^{EE} (A_{m,l+n}^{OE} - \text{sgn}(l-n)A_{m,|l-n|}^{OE}) \right] + \\
&+ \frac{1}{4} \left[\sum_{n=1}^{\infty} \sum_{l=1}^{\infty} Y_{l,n}^{EO} (A_{l+m,n}^{OO} - \text{sgn}(l-m)A_{|l-m|,n}^{OO}) - \sum_{n=1}^{\infty} \sum_{l=1}^{\infty} Y_{l,n}^{OE} (A_{l+m,n}^{EE} - A_{|l-m|,n}^{EE}) - \sum_{n=1}^{\infty} \sum_{l=1}^{\infty} Y_{l,n}^{OO} (A_{l+m,n}^{EO} - A_{|l-m|,n}^{EO}) \right] \\
\mathbf{P}_{0,n}^{0O} &= (Y_{00}A_{0n}^{0O} + A_{00}Y_{0n}^{0O}) + \frac{1}{2} \left[\sum_{l=1}^{\infty} Y_{0,l}^{0E} (A_{0,l+n}^{OO} - \text{sgn}(l-n)A_{0,|l-n|}^{OO}) - \sum_{l=1}^{\infty} Y_{0,l}^{0O} (A_{0,l+n}^{OE} - A_{0,|l-n|}^{OE}) \right] + \\
&+ \frac{1}{2} \left[\sum_{m=1}^{\infty} Y_{mn}^{EO} A_{m0}^{E0} + \sum_{m=1}^{\infty} Y_{mn}^{OO} A_{m0}^{O0} + \sum_{m=1}^{\infty} Y_{m0}^{OO} A_{mn}^{OO} + \sum_{m=1}^{\infty} Y_{m0}^{EO} A_{mn}^{EO} \right] - \frac{1}{4} \left[\sum_{m=1}^{\infty} \sum_{l=1}^{\infty} Y_{m,l}^{EO} (A_{m,l+n}^{EE} - A_{m,|l-n|}^{EE}) \right] + \\
&+ \frac{1}{4} \left[\sum_{m=1}^{\infty} \sum_{l=1}^{\infty} Y_{m,l}^{EE} [A_{m,l+n}^{EO} - \text{sgn}(l-n)A_{m,|l-n|}^{EO}] + \sum_{m=1}^{\infty} \sum_{l=1}^{\infty} Y_{m,l}^{OE} (A_{m,l+n}^{OO} - \text{sgn}(l-n)A_{m,|l-n|}^{OO}) - \sum_{m=1}^{\infty} \sum_{l=1}^{\infty} Y_{m,l}^{OO} (A_{m,l+n}^{OE} - A_{m,|l-n|}^{OE}) \right] \\
\mathbf{P}_{mn}^{EE} &= (Y_{00}A_{mn}^{EE} + A_{00}Y_{mn}^{EE} + Y_{m0}^{E0}A_{0n}^{OE} + A_{m0}^{E0}Y_{0n}^{OE}) + \frac{1}{2} \left[\sum_{l=1}^{\infty} Y_{l,n}^{EE} (A_{l+m,0}^{E0} + A_{|l-m|,0}^{E0}) + \sum_{l=1}^{\infty} Y_{l,n}^{OE} (A_{l+m,0}^{OO} + \text{sgn}(l-m)A_{|l-m|,0}^{OO}) \right] + \\
&+ \frac{1}{2} \left[\sum_{l=1}^{\infty} Y_{m,l}^{EE} (A_{0,l+n}^{OE} + A_{0,|l-n|}^{OE}) + \sum_{l=1}^{\infty} Y_{m,l}^{EO} (A_{0,l+n}^{OO} + \text{sgn}(l-n)A_{0,|l-n|}^{OO}) + \sum_{l=1}^{\infty} Y_{l,0}^{EO} (A_{l+m,n}^{EE} + A_{|l-m|,n}^{EE}) \right] + \\
&+ \frac{1}{2} \left[\sum_{l=1}^{\infty} Y_{l,0}^{OO} (A_{l+m,n}^{OE} + \text{sgn}(l-m)A_{|l-m|,n}^{OE}) + \sum_{l=1}^{\infty} Y_{0,l}^{OE} (A_{m,l+n}^{EE} + A_{m,|l-n|}^{EE}) + \sum_{l=1}^{\infty} Y_{m,l}^{OO} (A_{0,l+n}^{EO} + \text{sgn}(l-n)A_{0,|l-n|}^{EO}) \right] + \\
&+ \frac{1}{4} \left[\sum_{l=1}^{\infty} \sum_{r=1}^{\infty} (Y_{r+m,l}^{EE} + Y_{|r-m|,l}^{EE}) (A_{r,l+n}^{EE} + A_{r,|l-n|}^{EE}) + \sum_{l=1}^{\infty} \sum_{r=1}^{\infty} (Y_{r+m,l}^{EO} + Y_{|r-m|,l}^{EO}) (A_{r,l+n}^{EO} + \text{sgn}(l-n)A_{r,|l-n|}^{EO}) \right] + \\
&+ \frac{1}{4} \left[\sum_{l=1}^{\infty} \sum_{r=1}^{\infty} (Y_{r+m,l}^{OE} + \text{sgn}(r-m)Y_{|r-m|,l}^{OE}) (A_{r,l+n}^{OE} + A_{r,|l-n|}^{OE}) + \sum_{l=1}^{\infty} \sum_{r=1}^{\infty} (Y_{r+m,l}^{OO} + \text{sgn}(r-m)Y_{|r-m|,l}^{OO}) (A_{r,l+n}^{OO} + \text{sgn}(l-n)A_{r,|l-n|}^{OO}) \right] \\
\mathbf{P}_{mn}^{EO} &= (Y_{00}A_{mn}^{EO} + A_{00}Y_{mn}^{EO} + Y_{m0}^{E0}A_{0n}^{OO} + A_{m0}^{E0}Y_{0n}^{OO}) + \frac{1}{2} \left[\sum_{l=1}^{\infty} Y_{l,n}^{EO} (A_{l+m,0}^{E0} + A_{|l-m|,0}^{E0}) + \sum_{l=1}^{\infty} Y_{l,n}^{OO} (A_{l+m,0}^{OO} + \text{sgn}(l-m)A_{|l-m|,0}^{OO}) \right] + \\
&+ \frac{1}{2} \left[\sum_{l=1}^{\infty} Y_{m,l}^{EE} (A_{0,l+n}^{OO} - A_{0,|l-n|}^{OO}) - \sum_{l=1}^{\infty} Y_{m,l}^{EO} (A_{0,l+n}^{OE} - A_{0,|l-n|}^{OE}) + \sum_{l=1}^{\infty} Y_{l,0}^{EO} (A_{l+m,n}^{EO} + A_{|l-m|,n}^{EO}) \right] + \\
&+ \frac{1}{2} \left[\sum_{l=1}^{\infty} Y_{l,0}^{OO} (A_{l+m,n}^{OO} + \text{sgn}(l-m)A_{|l-m|,n}^{OO}) + \sum_{l=1}^{\infty} Y_{0,l}^{OO} (A_{m,l+n}^{EE} - A_{m,|l-n|}^{EE}) + \sum_{l=1}^{\infty} Y_{0,l}^{OE} (A_{m,l+n}^{EO} - \text{sgn}(l-n)A_{m,|l-n|}^{EO}) \right] + \\
&+ \frac{1}{4} \left[\sum_{l=1}^{\infty} \sum_{r=1}^{\infty} (Y_{r+m,l}^{EE} + Y_{|r-m|,l}^{EE}) (A_{r,l+n}^{EO} - \text{sgn}(l-n)A_{r,|l-n|}^{EO}) + \sum_{l=1}^{\infty} \sum_{r=1}^{\infty} (Y_{r+m,l}^{EO} + Y_{|r-m|,l}^{EO}) (A_{r,l+n}^{EE} - A_{r,|l-n|}^{EE}) \right] + \\
&+ \frac{1}{4} \left[\sum_{l=1}^{\infty} \sum_{r=1}^{\infty} (Y_{r+m,l}^{OE} + \text{sgn}(r-m)Y_{|r-m|,l}^{OE}) (A_{r,l+n}^{OO} - \text{sgn}(l-n)A_{r,|l-n|}^{OO}) + \sum_{l=1}^{\infty} \sum_{r=1}^{\infty} (Y_{r+m,l}^{OO} + \text{sgn}(r-m)Y_{|r-m|,l}^{OO}) (A_{r,l+n}^{OE} - A_{r,|l-n|}^{OE}) \right] \\
\mathbf{P}_{mn}^{OE} &= (Y_{00}A_{mn}^{OE} + A_{00}Y_{mn}^{OE} + Y_{m0}^{O0}A_{0n}^{OE} + A_{m0}^{O0}Y_{0n}^{OE}) + \frac{1}{2} \left[\sum_{l=1}^{\infty} Y_{l,n}^{EE} (A_{l+m,0}^{OO} - \text{sgn}(l-m)A_{|l-m|,0}^{OO}) - \sum_{l=1}^{\infty} Y_{l,n}^{OE} (A_{l+m,0}^{EO} - A_{|l-m|,0}^{EO}) \right] + \\
&+ \frac{1}{2} \left[\sum_{l=1}^{\infty} Y_{m,l}^{OE} (A_{0,l+n}^{OE} + A_{0,|l-n|}^{OE}) + \sum_{l=1}^{\infty} Y_{m,l}^{OO} (A_{0,l+n}^{OO} + \text{sgn}(l-n)A_{0,|l-n|}^{OO}) - \sum_{l=1}^{\infty} Y_{l,0}^{OO} (A_{l+m,n}^{EE} - A_{|l-m|,n}^{EE}) \right] + \\
&+ \frac{1}{2} \left[\sum_{l=1}^{\infty} Y_{l,0}^{EO} (A_{l+m,n}^{EO} - \text{sgn}(l-m)A_{|l-m|,n}^{EO}) + \sum_{l=1}^{\infty} Y_{0,l}^{OO} (A_{m,l+n}^{OO} + \text{sgn}(l-n)A_{0,|l-n|}^{OO}) + \sum_{l=1}^{\infty} Y_{m,l}^{OE} (A_{0,l+n}^{OE} + A_{0,|l-n|}^{OE}) \right] + \\
&+ \frac{1}{4} \left[\sum_{l=1}^{\infty} \sum_{r=1}^{\infty} (Y_{r+m,l}^{EO} + \text{sgn}(r-m)Y_{|r-m|,l}^{EO}) (A_{r,l+n}^{OO} - \text{sgn}(l-n)A_{r,|l-n|}^{OO}) + \sum_{l=1}^{\infty} \sum_{r=1}^{\infty} (Y_{r+m,l}^{OO} + \text{sgn}(r-m)Y_{|r-m|,l}^{OO}) (A_{r,l+n}^{OE} - A_{r,|l-n|}^{OE}) \right]
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{4} \left[- \sum_{l=1}^{\infty} \sum_{r=1}^{\infty} (Y_{r+m,l}^{EE} - Y_{|r-m|,l}^{EE}) (A_{r,l+n}^{OE} + A_{r,|l-n|}^{OE}) - \sum_{l=1}^{\infty} \sum_{r=1}^{\infty} (Y_{r+m,l}^{EO} - Y_{|r-m|,l}^{EO}) (A_{r,l+n}^{OO} + \text{sgn}(l-n) A_{r,|l-n|}^{OO}) \right] + \\
& + \frac{1}{4} \left[\sum_{l=1}^{\infty} \sum_{r=1}^{\infty} (Y_{r+m,l}^{OE} - \text{sgn}(r-m) Y_{|r-m|,l}^{OE}) (A_{r,l+n}^{EE} + A_{r,|l-n|}^{EE}) + \sum_{l=1}^{\infty} \sum_{r=1}^{\infty} (Y_{r+m,l}^{OO} - \text{sgn}(r-m) Y_{|r-m|,l}^{OO}) (A_{r,l+n}^{EO} + \text{sgn}(l-n) A_{r,|l-n|}^{EO}) \right] \\
\mathbf{P}_{mn}^{OO} & = (Y_{00} A_{mn}^{OO} + A_{00} Y_{mn}^{OO} + Y_{m0} A_{0n}^{OO} + A_{m0} Y_{0n}^{OO}) + \frac{1}{2} \left[\sum_{l=1}^{\infty} Y_{l,n}^{EO} (A_{l+m,0}^{OO} - \text{sgn}(l-m) A_{|l-m|,0}^{OO}) - \sum_{l=1}^{\infty} Y_{l,n}^{OO} (A_{l+m,0}^{EO} - A_{|l-m|,0}^{EO}) \right] + \\
& + \frac{1}{2} \left[\sum_{l=1}^{\infty} Y_{m,l}^{OE} (A_{0,l+n}^{OO} - A_{0,|l-n|}^{OO}) - \sum_{l=1}^{\infty} Y_{m,l}^{OO} (A_{0,l+n}^{OE} - A_{0,|l-n|}^{OE}) - \sum_{l=1}^{\infty} Y_{l,0}^{OO} (A_{l+m,n}^{EO} - A_{|l-m|,n}^{EO}) \right] + \\
& + \frac{1}{2} \left[\sum_{l=1}^{\infty} Y_{l,0}^{EO} (A_{l+m,n}^{OO} - \text{sgn}(l-m) A_{|l-m|,n}^{OO}) - \sum_{l=1}^{\infty} Y_{m,l}^{OO} (A_{0,l+n}^{OE} - A_{0,|l-n|}^{OE}) + \sum_{l=1}^{\infty} Y_{m,l}^{OE} (A_{0,l+n}^{OO} - \text{sgn}(l-n) A_{0,|l-n|}^{OO}) \right] + \\
& + \frac{1}{4} \left[- \sum_{l=1}^{\infty} \sum_{r=1}^{\infty} (Y_{r+m,l}^{EE} - Y_{|r-m|,l}^{EE}) (A_{r,l+n}^{OO} - \text{sgn}(l-n) A_{r,|l-n|}^{OO}) - \sum_{l=1}^{\infty} \sum_{r=1}^{\infty} (Y_{r+m,l}^{EO} - Y_{|r-m|,l}^{EO}) (A_{r,l+n}^{OE} - A_{r,|l-n|}^{OE}) \right] + \\
& + \frac{1}{4} \left[\sum_{l=1}^{\infty} \sum_{r=1}^{\infty} (Y_{r+m,l}^{OE} - \text{sgn}(r-m) Y_{|r-m|,l}^{OE}) (A_{r,l+n}^{EE} + A_{r,|l-n|}^{EE}) + \sum_{l=1}^{\infty} \sum_{r=1}^{\infty} (Y_{r+m,l}^{OO} - \text{sgn}(r-m) Y_{|r-m|,l}^{OO}) (A_{r,l+n}^{EO} + \text{sgn}(l-n) A_{r,|l-n|}^{EO}) \right]
\end{aligned}$$

Les coefficients \mathbf{P}_{mn} (où $m, n=0, 1, 2, \dots, \infty$) du développement en série du produit des deux fonctions contiennent chacun, des expressions d'une ou plusieurs séries numériques infinies. Si les coefficients du développement de chacun des termes du produit sont connus, dans certains cas particuliers la limite de ces séries peut être déterminée exactement. Dans les autres cas, la limite est calculée avec une approximation en fonction du nombre de termes calculés pour les sommes partiels. Compte tenu du nombre assez important des coefficients et des séries qui apparaissent dans leurs expressions (notamment dans le cas des fonctions à grand nombre de variables), un volume important de calculs est nécessaire, ce qui nécessite l'utilisation de la technologie de calcul moderne.

Outre leur utilité pour résoudre les PDE, les formules précédemment déterminées peuvent être utiles dans l'étude des séries numériques et des séries de fonctions. Si l'expression d'un coefficient \mathbf{P}_{mn} du développement en série des certaines fonctions, calculées en appliquant les formules (4.a) (ou celles correspondant à un plus grand nombre des variables) est comparée à l'expression du même coefficient déterminé sur la base des formules déduites dans cette section, peut être établi une relation d'égalité entre l'intégrale fourni par (4.a) et la série correspondante. De plus, nous avons constaté que le même coefficient peut être exprimé de plusieurs façons, selon l'ordre dans lequel ses composantes sont additionnées. Ces équations sont également valides lorsque l'ensemble de ces variables acquiert certaines valeurs particulières.

Dans le cas d'une fonction de $N=3$ variables, le nombre de types de coefficients du développement en série SFS est de 27 (N^3), et le produit de leurs développements en série est composé de 729 (N^6) fonctions élémentaires, donc les expressions mathématiques des coefficients peuvent contenir un nombre considérable des séries numériques. Toutefois, en présence d'un algorithme de calcul correctement conçu, ces problèmes sont facilement surmontés par un ordinateur aux performances habituelles.

6. Conclusions

Une équation PDEs est constituée d'une série de termes de la forme $y(\mathbf{x})$, $y''(\mathbf{x})$, $\partial y(\mathbf{x})/\partial x_i$, $\partial^2 y(\mathbf{x})/\partial x_i \partial x_j$, $\partial^n y(\mathbf{x})/\partial x_i \dots \partial x_j$, $\partial^{-n} y(\mathbf{x})/\partial x_i \dots \partial x_j$, $y^n(\mathbf{x}) \partial^n y(\mathbf{x})/\partial x_i \dots \partial x_j$, et de leurs produits avec d'autres fonctions multivariées $p(\mathbf{x})$, $q(\mathbf{x})$, ..., où \mathbf{x} est une d -uplet, et x_i , x_j sont ses variables. Le remplacement de ces termes, lorsqu'ils apparaissent dans une équation différentielle, par leur développement en série de Fourier, ou par leurs sommes de Fourier S_N lesquelles les approximent (développement contenant des coefficients de Fourier \mathbf{P}_{mn}

inconnus de la fonction $y(\mathbf{x})$), conduit à un système avec $2N+1$ ($N \rightarrow \infty$) équations algébriques. En le résolvant, nous pouvons trouver les coefficients P_{mn} , et de cette manière, la fonction inconnue (ou des fonctions, s'il s'agit d'un système d'équations).

Comme pour la méthode de développement en séries entières, les opérations de dérivation et d'intégration effectuées sur les développements en série, ainsi que le produit entre les fonctions multivariées sont transformés en opérations algébriques effectuées sur les coefficients du développement. Par conséquent, la résolution des PDE se transforme en résolution d'équations algébriques. De ce point de vue, la méthode de résolution par détermination des coefficients de développement en série sinusoïdale est plus avantageuse, en raison de la facilité avec laquelle les coefficients du développement peuvent être calculés et en raison de la vitesse de convergence élevée (ce qui les rend idéales pour les méthodes d'approximation numérique).

Cette méthode de résolution d'équations différentielles ordinaires, ou avec des dérivées partielles, en déterminant les coefficients du développement en séries sinusoïdales des fonctions inconnues peut devenir une méthode extrêmement efficace, applicable à tous les types d'équations différentielles et intégral-différentielles, linéaires et non linéaires, aux équations aux dérivées partielles linéaires et non linéaires, aux systèmes de telles équations, quels que soient leur ordre et la complexité de leurs coefficients, en particulier pour les situations rencontrées en physique et en génie. Il s'ensuit que cette méthode sera étudiée en continuation, en termes de conditions d'application et de conditionnalités que doivent satisfaire les fonctions impliquées aux limites du domaine de définition. En outre, les implications qui apparaissent en le comparant avec d'autres méthodes de résolution des PDE doivent être étudiées.

7. Bibliografie

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ANNEX. Summary of calculation

1. The coefficients of the product of two monovariate functions $y(x)a(x)$

Consider the functions $y(x)$ and $a(x)$ with the following SFS expansions:

$$\bar{y}(x) = y_0 + \bar{y}_E + \bar{y}_O = y_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} \quad \text{and}$$

$$\bar{a}(x) = \bar{a}_0 + \bar{a}_E + \bar{a}_O = a_0 + \sum_{n=1}^{\infty} C_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} D_n \sin \frac{n\pi x}{L},$$

where the mean values y_{E0} and a_{E0} are zero. The coefficients of the SFS expansion of the product ya are:

$$\begin{aligned} (ya)^{SFS} &= [y_0 a_0 + (y_0 a_E + a_0 y_E + y_E a_E + y_O a_O) + (y_0 a_O + a_0 y_O + y_E a_O + y_O a_E)]^{SFS} = \\ &= P_0 + \sum_{n=1}^{\infty} P_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} Q_n \sin \frac{n\pi x}{L} \end{aligned}$$

Here:

$$\begin{aligned} P_0 &= \frac{1}{2L} \int_{-L}^L (ya)^{SFS} dx = y_0 a_0 + \frac{1}{2L} \int_{-L}^L y_E a_E dx + \frac{1}{2L} \int_{-L}^L y_O a_O dx = \\ &= y_0 a_0 + \frac{1}{2L} \int_{-L}^L \left(\sum_{l=1}^{\infty} A_l \cos \frac{l\pi x}{L} \right) \left(\sum_{m=1}^{\infty} C_m \cos \frac{m\pi x}{L} \right) dx + \frac{1}{2L} \int_{-L}^L \left(\sum_{l=1}^{\infty} B_l \sin \frac{l\pi x}{L} \right) \left(\sum_{m=1}^{\infty} D_m \sin \frac{m\pi x}{L} \right) dx = \\ &= y_0 a_0 + \frac{1}{2L} \left(\sum_{l=1}^{\infty} A_l C_l \cos^2 \frac{l\pi x}{L} \right) + \frac{1}{2L} \left(\sum_{l=1}^{\infty} B_l D_l \sin^2 \frac{l\pi x}{L} \right) = y_0 a_0 + \frac{1}{2} \left(\sum_{l=1}^{\infty} A_l C_l + \sum_{l=1}^{\infty} B_l D_l \right). \end{aligned}$$

$$\begin{aligned} P_n &= \frac{1}{L} \int_{-L}^L (y_0 a_E + a_0 y_E + y_E a_E + y_O a_O) \cos \frac{n\pi x}{L} dx = \frac{y_0}{L} \int_{-L}^L \left(\sum_{m=1}^{\infty} C_m \cos \frac{m\pi x}{L} \right) \cos \frac{n\pi x}{L} dx + \\ &+ \frac{a_0}{L} \int_{-L}^L \left(\sum_{l=1}^{\infty} A_l \cos \frac{l\pi x}{L} \right) \cos \frac{n\pi x}{L} dx + \frac{1}{L} \int_{-L}^L \left(\sum_{l=1}^{\infty} A_l \cos \frac{l\pi x}{L} \right) \left(\sum_{m=1}^{\infty} C_m \cos \frac{m\pi x}{L} \right) \cos \frac{n\pi x}{L} dx + \\ &+ \frac{1}{L} \int_{-L}^L \left(\sum_{l=1}^{\infty} B_l \sin \frac{l\pi x}{L} \right) \left(\sum_{m=1}^{\infty} D_m \sin \frac{m\pi x}{L} \right) \cos \frac{n\pi x}{L} dx \\ &\rightarrow P_n = y_0 C_n + a_0 A_n + \frac{1}{2} \sum_{l=1}^{\infty} A_l (C_{|l-n|} + C_{l+n}) + \frac{1}{2} \sum_{l=1}^{\infty} B_l [D_{l+n} + \text{sgn}(l-n) D_{|l-n|}] \end{aligned}$$

$$\begin{aligned} Q_n &= \frac{1}{L} \int_{-L}^L (y_0 a_O + a_0 y_O + y_E a_O + y_O a_E) \sin \frac{n\pi x}{L} dx = \frac{y_0}{L} \int_{-L}^L \left(\sum_{m=1}^{\infty} D_m \sin \frac{m\pi x}{L} \right) \sin \frac{n\pi x}{L} dx + \\ &+ \frac{a_0}{L} \int_{-L}^L \left(\sum_{l=1}^{\infty} B_l \sin \frac{l\pi x}{L} \right) \sin \frac{n\pi x}{L} dx + \frac{1}{L} \int_{-L}^L \left(\sum_{l=1}^{\infty} A_l \cos \frac{l\pi x}{L} \right) \left(\sum_{m=1}^{\infty} D_m \sin \frac{m\pi x}{L} \right) \sin \frac{n\pi x}{L} dx + \\ &+ \frac{1}{L} \int_{-L}^L \left(\sum_{l=1}^{\infty} B_l \sin \frac{l\pi x}{L} \right) \left(\sum_{m=1}^{\infty} C_m \cos \frac{m\pi x}{L} \right) \sin \frac{n\pi x}{L} dx \\ &\rightarrow Q_n = y_0 D_n + a_0 B_n + \frac{1}{2} \sum_{l=1}^{\infty} A_l (D_{l+n} - \text{sgn}(l-n) D_{|l-n|}) - \frac{1}{2} \sum_{l=1}^{\infty} B_l [C_{l+n} - C_{|l-n|}] \end{aligned}$$

where: $A_0 = B_0 = C_0 = D_0 = 0$. Therefore:

$$P_0 = \frac{1}{2L} \int_{-L}^L (yp)^{SFS} dx = y_0 p_0 + \frac{1}{2} \left(\sum_{l=1}^{\infty} A_l C_l + \sum_{l=1}^{\infty} B_l D_l \right)$$

$$P_n = y_0 C_n + p_0 A_n + \frac{1}{2} \sum_{l=1}^{\infty} A_l (C_{n+l} + C_{|n-l|}) + \frac{1}{2} \sum_{l=1}^{\infty} B_l [D_{n+l} + \text{sgn}(l-n) D_{|n-l|}]$$

$$Q_n = y_0 D_n + p_0 B_n + \frac{1}{2} \sum_{l=1}^{\infty} A_l \left[D_{n+l} - \operatorname{sgn}(l-n) D_{|n-l|} \right] - \frac{1}{2} \sum_{l=1}^{\infty} B_l (C_{n+l} - C_{|n-l|})$$

We used the following intermediate calculations:

$$\begin{aligned} \int_{-L}^L \sum_{l=1}^{\infty} \left(A_l \cos \frac{l\pi x}{L} \right) \sum_{m=1}^{\infty} \left(C_m \cos \frac{m\pi x}{L} \right) \cos \frac{n\pi x}{L} dx &= \int_{-L}^L \sum_{l=1}^{\infty} \left(A_l \cos \frac{l\pi x}{L} \right) \sum_{m=1}^{\infty} \frac{C_m}{2} \left[\cos \frac{(m-n)\pi x}{L} + \cos \frac{(m+n)\pi x}{L} \right] dx = \\ &= \int_{-L}^L \sum_{l=1}^{\infty} \left(A_l \cos \frac{l\pi x}{L} \right) \left[\sum_{m=1}^{n-1} \frac{C_m}{2} \cos \frac{(m-n)\pi x}{L} + \frac{C_n}{2} + \sum_{m=n+1}^{\infty} \frac{C_m}{2} \cos \frac{(m-n)\pi x}{L} + \sum_{m=1}^{\infty} \frac{C_m}{2} \cos \frac{(m+n)\pi x}{L} \right] dx = \\ &= \frac{L}{2} \left(\sum_{l=1}^{n-1} A_l C_{n-l} + \sum_{l=1}^{\infty} A_l C_{l+n} + \sum_{l=n+1}^{\infty} A_l C_{l-n} \right) = \frac{L}{2} \sum_{l=1}^{\infty} A_l (C_{l+n} + C_{|l-n|}) \end{aligned}$$

where $C_{n-l=0}=0$. Obviously: $\sum_{l=1}^{\infty} A_l (C_{l+n} + C_{|l-n|}) = \sum_{l=1}^{\infty} C_l (A_{l+n} + A_{|l-n|})$

$$\begin{aligned} \int_{-L}^L \sum_{l=1}^{\infty} \left(B_l \sin \frac{l\pi x}{L} \right) \sum_{m=1}^{\infty} \left(D_m \sin \frac{m\pi x}{L} \right) \cos \frac{n\pi x}{L} dx &= \int_{-L}^L \sum_{l=1}^{\infty} \left(B_l \sin \frac{l\pi x}{L} \right) \sum_{m=1}^{\infty} \frac{D_m}{2} \left[\sin \frac{(m-n)\pi x}{L} + \sin \frac{(m+n)\pi x}{L} \right] dx = \\ &= \int_{-L}^L \sum_{l=1}^{\infty} \left(B_l \sin \frac{l\pi x}{L} \right) \left[\sum_{m=1}^{n-1} \frac{D_m}{2} \sin \frac{(m-n)\pi x}{L} + \frac{D_n}{2} + \sum_{m=n+1}^{\infty} \frac{D_m}{2} \sin \frac{(m-n)\pi x}{L} + \sum_{m=1}^{\infty} \frac{D_m}{2} \sin \frac{(m+n)\pi x}{L} \right] dx = \\ &= \frac{L}{2} \left[\sum_{l=1}^{n-1} -B_l D_{n-l} + \sum_{l=1}^{\infty} B_l D_{l+n} + \sum_{l=n+1}^{\infty} B_l D_{l-n} \right] = \frac{L}{2} \sum_{l=1}^{\infty} B_l [D_{l+n} + \operatorname{sgn}(l-n) D_{|l-n|}] \end{aligned}$$

where $D_{n-l=0}=0$. In the same way: $\sum_{l=1}^{\infty} B_l [D_{l+n} + \operatorname{sgn}(l-n) D_{|l-n|}] = \sum_{l=1}^{\infty} D_l [B_{l+n} + \operatorname{sgn}(l-n) B_{|l-n|}]$

$$\begin{aligned} \int_{-L}^L \left(\sum_{l=1}^{\infty} A_l \cos \frac{l\pi x}{L} \right) \left(\sum_{m=1}^{\infty} D_m \sin \frac{m\pi x}{L} \right) \sin \frac{n\pi x}{L} dx &= \int_{-L}^L \sum_{l=1}^{\infty} \left(A_l \cos \frac{l\pi x}{L} \right) \sum_{m=1}^{\infty} \frac{D_m}{2} \left[\cos \frac{(m-n)\pi x}{L} - \cos \frac{(m+n)\pi x}{L} \right] dx = \\ &= \int_{-L}^L \sum_{l=1}^{\infty} \left(A_l \cos \frac{l\pi x}{L} \right) \left[\sum_{m=1}^{n-1} \frac{D_m}{2} \cos \frac{(n-m)\pi x}{L} + \frac{D_n}{2} + \sum_{m=n+1}^{\infty} \frac{D_m}{2} \cos \frac{(m-n)\pi x}{L} - \sum_{m=1}^{\infty} \frac{D_m}{2} \cos \frac{(m+n)\pi x}{L} \right] dx = \\ &= \frac{L}{2} \left[\sum_{l=1}^{n-1} A_l D_{n-l} + \sum_{l=1}^{\infty} A_l D_{l+n} - \sum_{l=n+1}^{\infty} A_l D_{l-n} \right] = \frac{L}{2} \sum_{l=1}^{\infty} A_l [D_{l+n} - \operatorname{sgn}(l-n) D_{|l-n|}] \end{aligned}$$

where $D_{n-l=0}=0$. In the same way: $\sum_{l=1}^{\infty} A_l [D_{l+n} - \operatorname{sgn}(l-n) D_{|l-n|}] = -\sum_{l=1}^{\infty} D_l [A_{l+n} - A_{|l-n|}]$

$$\begin{aligned} \int_{-L}^L \left(\sum_{l=1}^{\infty} B_l \sin \frac{l\pi x}{L} \right) \left(\sum_{m=1}^{\infty} C_m \cos \frac{m\pi x}{L} \right) \sin \frac{n\pi x}{L} dx &= \sum_{l=1}^{\infty} \left(B_l \sin \frac{l\pi x}{L} \right) \sum_{m=1}^{\infty} \frac{C_m}{2} \left[\sin \frac{(n-m)\pi x}{L} + \sin \frac{(n+m)\pi x}{L} \right] dx = \\ &= \int_{-L}^L \sum_{l=1}^{\infty} \left(B_l \sin \frac{l\pi x}{L} \right) \left[\sum_{m=1}^{n-1} \frac{C_m}{2} \sin \frac{(n-m)\pi x}{L} + \frac{C_n}{2} + \sum_{m=n+1}^{\infty} \frac{C_m}{2} \sin \frac{(n-m)\pi x}{L} + \sum_{m=1}^{\infty} \frac{C_m}{2} \sin \frac{(m+n)\pi x}{L} \right] dx = \\ &= \frac{L}{2} \left[\sum_{l=1}^{n-1} B_l C_{n-l} - \sum_{l=1}^{\infty} B_l C_{l+n} + \sum_{l=n+1}^{\infty} B_l C_{l-n} \right] = -\frac{L}{2} \sum_{l=1}^{\infty} B_l [C_{l+n} - C_{|l-n|}] \end{aligned}$$

where $C_{n-l=0}=0$. In the same way: $-\sum_{l=1}^{\infty} B_l [C_{l+n} - C_{|l-n|}] = \sum_{l=1}^{\infty} C_l (B_{l+n} - \operatorname{sgn}(l-n) B_{|l-n|})$

2. The product of two monovariable functions depending on the same variable: $y(x_1)a(x_1)$

Consider the functions $y(x_1, x_2)$ and $a(x_1, x_2)$ with the following SFS expansions:

$$\begin{aligned} [y(x_1, x_2)]^{\text{SFS}} &= Y_{00} + \sum_{m=1}^{\infty} (Y_{m0}^{E0} \cos \omega_{m1} x_1 + Y_{m0}^{O0} \sin \omega_{m1} x_1) + \sum_{n=1}^{\infty} (Y_{0n}^{OE} \cos \omega_{n2} x_2 + Y_{0n}^{OO} \sin \omega_{n2} x_2) + \\ &+ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (Y_{mn}^{EE} \cos \omega_{m1} x_1 \cos \omega_{n2} x_2 + Y_{mn}^{EO} \cos \omega_{m1} x_1 \sin \omega_{n2} x_2 + Y_{mn}^{OE} \sin \omega_{m1} x_1 \cos \omega_{n2} x_2 + Y_{mn}^{OO} \sin \omega_{m1} x_1 \sin \omega_{n2} x_2) \end{aligned}$$

$$[a(x_1, x_2)]^{SFS} = A_{00} + \sum_{m=1}^{\infty} (A_{m0}^{E0} \cos \omega_{m1} x_1 + A_{m0}^{O0} \sin \omega_{m1} x_1) + \sum_{n=1}^{\infty} (A_{0n}^{OE} \cos \omega_{n2} x_2 + A_{0n}^{OO} \sin \omega_{n2} x_2) +$$

$$+ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (A_{mn}^{EE} \cos \omega_{m1} x_1 \cos \omega_{n2} x_2 + A_{mn}^{EO} \cos \omega_{m1} x_1 \sin \omega_{n2} x_2 + A_{mn}^{OE} \sin \omega_{m1} x_1 \cos \omega_{n2} x_2 + A_{mn}^{OO} \sin \omega_{m1} x_1 \sin \omega_{n2} x_2)$$

Among the elementary functions which are the terms of the product of these two expressions, there are also products between two monovariate functions depending on the same variable, of the form $y(x_l)a(x_l)$, analyzed in the previous paragraph:

$$\left(\sum_{m=1}^{\infty} Y_{m0}^{E0} \cos \omega_{m1} x_1 \right) \left(\sum_{m=1}^{\infty} A_{m0}^{E0} \cos \omega_{m1} x_1 \right) = \frac{1}{2} \sum_{m=1}^{\infty} Y_{m0}^{E0} A_{m0}^{E0} + \frac{1}{2} \sum_{m=1}^{\infty} \left[\sum_{l=1}^{\infty} Y_{l0}^{E0} (A_{l+m,0}^{E0} + A_{|l-m|,0}^{E0}) \right] \cos \omega_{m1} x_1$$

$$\left(\sum_{m=1}^{\infty} Y_{m0}^{E0} \cos \omega_{m1} x_1 \right) \left(\sum_{m=1}^{\infty} A_{m0}^{O0} \sin \omega_{m1} x_1 \right) = \frac{1}{2} \sum_{m=1}^{\infty} \left[\sum_{l=1}^{\infty} Y_{l0}^{E0} (A_{l+m,0}^{O0} - \text{sgn}(l-m) A_{|l-m|,0}^{O0}) \right] \sin \omega_{m1} x_1$$

$$\left(\sum_{m=1}^{\infty} Y_{m0}^{O0} \sin \omega_{m1} x_1 \right) \left(\sum_{m=1}^{\infty} A_{m0}^{E0} \cos \omega_{m1} x_1 \right) = -\frac{1}{2} \sum_{m=1}^{\infty} \left[\sum_{l=1}^{\infty} Y_{l0}^{O0} (A_{l+m,0}^{E0} - A_{|l-m|,0}^{E0}) \right] \sin \omega_{m1} x_1$$

$$\left(\sum_{m=1}^{\infty} Y_{m0}^{O0} \sin \omega_{m1} x_1 \right) \left(\sum_{m=1}^{\infty} A_{m0}^{O0} \sin \omega_{m1} x_1 \right) = \frac{1}{2} \sum_{m=1}^{\infty} Y_{m0}^{O0} A_{m0}^{O0} + \frac{1}{2} \sum_{m=1}^{\infty} \left[\sum_{l=1}^{\infty} Y_{l0}^{O0} (A_{l+m,0}^{O0} + \text{sgn}(l-m) A_{|l-m|,0}^{O0}) \right] \cos \omega_{m1} x_1$$

$$\left(\sum_{n=1}^{\infty} Y_{0n}^{OE} \cos \omega_{n2} x_2 \right) \left(\sum_{n=1}^{\infty} A_{0n}^{OE} \cos \omega_{n2} x_2 \right) = \frac{1}{2} \sum_{n=1}^{\infty} Y_{0n}^{OE} A_{0n}^{OE} + \frac{1}{2} \sum_{n=1}^{\infty} \left[\sum_{l=1}^{\infty} Y_{0,l}^{OE} (A_{0,l+n}^{OE} + A_{0,|l-n|}^{OE}) \right] \cos \omega_{n2} x_2$$

$$\left(\sum_{n=1}^{\infty} Y_{0n}^{OE} \cos \omega_{n2} x_2 \right) \left(\sum_{n=1}^{\infty} A_{0n}^{O0} \sin \omega_{n2} x_2 \right) = \frac{1}{2} \sum_{n=1}^{\infty} \left[\sum_{l=1}^{\infty} Y_{0,l}^{OE} (A_{0,l+n}^{O0} - \text{sgn}(l-n) A_{0,|l-n|}^{O0}) \right] \sin \omega_{n2} x_2$$

$$\left(\sum_{n=1}^{\infty} Y_{0n}^{O0} \sin \omega_{n2} x_2 \right) \left(\sum_{n=1}^{\infty} A_{0n}^{OE} \cos \omega_{n2} x_2 \right) = -\frac{1}{2} \sum_{n=1}^{\infty} \left[\sum_{l=1}^{\infty} Y_{0,l}^{O0} (A_{0,l+n}^{OE} - A_{0,|l-n|}^{OE}) \right] \sin \omega_{n2} x_2$$

$$\left(\sum_{n=1}^{\infty} Y_{0n}^{O0} \sin \omega_{n2} x_2 \right) \left(\sum_{n=1}^{\infty} A_{0n}^{O0} \sin \omega_{n2} x_2 \right) = \frac{1}{2} \sum_{n=1}^{\infty} Y_{0n}^{O0} A_{0n}^{O0} + \frac{1}{2} \sum_{n=1}^{\infty} \left[\sum_{l=1}^{\infty} Y_{0,l}^{O0} (A_{0,l+n}^{O0} + \text{sgn}(l-n) A_{0,|l-n|}^{O0}) \right] \cos \omega_{n2} x_2$$

3. The product of a monovariate function with a two-variable function:

$y(x_l, x_2)a(x_l)$

If we treat the function $y(x_l, x_2)$ as an x_l -monovariate function $y(x_l, C)$, we can again apply the deduced formulas for the product of the monovariate functions:

$$\left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} Y_{mn}^{EE} \cos \omega_{m1} x_1 \cos \omega_{n2} x_2 \right) \left(\sum_{m=1}^{\infty} A_{m0}^{E0} \cos \omega_{m1} x_1 \right) = \sum_{n=1}^{\infty} \left[\left(\sum_{m=1}^{\infty} Y_{mn}^{EE} \cos \omega_{m1} x_1 \right) \left(\sum_{m=1}^{\infty} A_{m0}^{E0} \cos \omega_{m1} x_1 \right) \right] \cos \omega_{n2} x_2 =$$

$$= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[\frac{1}{2} Y_{mn}^{EE} A_{m0}^{E0} + \frac{1}{2} \sum_{l=1}^{\infty} Y_{l,n}^{EE} (A_{l+m,0}^{E0} + A_{|l-m|,0}^{E0}) \cos \omega_{m1} x_1 \right] \cos \omega_{n2} x_2 =$$

$$= \frac{1}{2} \sum_{n=1}^{\infty} \left\{ \sum_{m=1}^{\infty} Y_{mn}^{EE} A_{m0}^{E0} \right\} \cos \omega_{n2} x_2 + \frac{1}{2} \sum_{n=1}^{\infty} \left\{ \sum_{m=1}^{\infty} \left[\sum_{l=1}^{\infty} Y_{l,n}^{EE} (A_{l+m,0}^{E0} + A_{|l-m|,0}^{E0}) \right] \cos \omega_{m1} x_1 \right\} \cos \omega_{n2} x_2$$

$$\left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} Y_{mn}^{EE} \cos \omega_{m1} x_1 \cos \omega_{n2} x_2 \right) \left(\sum_{m=1}^{\infty} A_{m0}^{O0} \sin \omega_{m1} x_1 \right) = \sum_{n=1}^{\infty} \left[\left(\sum_{m=1}^{\infty} Y_{mn}^{EE} \cos \omega_{m1} x_1 \right) \left(\sum_{m=1}^{\infty} A_{m0}^{O0} \sin \omega_{m1} x_1 \right) \right] \cos \omega_{n2} x_2 =$$

$$= \frac{1}{2} \sum_{n=1}^{\infty} \left\{ \sum_{m=1}^{\infty} \left[\sum_{l=1}^{\infty} Y_{l,n}^{EE} (A_{l+m,0}^{O0} - \text{sgn}(l-m) A_{|l-m|,0}^{O0}) \right] \sin \omega_{m1} x_1 \right\} \cos \omega_{n2} x_2$$

$$\left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} Y_{mn}^{EO} \cos \omega_{m1} x_1 \sin \omega_{n2} x_2 \right) \left(\sum_{m=1}^{\infty} A_{m0}^{E0} \cos \omega_{m1} x_1 \right) = \sum_{n=1}^{\infty} \left[\left(\sum_{m=1}^{\infty} Y_{mn}^{EO} \cos \omega_{m1} x_1 \right) \left(\sum_{m=1}^{\infty} A_{m0}^{E0} \cos \omega_{m1} x_1 \right) \right] \sin \omega_{n2} x_2 =$$

$$= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[\frac{1}{2} Y_{mn}^{EO} A_{m0}^{E0} + \frac{1}{2} \sum_{l=1}^{\infty} Y_{l,n}^{EO} (A_{l+m,0}^{E0} + A_{|l-m|,0}^{E0}) \cos \omega_{m1} x_1 \right] \sin \omega_{n2} x_2 =$$

$$\begin{aligned}
&= \frac{1}{2} \sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} Y_{m0}^{E0} A_{mn}^{EO} \right) \sin \omega_{n2} x_2 + \frac{1}{2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[\sum_{l=1}^{\infty} Y_{l,0}^{E0} \left(A_{l+m,n}^{EO} + A_{|l-m|,n}^{EO} \right) \right] \cos \omega_{m1} x_1 \sin \omega_{n2} x_2 \\
&\left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn}^{EO} \cos \omega_{m1} x_1 \sin \omega_{n2} x_2 \right) \left(\sum_{m=1}^{\infty} Y_{m0}^{O0} \sin \omega_{m1} x_1 \right) = \sum_{n=1}^{\infty} \left[\left(\sum_{m=1}^{\infty} A_{mn}^{EO} \cos \omega_{m1} x_1 \right) \left(\sum_{m=1}^{\infty} Y_{m0}^{O0} \sin \omega_{m1} x_1 \right) \right] \sin \omega_{n2} x_2 = \\
&= \frac{1}{2} \sum_{n=1}^{\infty} \left\{ \sum_{m=1}^{\infty} \left[\sum_{l=1}^{\infty} A_{l,n}^{EO} \left(Y_{l+m,0}^{O0} - \operatorname{sgn}(l-m) Y_{|l-m|,0}^{O0} \right) \right] \sin \omega_{m1} x_1 \right\} \sin \omega_{n2} x_2 = \\
&= -\frac{1}{2} \sum_{n=1}^{\infty} \left\{ \sum_{m=1}^{\infty} \left[\sum_{l=1}^{\infty} Y_{l,0}^{O0} \left(A_{l+m,n}^{EO} - A_{|l-m|,n}^{EO} \right) \right] \sin \omega_{m1} x_1 \right\} \sin \omega_{n2} x_2 \\
&\left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn}^{OE} \sin \omega_{m1} x_1 \cos \omega_{n2} x_2 \right) \left(\sum_{n=1}^{\infty} Y_{0n}^{E0} \cos \omega_{m1} x \right) = \sum_{m=1}^{\infty} \left[\left(\sum_{n=1}^{\infty} A_{mn}^{OE} \sin \omega_{m1} x \right) \left(\sum_{n=1}^{\infty} Y_{m0}^{E0} \cos \omega_{m1} x \right) \right] \cos \omega_{n2} x_2 = \\
&= -\frac{1}{2} \sum_{n=1}^{\infty} \left\{ \sum_{m=1}^{\infty} \left[\sum_{l=1}^{\infty} A_{l,n}^{OE} \left(Y_{l+m,0}^{E0} - Y_{|l-m|,0}^{E0} \right) \right] \sin \omega_{m1} x_1 \right\} \cos \omega_{n2} x_2 = \\
&= \frac{1}{2} \sum_{n=1}^{\infty} \left\{ \sum_{m=1}^{\infty} \left[\sum_{l=1}^{\infty} Y_{l,0}^{E0} \left(A_{l+m,n}^{OE} - \operatorname{sgn}(l-m) A_{|l-m|,n}^{OE} \right) \right] \sin \omega_{m1} x_1 \right\} \cos \omega_{n2} x_2 \\
&\left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn}^{OE} \sin \omega_{m1} x_1 \cos \omega_{n2} x_2 \right) \left(\sum_{m=1}^{\infty} Y_{m0}^{O0} \sin \omega_{m1} x_1 \right) = \sum_{n=1}^{\infty} \left[\left(\sum_{m=1}^{\infty} A_{mn}^{OE} \sin \omega_{m1} x_1 \right) \left(\sum_{m=1}^{\infty} Y_{m0}^{O0} \sin \omega_{m1} x_1 \right) \right] \cos \omega_{n2} x_2 = \\
&= \sum_{n=1}^{\infty} \left[\frac{1}{2} \sum_{m=1}^{\infty} A_{mn}^{OE} Y_{m0}^{O0} + \frac{1}{2} \sum_{m=1}^{\infty} \left[\sum_{l=1}^{\infty} A_{l,n}^{OE} \left(Y_{l+m,0}^{O0} + \operatorname{sgn}(l-m) Y_{|l-m|,0}^{O0} \right) \right] \cos \omega_{m1} x_1 \right] \cos \omega_{n2} x_2 = \\
&= \frac{1}{2} \sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} Y_{m0}^{O0} A_{mn}^{OE} \right) \cos \omega_{n2} x_2 + \frac{1}{2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[\sum_{l=1}^{\infty} Y_{l,0}^{O0} \left(A_{l+m,n}^{OE} + \operatorname{sgn}(l-m) A_{|l-m|,n}^{OE} \right) \right] \cos \omega_{m1} x_1 \cos \omega_{n2} x_2 \\
&\left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn}^{OO} \sin \omega_{m1} x_1 \sin \omega_{n2} x_2 \right) \left(\sum_{m=1}^{\infty} Y_{m0}^{E0} \cos \omega_{m1} x_1 \right) = \sum_{n=1}^{\infty} \left[\left(\sum_{m=1}^{\infty} A_{mn}^{OO} \sin \omega_{m1} x_1 \right) \left(\sum_{m=1}^{\infty} Y_{m0}^{E0} \cos \omega_{m1} x_1 \right) \right] \sin \omega_{n2} x_2 = \\
&= -\frac{1}{2} \sum_{n=1}^{\infty} \left\{ \sum_{m=1}^{\infty} \left[\sum_{l=1}^{\infty} A_{l,n}^{OO} \left(Y_{l+m,0}^{E0} - Y_{|l-m|,0}^{E0} \right) \right] \sin \omega_{m1} x_1 \right\} \sin \omega_{n2} x_2 = \\
&= \frac{1}{2} \sum_{n=1}^{\infty} \left\{ \sum_{m=1}^{\infty} \left[\sum_{l=1}^{\infty} Y_{l,0}^{E0} \left(A_{l+m,n}^{OO} - \operatorname{sgn}(l-m) A_{|l-m|,n}^{OO} \right) \right] \sin \omega_{m1} x_1 \right\} \sin \omega_{n2} x_2 \\
&\left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn}^{OO} \sin \omega_{m1} x_1 \sin \omega_{n2} x_2 \right) \left(\sum_{m=1}^{\infty} Y_{m0}^{O0} \sin \omega_{m1} x_1 \right) = \sum_{n=1}^{\infty} \left[\left(\sum_{m=1}^{\infty} A_{mn}^{OO} \sin \omega_{m1} x_1 \right) \left(\sum_{m=1}^{\infty} Y_{m0}^{O0} \sin \omega_{m1} x_1 \right) \right] \sin \omega_{n2} x_2 = \\
&= \sum_{n=1}^{\infty} \left[\frac{1}{2} \sum_{m=1}^{\infty} A_{mn}^{OO} Y_{m0}^{O0} + \frac{1}{2} \sum_{m=1}^{\infty} \left[\sum_{l=1}^{\infty} A_{l,n}^{OO} \left(Y_{l+m,0}^{O0} + \operatorname{sgn}(l-m) Y_{|l-m|,0}^{O0} \right) \right] \cos \omega_{m1} x_1 \right] \sin \omega_{n2} x_2 = \\
&= \frac{1}{2} \sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} Y_{m0}^{O0} A_{mn}^{OO} \right) \sin \omega_{n2} x_2 + \frac{1}{2} \sum_{n=1}^{\infty} \left\{ \sum_{m=1}^{\infty} \left[\sum_{l=1}^{\infty} Y_{l,0}^{O0} \left(A_{l+m,n}^{OO} + \operatorname{sgn}(l-m) A_{|l-m|,n}^{OO} \right) \right] \cos \omega_{m1} x_1 \right\} \sin \omega_{n2} x_2 \\
&\left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn}^{EE} \cos \omega_{m1} x_1 \cos \omega_{n2} x_2 \right) \left(\sum_{n=1}^{\infty} Y_{0n}^{OE} \cos \omega_{n2} x_2 \right) = \sum_{m=1}^{\infty} \left[\left(\sum_{n=1}^{\infty} A_{mn}^{EE} \cos \omega_{n2} x_2 \right) \left(\sum_{n=1}^{\infty} Y_{0n}^{OE} \cos \omega_{n2} x_2 \right) \right] \cos \omega_{m1} x_1 = \\
&= \sum_{m=1}^{\infty} \left\{ \frac{1}{2} \sum_{n=1}^{\infty} A_{mn}^{EE} Y_{0n}^{OE} + \frac{1}{2} \sum_{n=1}^{\infty} \left[\sum_{l=1}^{\infty} A_{m,l}^{EE} \left(Y_{0,l+n}^{OE} + Y_{0,|l-n|}^{OE} \right) \right] \cos \omega_{n2} x_2 \right\} \cos \omega_{m1} x_1 = \\
&= \frac{1}{2} \sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} Y_{0n}^{OE} A_{mn}^{EE} \right) \cos \omega_{m1} x_1 + \frac{1}{2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[\sum_{l=1}^{\infty} Y_{0,l}^{OE} \left(A_{m,l+n}^{EE} + A_{m,|l-n|}^{EE} \right) \right] \cos \omega_{m1} x_1 \cos \omega_{n2} x_2 \\
&\left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn}^{EE} \cos \omega_{m1} x_1 \cos \omega_{n2} x_2 \right) \left(\sum_{n=1}^{\infty} Y_{0n}^{O0} \sin \omega_{n2} x_2 \right) = \sum_{m=1}^{\infty} \left[\left(\sum_{n=1}^{\infty} A_{mn}^{EE} \cos \omega_{n2} x_2 \right) \left(\sum_{n=1}^{\infty} Y_{0n}^{O0} \sin \omega_{n2} x_2 \right) \right] \cos \omega_{m1} x_1 = \\
&= \frac{1}{2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[\sum_{l=1}^{\infty} A_{m,l}^{EE} \left(Y_{0,l+n}^{O0} - \operatorname{sgn}(l-n) Y_{0,|l-n|}^{O0} \right) \right] \cos \omega_{m1} x_1 \sin \omega_{n2} x_2 =
\end{aligned}$$

$$\begin{aligned}
&= -\frac{1}{2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[\sum_{l=1}^{\infty} Y_{0,l}^{00} (A_{m,l+n}^{EE} - A_{m,|l-n|}^{EE}) \right] \cos \omega_{m1} x_1 \sin \omega_{n2} x_2 \\
&\left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn}^{EO} \cos \omega_{m1} x_1 \sin \omega_{n2} x_2 \right) \left(\sum_{n=1}^{\infty} Y_{0n}^{0E} \cos \omega_{n2} x_2 \right) = \sum_{m=1}^{\infty} \left[\left(\sum_{n=1}^{\infty} A_{mn}^{EO} \sin \omega_{n2} x_2 \right) \left(\sum_{n=1}^{\infty} Y_{0n}^{0E} \cos \omega_{n2} x_2 \right) \right] \cos \omega_{m1} x_1 = \\
&= -\frac{1}{2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[\sum_{l=1}^{\infty} A_{m,l}^{EO} (Y_{0,l+n}^{0E} - Y_{0,|l-n|}^{0E}) \right] \cos \omega_{m1} x_1 \sin \omega_{n2} x_2 = \\
&= \frac{1}{2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[\sum_{l=1}^{\infty} Y_{0,l}^{0E} (A_{m,l+n}^{EO} - \operatorname{sgn}(l-n) A_{m,|l-n|}^{EO}) \right] \cos \omega_{m1} x_1 \sin \omega_{n2} x_2 \\
&\left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn}^{EO} \cos \omega_{m1} x_1 \sin \omega_{n2} x_2 \right) \left(\sum_{n=1}^{\infty} Y_{0n}^{00} \sin \omega_{n2} x_2 \right) = \sum_{m=1}^{\infty} \left[\left(\sum_{n=1}^{\infty} A_{mn}^{EO} \sin \omega_{n2} x_2 \right) \left(\sum_{n=1}^{\infty} Y_{0n}^{00} \sin \omega_{n2} x_2 \right) \right] \cos \omega_{m1} x_1 = \\
&= \sum_{m=1}^{\infty} \left[\frac{1}{2} \sum_{n=1}^{\infty} A_{mn}^{EO} Y_{0,n}^{00} + \frac{1}{2} \sum_{n=1}^{\infty} \left[\sum_{l=1}^{\infty} A_{m,l}^{EO} (Y_{0,l+n}^{00} + \operatorname{sgn}(l-n) Y_{0,|l-n|}^{00}) \right] \cos \omega_{n2} x_2 \right] \cos \omega_{m1} x_1 = \\
&= \frac{1}{2} \sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} Y_{0n}^{00} A_{mn}^{EO} \right) \cos \omega_{m1} x_1 + \frac{1}{2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[\sum_{l=1}^{\infty} Y_{0,l}^{00} (A_{m,l+n}^{EO} + \operatorname{sgn}(l-n) A_{m,|l-n|}^{EO}) \right] \cos \omega_{m1} x_1 \cos \omega_{n2} x_2 \\
&\left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn}^{OE} \sin \omega_{m1} x_1 \cos \omega_{n2} x_2 \right) \left(\sum_{n=1}^{\infty} Y_{0n}^{0E} \cos \omega_{n2} x_2 \right) = \sum_{m=1}^{\infty} \left[\left(\sum_{n=1}^{\infty} A_{mn}^{OE} \cos \omega_{n2} x_2 \right) \left(\sum_{n=1}^{\infty} Y_{0n}^{0E} \cos \omega_{n2} x_2 \right) \right] \sin \omega_{m1} x_1 = \\
&= \sum_{m=1}^{\infty} \left\{ \frac{1}{2} \sum_{n=1}^{\infty} A_{mn}^{OE} Y_{0n}^{0E} + \frac{1}{2} \sum_{n=1}^{\infty} \left[\sum_{l=1}^{\infty} A_{m,l}^{OE} (Y_{0,l+n}^{0E} + Y_{0,|l-n|}^{0E}) \right] \cos \omega_{n2} x_2 \right\} \sin \omega_{m1} x_1 = \\
&= \frac{1}{2} \sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} Y_{0n}^{0E} A_{mn}^{OE} \right) \sin \omega_{m1} x_1 + \frac{1}{2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[\sum_{l=1}^{\infty} Y_{0,l}^{0E} (A_{m,l+n}^{OE} + A_{m,|l-n|}^{OE}) \right] \sin \omega_{m1} x_1 \cos \omega_{n2} x_2 \\
&\left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn}^{OE} \sin \omega_{m1} x_1 \cos \omega_{n2} x_2 \right) \left(\sum_{n=1}^{\infty} Y_{0n}^{00} \sin \omega_{n2} x_2 \right) = \sum_{m=1}^{\infty} \left[\left(\sum_{n=1}^{\infty} A_{mn}^{OE} \cos \omega_{n2} x_2 \right) \left(\sum_{n=1}^{\infty} Y_{0n}^{00} \sin \omega_{n2} x_2 \right) \right] \sin \omega_{m1} x_1 = \\
&= \frac{1}{2} \sum_{m=1}^{\infty} \left\{ \sum_{n=1}^{\infty} \left[\sum_{l=1}^{\infty} A_{m,l}^{OE} (Y_{0,l+n}^{00} - \operatorname{sgn}(l-n) Y_{0,|l-n|}^{00}) \right] \sin \omega_{n2} x_2 \right\} \sin \omega_{m1} x_1 = \\
&= -\frac{1}{2} \sum_{m=1}^{\infty} \left\{ \sum_{n=1}^{\infty} \left[\sum_{l=1}^{\infty} Y_{0,l}^{00} (A_{m,l+n}^{OE} - A_{m,|l-n|}^{OE}) \right] \sin \omega_{m1} x_1 \right\} \sin \omega_{n2} x_2 \\
&\left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn}^{OO} \sin \omega_{m1} x_1 \sin \omega_{n2} x_2 \right) \left(\sum_{n=1}^{\infty} Y_{0n}^{0E} \cos \omega_{n2} x_2 \right) = \sum_{m=1}^{\infty} \left[\left(\sum_{n=1}^{\infty} A_{mn}^{OO} \sin \omega_{n2} x_2 \right) \left(\sum_{n=1}^{\infty} Y_{0n}^{0E} \cos \omega_{n2} x_2 \right) \right] \sin \omega_{m1} x_1 = \\
&= -\frac{1}{2} \sum_{m=1}^{\infty} \left\{ \sum_{n=1}^{\infty} \left[\sum_{l=1}^{\infty} A_{m,l}^{OO} (Y_{0,l+n}^{0E} - Y_{0,|l-n|}^{0E}) \right] \sin \omega_{n2} x_2 \right\} \sin \omega_{m1} x_1 = \\
&= \frac{1}{2} \sum_{m=1}^{\infty} \left\{ \sum_{n=1}^{\infty} \left[\sum_{l=1}^{\infty} Y_{0,l}^{0E} (A_{m,l+n}^{OO} - \operatorname{sgn}(l-n) A_{m,|l-n|}^{OO}) \right] \sin \omega_{m1} x_1 \right\} \sin \omega_{n2} x_2 \\
&\left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn}^{OO} \sin \omega_{m1} x_1 \sin \omega_{n2} x_2 \right) \left(\sum_{n=1}^{\infty} Y_{0n}^{00} \sin \omega_{n2} x_2 \right) = \sum_{m=1}^{\infty} \left[\left(\sum_{n=1}^{\infty} A_{mn}^{OO} \sin \omega_{n2} x_2 \right) \left(\sum_{n=1}^{\infty} Y_{0n}^{00} \sin \omega_{n2} x_2 \right) \right] \sin \omega_{m1} x_1 = \\
&= \sum_{m=1}^{\infty} \left[\frac{1}{2} \sum_{n=1}^{\infty} A_{mn}^{OO} Y_{0,n}^{00} + \frac{1}{2} \sum_{n=1}^{\infty} \left[\sum_{l=1}^{\infty} A_{m,l}^{OO} (Y_{0,l+n}^{00} + \operatorname{sgn}(l-n) Y_{0,|l-n|}^{00}) \right] \cos \omega_{n2} x_2 \right] \sin \omega_{m1} x_1 = \\
&= \frac{1}{2} \sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} Y_{0n}^{00} A_{mn}^{OO} \right) \sin \omega_{m1} x_1 + \frac{1}{2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[\sum_{l=1}^{\infty} Y_{0,l}^{00} (A_{m,l+n}^{OO} + \operatorname{sgn}(l-n) A_{m,|l-n|}^{OO}) \right] \sin \omega_{m1} x_1 \cos \omega_{n2} x_2
\end{aligned}$$

4. The product of two functions of two variables: $y(x_1, x_2)a(x_1, x_2)$:

This problem (as well as those in which the given functions have a larger number of variables) can also be treated, started from the product of the x_l -monovariate functions

$y(x_1, C)$ and $a(x_1, C)$, a product of which the SFS expansion has coefficients in the form of a product between two functions, both dependent on the variable x_2 .

$$\begin{aligned}
& \left[\sum_{n=1}^{\infty} \left(\underbrace{\sum_{m=1}^{\infty} Y_{mn}^{EE} \cos \omega_{m1} x_1}_{Y_n^E} \right) \cos \omega_{n2} x_2 \right] \left[\sum_{n=1}^{\infty} \left(\underbrace{\sum_{m=1}^{\infty} A_{mn}^{EE} \cos \omega_{m1} x_1}_{A_n^E} \right) \cos \omega_{n2} x_2 \right] = \left(\sum_{n=1}^{\infty} Y_n^E \cos \omega_{n2} x_2 \right) \left(\sum_{n=1}^{\infty} A_n^E \cos \omega_{n2} x_2 \right) = \\
& = \frac{1}{2} \sum_{n=1}^{\infty} Y_n^E A_n^E + \frac{1}{2} \sum_{n=1}^{\infty} \sum_{l=1}^{\infty} \left[Y_l^E (A_{l+n}^E + A_{|l-n|}^E) \right] \cos \omega_{n2} x_2 = \frac{1}{2} \sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} Y_{m,n}^{EE} \cos \omega_{m1} x_1 \right) \left(\sum_{m=1}^{\infty} A_{m,n}^{EE} \cos \omega_{m1} x_1 \right) + \\
& + \frac{1}{2} \sum_{n=1}^{\infty} \left\{ \sum_{l=1}^{\infty} \left(\sum_{m=1}^{\infty} Y_{m,l}^{EE} \cos \omega_{m1} x_1 \right) \left[\left(\sum_{m=1}^{\infty} A_{m,l+n}^{EE} + \sum_{m=1}^{\infty} A_{m,|l-n|}^{EE} \right) \cos \omega_{m1} x_1 \right] \right\} \cos \omega_{n2} x_2 = \frac{1}{4} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} Y_{m,n}^{EE} A_{m,n}^{EE} + \\
& + \frac{1}{4} \sum_{n=1}^{\infty} \left\{ \sum_{m=1}^{\infty} \left[\sum_{l=1}^{\infty} Y_{l,n}^{EE} (A_{l+m,n}^{EE} + A_{|l-m|,n}^{EE}) \right] \right\} \cos \omega_{m1} x_1 + \frac{1}{4} \sum_{n=1}^{\infty} \left[\sum_{l=1}^{\infty} \sum_{m=1}^{\infty} Y_{m,l}^{EE} (A_{m,l+n}^{EE} + A_{m,|l-m|}^{EE}) \right] \cos \omega_{n2} x_2 + \\
& + \frac{1}{4} \sum_{n=1}^{\infty} \left\{ \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} \left[\sum_{r=1}^{\infty} (A_{r,l+n}^{EE} + A_{r,|l-n|}^{EE}) (Y_{r+m,l}^{EE} + Y_{|r-m|,l}^{EE}) \right] \cos \omega_{m1} x_1 \right\} \cos \omega_{n2} x_2 \\
& \left[\sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} Y_{mn}^{EO} \cos \omega_{m1} x_1 \right) \cos \omega_{n2} x_2 \right] \left[\sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} A_{mn}^{EO} \cos \omega_{m1} x_1 \right) \sin \omega_{n2} x_2 \right] = \\
& = \frac{1}{2} \sum_{n=1}^{\infty} \left\{ \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} (Y_{m,l}^{EO} \cos \omega_{m1} x_1) \left[(A_{m,l+n}^{EO} - \text{sgn}(l-n) A_{m,|l-n|}^{EO}) \cos \omega_{m1} x_1 \right] \right\} \sin \omega_{n2} x_2 = \\
& = \frac{1}{4} \sum_{n=1}^{\infty} \left\{ \sum_{m=1}^{\infty} \sum_{l=1}^{\infty} Y_{m,l}^{EO} \left[(A_{m,l+n}^{EO} - \text{sgn}(l-n) A_{m,|l-n|}^{EO}) \right] \right\} \sin \omega_{n2} x_2 + \\
& + \frac{1}{4} \sum_{n=1}^{\infty} \left\{ \sum_{m=1}^{\infty} \left[\sum_{l=1}^{\infty} \sum_{r=1}^{\infty} (Y_{r+m,l}^{EO} + Y_{|r-m|,l}^{EO}) (A_{r,l+n}^{EO} - \text{sgn}(l-n) A_{r,|l-n|}^{EO}) \right] \cos \omega_{m1} x_1 \right\} \sin \omega_{n2} x_2 \\
& \left[\sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} Y_{mn}^{EO} \cos \omega_{m1} x_1 \right) \sin \omega_{n2} x_2 \right] \left[\sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} A_{mn}^{EO} \cos \omega_{m1} x_1 \right) \cos \omega_{n2} x_2 \right] = \\
& = -\frac{1}{2} \sum_{n=1}^{\infty} \left\{ \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} (Y_{m,l}^{EO} \cos \omega_{m1} x_1) \left[(A_{m,l+n}^{EO} - A_{m,|l-n|}^{EO}) \cos \omega_{m1} x_1 \right] \right\} \sin \omega_{n2} x_2 = \\
& = -\frac{1}{4} \sum_{n=1}^{\infty} \left[\sum_{m=1}^{\infty} \sum_{l=1}^{\infty} Y_{m,l}^{EO} (A_{m,l+n}^{EO} - A_{m,|l-n|}^{EO}) \right] \sin \omega_{n2} x_2 - \\
& - \frac{1}{4} \sum_{n=1}^{\infty} \left\{ \sum_{m=1}^{\infty} \left[\sum_{l=1}^{\infty} \sum_{r=1}^{\infty} (Y_{r+m,l}^{EO} + Y_{|r-m|,l}^{EO}) (A_{r,l+n}^{EO} - A_{r,|l-n|}^{EO}) \right] \cos \omega_{m1} x_1 \right\} \sin \omega_{n2} x_2 \\
& \left[\sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} Y_{mn}^{EO} \cos \omega_{m1} x_1 \right) \sin \omega_{n2} x_2 \right] \left[\sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} A_{mn}^{EO} \cos \omega_{m1} x_1 \right) \sin \omega_{n2} x_2 \right] = \frac{1}{2} \sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} Y_{m,n}^{EO} \cos \omega_{m1} x_1 \right) \left(\sum_{m=1}^{\infty} A_{m,n}^{EO} \cos \omega_{m1} x_1 \right) + \\
& + \frac{1}{2} \sum_{n=1}^{\infty} \left\{ \sum_{l=1}^{\infty} \left(\sum_{m=1}^{\infty} Y_{m,l}^{EO} \cos \omega_{m1} x_1 \right) \left[\sum_{m=1}^{\infty} (A_{m,l+n}^{EO} + \text{sgn}(l-n) A_{m,|l-n|}^{EO}) \cos \omega_{m1} x_1 \right] \right\} \cos \omega_{n2} x_2 = \frac{1}{4} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} Y_{m,n}^{EO} A_{m,n}^{EO} + \\
& + \frac{1}{4} \sum_{m=1}^{\infty} \left[\sum_{l=1}^{\infty} \sum_{n=1}^{\infty} Y_{l,n}^{EO} (A_{l+m,n}^{EO} + A_{|l-m|,n}^{EO}) \right] \cos \omega_{m1} x_1 + \frac{1}{4} \sum_{n=1}^{\infty} \left[\sum_{m=1}^{\infty} \sum_{l=1}^{\infty} Y_{m,l}^{EO} (A_{m,l+n}^{EO} + \text{sgn}(l-n) A_{m,|l-n|}^{EO}) \right] \cos \omega_{n2} x_2 + \\
& + \frac{1}{4} \sum_{n=1}^{\infty} \left\{ \sum_{m=1}^{\infty} \left[\sum_{l=1}^{\infty} \sum_{r=1}^{\infty} (Y_{r+m,l}^{EO} + Y_{|r-m|,l}^{EO}) (A_{r,l+n}^{EO} + \text{sgn}(l-n) A_{r,|l-n|}^{EO}) \right] \cos \omega_{m1} x_1 \right\} \cos \omega_{n2} x_2 \\
& + \frac{1}{4} \sum_{m=1}^{\infty} \left[\sum_{l=1}^{\infty} \sum_{n=1}^{\infty} Y_{l,n}^{EO} (A_{l+m,n}^{EO} + A_{|l-m|,n}^{EO}) \right] \cos \omega_{m1} x_1 + \frac{1}{4} \sum_{n=1}^{\infty} \left[\sum_{m=1}^{\infty} \sum_{l=1}^{\infty} Y_{m,l}^{EO} (A_{m,l+n}^{EO} + \text{sgn}(l-n) A_{m,|l-n|}^{EO}) \right] \cos \omega_{n2} x_2 + \\
& + \frac{1}{4} \sum_{n=1}^{\infty} \left\{ \sum_{m=1}^{\infty} \left[\sum_{l=1}^{\infty} \sum_{r=1}^{\infty} (Y_{r+m,l}^{EO} + Y_{|r-m|,l}^{EO}) (A_{r,l+n}^{EO} + \text{sgn}(l-n) A_{r,|l-n|}^{EO}) \right] \cos \omega_{m1} x_1 \right\} \cos \omega_{n2} x_2
\end{aligned}$$

$$\begin{aligned}
& \left[\sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} Y_{mn}^{OE} \sin \omega_{m1} x_1 \right) \cos \omega_{n2} x_2 \right] \left[\sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} A_{mn}^{OE} \sin \omega_{m1} x_1 \right) \cos \omega_{n2} x_2 \right] = \frac{1}{2} \sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} Y_{m,n}^{OE} \sin \omega_{m1} x_1 \right) \left(\sum_{m=1}^{\infty} A_{m,n}^{OE} \sin \omega_{m1} x_1 \right) + \\
& + \frac{1}{2} \sum_{n=1}^{\infty} \left\{ \sum_{l=1}^{\infty} \left(\sum_{m=1}^{\infty} Y_{m,l}^{OE} \sin \omega_{m1} x_1 \right) \left[\sum_{m=1}^{\infty} \left(A_{m,l+n}^{OE} + A_{m,|l-n|}^{OE} \right) \sin \omega_{m1} x_1 \right] \right\} \cos \omega_{n2} x_2 = \frac{1}{4} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} Y_{m,n}^{OE} A_{m,n}^{OE} + \\
& + \frac{1}{4} \sum_{m=1}^{\infty} \left[\sum_{n=1}^{\infty} \sum_{l=1}^{\infty} Y_{l,n}^{OE} \left(A_{l+m,n}^{OE} + \operatorname{sgn}(l-m) A_{|l-m|,n}^{OE} \right) \cos \omega_{m1} x_1 + \frac{1}{4} \sum_{n=1}^{\infty} \left[\sum_{m=1}^{\infty} \sum_{l=1}^{\infty} Y_{m,l}^{OE} \left(A_{m,l+n}^{OE} + A_{m,|l-n|}^{OE} \right) \right] \cos \omega_{n2} x_2 + \right. \\
& \left. + \frac{1}{4} \sum_{n=1}^{\infty} \left\{ \sum_{m=1}^{\infty} \left[\sum_{r=1}^{\infty} \left(Y_{r+m,l}^{OE} + \operatorname{sgn}(r-m) Y_{|r-m|,l}^{OE} \right) \left(A_{r,l+n}^{OE} + A_{r,|l-n|}^{OE} \right) \right] \cos \omega_{m1} x_1 \right\} \cos \omega_{n2} x_2 \right. \\
& \left[\sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} Y_{mn}^{OE} \sin \omega_{m1} x_1 \right) \cos \omega_{n2} x_2 \right] \left[\sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} A_{mn}^{OO} \sin \omega_{m1} x_1 \right) \sin \omega_{n2} x_2 \right] = \\
& = \frac{1}{2} \sum_{n=1}^{\infty} \left\{ \sum_{l=1}^{\infty} \left(\sum_{m=1}^{\infty} Y_{m,l}^{OE} \sin \omega_{m1} x_1 \right) \left[\sum_{m=1}^{\infty} \left(A_{m,l+n}^{OO} - \operatorname{sgn}(l-n) A_{m,|l-n|}^{OO} \right) \sin \omega_{m1} x_1 \right] \right\} \sin \omega_{n2} x_2 = \\
& = \frac{1}{4} \sum_{n=1}^{\infty} \left[\sum_{m=1}^{\infty} \sum_{l=1}^{\infty} Y_{m,l}^{OE} \left(A_{m,l+n}^{OO} - \operatorname{sgn}(l-n) A_{m,|l-n|}^{OO} \right) \right] \sin \omega_{n2} x_2 + \\
& + \frac{1}{4} \sum_{n=1}^{\infty} \left\{ \sum_{m=1}^{\infty} \left[\sum_{l=1}^{\infty} \sum_{r=1}^{\infty} \left(Y_{r+m,l}^{OE} + \operatorname{sgn}(r-m) Y_{|r-m|,l}^{OE} \right) \left(A_{r,l+n}^{OO} - \operatorname{sgn}(l-n) A_{r,|l-n|}^{OO} \right) \right] \cos \omega_{m1} x_1 \right\} \sin \omega_{n2} x_2 \\
& \left[\sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} Y_{mn}^{OO} \sin \omega_{m1} x_1 \right) \sin \omega_{n2} x_2 \right] \left[\sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} A_{mn}^{OE} \sin \omega_{m1} x_1 \right) \cos \omega_{n2} x_2 \right] = \\
& = -\frac{1}{2} \sum_{n=1}^{\infty} \left[\sum_{l=1}^{\infty} \left(\sum_{m=1}^{\infty} Y_{m,l}^{OO} \sin \omega_{m1} x_1 \right) \left(\sum_{m=1}^{\infty} \left(A_{m,l+n}^{OE} - A_{m,|l-n|}^{OE} \right) \sin \omega_{m1} x_1 \right) \right] \sin \omega_{n2} x_2 = \\
& = -\frac{1}{4} \sum_{n=1}^{\infty} \left[\sum_{m=1}^{\infty} \sum_{l=1}^{\infty} Y_{m,l}^{OO} \left(A_{m,l+n}^{OE} - A_{m,|l-n|}^{OE} \right) \right] \sin \omega_{n2} x_2 - \\
& - \frac{1}{4} \sum_{n=1}^{\infty} \left\{ \sum_{m=1}^{\infty} \left[\sum_{l=1}^{\infty} \sum_{r=1}^{\infty} \left(Y_{r+m,l}^{OO} + \operatorname{sgn}(r-m) Y_{|r-m|,l}^{OO} \right) \left(A_{r,l+n}^{OE} - A_{r,|l-n|}^{OE} \right) \right] \cos \omega_{m1} x_1 \right\} \sin \omega_{n2} x_2 \\
& \left[\sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} Y_{mn}^{OO} \sin \omega_{m1} x_1 \right) \sin \omega_{n2} x_2 \right] \left[\sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} A_{mn}^{OO} \sin \omega_{m1} x_1 \right) \sin \omega_{n2} x_2 \right] = \frac{1}{2} \sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} Y_{m,n}^{OO} \sin \omega_{m1} x_1 \right) \left(\sum_{m=1}^{\infty} A_{m,n}^{OO} \sin \omega_{m1} x_1 \right) + \\
& + \frac{1}{2} \sum_{n=1}^{\infty} \left\{ \sum_{l=1}^{\infty} \left(\sum_{m=1}^{\infty} Y_{m,l}^{OO} \sin \omega_{m1} x_1 \right) \left[\sum_{m=1}^{\infty} \left(A_{m,l+n}^{OO} + \operatorname{sgn}(l-n) A_{m,|l-n|}^{OO} \right) \sin \omega_{m1} x_1 \right] \right\} \cos \omega_{n2} x_2 = \frac{1}{4} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} Y_{m,n}^{OO} A_{m,n}^{OO} + \\
& + \frac{1}{4} \sum_{m=1}^{\infty} \left[\sum_{n=1}^{\infty} \sum_{l=1}^{\infty} Y_{l,n}^{OO} \left(A_{l+m,n}^{OO} + \operatorname{sgn}(l-m) A_{|l-m|,n}^{OO} \right) \cos \omega_{m1} x_1 \right] + \frac{1}{4} \sum_{n=1}^{\infty} \left[\sum_{m=1}^{\infty} \sum_{l=1}^{\infty} Y_{m,l}^{OO} \left(A_{m,l+n}^{OO} + \operatorname{sgn}(l-n) A_{m,|l-n|}^{OO} \right) \right] \cos \omega_{n2} x_2 + \\
& + \frac{1}{4} \sum_{n=1}^{\infty} \left\{ \sum_{m=1}^{\infty} \left[\sum_{l=1}^{\infty} \sum_{r=1}^{\infty} \left(Y_{r+m,l}^{OO} + \operatorname{sgn}(r-m) Y_{|r-m|,l}^{OO} \right) \left(A_{r,l+n}^{OO} + \operatorname{sgn}(l-n) A_{r,|l-n|}^{OO} \right) \right] \cos \omega_{m1} x_1 \right\} \cos \omega_{n2} x_2 \\
& \left[\sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} Y_{mn}^{EE} \cos \omega_{m1} x_1 \right) \cos \omega_{n2} x_2 \right] \left[\sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} A_{mn}^{OE} \sin \omega_{m1} x_1 \right) \cos \omega_{n2} x_2 \right] = \left(\sum_{m=1}^{\infty} Y_{mn}^{EE} \cos \omega_{m1} x_1 \right) \left(\sum_{m=1}^{\infty} A_{mn}^{OE} \sin \omega_{m1} x_1 \right) + \\
& = \frac{1}{2} \sum_{n=1}^{\infty} \left\{ \sum_{l=1}^{\infty} \left(\sum_{m=1}^{\infty} Y_{m,l}^{EE} \cos \omega_{m1} x_1 \right) \left[\sum_{m=1}^{\infty} \left(A_{m,l+n}^{OE} + A_{m,|l-n|}^{OE} \right) \sin \omega_{m1} x_1 \right] \right\} \cos \omega_{n2} x_2 = \\
& = \frac{1}{4} \sum_{n=1}^{\infty} \left[\sum_{m=1}^{\infty} \sum_{l=1}^{\infty} Y_{m,l}^{EE} \left(A_{m,l+n}^{OE} - \operatorname{sgn}(l-n) A_{m,|l-n|}^{OE} \right) \right] \sin \omega_{m1} x_1 - \\
& - \frac{1}{4} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_{l=1}^{\infty} \left[\sum_{r=1}^{\infty} \left(Y_{r+m,l}^{EE} - Y_{|r-m|,l}^{EE} \right) \left(A_{r,l+n}^{OE} + A_{r,|l-n|}^{OE} \right) \right] \sin \omega_{m1} x_1 \cos \omega_{n2} x_2
\end{aligned}$$

$$\begin{aligned}
& \left[\sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} Y_{mn}^{EE} \cos \omega_{m1} x_1 \right) \cos \omega_{n2} x_2 \right] \left[\sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} A_{mn}^{OO} \sin \omega_{m1} x_1 \right) \sin \omega_{n2} x_2 \right] = \\
& = \frac{1}{2} \sum_{n=1}^{\infty} \left\{ \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} \left(Y_{m,l}^{EE} \cos \omega_{m1} x_1 \right) \left[A_{m,l+n}^{OO} - \operatorname{sgn}(l-n) A_{m,|l-n|}^{OO} \right] \sin \omega_{m1} x_1 \right\} \sin \omega_{n2} x_2 = \\
& = -\frac{1}{4} \sum_{n=1}^{\infty} \left\{ \sum_{m=1}^{\infty} \sum_{l=1}^{\infty} \left[\sum_{r=1}^{\infty} \left(Y_{r+m,l}^{EE} - Y_{|r-m|,l}^{EE} \right) \left(A_{r,l+n}^{OO} - \operatorname{sgn}(l-n) A_{r,|l-n|}^{OO} \right) \right] \sin \omega_{m1} x_1 \right\} \sin \omega_{n2} x_2 \\
& \left[\sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} Y_{mn}^{EO} \cos \omega_{m1} x_1 \right) \sin \omega_{n2} x_2 \right] \left[\sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} A_{mn}^{OE} \sin \omega_{m1} x_1 \right) \cos \omega_{n2} x_2 \right] = \\
& = -\frac{1}{2} \sum_{n=1}^{\infty} \left\{ \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} \left(Y_{m,l}^{EO} \cos \omega_{m1} x_1 \right) \left[A_{m,l+n}^{OE} - A_{m,|l-n|}^{OE} \right] \sin \omega_{m1} x_1 \right\} \sin \omega_{n2} x_2 = \\
& = \frac{1}{4} \sum_{n=1}^{\infty} \left\{ \sum_{m=1}^{\infty} \sum_{l=1}^{\infty} \left[\sum_{r=1}^{\infty} \left(Y_{r+m,l}^{EO} - Y_{|r-m|,l}^{EO} \right) \left(A_{r,l+n}^{OE} - A_{r,|l-n|}^{OE} \right) \right] \sin \omega_{m1} x_1 \right\} \sin \omega_{n2} x_2 \\
& \left[\sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} Y_{mn}^{EO} \cos \omega_{m1} x_1 \right) \sin \omega_{n2} x_2 \right] \left[\sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} A_{mn}^{OO} \sin \omega_{m1} x_1 \right) \sin \omega_{n2} x_2 \right] = \frac{1}{2} \sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} Y_{m,n}^{EO} \cos \omega_{m1} x_1 \right) \left(\sum_{m=1}^{\infty} A_{m,n}^{OO} \sin \omega_{m1} x_1 \right) + \\
& + \frac{1}{2} \sum_{n=1}^{\infty} \left\{ \sum_{l=1}^{\infty} \left(\sum_{m=1}^{\infty} Y_{m,l}^{EO} \cos \omega_{m1} x_1 \right) \left[\sum_{m=1}^{\infty} \left(A_{m,l+n}^{OO} + \operatorname{sgn}(l-n) A_{m,|l-n|}^{OO} \right) \sin \omega_{m1} x_1 \right] \right\} \cos \omega_{n2} x_2 = \\
& = \frac{1}{4} \sum_{n=1}^{\infty} \left[\sum_{l=1}^{\infty} \sum_{m=1}^{\infty} Y_{l,n}^{EO} \left(A_{l+m,n}^{OO} - \operatorname{sgn}(l-m) A_{l,|l-m|}^{OO} \right) \right] \sin \omega_{m1} x_1 - \\
& - \frac{1}{4} \sum_{n=1}^{\infty} \left\{ \sum_{m=1}^{\infty} \sum_{l=1}^{\infty} \left[\sum_{r=1}^{\infty} \left(A_{r,l+n}^{OO} + \operatorname{sgn}(l-n) A_{r,|l-n|}^{OO} \right) \left(Y_{r+m,l}^{EO} - Y_{|r-m|,l}^{EO} \right) \right] \sin \omega_{m1} x_1 \right\} \cos \omega_{n2} x_2 \\
& \left[\sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} Y_{mn}^{OE} \sin \omega_{m1} x_1 \right) \cos \omega_{n2} x_2 \right] \left[\sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} A_{mn}^{EE} \cos \omega_{m1} x_1 \right) \cos \omega_{n2} x_2 \right] = \frac{1}{2} \sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} Y_{m,n}^{OE} \sin \omega_{m1} x_1 \right) \left(\sum_{m=1}^{\infty} A_{m,n}^{EE} \cos \omega_{m1} x_1 \right) + \\
& + \frac{1}{2} \sum_{n=1}^{\infty} \left\{ \sum_{l=1}^{\infty} \left(\sum_{m=1}^{\infty} Y_{m,l}^{OE} \sin \omega_{m1} x_1 \right) \left[\sum_{m=1}^{\infty} \left(A_{m,l+n}^{EE} + A_{m,|l-n|}^{EE} \right) \cos \omega_{m1} x_1 \right] \right\} \cos \omega_{n2} x_2 = \\
& = -\frac{1}{4} \sum_{n=1}^{\infty} \left\{ \sum_{m=1}^{\infty} \left[\sum_{l=1}^{\infty} Y_{l,n}^{OE} \left(A_{l+m,n}^{EE} - A_{l,|l-m|}^{EE} \right) \right] \sin \omega_{m1} x_1 \right\} + \\
& + \frac{1}{4} \sum_{n=1}^{\infty} \left\{ \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} \left[\sum_{r=1}^{\infty} \left(Y_{r+m,l}^{OE} + \operatorname{sgn}(r-m) Y_{|r-m|,l}^{OE} \right) \left(A_{r,l+n}^{EE} + A_{r,|l-n|}^{EE} \right) \right] \sin \omega_{m1} x_1 \right\} \cos \omega_{n2} x_2 \\
& \left[\sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} Y_{mn}^{OE} \sin \omega_{m1} x_1 \right) \cos \omega_{n2} x_2 \right] \left[\sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} A_{mn}^{EO} \cos \omega_{m1} x_1 \right) \sin \omega_{n2} x_2 \right] = \\
& = \frac{1}{2} \sum_{n=1}^{\infty} \left\{ \sum_{l=1}^{\infty} \left(\sum_{m=1}^{\infty} Y_{m,l}^{OE} \sin \omega_{m1} x_1 \right) \left[\sum_{m=1}^{\infty} \left(A_{m,l+n}^{EO} - \operatorname{sgn}(l-n) A_{m,|l-n|}^{EO} \right) \cos \omega_{m1} x_1 \right] \right\} \sin \omega_{n2} x_2 = \\
& = \frac{1}{4} \sum_{n=1}^{\infty} \left\{ \sum_{m=1}^{\infty} \left[\sum_{l=1}^{\infty} \sum_{r=1}^{\infty} \left(Y_{r+m,l}^{OE} - \operatorname{sgn}(r-m) Y_{|r-m|,l}^{OE} \right) \left(A_{r,l+n}^{EO} - \operatorname{sgn}(l-n) A_{r,|l-n|}^{EO} \right) \right] \sin \omega_{m1} x_1 \right\} \sin \omega_{n2} x_2 \\
& \left[\sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} Y_{mn}^{OO} \sin \omega_{m1} x_1 \right) \sin \omega_{n2} x_2 \right] \left[\sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} A_{mn}^{EE} \cos \omega_{m1} x_1 \right) \cos \omega_{n2} x_2 \right] = \\
& = -\frac{1}{2} \sum_{n=1}^{\infty} \left[\sum_{l=1}^{\infty} \left(\sum_{m=1}^{\infty} Y_{m,l}^{OO} \sin \omega_{m1} x_1 \right) \left(\sum_{m=1}^{\infty} \left(A_{m,l+n}^{EE} - A_{m,|l-n|}^{EE} \right) \cos \omega_{m1} x_1 \right) \right] \sin \omega_{n2} x_2 = \\
& = -\frac{1}{4} \sum_{n=1}^{\infty} \left\{ \sum_{m=1}^{\infty} \left[\sum_{l=1}^{\infty} \sum_{r=1}^{\infty} \left(Y_{r+m,l}^{OO} - \operatorname{sgn}(r-m) Y_{|r-m|,l}^{OO} \right) \left(A_{r,l+n}^{EE} - A_{r,|l-n|}^{EE} \right) \right] \sin \omega_{m1} x_1 \right\} \sin \omega_{n2} x_2
\end{aligned}$$

$$\begin{aligned}
& \left[\sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} Y_{mn}^{OO} \sin \omega_m x_1 \right) \sin \omega_n x_2 \right] \left[\sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} A_{mn}^{EO} \cos \omega_m x_1 \right) \sin \omega_n x_2 \right] = \frac{1}{2} \sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} Y_{m,n}^{OO} \sin \omega_m x_1 \right) \left(\sum_{m=1}^{\infty} A_{m,n}^{EO} \cos \omega_m x_1 \right) + \\
& + \frac{1}{2} \sum_{n=1}^{\infty} \left\{ \sum_{l=1}^{\infty} \left(\sum_{m=1}^{\infty} Y_{m,l}^{OO} \sin \omega_m x_1 \right) \left[\sum_{m=1}^{\infty} \left(A_{m,l+n}^{EO} + \operatorname{sgn}(l-n) A_{m,|l-n|}^{EO} \right) \cos \omega_m x_1 \right] \right\} \cos \omega_n x_2 = \\
& = -\frac{1}{4} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[\sum_{l=1}^{\infty} Y_{l,n}^{OO} \left(A_{l+m,n}^{EO} - A_{|l-m|,n}^{EO} \right) \right] \sin \omega_m x_1 + \\
& + \frac{1}{4} \sum_{n=1}^{\infty} \left\{ \sum_{m=1}^{\infty} \left[\sum_{l=1}^{\infty} \sum_{r=1}^{\infty} \left(Y_{r+m,l}^{OO} - \operatorname{sgn}(r-m) Y_{|r-m|,l}^{OO} \right) \left(A_{r,l+n}^{EO} + \operatorname{sgn}(l-n) A_{r,|l-n|}^{EO} \right) \right] \sin \omega_m x_1 \right\} \cos \omega_n x_2
\end{aligned}$$

As we have already noticed at the coefficients of the product of the two monovariate functions, the expressions of these coefficients can have different forms, depending on the order in which the summation of the coefficients of the secondary harmonics is carried out. Here are some conclusive examples:

$$\begin{aligned}
& \sum_{l=1}^{\infty} \left[\left(\sum_{m=1}^{\infty} Y_{m,l}^{EE} \cos \omega_m x_1 \right) \sum_{m=1}^{\infty} \left(A_{m,l+n}^{EE} + A_{m,|l-n|}^{EE} \right) \cos \omega_m x_1 \right] = \\
& = \sum_{l=1}^{\infty} \left[\sum_{m=1}^{\infty} \left(A_{m,l+n}^{EE} + A_{m,|l-n|}^{EE} \right) \cos \omega_m x_1 \left(\sum_{m=1}^{\infty} Y_{m,l}^{EE} \cos \omega_m x_1 \right) \right] = \frac{1}{2} \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} Y_{m,l}^{EE} \left(A_{m,l+n}^{EE} + A_{m,|l-n|}^{EE} \right) + \\
& + \frac{1}{2} \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} \left[\sum_{r=1}^{\infty} Y_{r,l}^{EE} \left(A_{r+m,l+n}^{EE} + A_{r+m,|l-n|}^{EE} + A_{|r-m|,l+n}^{EE} + A_{|r-m|,|l-n|}^{EE} \right) \right] \cos \omega_m x_1 = \\
& = \frac{1}{2} \sum_{l=1}^{\infty} \left[\sum_{m=1}^{\infty} \left(A_{m,l+n}^{EE} + A_{m,|l-n|}^{EE} \right) Y_{m,l}^{EE} \right] + \frac{1}{2} \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} \left[\sum_{r=1}^{\infty} \left(A_{r,l+n}^{EE} + A_{r,|l-n|}^{EE} \right) \left(Y_{r+m,l}^{EE} + Y_{|r-m|,l}^{EE} \right) \right] \\
& \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} Y_{mn}^{EE} \cos \omega_m x_1 \cos \omega_n x_2 \right) \left(\sum_{n=1}^{\infty} A_{0n}^{OE} \cos \omega_n x_2 \right) = \sum_{m=1}^{\infty} \left[\left(\sum_{n=1}^{\infty} Y_{mn}^{EE} \cos \omega_n x_2 \right) \left(\sum_{n=1}^{\infty} A_{0n}^{OE} \cos \omega_n x_2 \right) \right] \cos \omega_m x_1 = \\
& = \sum_{m=1}^{\infty} \left\{ \frac{1}{2} \sum_{n=1}^{\infty} Y_{mn}^{EE} A_{0n}^{OE} + \frac{1}{2} \sum_{n=1}^{\infty} \left[\sum_{l=1}^{\infty} Y_{m,l}^{EE} \left(A_{0,l+n}^{OE} + A_{0,|l-n|}^{OE} \right) \right] \cos \omega_n x_2 \right\} \cos \omega_m x_1 = \\
& = \frac{1}{2} \sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} Y_{mn}^{EE} A_{0n}^{OE} \right) \cos \omega_m x_1 + \frac{1}{2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[\sum_{l=1}^{\infty} Y_{m,l}^{EE} \left(A_{0,l+n}^{OE} + A_{0,|l-n|}^{OE} \right) \right] \cos \omega_m x_1 \cos \omega_n x_2 = \\
& = \sum_{m=1}^{\infty} \left\{ \frac{1}{2} \sum_{n=1}^{\infty} Y_{mn}^{EE} A_{0n}^{OE} + \frac{1}{2} \sum_{n=1}^{\infty} \left[\sum_{l=1}^{\infty} A_{0,l}^{OE} \left(Y_{m,l+n}^{EE} + Y_{m,|l-n|}^{EE} \right) \right] \cos \omega_n x_2 \right\} \cos \omega_m x_1 = \\
& = \frac{1}{2} \sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} Y_{mn}^{EE} A_{0n}^{OE} \right) \cos \omega_m x_1 + \frac{1}{2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[\sum_{l=1}^{\infty} \left(Y_{m,l+n}^{EE} + Y_{m,|l-n|}^{EE} \right) A_{0,l}^{OE} \right] \cos \omega_m x_1 \cos \omega_n x_2
\end{aligned}$$