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# On the estimation of the public transport demand

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## Abstract

For the case of public transport, we consider the problem of demand estimation. Given an origin-destination matrix representing the public transport demand, the distribution of flow among different lines can be obtained assuming that it corresponds to a certain equilibrium characterized by an optimization problem. The knowledge of that matrix is expensive and sometimes unaffordable in practice. In this work, we explore its estimation through the numerical solution of a bilevel optimization problem.

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*Keywords:* transit assignment; public-transport demand

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## 1. Introduction

Transit assignment models have become an interesting research area because knowing the passenger behavior allows comparing different planning scenarios in terms of network performance, always assuming that the transport demand is known.

Many models for passenger behavior have been proposed. Most of them consider that when a passenger decides to travel between certain O-D pairs and is waiting for a vehicle at a stop, he must decide which transit line should he take to minimize his total expected travel time. Among the first models that considered congestion effects, we can cite (10) that work with the concept of hyperpath composed by "strategies of attractive lines", but failed to be realistic in cases of high demand.

De Cea and Fernandez (11) began to consider the congestion effects at bus stops and inside the bus. This model was improved in (12) formulating a transit equilibrium problem that uses effective frequencies functions that vanish if the in-vehicle flow exceeds its capacity (see 7). The main limitation of these methods is that the technical assumptions are very limiting in the first case and there no efficient algorithms to compute the solution in both cases.

Cepeda et al (13) decided to continue this idea and reformulated the equilibrium problem as the minimization of a nonconvex and nondifferentiable gap function. To solve this problem a heuristic method was proposed, using an adaptation of the Method of Successive Averages (MSA) and obtaining the lines flow vector. This method can

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be applied on high scales networks without computational drawbacks but can generate lines flows that exceed the capacity when the demands are high. To improve this method, Codina and Rosell (14) presented an algorithm with strict capacities that find the solution of the fixed point inclusion formulation derived from the problem of variational inequality proposed by Codina (15). At each iteration an assignment problem is solved, using Lagrangian duality and the cutting-planes method.

The use of the previous models of transit assignment in any planning study requires the knowledge of the transport demand, commonly known as the origin-destination matrix. To obtain that matrix could be very expensive and sometimes unaffordable in practice. As has been made for the case of traffic assignment (see (16)), in this work we explore its estimation through some directly measurable quantities like the real frequencies of the buses. As we know how to compute, given the demand, the flows and hence the frequencies, we pose a kind of inverse problem whose solution estimates the actual demand. As far as we know there are no previous work about public transport estimation using this approach. Most of the articles devoted to public transport demand estimation are based on statistical or econometrical considerations **INSET REFERENCES**.

In the next section we present a detailed description of the assignment model following the one presented in (13). In section 3 we pose the inverse problem used for demand estimation and in section 4 we present the numerical experiments made with the example given in (13).

## 2. Transit assignment model

Following the notation of previous works as (10; 12; 13; 15) we consider a directed graph  $G = (N, A)$  where  $N$  is the node set and  $A$  the link set, each one with cardinality  $N_N$  and  $N_A$ . The set of nodes is composed of the bus-stop nodes  $N_s$  and the line nodes  $N_l$ . The arcs are divided in the alighting and boarding arcs connecting the bus-stop nodes with the line nodes, the on-board arcs (or line segments) connecting line-nodes and the walk arcs connecting bus-stop nodes, see figure 1 for a sketch.

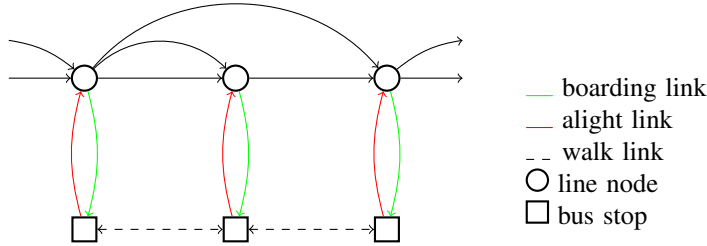


Fig. 1. Public transport network.

For some origin-destination (od) pairs  $(i, d) \in W \subset N \times N$ , there is a transport demand called  $g_i^d$ , and we call  $D$  the set of all nodes that are destinations of some od pair. For a node  $i$  we call  $E(i)$  the set of emerging links and  $I(i)$  the incoming link set. We also define the node-link incidence matrix  $A \in \mathbb{R}^{N_N \times N_A}$  where  $A_{ia} = 1$  iff  $a \in E(i)$ ,  $A_{ia} = -1$  iff  $a \in I(i)$  and otherwise zero.

We call  $v_a^d$  the flow trough link  $a$  with destination  $d$ . For each destination  $d$  we define the set of feasible flows with destination  $d$  and the set of total feasible flows as

$$V^d = \{v^d \in \mathbb{R}_+^{N_A} : Av^d = g^d\}, \quad V = \left\{v \in \mathbb{R}_+^{N_A} : v = \sum_d v^d, v^d \in V^d, \forall d\right\}. \quad (1)$$

We call  $v_a^d$  the flow trough link  $a$  with destination  $d$ , and we call  $V(g)$  the set of feasible flows for the demand  $g$ , that is the set of all  $v_a^d \geq 0$  such that  $v_a^d = 0$  for all  $a \in A_a^+$ ,

$$g_i^d + \sum_{a \in E(i)} v_a^d = \sum_{a \in I(i)} v_a^d, \quad \forall i \neq d, \quad (2)$$

where  $A_i^+ = \{a \in A : a = (i, j) \text{ for some } j \in N\}$  and  $A_i^- = \{a \in A : a = (j, i) \text{ for some } j \in N\}$ . Two functions of the full flow vector  $v$  are associated to each arc, the travel time function  $t_a(v)$  and the effective frequencies  $f_a(v)$ . Both

have non negative values and the frequencies can have the constant value  $+\infty$ . As mentioned in (13) the case when  $t_a$  and  $f_a$  are constants is called the *uncongested* case and the case where only the travel time  $T_a$  is fixed is called the *semicongested* case. Here we will consider a third case where the travel time function are constant but the frequencies are not. To model the impact of the bus load on the frequency the function 7 is used.

$$f_a(v) \begin{cases} \mu \left[ 1 - \left( \frac{v_a}{\mu c - v_{a'} + v_a} \right)^\beta \right] & \text{if } v_{a'} < \mu c \\ 0 & \text{otherwise} \end{cases} \quad (3)$$

where ....complete definitions: donde  $v_a = \sum_{d \in D} v_a^d$  representa al flujo abordando en la parada (sumado sobre todos los destinos) y  $v_{a'}$  es el flujo a bordo justo después de la parada ( $v_{a'} \geq v_a$ ). El parámetro  $\mu$  es la frecuencia nominal de la línea correspondiente y  $c$  es la capacidad física de los colectivos, con lo cual  $\mu c - v_{a'}$  es la capacidad residual esperada en la parada.

The rationale behind the model is that each passenger at each node chooses the arc to continue its trip. The decision is based on minimizing the total travel time. Thus, at each node a Common Line Problem should be solved, where, nnow, the frequencies depend on the flows. In (13) it is shown that the corresponding (equilibrium) flow  $v \in V_0^*$  is the global minimizer of the so called gap function  $G$  of the flow  $v$ , that we write here also as a function also of the demand  $g$

$$G(v, g) = \sum_{\substack{d \in D \\ i \neq d}} \left[ \sum_{a \in A_i^+} [t_a(v) + \tau_{j_a}^d(v)] v_a^d + \max_{a \in A_i^+} \frac{v_a^d}{f_a(v)} + \tau_i^d(d) \sum_{a \in A_i^+} v_a^d \right] \quad (4)$$

where  $t_a$  is the travel time,  $\tau_j^d$  is the total expected travel time from  $j$  to  $d$ ,  $A_i^+$  is the set of links emerging from  $i$ ,  $f_a$  models the impact of the congestion on the frequency,  $\mu$  is the nominal frequency of the line and  $c$  its capacity,  $\beta$  is a calibrated parameter and  $v_a'$  is the on-board flow right after the stop.

Then the transit assignment for a given demand  $g$  is obtained minimizing  $G(v, g)$  over the flows in  $V(g)$ . It is known, also by (13), that the optimal value is 0.

**TRANSLATE:** Para resolver este problema propone utilizar el método de promedios sucesivos para calcular, en cada iteración, un vector de flujos. Con esta solución actual se calculan el tiempo y la frecuencia, obteniendo así valores fijos. Luego se utiliza MSA para calcular la nueva solución. Se procede así hasta que  $G(v^n)$  es cercano a 0 (el valor óptimo). Como en cada iteración se van a tener tiempos y frecuencias fijos, lo que se debe hacer es calcular el hyperpath más corto. Esto puede hacerse con el método de Florian-Spiess o el de Cominetti-Cepeda (2001).

### Algoritmo

Sea  $\alpha_k \in (0, 1)$  tal que  $\alpha_k \rightarrow 0$  y  $\sum_{k=0}^{\infty} \alpha_k = \infty$

**Inicializar:** hallar  $v^0 \in V_0$  y hacer  $k = 0$ .

**While**  $G(v^k) > \epsilon$

- Calcular  $t_a = t_a(v^k)$  y  $f_a = f_a(v^k)$

- Calcular el hyperpath más corto para cada  $d \in D$

- Determinar los flujos inducidos  $\hat{v}_a^d$

- Actualizar  $v^{k+1} = (1 - \alpha_k)v^k + \alpha_k \hat{v}$

- Hacer  $k = k + 1$

**end**

**Parar:** Si  $v^k$  es una solución tal que  $G(v^k) < \epsilon$ , frenar.

Para obtener una solución inicial  $v^0$  se puede hacer una asignación todo o nada usando los hyperpaths más cortos obtenidos con  $t_a = t_a(0)$  y  $f_a = f_a(0)$ . Posible problema: si  $f_a(v^0) = 0$  para algunos arcos, entonces en la siguiente iteración el problema será infactible. Una forma de evitar esto es truncando las funciones de frecuencia efectiva:  $\tilde{f}_a(v) = \max\{f_a(v), \epsilon\}$ . De esta forma lo que se hace es asegurar que aunque el flujo sea mucho, siempre habrá un arco que puede llevarlo (por ejemplo, puede tratarse de arcos para peatones, lo cuales tienen capacidad infinita).

### 3. Demand estimation problem

Assuming that the model carefully represents the real dynamic of the passengers, then it is possible to use it to detect anomalies or changes in the demand data when the observed flow or frequencies are different from the computed one. Here we focus on correcting the given demand to comply with the observed frequencies. That is, given a nominal demand  $\bar{g}$  and observed (measured) frequencies  $\bar{f}$  we look for the demand  $g$  that minimizes

$$\begin{aligned} \min_{g,v} \quad & \sum_{a \in A_{obs}} (\bar{f}_a - f_a(v))^2 + \gamma \sum_{a \in A} (\bar{g}_a - g_a)^2 \\ \text{s.t.} \quad & v \in V(g), \\ & G(v, g) = 0. \end{aligned} \tag{5}$$

### 4. Numerical experiments

Cepeda et. al (13) proposed a small example (Section 4.1.1) to obtain the equilibrium assignment using their method. In our case, we assume that we have the real bus frequency data and the objective is to estimate the O-D matrix that induces these frequencies (or at least approximates them).

To find the minimizers in 5 we use the Nelder-Mead method (see (17)) as it is included in matlab through the command `fminsearch` of Matlab ((18)). For the solution of the simple example given in Cepeda et. al, we consider that the effective frequency of the express line is equal to 0.0107 and search the demand from node  $A$  to  $C$  that induce this frequency. We start the search in an initial point such that  $g_A^B = 10$ ,  $g_B^C = 10$  and  $g_A^C = 340$ , and we obtain that the demands that induce  $f_E = 0.0107$  are  $g_A^B = 10$ ,  $g_B^C = 10$  and  $g_A^C = 350.2686$ , which is very close to the demand used by the authors in their example.

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## 5. Transit assignment model

For solve this situation, Spiess and Florian (10) propose the concept of *attractive lines* and *strategy*. The main assumption is that each passenger will board the first vehicle that arrives at the stop and belongs to the attractive set. They propose an algorithm that allows obtaining the optimal strategy (the one that minimizes the total expected travel time) and the flow assignment on each line, proportionally to the line frequencies. A similar method for calculating the shortest hyperpath was presented by Cominetti and Correa (12), without ensuring that one is better than the other. The main limitation of these methods is that they don't consider congestion effects, and this is very important when networks with high demand are studied. In this sense, Spiess and Florian (10) propose to use a travel time function depending on the flows, but this reduces the congestion phenomenon to a comfort problem.

De Cea and Fernandez (11) began to consider the congestion effects formulating a model based on *transit routes* and using cost functions in which waiting times at stop increase when the flow inside the vehicle approach to its capacity. With these functions they formulated an equilibrium problem whose solution represents the route section flows, and with them, line flows can be obtained making an assignment proportional to its frequencies. Also, they introduce the concept of *effective frequencies*, which are decreasing functions depending on the flows generating higher waiting times at stops when the flow increases. They proposed two methods, considering nominal and effective frequencies. The problem is that cost functions were presented heuristically and also the algorithms could yield infeasible solutions. On the other hand, the problem resolution requires a high computational effort.

Nevertheless, they only prove the existence and present the conditions for the uniqueness of the solution, without proposing an algorithm that solves the problem.

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They proposed two methods, considering nominal and effective frequencies. The problem is that cost functions were presented heuristically and also the algorithms could yield infeasible solutions. On the other hand, the problem resolution requires a high computational effort.

Cominetti and Correa (12) formulated a transit equilibrium problem using effective frequencies functions that vanishes if the in-vehicle flow exceeds its capacity. Nevertheless, they only prove the existence and present the conditions for the uniqueness of the solution, without proposing an algorithm that solves the problem.

## 6. Transit assignment model

Following the notation in (15), let the transit network be represented by means of a directed graph  $G = (N, A)$ , where  $N$  is the node set and  $A$  the link set, each one with cardinality  $N_N$  and  $N_A$ . For some origin-destination pairs  $(i, d) \in W \subset N \times N$  there is a transport demand called  $g_i^d$ . For a node  $i$  we call  $E(i)$  the set of emerging links and  $I(i)$  the incoming link set. We also define the node-link incidence matrix  $A \in \mathbb{R}^{N_N \times N_A}$  where  $A_{ia} = 1$  iff  $a \in E(i)$ ,  $A_{ia} = -1$  iff  $a \in I(i)$  and otherwise zero. We call  $v_a^d$  the flow through link  $a$  with destination  $d$ .

For each destination  $d$  we define the set of feasible flow with destination  $d$  and the set of feasible flows as

$$V^d = \left\{ v^d \in \mathbb{R}_+^{N_A} : Av^d = g^d \right\}, \quad V = \left\{ v \in \mathbb{R}_+^{N_A} : v = \sum_d v^d, v^d \in V^d, \forall d \right\}.$$

To model the impact of the bus load on the frequency the function 7 is used.  
waiting time at a stop for a strategy

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given a transport network represented by a graph with nodes  $i$  and arcs  $a$  we have a transport demand matrix  $g$ . For a node  $i$  we call  $E(i)$  the set of emerging links and  $I(i)$  the incoming link set. We also define the node-link incidence matrix  $A \in \mathbb{R}^{N_N \times N_A}$  where  $A_{ia} = 1$  iff  $a \in E(i)$ ,  $A_{ia} = -1$  iff  $a \in I(i)$  and otherwise zero.

La idea general es que, en cada nodo, el pasajero decide por qué arco (saliente de este nodo) continuará su viaje. La decisión se basa en intentar minimizar su tiempo total de viaje. Por lo tanto en cada nodo debe resolverse un problema de líneas comunes, sólo que ahora se consideran frecuencias dependientes del flujo. Basándose en esto y en resultados teóricos presentados en el trabajo anterior, llegan a la conclusión de que los flujos en equilibrio  $v \in V_0^*$  son el mínimo global de la función:

$$\sum_{\substack{d \in D \\ i \neq d}} \left[ \sum_{a \in A_i^+} [t_a(v) + \tau_{j_a}^d(v)] v_a^d + \max_{a \in A_i^+} \frac{v_a^d}{f_a(v)} + \tau_i^d(d) \sum_{a \in A_i^+} v_a^d \right] \quad (6)$$

cuyo valor óptimo es 0.

$$f_a(v) \begin{cases} \mu \left[ 1 - \left( \frac{v_a}{\mu c - v_{a'} + v_a} \right)^\beta \right] & \text{si } v_{a'} < \mu c \\ 0 & \text{otro caso} \end{cases} \quad (7)$$

Para poder aplicar el modelo de asignación es necesario contar con un input fundamental: OD-matrix. This matrix is generally estimated by indirect methods because asking each potential traveller is not affordable. Mediciones the uso de algunas líneas a ciertas horas pueden usarse para actualizar la matrix OD, pero para esto es necesario saber cómo resolver el problema inverso de dado el flujo obtener la matriz OD. En el caso de la afectación de tráfico varios trabajos han estudiado el problema de sensibilidad de la solución respecto de la demanda, diseñando distintos métodos para resolver dicho problema inverso. En este trabajo proponemos utilizar la formulación de Codina, del equilibrio de tránsito como IV y estudiar la sensibilidad de la solución respecto de la demanda, con el objetivo de utilizar esta sensibilidad en el desarrollo de un algoritmo de estimación de la matrix OD.

La idea general es que, en cada nodo, el pasajero decide por qué arco (saliente de este nodo) continuará su viaje. La decisión se basa en intentar minimizar su tiempo total de viaje. Por lo tanto en cada nodo debe resolverse un problema de líneas comunes, sólo que ahora se consideran frecuencias dependientes del flujo. Basándose en esto y en resultados teóricos presentados en el trabajo anterior, llegan a la conclusión de que los flujos en equilibrio  $v \in V_0^*$  son el mínimo global de la función:

$$\sum_{\substack{d \in D \\ i \neq d}} \left[ \sum_{a \in A_i^+} [t_a(v) + \tau_{j_a}^d(v)] v_a^d + \max_{a \in A_i^+} \frac{v_a^d}{f_a(v)} + \tau_i^d(d) \sum_{a \in A_i^+} v_a^d \right] \quad (8)$$

cuyo valor óptimo es 0. Por lo tanto, para hallar los flujos de equilibrio se debe resolver el siguiente problema:

$$\min_{v \in V_0} G(v) \quad (9)$$

donde

$$G(v) \triangleq \sum_{d \in D} \left[ \sum_{a \in A} t_a(v) v_a^d + \sum_{i \neq d} \max_{a \in A_i^+} \frac{v_a^d}{f_a(v)} - \sum_{i \neq d} g_i^d \tau_i^d(v) \right] \quad (10)$$

Para resolver este problema propone utilizar el método de promedios sucesivos para calcular, en cada iteración, un vector de flujos. Con esta solución actual se calculan el tiempo y la frecuencia, obteniendo así valores fijos. Luego se utiliza MSA para calcular la nueva solución. Se procede así hasta que  $G(v^n)$  es cercano a 0 (el valor óptimo). Como en cada iteración se van a tener tiempos y frecuencias fijos, lo que se debe hacer es calcular el hyperpath más corto. Esto puede hacerse con el método de Florian-Spiess o el de Cominetti-Cepeda (2001).

### Algoritmo

Sea  $\alpha_k \in (0, 1)$  tal que  $\alpha_k \rightarrow 0$  y  $\sum_{k=0}^{\infty} \alpha_k = \infty$

*Inicializar:* hallar  $v^0 \in V_0$  y hacer  $k = 0$ .

**While**  $G(v^k) > \epsilon$

- Calcular  $t_a = t_a(v^k)$  y  $f_a = f_a(v^k)$
- Calcular el hyperpath más corto para cada  $d \in D$
- Determinar los flujos inducidos  $\hat{v}_a^d$
- Actualizar  $v^{k+1} = (1 - \alpha_k)v^k + \alpha_k \hat{v}$
- Hacer  $k = k + 1$

**end**

*Parar:* Si  $v^k$  es una solución tal que  $G(v^k) < \epsilon$ , frenar.

Para obtener una solución inicial  $v^0$  se puede hacer una asignación todo o nada usando los hyperpaths más cortos obtenidos con  $t_a = t_a(0)$  y  $f_a = f_a(0)$ . Posible problema: si  $f_a(v^0) = 0$  para algunos arcos, entonces en la siguiente iteración el problema será infactible. Una forma de evitar esto es truncando las funciones de frecuencia efectiva:  $\tilde{f}_a(v) = \max\{f_a(v), \epsilon\}$ . De esta forma lo que se hace es asegurar que aunque el flujo sea mucho, siempre habrá un arco que puede llevarlo (por ejemplo, puede tratarse de arcos para peatones, lo cuales tienen capacidad infinita).

En la práctica, consideran tiempos de viaje constantes  $t_a(v) = t_a$  y frecuencias efectivas:

$$f_a(v) \begin{cases} \mu \left[ 1 - \left( \frac{v_a}{\mu c - v_{a'} + v_a} \right)^\beta \right] & \text{si } v_{a'} < \mu c \\ 0 & \text{otro caso} \end{cases} \quad (11)$$

donde  $v_a = \sum_{d \in D} v_a^d$  representa al flujo abordando en la parada (sumado sobre todos los destinos) y  $v_{a'}$  es el flujo a bordo justo después de la parada ( $v_{a'} \geq v_a$ ). El parámetro  $\mu$  es la frecuencia nominal de la línea correspondiente y  $c$  es la capacidad física de los colectivos, con lo cual  $\mu c - v_{a'}$  es la capacidad residual esperada en la parada.

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