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Generalized gaussian bounds for discrete convolution powers

Jean-François COULOMBEL & Grégory FAYE*

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Abstract

We prove a uniform generalized gaussian bound for the powers of a discrete convolution operator in one space dimension. Our bound is derived under the assumption that the Fourier transform of the coefficients of the convolution operator is a trigonometric rational function, which generalizes previous results that were restricted to trigonometric polynomials. We also allow the modulus of the Fourier transform to attain its maximum at finitely many points over a period.

AMS classification: 42A85, 35K25, 60F99, 65M12.

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For $1 \leq q < +\infty$, we let $\ell^q(\mathbb{Z}; \mathbb{C})$ denote the Banach space of complex valued sequences indexed by \mathbb{Z} and such that the norm:

$$\|u\|_{\ell^q} := \left(\sum_{j \in \mathbb{Z}} |u_j|^q \right)^{1/q},$$

is finite. We also let $\ell^\infty(\mathbb{Z}; \mathbb{C})$ denote the Banach space of bounded complex valued sequences indexed by \mathbb{Z} and equipped with the norm:

$$\|u\|_{\ell^\infty} := \sup_{j \in \mathbb{Z}} |u_j|.$$

Throughout this article, we use the notation

$$\begin{aligned} \mathcal{U} &:= \{\zeta \in \mathbb{C}, |\zeta| > 1\}, & \mathbb{D} &:= \{\zeta \in \mathbb{C}, |\zeta| < 1\}, & \mathbb{S}^1 &:= \{\zeta \in \mathbb{C}, |\zeta| = 1\}, \\ \overline{\mathcal{U}} &:= \mathcal{U} \cup \mathbb{S}^1, & \overline{\mathbb{D}} &:= \mathbb{D} \cup \mathbb{S}^1. \end{aligned}$$

If w is a complex number, the notation $B_r(w)$ stands for the open ball in \mathbb{C} centered at w and with radius $r > 0$, that is $B_r(w) := \{z \in \mathbb{C} / |z - w| < r\}$.

The notation $\sigma(T)$ stands for the spectrum of a bounded operator T acting on a Banach space E .

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1 Introduction and main result

1.1 A (brief) reminder on Laurent operators

Let us recall a few facts about Laurent operators on $\ell^2(\mathbb{Z}; \mathbb{C})$. If $a \in \ell^1(\mathbb{Z}; \mathbb{C})$, we let L_a denote the so-called Laurent operator associated with the sequence a [TE05, Nik02], which is defined by:

$$L_a : (u_j)_{j \in \mathbb{Z}} \mapsto \left(\sum_{\ell \in \mathbb{Z}} a_\ell u_{j-\ell} \right)_{j \in \mathbb{Z}}, \quad (1)$$

whenever the defining formula (1) for the sequence $L_a u$ makes sense. In particular, Young's inequality shows that L_a acts boundedly on $\ell^q(\mathbb{Z}; \mathbb{C})$ for any $1 \leq q \leq +\infty$. The spectrum of L_a is also well-understood since the celebrated Wiener Theorem [New75] characterizes the invertible elements of $\ell^1(\mathbb{Z}; \mathbb{C})$ for the convolution product (and we have the morphism property $L_a \circ L_b = L_{a \star b}$, where \star denotes the convolution product on $\ell^1(\mathbb{Z}; \mathbb{C})$). Namely, the spectrum of L_a as an operator acting on $\ell^q(\mathbb{Z}; \mathbb{C})$ does not depend on q and is nothing but the image of the Fourier transform of the sequence a :

$$\sigma(L_a) = \left\{ \sum_{\ell \in \mathbb{Z}} a_\ell e^{i\ell\xi} / \xi \in \mathbb{R} \right\}.$$

Since a belongs to $\ell^1(\mathbb{Z}; \mathbb{C})$, its Fourier transform is continuous on \mathbb{R} (it actually belongs to the so-called Wiener algebra).

Following, among other works, [Tho65, DSC14], we are interested here in giving uniform pointwise bounds for the n -th iterated convolution product $a \star \dots \star a = a^{\star n}$ as the number n gets large. The fundamental result in [Tho65] characterizes, under suitable regularity assumptions on their Fourier transform, the elements $a \in \ell^1(\mathbb{Z}; \mathbb{C})$ such that the geometric sequence $(a^{\star n})_{n \in \mathbb{N}}$ is bounded in $\ell^1(\mathbb{Z}; \mathbb{C})$, see also [Hed66, Des08, DSC14] and references therein for further developments. The latter characterization is performed in [Tho65], see also [Des08], by deriving a suitable ‘‘algebraic’’ pointwise bound for the coefficient $(a^{\star n})_j$, and the latter bound is obtained by integrating by parts the Fourier transform of $a^{\star n}$. Refining and optimizing this approach, the algebraic bound in [Tho65] was turned in [DSC14] into a generalized gaussian bound thanks to a suitable contour deformation. The contours chosen in [DSC14] can go arbitrarily far away from the real line (where the Fourier transform of a is defined at first), which is the reason why the authors in [DSC14] assume a to have finite support (so that its Fourier transform extends as a holomorphic function of ξ in the whole complex plane).

Our goal in this article is to extend the results in [DSC14] in two directions: we first wish to consider sequences a with *infinite* support, since such sequences arise when one considers *implicit* discretizations of partial differential equations. We also wish to relax the (crucial) assumption made in [DSC14] that the modulus of the Fourier transform of a attains its maximum at only one point over each period (say, at 0, in the interval $[-\pi, \pi]$). When the modulus of the Fourier transform attains its maximum at more than one point over a period, the arguments in either [Tho65] or [DSC14], rely on partitions of unity which destroy the holomorphy of the Fourier transform. This is the reason why the bounds obtained in [DSC14] in that situation are only of ‘‘sub-gaussian’’ type (compare for instance [DSC14, Theorem 3.1] with [DSC14, Theorem 3.5]). The results in [DSC14] were refined and extended in [RSC15, RSC17] but the approach there does not seem to deal with situations where the modulus of the Fourier transform attains its maximum at several points.

Our approach is quite different from the one in [Tho65, Des08, DSC14] where the coefficient $(a^{\star n})_j$ is represented by an integral involving the Fourier transform of a (to the n -th power). We rather follow

here an approach which is usually referred to in the partial differential equation community as “spatial dynamics”, which amounts to representing $(a^{*n})_j$ in terms of the resolvent of the operator L_a . The link between the two comes from the so-called functional calculus [Con90, Chapter VII] which expresses the *temporal* Green’s function (here the coefficient $(a^{*n})_j$) in terms of the *spatial* Green’s function, which is the solution to the resolvent equation:

$$(zI - L_a)u = \delta, \quad z \notin \sigma(L_a),$$

where δ stands for the “discrete” Dirac mass ($\delta_j = 1$ if $j = 0$, and 0 otherwise). A detailed analysis of the spatial Green’s function with sharp holomorphic extensions and bounds is provided below under conditions that are similar to but less restrictive than those in [DSC14]. With this preliminary analysis at our disposal, the final argument relies on a suitable choice of contours in the defining expression of the temporal Green’s function. A fundamental contribution in this direction is [ZH98] (for parabolic partial differential equations) and we also refer to [God03] for an application of this method to the stability analysis of discrete shock profiles in the theory of hyperbolic conservation laws. We now make several assumptions and state our main result.

1.2 Assumptions

First, we consider two convolution¹ operators on \mathbb{Z} with finite support :

$$\forall \sigma = 0, 1, \quad \forall j \in \mathbb{Z}, \quad (Q_\sigma u)_j := \sum_{\ell=-r}^p a_{\ell,\sigma} u_{j+\ell}. \quad (2)$$

Both operators Q_0 and Q_1 act boundedly on any $\ell^q(\mathbb{Z}; \mathbb{C})$, $1 \leq q \leq +\infty$. Our main focus below is on the three cases $q = 1$, $q = 2$ and $q = +\infty$. The integers r, p in (2) define the common “stencil” of the operators Q_0, Q_1 .

Our first assumption is the following.

Assumption 1. *The operator Q_1 is an isomorphism on $\ell^2(\mathbb{Z}; \mathbb{C})$, that is:*

$$\forall \kappa \in \mathbb{S}^1, \quad \widehat{Q}_1(\kappa) := \sum_{\ell=-r}^p a_{\ell,1} \kappa^\ell \neq 0, \quad (3)$$

and it satisfies furthermore the index condition:

$$\frac{1}{2\pi \mathbf{i}} \int_{\mathbb{S}^1} \frac{\widehat{Q}'_1(\kappa)}{\widehat{Q}_1(\kappa)} d\kappa = 0. \quad (4)$$

The function \widehat{Q}_1 in (3) is referred to below as the symbol of the convolution operator Q_1 . We can similarly define the symbol \widehat{Q}_0 associated with Q_0 :

$$\forall \kappa \in \mathbb{S}^1, \quad \widehat{Q}_0(\kappa) := \sum_{\ell=-r}^p a_{\ell,0} \kappa^\ell.$$

¹Since this work is much inspired by the theory of partial differential equations and its numerical approximations, we rather stick to the standard notation from the literature in that field and thus consider Laurent operators written in the form $\sum_\ell a_\ell u_{j+\ell}$ rather than the more standard convolution type convention $\sum_\ell a_\ell u_{j-\ell}$. This is of little consequence, of course, since the two expressions are equivalent up to considering the sequence $(a_{-\ell})_{\ell \in \mathbb{Z}}$ instead of $(a_\ell)_{\ell \in \mathbb{Z}}$.

We are then interested in the operator $\mathcal{L} := Q_1^{-1} Q_0$ and more specifically in its powers \mathcal{L}^n as n becomes large. Since \mathcal{L} involves the inverse of Q_1 , we can always write \mathcal{L} as a Laurent operator L_ϕ , just as in [DSC14], but now, unlike [DSC14], the support of the sequence ϕ is infinite (unless Q_1 is a pure shift, that is, only one coefficient $a_{\ell,1}$ is nonzero, but in that case the index condition (4) makes Q_1 to be the identity operator).

A crucial role is played below by the so-called symbol (later on denoted F) of \mathcal{L} which is defined by:

$$\forall \kappa \in \mathbb{S}^1, \quad F(\kappa) := \frac{\widehat{Q}_0(\kappa)}{\widehat{Q}_1(\kappa)}. \quad (5)$$

The main difference between [DSC14] and the present work is that we allow $F(\exp(i\xi))$ to be a trigonometric rational function of ξ rather than just a trigonometric polynomial in ξ . In other words, we deal here with the class of sequences in ℓ^1 whose Fourier transform is a trigonometric rational function. The following assumption on F is inspired by the fundamental contribution [Tho65]. Because F is a holomorphic function of κ on an open neighborhood of the unit circle, the first part of Assumption 2 below is equivalent to assuming that $F(\kappa)$ does not have constant modulus on \mathbb{S}^1 . The second part of Assumption 2 gives a precise, and not necessarily automatic, description of the behavior of F near every point where $|F|$ achieves its maximum on the unit circle. As in [Tho65, DSC14], the maximum is normalized to be 1.

Assumption 2. *There exists a finite set of points $\{\underline{\kappa}_1, \dots, \underline{\kappa}_K\}$, $K \geq 1$, in \mathbb{S}^1 such that:*

$$\forall \kappa \in \mathbb{S}^1 \setminus \{\underline{\kappa}_1, \dots, \underline{\kappa}_K\}, \quad |F(\kappa)| < 1. \quad (6)$$

Moreover, for all index $k = 1, \dots, K$, $F(\underline{\kappa}_k)$ belongs to \mathbb{S}^1 and there exist a nonzero real number α_k , an even integer $2\mu_k \geq 2$ and a positive real number β_k such that:

$$\frac{F(\underline{\kappa}_k e^{i\xi})}{F(\underline{\kappa}_k)} = \exp\left(-i\alpha_k \xi - \beta_k \xi^{2\mu_k} + O(\xi^{2\mu_k+1})\right), \quad (7)$$

as ξ tends to 0.

The number α_k in (7) is necessarily real since $F(\kappa)$ belongs to $\overline{\mathbb{D}}$ for all $\kappa \in \mathbb{S}^1$. The fact that all real numbers α_k are nonzero is a major assumption that is fundamental in the description below of the so-called spatial Green's function. We shall deal with the case where at least one α_k vanishes in a future work. Examples of operators Q_0, Q_1 for which Assumptions 1 and 2 are satisfied are provided in the final Section of this article.

Since $F(\kappa)$ belongs to $\overline{\mathbb{D}}$ for all $\kappa \in \mathbb{S}^1$, the operator \mathcal{L} is a contraction on $\ell^2(\mathbb{Z}; \mathbb{C})$, that is:

$$\forall u \in \ell^2(\mathbb{Z}; \mathbb{C}), \quad \|\mathcal{L}u\|_{\ell^2} \leq \|u\|_{\ell^2},$$

since the ℓ^2 norms on both sides can be computed by the Parseval-Bessel identity. Of course, this implies that every power of \mathcal{L} is also a contraction on $\ell^2(\mathbb{Z}; \mathbb{C})$. In the field of numerical analysis, this property is referred to as ℓ^2 -stability, or strong stability [Str68, Tad86], for the “numerical scheme”:

$$\begin{cases} Q_1 u^{n+1} = Q_0 u^n, & n \in \mathbb{N}, \\ u^0 \in \ell^2(\mathbb{Z}). \end{cases}$$

Let us now define the quantities:

$$\forall z \in \mathbb{C}, \quad \forall \ell = -r, \dots, p, \quad \mathbb{A}_\ell(z) := z a_{\ell,1} - a_{\ell,0}. \quad (8)$$

The following assumption already appears in several works devoted to the stability analysis of numerical boundary conditions for discretized hyperbolic equations, see, e.g., [Kre68, Osh69, GKS72, GT81, Cou13] and references therein. Not only does it determine the minimal integers r and p in (2) (by prohibiting to add artificial zero coefficients), but it is also crucially used below to analyze the so-called resolvent equation (12). It might be relaxed though, but a more elaborate analysis would be required.

Assumption 3. *The functions \mathbb{A}_{-r} and \mathbb{A}_p defined in (8) do not vanish on $\overline{\mathcal{U}}$.*

Thanks to Assumption 3, we can define the following matrix:

$$\mathbb{M}(z) := \begin{bmatrix} -\frac{\mathbb{A}_{p-1}(z)}{\mathbb{A}_p(z)} & \cdots & \cdots & -\frac{\mathbb{A}_{-r}(z)}{\mathbb{A}_p(z)} \\ 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ 0 & 0 & 1 & 0 \end{bmatrix} \in \mathcal{M}_{p+r}(\mathbb{C}), \quad (9)$$

which is holomorphic on the set $\{z \in \mathbb{C} / |z| > \exp(-\eta)\}$ for some parameter $\eta > 0$ which only depends on the location of the root of \mathbb{A}_p (if it exists). A crucial observation is that the upper right coefficient of $\mathbb{M}(z)$ is always nonzero, because of Assumption 3 and up to restricting η , so the matrix $\mathbb{M}(z)$ is invertible for all relevant values of z . We shall repeatedly use the inverse matrix $\mathbb{M}(z)^{-1}$ in what follows.

The analysis in this article heavily relies on a precise description of the spectrum of $\mathbb{M}(z)$ as z runs through $\overline{\mathcal{U}}$ (and even sometimes slightly through \mathbb{D}). This description is given in Lemma 1 below, and uses the following two Assumptions.

Assumption 4. *Either Q_1 is the identity, or $a_{-r,1}$ and $a_{p,1}$ are nonzero.*

Assumption 5. *For all index $k = 1, \dots, K$, let us define $z_k := F(\kappa_k) \in \mathbb{S}^1$. Then for any $k = 1, \dots, K$, the set:*

$$\mathcal{I}_k := \left\{ \nu \in \{1, \dots, K\} / z_\nu = z_k \right\} \quad (10)$$

has either one or two elements. Furthermore, in case it has two elements, which we denote ν_1, ν_2 , then $\alpha_{\nu_1} \alpha_{\nu_2} < 0$. (Let us recall that the drift parameters α_k are given in Assumption 2.)

From now on, we always make Assumptions 1, 2, 3, 4 and 5. Our main result is an extension of [DSC14, Theorem 3.1]. It gives a uniform, generalized gaussian bound for the convolution coefficients of the powers \mathcal{L}^n . A precise statement is the following.

Theorem 1. *Let the operators Q_0, Q_1 in (2) satisfy Assumptions 1, 2, 3, 4 and 5. Then there exist two constants $C > 0$ and $c > 0$ such that the operator $\mathcal{L} = Q_1^{-1} Q_0$ satisfies the uniform generalized gaussian bound:*

$$\forall n \in \mathbb{N}^*, \quad \forall j \in \mathbb{Z}, \quad |(\mathcal{L}^n \delta)_j| \leq C \sum_{k=1}^K \frac{1}{n^{1/(2\mu_k)}} \exp \left(-c \left(\frac{|j - \alpha_k n|}{n^{1/(2\mu_k)}} \right)^{\frac{2\mu_k}{2\mu_k-1}} \right), \quad (11)$$

where δ denotes the discrete Dirac mass defined by $\delta_j = 1$ if $j = 0$ and $\delta_j = 0$ otherwise.

The rest of this article is organized as follows. In Section 2, we prove sharp bounds on the so-called spatial Green's function. This is where Assumptions 3, 4 and 5 are used. Then we use these preliminary bounds in Section 3 to obtain the uniform bound (11) for what we call the temporal Green's function. Examples and possible extensions are given in Section 4.

2 The spatial Green's function

The spectrum of \mathcal{L} as an operator on $\ell^2(\mathbb{Z}; \mathbb{C})$ is the parametrized curve $F(\mathbb{S}^1)$. We know from Assumption 2 that this curve touches the unit circle \mathbb{S}^1 at the points \underline{z}_k , $k = 1, \dots, K$, and that it is located inside the open unit disk \mathbb{D} otherwise. Hence the resolvent set of \mathcal{L} contains at least $\overline{\mathcal{U}} \setminus \{\underline{z}_1, \dots, \underline{z}_K\}$. For such values of z , we can thus define the sequence $G_z \in \ell^2(\mathbb{Z}; \mathbb{C})$ (the capital G letter stands for Green, as in Green's function) by the formula:

$$(zI - \mathcal{L})G_z = \delta, \quad (12)$$

where we recall that δ stands for the Dirac mass ($\delta_j = 1$ if $j = 0$ and $\delta_j = 0$ if $j \in \mathbb{Z} \setminus \{0\}$).

From the definition $\mathcal{L} = Q_1^{-1}Q_0$, the equation (12) can be equivalently rewritten:

$$(zQ_1 - Q_0)G_z = Q_1\delta,$$

and the definitions (2), (8) give the final form:

$$\forall j \in \mathbb{Z}, \quad \sum_{\ell=-r}^p \mathbb{A}_\ell(z)G_z(j+\ell) = (Q_1\delta)_j, \quad (13)$$

together with the integrability conditions at infinity $G_z \in \ell^2(\mathbb{Z}; \mathbb{C})$.

We introduce the augmented vectors:

$$\forall j \in \mathbb{Z}, \quad W_j := \begin{bmatrix} G_z(j+p-1) \\ \vdots \\ G_z(j-r) \end{bmatrix} \in \mathbb{C}^{p+r}, \quad \mathbf{e} := \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{C}^{p+r},$$

and rewrite equivalently (13) as:

$$\forall j \in \mathbb{Z}, \quad W_{j+1} - \mathbb{M}(z)W_j = \frac{(Q_1\delta)_j}{\mathbb{A}_p(z)}\mathbf{e}. \quad (14)$$

The construction and analysis of the solution to the recurrence relation (14) relies on the following spectral splitting lemma, which is originally due to Kreiss [Kre68] in the context of finite difference approximations.

Lemma 1 (Spectral splitting). *Let $z \in \overline{\mathcal{U}}$ and let the matrix $\mathbb{M}(z)$ be defined as in (9). Then for $z \in \overline{\mathcal{U}} \setminus \{\underline{z}_1, \dots, \underline{z}_K\}$, $\mathbb{M}(z)$ has:*

- no eigenvalue on \mathbb{S}^1 ,
- r eigenvalues in $\mathbb{D} \setminus \{0\}$,
- p eigenvalues in \mathcal{U} (eigenvalues are counted with multiplicity).

Let now $k \in \{1, \dots, K\}$ be such that the set \mathcal{I}_k in (10) is the singleton $\{k\}$. Then if $\alpha_k > 0$, the matrix $\mathbb{M}(\underline{z}_k)$ has $\underline{\kappa}_k$ as a simple eigenvalue, it has $r-1$ eigenvalues in \mathbb{D} and p eigenvalues in \mathcal{U} . If $\alpha_k < 0$, the matrix $\mathbb{M}(\underline{z}_k)$ has $\underline{\kappa}_k$ as a simple eigenvalue, it has r eigenvalues in \mathbb{D} and $p-1$ eigenvalues in \mathcal{U} .

Eventually, let now $k \in \{1, \dots, K\}$ be such that the set \mathcal{I}_k in (10) has two elements ν_1, ν_2 . Then the matrix $\mathbb{M}(\underline{z}_k)$ has $\underline{\kappa}_{\nu_1}$ and $\underline{\kappa}_{\nu_2}$ as simple eigenvalues, it has $r-1$ eigenvalues in \mathbb{D} and $p-1$ eigenvalues in \mathcal{U} .

The arguments are basically the same as in [Kre68] but we give them here for the sake of completeness.

Proof of Lemma 1. We first recall that the matrix $\mathbb{M}(z)$ is given by (9) and that it is invertible for all z satisfying $|z| > \exp(-\underline{\eta})$. Let us then observe that κ is an eigenvalue of $\mathbb{M}(z)$ if and only if z and κ satisfies the so-called dispersion relation:

$$\sum_{\ell=-r}^p \mathbb{A}_\ell(z) \kappa^\ell = 0,$$

and the definition (8) of the functions \mathbb{A}_ℓ yields the equivalent form:

$$\widehat{Q}_1(\kappa) z = \widehat{Q}_0(\kappa). \quad (15)$$

In particular, for any $z \in \overline{\mathcal{U}} \setminus \{z_1, \dots, z_K\}$, $\mathbb{M}(z)$ has no eigenvalue on the unit circle \mathbb{S}^1 for otherwise we would have $z = F(\kappa)$ for some $\kappa \in \mathbb{S}^1$ which is precluded by Assumption 2. To obtain the first statement of Lemma 1, it remains to count the number of eigenvalues of $\mathbb{M}(z)$ in \mathbb{D} , which is done by letting z tend to infinity. The argument, however, now depends on the form of the operator Q_1 (for this dictates the limit of $\mathbb{M}(z)$ at infinity).

The explicit case ($Q_1 = I$). The dispersion relation (15) then reduces to:

$$z = \sum_{\ell=-r}^p a_{\ell,0} \kappa^\ell. \quad (16)$$

Since the set $\overline{\mathcal{U}} \setminus \{z_1, \dots, z_K\}$ is connected, the number of eigenvalues of $\mathbb{M}(z)$ in \mathbb{D} is independent of z in that set (same for the number of eigenvalues in \mathcal{U}). If $r = 0$, then there are no eigenvalues in \mathbb{D} for any z for otherwise there would be at least one eigenvalue in \mathbb{D} for all z and the triangle inequality in (16) would imply:

$$|z| \leq \sum_{\ell=0}^p |a_{\ell,0}|.$$

which is impossible because z can be arbitrarily large. The result is thus proved in the case $r = 0$ so we assume $r \geq 1$ from now on. Following [Kre68] (see also [Cou13] for the complete details), the number of eigenvalues of $\mathbb{M}(z)$ in \mathbb{D} is computed by letting z tend to infinity for in that case, all such (stable) eigenvalues of $\mathbb{M}(z)$ collapse to zero (an eigenvalue in \mathbb{D} cannot remain uniformly away from the origin for otherwise the right hand side of (16) would remain bounded while the left hand side tends to infinity).

The final argument is the following. For any $z \in \overline{\mathcal{U}} \setminus \{z_1, \dots, z_K\}$, the eigenvalues of $\mathbb{M}(z)$ are those $\kappa \neq 0$ such that:

$$\kappa^r = \frac{1}{z} \sum_{\ell=-r}^p a_{\ell,0} \kappa^{r+\ell},$$

which is just an equivalent way of writing (16). Hence for z large, the small eigenvalues of $\mathbb{M}(z)$ behave at the leading order like the roots of the reduced equation:

$$\kappa^r = \frac{a_{-r,0}}{z},$$

and there are exactly r distinct roots close to 0 of that equation. Hence $\mathbb{M}(z)$ has r eigenvalues in \mathbb{D} for any $z \in \overline{\mathcal{U}} \setminus \{z_1, \dots, z_K\}$.

The implicit case ($Q_1 \neq I$). We know that $a_{-r,1} \neq 0$ and $a_{p,1} \neq 0$. Moreover, the function \widehat{Q}_1 satisfies the index condition (4). By the residue Theorem, this means that \widehat{Q}_1 has as many poles as roots in \mathbb{D} and since it only has a pole of order r at 0, we can conclude that \widehat{Q}_1 has r roots in \mathbb{D} . Since $\kappa^r \widehat{Q}_1(\kappa)$ is a polynomial of degree $p+r$, we can also conclude that \widehat{Q}_1 has p roots in \mathcal{U} .

From the definition (8), we compute:

$$\lim_{z \rightarrow \infty} \mathbb{M}(z) = \begin{bmatrix} -\frac{a_{p-1,1}}{a_{p,1}} & \cdots & \cdots & -\frac{a_{-r,1}}{a_{p,1}} \\ 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

and the eigenvalues of that (invertible) matrix are exactly those κ that satisfy $\widehat{Q}_1(\kappa) = 0$. Hence for any sufficiently large z , $\mathbb{M}(z)$ has r eigenvalues in \mathbb{D} and p eigenvalues in \mathcal{U} (which are close to the roots of \widehat{Q}_1). This completes the proof of the first statement in Lemma 1. It now remains to examine the situation at the points \underline{z}_k , $k = 1, \dots, K$.

Case I. We assume that the index $k \in \{1, \dots, K\}$ is such that the set \mathcal{J}_k in (10) is the singleton $\{k\}$ and we assume for now $\alpha_k > 0$ in (7). Since the eigenvalues of $\mathbb{M}(\underline{z}_k)$ are the roots of the dispersion relation:

$$\widehat{Q}_1(\kappa) \underline{z}_k = \widehat{Q}_0(\kappa),$$

we first observe that the only eigenvalue of $\mathbb{M}(\underline{z}_k)$ on \mathbb{S}^1 is $\underline{\kappa}_k$ and we are now going to show that this eigenvalue is algebraically (and therefore geometrically) simple. The relation (7) gives:

$$F'(\underline{\kappa}_k) = -\frac{\underline{z}_k \alpha_k}{\underline{\kappa}_k} \neq 0.$$

Moreover, the characteristic polynomial of $\mathbb{M}(z)$ at κ equals $z - F(\kappa)$ up to a nonvanishing holomorphic function of (z, κ) close to $(\underline{z}_k, \underline{\kappa}_k)$. This means that $\underline{\kappa}_k$ is an algebraically simple eigenvalue of $\mathbb{M}(\underline{z}_k)$ and can therefore be extended holomorphically with respect to z in a sufficiently small neighborhood of \underline{z}_k . We let $\kappa_k(z)$ denote this holomorphic extension, which satisfies $z = F(\kappa_k(z))$ for any z close to \underline{z}_k . Performing a Taylor expansion, we compute:

$$\frac{\kappa_k(\underline{z}_k(1+\varepsilon))}{\underline{\kappa}_k} = 1 - \frac{\varepsilon}{\alpha_k} + O(\varepsilon^2).$$

In particular, $\kappa_k(\underline{z}_k(1+\varepsilon))$ belongs to \mathbb{D} for $\varepsilon > 0$ small enough.

To conclude, we observe that the $p+r-1$ eigenvalues of $\mathbb{M}(\underline{z}_k)$ which differ from $\underline{\kappa}_k$ lie either in $\mathbb{D} \cup \mathcal{U}$. Those eigenvalues remain in $\mathbb{D} \cup \mathcal{U}$ as \underline{z}_k is perturbed into $\underline{z}_k(1+\varepsilon)$ for a sufficiently small $\varepsilon > 0$. Using the previous step of the analysis, we know that $\mathbb{M}(\underline{z}_k(1+\varepsilon))$ has r eigenvalues in \mathbb{D} and p eigenvalues in \mathcal{U} so the reader will easily get convinced that the only possible situation for the location of the eigenvalues of $\mathbb{M}(\underline{z}_k)$ is the one stated in Lemma 1.

Cases II and III. It remains to deal with the case where \mathcal{S}_k is the singleton $\{k\}$ and $\alpha_k < 0$ (Case II), and the final case where \mathcal{S}_k has two elements (Case III). The argument for Case II is the same as for Case I except that now the Taylor expansion of κ_k shows that $\kappa_k(\underline{z}_k(1+\varepsilon))$ belongs to \mathcal{U} for $\varepsilon > 0$ small enough. The remaining details for that case are easily filled in. For case III, $\mathbb{M}(\underline{z}_k)$ has two eigenvalues on \mathbb{S}^1 , which are, in our usual notation, $\underline{\kappa}_{\nu_1}$ and $\underline{\kappa}_{\nu_2}$. The same argument as in Case I or Case II shows that one of these eigenvalues moves into \mathbb{D} as z is perturbed from \underline{z}_k to $\underline{z}_k(1+\varepsilon)$, and the other eigenvalue moves into \mathcal{U} . This situation thus mixes Cases I and II. The conclusion follows and the proof of Lemma 1 is now complete. \square

The rest of this Section is devoted to the analysis of the solution to the recurrence relation (14), which we recall is an equivalent formulation of (13). For the sake of clarity, we split the analysis between the explicit and implicit cases since there is an extra ingredient that is required in Section 3 for dealing with the implicit case. The tools, however, are exactly the same in either case.

2.1 The explicit case

We start with the following result.

Lemma 2 (Bounds away from the tangency points). *Let $\underline{z} \in \overline{\mathcal{U}} \setminus \{\underline{z}_1, \dots, \underline{z}_K\}$. Then there exists an open ball $B_\delta(\underline{z})$ centered at \underline{z} and there exist two constants $C > 0$, $c > 0$ such that:*

$$\forall z \in B_\delta(\underline{z}), \quad \forall j \in \mathbb{Z}, \quad |G_z(j)| \leq C \exp(-c|j|).$$

Proof. We first introduce some notation. For $z \in \overline{\mathcal{U}} \setminus \{\underline{z}_1, \dots, \underline{z}_K\}$, Lemma 1 shows that the so-called stable subspace, which is spanned by the generalized eigenvectors of $\mathbb{M}(z)$ associated with eigenvalues in \mathbb{D} , has constant dimension r . Similarly, the unstable subspace, which is spanned by the generalized eigenvectors of $\mathbb{M}(z)$ associated with eigenvalues in \mathcal{U} , has constant dimension p . We let $\mathbb{E}^s(z)$, resp. $\mathbb{E}^u(z)$, denote the stable, resp. unstable, subspace of $\mathbb{M}(z)$ for $z \in \overline{\mathcal{U}} \setminus \{\underline{z}_1, \dots, \underline{z}_K\}$. These spaces depend holomorphically on z thanks to Dunford's formula for the eigenprojectors.

Let now $\underline{z} \in \overline{\mathcal{U}} \setminus \{\underline{z}_1, \dots, \underline{z}_K\}$ be fixed. We first know that $\mathbb{M}(z)$ in (9) is well-defined and holomorphic in a sufficiently small neighborhood of \underline{z} (including in the case where \underline{z} belongs to \mathbb{S}^1). Moreover, because of the spectral splitting shown in Lemma 1, the matrix $\mathbb{M}(z)$ has no eigenvalue on \mathbb{S}^1 for z close to \underline{z} , and it has r , resp. p , eigenvalues in \mathbb{D} , resp. \mathcal{U} , for z close to \underline{z} . Hence $\mathbb{E}^s(z)$ and $\mathbb{E}^u(z)$ depend holomorphically on z close to \underline{z} and we have the decomposition:

$$\forall z \in B_\delta(\underline{z}), \quad \mathbb{C}^{p+r} = \mathbb{E}^s(z) \oplus \mathbb{E}^u(z),$$

for some sufficiently small radius $\delta > 0$. The associated projectors are denoted $\pi^s(z)$ and $\pi^u(z)$. The dynamics of (14) therefore is of hyperbolic type for any $z \in B_\delta(\underline{z})$.

Up to restricting δ , any complex number z in the open ball $B_\delta(\underline{z})$ lies in the resolvent set of the operator \mathcal{L} , hence there exists a unique solution $(W_j)_{j \in \mathbb{Z}} \in \ell^2(\mathbb{Z}; \mathbb{C})$ to (14). Since the dynamics of the iteration (14) for such z enjoys a hyperbolic dichotomy, the solution to (14) is given by integrating either from $+\infty$ to j , or from $-\infty$ to j , depending on whether we compute the stable or unstable components of the vector W_j . This leads to the expression:

$$\forall j \in \mathbb{Z}, \quad \pi^u(z) W_j = -\frac{1}{\mathbb{A}_p(z)} \sum_{\ell \geq 0} (Q_1 \boldsymbol{\delta})_{j+\ell} \mathbb{M}(z)^{-1-\ell} \pi^u(z) \mathbf{e}, \quad (17)$$

for the unstable components, and to the expression:

$$\forall j \in \mathbb{Z}, \quad \pi^s(z) W_j = \frac{1}{\mathbb{A}_p(z)} \sum_{\ell=-\infty}^j (Q_1 \boldsymbol{\delta})_\ell \mathbb{M}(z)^{j-1-\ell} \pi^s(z) \mathbf{e}, \quad (18)$$

for the stable components.

At this stage, we observe that the sequence $Q_1 \boldsymbol{\delta}$ only has finitely many nonzero coefficients, which are given by:

$$\forall j \in \mathbb{Z}, \quad (Q_1 \boldsymbol{\delta})_j = \begin{cases} a_{-j,1}, & \text{if } j \in \{-p, \dots, r\}, \\ 0, & \text{otherwise.} \end{cases}$$

Hence we see from (17) that $\pi^u(z) W_j$ vanishes for $j \geq r+1$, and we see from (18) that $\pi^s(z) W_j$ vanishes for $j \leq -p-1$. For $j \leq r$, we get:

$$\pi^u(z) W_j = -\frac{1}{\mathbb{A}_p(z)} \sum_{\ell=\max(-p-j,0)}^{r-j} a_{-j-\ell,1} \mathbb{M}(z)^{-1-\ell} \pi^u(z) \mathbf{e},$$

and since the sequence $(\mathbb{M}(z)^{-\ell} \pi^u(z))_{\ell \geq 1}$ is exponentially decreasing, uniformly with respect to $z \in B_\delta(\underline{z})$, we get the uniform bound²:

$$\forall z \in B_\delta(\underline{z}), \quad \forall j \in \mathbb{Z}, \quad \left| \pi^u(z) W_j \right| \leq \begin{cases} 0, & \text{if } j \geq r+1, \\ C \exp(-c|j|), & \text{if } j \leq r. \end{cases} \quad (19)$$

Similar arguments, using the uniform exponential decay of the sequence $(\mathbb{M}(z)^\ell \pi^s(z))_{\ell \geq 1}$, yield the bound:

$$\forall z \in B_\delta(\underline{z}), \quad \forall j \geq 1, \quad \left| \pi^s(z) W_j \right| \leq \begin{cases} 0, & \text{if } j \leq -p-1, \\ C \exp(-c|j|), & \text{if } j \geq -p. \end{cases} \quad (20)$$

Adding (19) and (20) gives the claim of Lemma 2 since the spatial Green's function $G_z(j)$ is just one coordinate of the vector $W_j \in \mathbb{C}^{p+r}$. \square

We are now going to examine the behavior of the spatial Green's function G_z close to any of the points \underline{z}_k , $k = 1, \dots, K$, where the spectrum of \mathcal{L} is tangent to the unit circle. Let us first recall that the exterior \mathcal{U} of the unit disk belongs to the resolvent set of \mathcal{L} hence the spatial Green's function G_z is well-defined in the "half-ball" $B_\delta(\underline{z}_k) \cap \mathcal{U}$ for any radius $\delta > 0$. Our goal below is to extend holomorphically G_z to a whole neighborhood of \underline{z}_k for each k , which amounts to passing through the essential spectrum of \mathcal{L} . Our results are the following two Lemmas.

Lemma 3 (Bounds close to the tangency points - cases I and II). *Let $k \in \{1, \dots, K\}$ be such that the set \mathcal{S}_k in (10) is the singleton $\{k\}$. Then there exists an open ball $B_\varepsilon(\underline{z}_k)$ and there exist two constants $C > 0$ and $c > 0$ such that, for any integer $j \in \mathbb{Z}$, the component $G_z(j)$ defined on $B_\varepsilon(\underline{z}_k) \cap \mathcal{U}$ extends holomorphically to the whole ball $B_\varepsilon(\underline{z}_k)$ with respect to z , and the holomorphic extension satisfies the bound:*

$$\forall z \in B_\varepsilon(\underline{z}_k), \quad \forall j \in \mathbb{Z}, \quad \left| G_z(j) \right| \leq \begin{cases} C \exp(-c|j|), & \text{if } j \leq 0, \\ C \left| \kappa_k(z) \right|^j, & \text{if } j \geq 1, \end{cases} \quad \text{if } \alpha_k > 0 \quad (\text{case I}),$$

²Here we also use Assumption 3 to get a uniform local bound for $\mathbb{A}_p(z)^{-1}$, including in the case $\underline{z} \in \mathbb{S}^1$ for which z can come inside the unit disk.

and

$$\forall z \in B_\varepsilon(\underline{z}_k), \quad \forall j \in \mathbb{Z}, \quad |G_z(j)| \leq \begin{cases} C |\kappa_k(z)|^{|j|}, & \text{if } j \leq 0, \\ C \exp(-cj), & \text{if } j \geq 1, \end{cases} \quad \text{if } \alpha_k < 0 \quad (\text{case II}),$$

where, in either case, $\kappa_k(z)$ denotes the (unique) holomorphic eigenvalue of $\mathbb{M}(z)$ defined close to \underline{z}_k and that satisfies $\kappa_k(\underline{z}_k) = \underline{\kappa}_k$.

Lemma 4 (Bounds close to the tangency points - case III). *Let now $k \in \{1, \dots, K\}$ be such that the set \mathcal{I}_k in (10) has two elements $\{\nu_1, \nu_2\}$ which are fixed by the convention $\alpha_{\nu_1} < 0 < \alpha_{\nu_2}$. Then there exists an open ball $B_\varepsilon(\underline{z}_k)$ centered at \underline{z}_k and there exist two constants $C > 0$ and $c > 0$ such that, for any integer $j \in \mathbb{Z}$, the component $G_z(j)$ defined on $B_\varepsilon(\underline{z}_k) \cap \mathcal{U}$ extends holomorphically to the whole ball $B_\varepsilon(\underline{z}_k)$ with respect to z , and the holomorphic extension satisfies the bound:*

$$\forall z \in B_\varepsilon(\underline{z}_k), \quad \forall j \in \mathbb{Z}, \quad |G_z(j)| \leq \begin{cases} C |\kappa_{\nu_1}(z)|^{|j|}, & \text{if } j \leq 0, \\ C |\kappa_{\nu_2}(z)|^j, & \text{if } j \geq 1, \end{cases} \quad (\text{case III}),$$

where $\kappa_{\nu_1}(z)$, resp. $\kappa_{\nu_2}(z)$, denotes the (unique) holomorphic eigenvalue of $\mathbb{M}(z)$ defined close to \underline{z}_k and that satisfies $\kappa_{\nu_1}(\underline{z}_k) = \underline{\kappa}_{\nu_1}$, resp. $\kappa_{\nu_2}(\underline{z}_k) = \underline{\kappa}_{\nu_2}$.

The proofs of Lemma 3 and Lemma 4 are mostly identical so we just give the proof of Lemma 3 and indicate the minor refinements for the proof of Lemma 4.

Proof of Lemma 3. Most ingredients of the proof are similar to what we have already done in the proof of Lemma 2. We assume from now on $\alpha_k > 0$, the case $\alpha_k < 0$ being left to the interested reader. We just need to slightly adapt the notation used in the proof of Lemma 2 since the hyperbolic dichotomy of $\mathbb{M}(z)$ does not hold any longer in a whole neighborhood of \underline{z}_k . Since $\underline{\kappa}_k$ is a simple eigenvalue of $\mathbb{M}(\underline{z}_k)$, we can extend it holomorphically to a simple eigenvalue $\kappa_k(z)$ of $\mathbb{M}(z)$ in a neighborhood of \underline{z}_k . This eigenvalue is associated with the eigenvector:

$$E_k(z) := \begin{bmatrix} 1 \\ \kappa_k(z) \\ \vdots \\ \kappa_k(z)^{p+r-1} \end{bmatrix} \in \mathbb{C}^{p+r},$$

which also depends holomorphically on z in a neighborhood of \underline{z}_k . The vector $E_k(z)$ contributes to the stable subspace of $\mathbb{M}(z)$ for $z \in \mathcal{U}$ close to \underline{z}_k but the situation is unclear as z goes inside \mathbb{D} (it actually depends on the position of z with respect to the spectrum of \mathcal{L}). The remaining $p+r-1$ eigenvalues of $\mathbb{M}(z)$ enjoy the now familiar hyperbolic dichotomy, uniformly with respect to z close to \underline{z}_k . We let below $\mathbb{E}^{ss}(z)$, resp. $\mathbb{E}^u(z)$, denote the strongly stable, resp. unstable, subspace of $\mathbb{M}(z)$ associated with those eigenvalues that remain uniformly inside \mathbb{D} , resp. \mathcal{U} , as z belongs to a neighborhood of \underline{z}_k . In particular, $\mathbb{E}^{ss}(z)$, resp. $\mathbb{E}^u(z)$, has dimension $r-1$, resp. p , thanks to Lemma 1, and we have the decomposition:

$$\forall z \in B_\varepsilon(\underline{z}_k), \quad \mathbb{C}^{p+r} = \mathbb{E}^{ss}(z) \oplus \text{Span } E_k(z) \oplus \mathbb{E}^u(z), \quad (21)$$

for a sufficiently small radius $\varepsilon > 0$ sufficiently small. We let below $\pi^{ss}(z)$, $\pi_k(z)$ and $\pi^u(z)$ denote the projectors associated with the decomposition (21).

We first consider a point $z \in B_\varepsilon(\underline{z}_k) \cap \mathcal{U}$ so that the decomposition (21) holds and the Green's function $G_z \in \ell^2(\mathbb{Z}; \mathbb{C})$ is well-defined as the only solution to (13). We use the equivalent formulation (14) and derive the following expressions that are entirely similar to those found in the proof of Lemma 2:

$$\forall j \in \mathbb{Z}, \quad \pi^u(z) W_j = -\frac{1}{\mathbb{A}_p(z)} \sum_{\ell \geq 0} (Q_1 \boldsymbol{\delta})_{j+\ell} \mathbb{M}(z)^{-1-\ell} \pi^u(z) \mathbf{e}, \quad (22a)$$

$$\pi^{ss}(z) W_j = \frac{1}{\mathbb{A}_p(z)} \sum_{\ell=-\infty}^j (Q_1 \boldsymbol{\delta})_\ell \mathbb{M}(z)^{j-1-\ell} \pi^{ss}(z) \mathbf{e}, \quad (22b)$$

$$\pi_k(z) W_j = \frac{1}{\mathbb{A}_p(z)} \sum_{\ell=-\infty}^j (Q_1 \boldsymbol{\delta})_\ell \kappa_k(z)^{j-1-\ell} \pi_k(z) \mathbf{e}. \quad (22c)$$

The strongly stable ($\pi^{ss}(z)W_j$) and unstable ($\pi^u(z)W_j$) components obviously extend holomorphically to the whole neighborhood $B_\varepsilon(\underline{z}_k)$ of \underline{z}_k since the projectors $\pi^{ss}(z)$ and $\pi^u(z)$ depend holomorphically on z on that set and the sums on the right hand side of (22a) and (22b) are at most finite. Furthermore, by using the same type of bounds as in the proof of Lemma 2, we obtain:

$$\forall z \in B_\varepsilon(\underline{z}_k), \quad \forall j \in \mathbb{Z}, \quad |\pi^u(z) W_j + \pi^{ss}(z) W_j| \leq C \exp(-c|j|), \quad (23)$$

for some appropriate constants $C > 0$ and $c > 0$. We now focus on the vector $\pi_k(z)W_j$ in (22c) which is aligned with the eigenvector $E_k(z)$. We see from (22c) that $\pi_k(z)W_j$ vanishes for $j \leq -p-1$. For j in the finite set $\{-p, \dots, r\}$, we have:

$$\pi_k(z) W_j = \frac{1}{\mathbb{A}_p(z)} \sum_{\ell=-p}^j a_{-\ell,1} \kappa_k(z)^{j-1-\ell} \pi_k(z) \mathbf{e},$$

and for $j \geq r+1$, we have:

$$\pi_k(z) W_j = \frac{1}{\mathbb{A}_p(z)} \sum_{\ell=-p}^r a_{-\ell,1} \kappa_k(z)^{j-1-\ell} \pi_k(z) \mathbf{e}.$$

In either case, we see that the component $\pi_k(z)W_j$ extends holomorphically to the whole neighborhood $B_\varepsilon(\underline{z}_k)$ of \underline{z}_k and we have a bound of the form³:

$$\forall z \in B_\varepsilon(\underline{z}_k), \quad \forall j \in \mathbb{Z}, \quad |\pi_k(z) W_j| \leq \begin{cases} 0, & \text{if } j \leq -p-1, \\ C |\kappa_k(z)|^j, & \text{if } j \geq -p. \end{cases} \quad (24)$$

In order to conclude, we can always assume that the ball $B_\varepsilon(\underline{z}_k)$ is so small that the modulus $|\kappa_k(z)|$ belongs to the interval⁴ $[\exp(-c), \exp(c)]$ (it equals 1 at \underline{z}_k). It then remains to add the bounds in (23) and (24) and to compare which is the largest. This completes the proof of Lemma 3 in the case $\alpha_k > 0$. The remaining case II ($\alpha_k < 0$) is dealt with similarly except that now the eigenvector $E_k(z)$ contributes to the unstable subspace of $\mathbb{M}(z)$ for $|z| > 1$. The minor modifications are left to the reader. \square

³Here we use again that $\mathbb{A}_p(z)$ does not vanish in the ball $B_\varepsilon(\underline{z}_k)$ up to restricting the radius ε .

⁴The constant c here refers to the same one as in (23).

The proof of Lemma 4 is a mixture between cases I and II in which now $\mathbb{M}(z)$ has two (holomorphic) eigenvalues whose modulus equals 1 at z_k . One contributes to the stable subspace of $\mathbb{M}(z)$ and the other one contributes to the unstable subspace of $\mathbb{M}(z)$ for $|z| > 1$. The same ingredients as in the proof of Lemma 3 can then be applied with minor modifications.

Let us observe that we can extend holomorphically each scalar component $G_z(j)$ to a neighborhood $B_\varepsilon(z_k)$ of z_k , but that does not necessarily mean that the extended sequence G_z lies in $\ell^2(\mathbb{Z}; \mathbb{C})$. For instance, in case I of Lemma 3, the eigenvalue $\kappa_k(z)$ contributes to the stable subspace of $\mathbb{M}(z)$ for $|z| > 1$ but it starts contributing to the unstable subspace of $\mathbb{M}(z)$ as z crosses the spectrum of \mathcal{L} (which coincides with the curve $F(\mathbb{S}^1)$). Hence the holomorphic extension G_z ceases to be in ℓ^2 as z crosses the spectrum of \mathcal{L} for it then has an exponentially growing mode in j at $+\infty$.

We end this paragraph with the following Corollary, which is a direct consequence of Lemma 2, Lemma 3 and Lemma 4 above by applying a standard compactness argument.

Corollary 1. *There exist a radius $\varepsilon > 0$, some width $\eta_\varepsilon > 0$ and two constants $C > 0$, $c > 0$ such that, for all z in the set:*

$$\left\{ \zeta \in \mathbb{C} / e^{-\eta_\varepsilon} < |\zeta| \leq e^\pi \right\} \setminus \bigcup_{k=1}^K B_\varepsilon(z_k),$$

the Green's function $G_z \in \ell^2(\mathbb{Z}; \mathbb{C})$ solution to (13) satisfies the pointwise bound:

$$\forall j \in \mathbb{Z}, \quad |G_z(j)| \leq C \exp(-c|j|).$$

Moreover, for z inside the ball $B_\varepsilon(z_k)$, the Green's function component $G_z(j)$ depends holomorphically on z and satisfies one of the bounds in Lemma 3 or 4 depending on the cardinal of \mathcal{I}_k and the sign of α_k .

2.2 The implicit case

All the claims in Lemma 2, Lemma 3 and Lemma 4 remain valid in the implicit case since they only rely on Lemma 1 which does not distinguish between the explicit and implicit case. The only refinement we incorporate in the implicit case is the following result, which deals with large values of z .

Lemma 5 (Bounds at infinity). *There exist a radius $R \geq 2$ and two constants $C > 0$, $c > 0$ such that there holds:*

$$\forall z \notin B_R(0), \quad \forall j \in \mathbb{Z}, \quad |G_z(j)| \leq C \exp(-c|j|).$$

Proof. The proof is basically the same as that of Lemma 2. Indeed, we recall that in the implicit case, the matrix $\mathbb{M}(z)$ has a limit at infinity, given by:

$$\begin{bmatrix} -\frac{a_{p-1,1}}{a_{p,1}} & \dots & \dots & -\frac{a_{-r,1}}{a_{p,1}} \\ 1 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

and this matrix has a hyperbolic dichotomy because of Assumption 1. We can thus apply the same argument as in the proof of Lemma 2 for z in a neighborhood of infinity. In particular, we can use the fact that the sequences $(\mathbb{M}(z)^{-\ell} \pi^u(z))_{\ell \geq 1}$ and $(\mathbb{M}(z)^\ell \pi^s(z))_{\ell \geq 1}$ are exponentially decreasing, uniformly with respect to z in a neighborhood $B_R(0)^c$ of infinity. The conclusion of Lemma 5 follows. \square

The following result is the analogue of Corollary 1 and shows that in the implicit case, one can obtain a uniform exponential bound for the Green's function as z gets large. The proof combines Corollary 1 and Lemma 5.

Corollary 2. *There exist a radius $\varepsilon > 0$, some width $\eta_\varepsilon > 0$ and two constants $C > 0$, $c > 0$ such that, for all z in the set:*

$$\left\{ \zeta \in \mathbb{C} / e^{-\eta_\varepsilon} < |\zeta| \right\} \setminus \bigcup_{k=1}^K B_\varepsilon(\underline{z}_k),$$

the Green's function $G_z \in \ell^2(\mathbb{Z}; \mathbb{C})$ solution to (13) satisfies the pointwise bound:

$$\forall j \in \mathbb{Z}, \quad |G_z(j)| \leq C \exp(-c|j|).$$

Moreover, for z inside the ball $B_\varepsilon(\underline{z}_k)$, the Green's function component $G_z(j)$ depends holomorphically on z and satisfies one of the bounds in Lemma 3 or 4 depending on the cardinal of \mathcal{I}_k and the sign of α_k .

3 The temporal Green's function

The starting point of the analysis is to use the inverse Laplace transform formula to express the so-called Green's function $\mathcal{G}^n := \mathcal{L}^n \delta$ as the following contour integral

$$\forall n \in \mathbb{N}^*, \quad \forall j \in \mathbb{Z}, \quad \mathcal{G}_j^n = (\mathcal{L}^n \delta)_j = \frac{1}{2\pi \mathbf{i}} \int_{\tilde{\Gamma}} z^n G_z(j) dz, \quad (25)$$

where $\tilde{\Gamma}$ is a closed curve in the complex plane surrounding the unit disk \mathbb{D} lying in the resolvent set of \mathcal{L} . The idea will be to deform $\tilde{\Gamma}$ in order to obtain sharp pointwise estimates on the temporal Green's function using our pointwise estimates on the spatial Green's function given in Lemma 2, Lemma 3, Lemma 4 and Lemma 5 above. To do so, we first change variable in (25), by setting $z = \exp(\tau)$, such that we get

$$\mathcal{G}_j^n = \frac{1}{2\pi \mathbf{i}} \int_{\Gamma} e^{n\tau} \mathbf{G}_\tau(j) d\tau, \quad (26)$$

where without loss of generality $\Gamma = \{s + \mathbf{i}\ell \mid \ell \in [-\pi, \pi]\}$ for some $s > 0$ (and actually *any* $s > 0$ thanks to Cauchy's formula), and $\mathbf{G}_\tau(j)$ is given by

$$\forall j \in \mathbb{Z}, \quad \mathbf{G}_\tau(j) := G_{e^\tau}(j) e^\tau.$$

The remaining of this Section is devoted to the proof of Theorem 1. For the sake of clarity, we first treat the explicit case with $K = 1$, and then deal with the implicit case still with $K = 1$. And finally, we explain how the results generalize to $K > 1$ in both cases. (Let us recall that K denotes the number of tangency points of the spectrum of \mathcal{L} with the unit circle \mathbb{S}^1).

3.1 The explicit case with $K = 1$

We first remark that, since \mathcal{L} is a recurrence operator with finite stencil, for each $n \geq 1$, there holds

$$\mathcal{G}_j^n = 0, \quad \text{for } j > rn \text{ or } j < -pn.$$

As a consequence, throughout this section, we assume that j and n satisfy

$$n \geq 1, \quad -pn \leq j \leq rn.$$

We also assume without loss of generality that $\kappa_1 = \underline{z}_1 = 1$ together with $\alpha_1 > 0$ (the case $\alpha_1 < 0$ being handled similarly). In that case, we have from (7) that

$$F(e^{i\xi}) = \exp\left(-\mathbf{i}\alpha\xi - \beta\xi^{2\mu} + O(\xi^{2\mu+1})\right) \text{ as } \xi \rightarrow 0, \quad (27)$$

where we dropped the index to simplify our notations. Now, using Lemma 3, bounds close to the tangency point $z = 1$ for $G_z(j)$ translate into bounds near the origin $\tau = 0$ for $\mathbf{G}_\tau(j)$. More precisely, we have the following Lemma which combines Lemma 3 and Corollary 1.

Lemma 6. *There exist an open ball $B_\varepsilon(0)$, some width $\eta_\varepsilon > 0$, and two constants, still denoted $C > 0$, $c > 0$, such that, for any integer $j \in \mathbb{Z}$, the component $\mathbf{G}_\tau(j)$ extends holomorphically on $B_\varepsilon(0)$ with bounds:*

$$\forall \tau \in B_\varepsilon(0), \quad \forall j \in \mathbb{Z}, \quad |\mathbf{G}_\tau(j)| \leq \begin{cases} C \exp(-c|j|), & \text{if } j \leq 0, \\ C e^{j \operatorname{Re}(\varpi(\tau))}, & \text{if } j \geq 1, \end{cases} \quad (28)$$

where ϖ is holomorphic on $B_\varepsilon(0)$ and has the Taylor expansion:

$$\varpi(\tau) = -\frac{1}{\alpha}\tau + (-1)^{\mu+1} \frac{\beta}{\alpha^{2\mu+1}} \tau^{2\mu} + O(|\tau|^{2\mu+1}), \quad \forall \tau \in B_\varepsilon(0).$$

Furthermore, we have

$$\forall \tau \in \Omega_\varepsilon := \{-\eta_\varepsilon < \operatorname{Re}(\tau) \leq \pi\} \setminus B_\varepsilon(0), \quad \forall j \in \mathbb{Z}, \quad |\mathbf{G}_\tau(j)| \leq C \exp(-c|j|). \quad (29)$$

Proof. The proof simply relies on writing $\kappa(z) = \exp(\omega(z))$ near $z = 1$ and using $z = \exp(\tau)$, such that after identification we have $\varpi(\tau) := \omega(\exp(\tau))$. Next, using (27), we obtain the desired expansion for $\varpi(\tau)$ near $\tau = 0$. \square

Using Lemma 6, there exists $0 < \eta < \eta_\varepsilon$ such that the segment $\{-\eta + \mathbf{i}\ell \mid \ell \in [-\pi, \pi]\}$ intersects the boundary $\partial B_\varepsilon(0)$ outside the curve of essential spectrum $\sigma(\mathcal{L})$ near the origin. And, we define Ω to be the following closed region in the complex plane

$$\Omega := \{\rho + \mathbf{i}\ell \mid \rho \in [-\eta, \pi], \ell \in [-\pi, \pi]\} \setminus B_\varepsilon(0),$$

for which estimate (29) still holds in this region. We will distinguish two cases:

- (a) $-np \leq j \leq 0$;
- (b) $1 \leq j \leq nr$.

Indeed, in the first case (a), we remark that as $j \leq 0$, our estimates (28)-(29) from Lemma 6 can be combined to

$$\forall \tau \in \Omega_\varepsilon \cup B_\varepsilon(0), \quad \forall j \leq 0, \quad |\mathbf{G}_\tau(j)| \leq C e^{-c|j|},$$

from which we readily obtain the following estimate, using the contour $\Gamma = \{-\eta + \mathbf{i}\ell \mid \ell \in [-\pi, \pi]\} \subset \Omega \cup B_\varepsilon(0)$ in (26). Modifying the contour in (26) is legitimate thanks to Cauchy's formula and also because the integrals on the segments $\{-v \pm \mathbf{i}\pi \mid v \in [-\eta, \pi]\}$ compensate one another.

Lemma 7. *There exist $C > 0$ and $c > 0$ such that for all $-np \leq j \leq 0$ with $n \geq 1$, there holds*

$$|\mathcal{G}_j^n| \leq C e^{-n\eta - c|j|}.$$

From now on, we assume that $1 \leq j \leq nr$. It turns out that we will need again to divide the analysis in two pieces. We will consider first the medium range where $1 \leq j \leq n\delta$ where $\delta := \frac{\alpha}{2}$. In that case we can prove the following Lemma.

Lemma 8. *There exists $C > 0$ such that for $n \geq 1$ and $1 \leq j \leq n\delta$, the following estimate holds:*

$$|\mathcal{G}_j^n| \leq C e^{-n \frac{\eta}{4}}.$$

Proof. We use again the segment $\Gamma = \{-\eta + \mathbf{i}\ell \mid \ell \in [-\pi, \pi]\} \subset \Omega \cup B_\epsilon(0)$ in (26). We denote by Γ^{in} and Γ^{out} the portions of the segment $\text{Re}(\tau) = -\eta$ which lie either inside $B_\epsilon(0)$ or outside $B_\epsilon(0)$ with $|\text{Im}(\tau)| \leq \pi$. Standard computations (using Lemma 6) lead to

$$\left| \frac{1}{2\pi \mathbf{i}} \int_{\Gamma^{out}} e^{n\tau} \mathbf{G}_\tau(j) d\tau \right| \leq C e^{-n\eta - cj},$$

and

$$\left| \frac{1}{2\pi \mathbf{i}} \int_{\Gamma^{in}} e^{n\tau} \mathbf{G}_\tau(j) d\tau \right| \leq C e^{-n\eta} \int_{\Gamma^{in}} e^{j \text{Re}(\varpi(\tau))} |d\tau|.$$

Next, we split Γ^{in} into two parts $\Gamma^{in} = \Gamma_{<0}^{in} \cup \Gamma_{\geq 0}^{in}$, where

$$\Gamma_{<0}^{in} := \{\tau = -\eta + \mathbf{i}\ell \mid \text{Re}(\varpi(\tau)) < 0\} \quad \text{and} \quad \Gamma_{\geq 0}^{in} := \{\tau = -\eta + \mathbf{i}\ell \mid \text{Re}(\varpi(\tau)) \geq 0\}.$$

Trivially we get

$$\left| \frac{1}{2\pi \mathbf{i}} \int_{\Gamma_{<0}^{in}} e^{n\tau} \mathbf{G}_\tau(j) d\tau \right| \leq C e^{-n\eta}.$$

Next, from the definition of $\varpi(\tau)$ and its expansion in Lemma 6, we have

$$\text{Re}(\varpi(\tau)) = -\frac{1}{\alpha} \text{Re}(\tau) - \frac{\beta}{\alpha^{2\mu+1}} \sum_{m=0}^{\mu} (-1)^m \binom{2\mu}{2m} \text{Re}(\tau)^{2m} \text{Im}(\tau)^{2(\mu-m)} + O(|\tau|^{2\mu+1}),$$

for all $\tau \in B_\epsilon(0)$. As a consequence, we can solve⁵ $\text{Re}(\varpi(\tau)) = 0$ for $\text{Re}(\tau)$ as a function of $\text{Im}(\tau)$ and get that

$$\text{Re}(\varpi(\tau)) = 0 \iff \text{Re}(\tau) = -\frac{\beta}{\alpha^{2\mu}} \text{Im}(\tau)^{2\mu} + O(|\text{Im}(\tau)|^{2\mu+1}) \text{ for } \tau \in B_\epsilon(0).$$

This in turn implies that there exists $c_0 > 0$ such that

$$|\text{Im}(\tau)| \leq c_0 \eta^{\frac{1}{2\mu}}, \quad \forall \tau \in \Gamma_{\geq 0}^{in},$$

from which we deduce that

$$\begin{aligned} \text{Re}(\varpi(\tau)) &= -\frac{1}{\alpha} \text{Re}(\tau) - \underbrace{\frac{\beta}{\alpha^{2\mu+1}} \text{Im}(\tau)^{2\mu}}_{\leq 0} - \frac{\beta}{\alpha^{2\mu+1}} \sum_{m=1}^{\mu} (-1)^m \binom{2\mu}{2m} \text{Re}(\tau)^{2m} \text{Im}(\tau)^{2(\mu-m)} + O(|\tau|^{2\mu+1}) \\ &\leq \frac{\eta}{\alpha} + c_1 \eta^{2+\frac{\mu-1}{\mu}} + c_2 \eta^{1+\frac{1}{2\mu}} \leq \frac{\eta}{\alpha} + (c_1 + c_2) \eta^{1+\frac{1}{2\mu}}, \end{aligned}$$

⁵This inversion process can be performed ahead of Lemma 6 in order to fix the radius $\epsilon > 0$ small enough.

for some $c_1, c_2 > 0$ and all $\tau \in \Gamma_{\geq 0}^{in}$. Here, we crucially used the fact that the term in $\text{Im}(\tau)^{2\mu}$ in $\text{Re}(\varpi(\tau))$ comes with a negative sign. As a consequence,

$$-n\eta + j\text{Re}(\varpi(\tau)) \leq n\eta \left(-1 + \frac{j}{n\alpha} + \frac{j}{n}(c_1 + c_2)\eta^{1+\frac{1}{2\mu}} \right) \leq -n\frac{\eta}{4},$$

provided that η is chosen small enough, which is always possible. The estimate of the integral along the contour $\Gamma_{\geq 0}^{in}$ follows. \square

We now turn to the last case where $n \geq 1$ and $n\delta \leq j \leq nr$. Our generalized Gaussian estimates will precisely come from this part of the analysis. In order to proceed, we follow the strategy developed in [ZH98] in the fully continuous case (see also [God03] in the fully discrete case), and introduce a family of parametrized curves given by

$$\Gamma_p := \left\{ \text{Re}(\tau) + \frac{\beta}{\alpha^{2\mu}} \sum_{m=0}^{\mu} (-1)^m \binom{2\mu}{2m} \text{Re}(\tau)^{2m} \text{Im}(\tau)^{2(\mu-m)} = \Psi(\tau_p) \mid -\eta \leq \text{Re}(\tau) \leq \tau_p \right\} \quad (30)$$

with $\Psi(\tau_p) := \tau_p + (-1)^\mu \frac{\beta}{\alpha^{2\mu}} \tau_p^{2\mu}$. These curves intersect the real axis at τ_p . The specific value of τ_p will be fixed depending on the ratio $\frac{\zeta}{\gamma}$ as follows

$$\tau_p := \begin{cases} \rho\left(\frac{\zeta}{\gamma}\right) & \text{if } -\frac{\eta}{2} \leq \rho\left(\frac{\zeta}{\gamma}\right) \leq \epsilon_0, \\ \epsilon_0 & \text{if } \rho\left(\frac{\zeta}{\gamma}\right) > \epsilon_0, \\ -\frac{\eta}{2} & \text{if } \rho\left(\frac{\zeta}{\gamma}\right) < -\frac{\eta}{2}, \end{cases}$$

where $0 < \epsilon_0 < \epsilon$ is chosen such that Γ_p with $\tau_p = \epsilon_0$ intersects the segment $\{-\eta + \mathbf{i}\ell \mid \ell \in [-\pi, \pi]\}$ precisely on the boundary⁶ of $B_\epsilon(0)$. There only remains to define ζ , γ and the function ρ . We let

$$\zeta := \frac{j - n\alpha}{2\mu n}, \quad \text{and} \quad \gamma := \frac{j}{n} \frac{\beta}{\alpha^{2\mu}} > 0,$$

and $\rho\left(\frac{\zeta}{\gamma}\right)$ is the unique real root to the equation

$$-\zeta + \gamma x^{2\mu-1} = 0,$$

that is

$$\rho\left(\frac{\zeta}{\gamma}\right) := \text{sgn}(\zeta) \left(\frac{|\zeta|}{\gamma} \right)^{\frac{1}{2\mu-1}}.$$

We remark that $-\frac{\alpha}{4\mu} \leq \zeta \leq \frac{r-\alpha}{2\mu}$, and our generalized Gaussian estimates will come from those values of $\zeta \approx 0$. Before proceeding with the analysis, we note that for all $\tau \in \Gamma_p$, we have

$$|\tau|^{2\mu+1} \leq C_* (|\tau_p|^{2\mu+1} + |\text{Im}(\tau)|^{2\mu+1}),$$

together with

$$\text{Re}(\tau) \leq \tau_p - c_* \text{Im}(\tau)^{2\mu},$$

for some constants $C_* > 0$ and $c_* > 0$.

⁶This is possible because the curves Γ_p are symmetric with respect to the real axis.

As a consequence, we can derive the following bound

$$\begin{aligned}
\left| \frac{1}{2\pi\mathbf{i}} \int_{\Gamma_p} e^{n\tau} \mathbf{G}_\tau(j) d\tau \right| &\lesssim \int_{\Gamma_p} e^{n\operatorname{Re}(\tau)+j\operatorname{Re}(\varpi(\tau))} |d\tau| \\
&\lesssim e^{-\frac{n}{\alpha} \frac{(2\mu-1)\gamma}{2} \left(\frac{|\zeta|}{\gamma}\right)^{\frac{2\mu}{2\mu-1}}} \int_{\Gamma_p} e^{-n\frac{c_*}{2} \operatorname{Im}(\tau)^{2\mu}} |d\tau| \\
&\lesssim \frac{e^{-\frac{n}{\alpha} \frac{(2\mu-1)\gamma}{2} \left(\frac{|\zeta|}{\gamma}\right)^{\frac{2\mu}{2\mu-1}}}}{n^{\frac{1}{2\mu}}}.
\end{aligned}$$

Next we denote by Γ_{\pm}^{in} and Γ_{\pm}^{out} the portions of the segment $\operatorname{Re}(\tau) = -\eta$ which lie either inside $B_\epsilon(0)$ or outside $B_\epsilon(0)$ with $|\operatorname{Im}(\tau)| \leq \pi$. We refer to Figure 1 for an illustration. Usual computations lead to

$$\left| \frac{1}{2\pi\mathbf{i}} \int_{\Gamma_{\pm}^{out}} e^{n\tau} \mathbf{G}_\tau(j) d\tau \right| \leq C e^{-n\eta - cj}.$$

For all $\tau \in \Gamma_{\pm}^{in}$, we use that $\operatorname{Im}(\tau)^2 \geq \operatorname{Im}(\tau_*)^2$ where $\tau_* = -\eta + \mathbf{i}\ell_*$ and $\ell_* > 0$ is the positive root of

$$-\eta + \frac{\beta}{\alpha^{2\mu}} \sum_{m=0}^{\mu} (-1)^m \binom{2\mu}{2m} \eta^{2m} \ell_*^{2(\mu-m)} = \tau_p + (-1)^\mu \frac{\beta}{\alpha^{2\mu}} \tau_p^{2\mu}.$$

That is $\tau_* = -\eta + \mathbf{i}\ell_*$ lies at the intersection of Γ_p and the segment $\{-\eta + \mathbf{i}\ell \mid \ell \in [-\pi, \pi]\}$ with $\tau_* \in B_\epsilon(0)$. As a consequence, for all $\tau \in \Gamma_{\pm}^{in}$ we have

$$\begin{aligned}
\operatorname{Re}(\varpi(\tau)) &= \frac{\eta}{\alpha} - \frac{\beta}{\alpha^{2\mu+1}} \sum_{m=0}^{\mu} (-1)^m \binom{2\mu}{2m} \eta^{2m} \operatorname{Im}(\tau)^{2(\mu-m)} + O(|\tau|^{2\mu+1}) \\
&= -\frac{\tau_p}{\alpha} - (-1)^\mu \frac{\beta}{\alpha^{2\mu+1}} \tau_p^{2\mu} - \frac{\beta}{\alpha^{2\mu+1}} (\operatorname{Im}(\tau)^{2\mu} - \ell_*^{2\mu}) \\
&\quad - \frac{\beta}{\alpha^{2\mu+1}} \sum_{m=1}^{\mu} (-1)^m \binom{2\mu}{2m} \eta^{2m} (\operatorname{Im}(\tau)^{2(\mu-m)} - \ell_*^{2(\mu-m)}) + O(|\tau|^{2\mu+1}) \\
&\leq -\frac{\tau_p}{\alpha} + \frac{\beta}{\alpha^{2\mu+1}} \tau_p^{2\mu} + c_1 \epsilon^{2\mu},
\end{aligned}$$

for some $c_1 > 0$. As a consequence, we have

$$\begin{aligned}
n\operatorname{Re}(\tau) + j\operatorname{Re}(\varpi(\tau)) &\leq -n\eta + j \left(-\frac{\tau_p}{\alpha} + \frac{\beta}{\alpha^{2\mu+1}} \tau_p^{2\mu} + c_1 \epsilon^{2\mu} \right), \\
&= \frac{n}{\alpha} \left[-\eta\alpha + \frac{j}{n} \left(-\tau_p + \frac{\beta}{\alpha^{2\mu}} \tau_p^{2\mu} + c_1 \alpha \epsilon^{2\mu} \right) \right] \\
&\leq \frac{n}{\alpha} \left[-(\eta + \tau_p - c_1 r \epsilon^{2\mu})\alpha - 2\mu\zeta\tau_p + \gamma\tau_p^{2\mu} \right] \\
&= \frac{n}{\alpha} \left[-(\eta + \tau_p - c_1 r \epsilon^{2\mu})\alpha + (-2\mu + 1)\gamma \left(\frac{|\zeta|}{\gamma} \right)^{\frac{2\mu}{2\mu-1}} \right]
\end{aligned}$$

for all $\tau \in \Gamma_{\pm}^{in}$. Finally, as $-\frac{\eta}{2} \leq \rho(\frac{\zeta}{\gamma}) = \tau_p$ we have $\eta + \tau_p \geq \frac{\eta}{2}$, and upon having $\epsilon > 0$ small enough, we get that

$$\eta + \tau_p - c_1 r \epsilon^{2\mu} \geq \frac{\eta}{4},$$

and we obtain an estimate of the form

$$\left| \frac{1}{2\pi \mathbf{i}} \int_{\Gamma_{\pm}^{in}} e^{n\tau} \mathbf{G}_{\tau}(j) d\tau \right| \leq C e^{-n \frac{\eta}{4} - \frac{n}{\alpha} (2\mu-1) \gamma \left(\frac{|\zeta|}{\gamma}\right)^{\frac{2\mu}{2\mu-1}}}.$$

- Next, we consider the case $\rho(\zeta/\gamma) > \epsilon_0$. The contour Γ is decomposed into $\Gamma_p \cup \Gamma_{\pm}$ where $\Gamma_{\pm} = \Gamma_{\pm}^{out}$ are the portions of the segment $\text{Re}(\tau) = -\eta$ which lie outside $B_{\epsilon}(0)$ with $|\text{Im}(\tau)| \leq \pi$. In that case, we have that for all $\tau \in \Gamma_p$

$$n\text{Re}(\tau) + j\text{Re}(\varpi(\tau)) \leq -nc_* \text{Im}(\tau)^{2\mu} \left(1 - \frac{C_*}{c_* \alpha} |\text{Im}(\tau)|\right) + \frac{n}{\alpha} \left(-2\mu\zeta\epsilon_0 + \gamma\epsilon_0^{2\mu} + C_*\epsilon_0^{2\mu+1}\right).$$

But as $\rho(\zeta/\gamma) > \epsilon_0$ we get that $\zeta > 0$ and $\zeta > \epsilon_0^{2\mu-1}\gamma$, the last term in the previous inequality is estimated via

$$-2\mu\zeta\epsilon_0 + \gamma\epsilon_0^{2\mu} + C_*\epsilon_0^{2\mu+1} < (-2\mu + 1)\mu\epsilon_0^{2\mu} + C_*\epsilon_0^{2\mu+1} \leq \frac{(-2\mu + 1)\gamma}{2} \epsilon_0^{2\mu},$$

as $\epsilon_0 < \epsilon$ and ϵ can be chosen small enough. As a consequence, we can derive the following bound

$$\left| \frac{1}{2\pi \mathbf{i}} \int_{\Gamma_p} e^{n\tau} \mathbf{G}_{\tau}(j) d\tau \right| \lesssim \frac{e^{-\frac{n}{\alpha} \frac{(2\mu-1)\gamma}{2} \epsilon_0^{2\mu}}}{n^{\frac{1}{2\mu}}}.$$

With our careful choice of $\epsilon_0 > 0$, the remaining contribution along segments Γ_{\pm} with $\text{Re}(\tau) = -\eta$ can be estimated as usual as

$$\left| \frac{1}{2\pi \mathbf{i}} \int_{\Gamma_{\pm}} e^{n\tau} \mathbf{G}_{\tau}(j, j_0) d\tau \right| \leq C e^{-n\eta - c_j},$$

as $|\tau| \geq \epsilon$ for $\tau \in \Gamma_{\pm}$.

- It remains to handle the last case $\rho(\zeta/\gamma) < -\eta/2$ and we readily note that in this setting $\zeta < 0$. The contour Γ is decomposed into $\Gamma_p \cup \Gamma_{\pm}^{out} \cup \Gamma_{\pm}^{in}$ where once again Γ_{\pm}^{in} and Γ_{\pm}^{out} are the portions of the segment $\text{Re}(\tau) = -\eta$ which lie either inside $B_{\epsilon}(0)$ or outside $B_{\epsilon}(0)$ with $|\text{Im}(\tau)| \leq \pi$. For all $\tau \in \Gamma_p \subset B_{\epsilon}(0)$, we find that

$$n\text{Re}(\tau) + j\text{Re}(\varpi(\tau)) \leq -nc_* \text{Im}(\tau)^{2\mu} \left(1 - \frac{C_*}{c_* \alpha} |\text{Im}(\tau)|\right) + \frac{n}{\alpha} \left(\mu\zeta\eta + \gamma \left(\frac{\eta}{2}\right)^{2\mu} + C_* \left(\frac{\eta}{2}\right)^{2\mu+1}\right).$$

Using that $\rho(\zeta/\gamma) < -\eta/2$ which is equivalent to $\zeta/\gamma < -\left(\frac{\eta}{2}\right)^{2\mu-1}$, we get that

$$\mu\zeta\eta + \gamma \left(\frac{\eta}{2}\right)^{2\mu} + C_* \left(\frac{\eta}{2}\right)^{2\mu+1} < (-2\mu + 1)\gamma \left(\frac{\eta}{2}\right)^{2\mu} + C_* \left(\frac{\eta}{2}\right)^{2\mu+1} \leq \frac{(-2\mu + 1)\gamma}{2} \left(\frac{\eta}{2}\right)^{2\mu},$$

provided that ϵ is small enough. As a consequence, we can derive the following bound

$$\left| \frac{1}{2\pi \mathbf{i}} \int_{\Gamma_p} e^{n\tau} \mathbf{G}_{\tau}(j) d\tau \right| \lesssim \frac{e^{-\frac{n}{\alpha} \frac{(2\mu-1)\gamma}{2} \left(\frac{\eta}{2}\right)^{2\mu}}}{n^{\frac{1}{2\mu}}}.$$

As usual, we have that

$$\left| \frac{1}{2\pi \mathbf{i}} \int_{\Gamma_{\pm}^{out}} e^{n\tau} \mathbf{G}_{\tau}(j) d\tau \right| \leq C e^{-n\eta - cj}.$$

It only remains to estimate the contribution on Γ_{\pm}^{in} . We proceed as before, and we have that for all $\tau \in \Gamma_{\pm}^{in}$

$$\begin{aligned} n\operatorname{Re}(\tau) + j\operatorname{Re}(\varpi(\tau)) &\leq \frac{n}{\alpha} [-(\eta + \tau_p - c_1 r \epsilon^{2\mu}) \alpha - 2\mu\zeta\tau_p + \gamma\tau_p^{2\mu}] \\ &= \frac{n}{\alpha} \left[-\left(\frac{\eta}{2} - c_1 r \epsilon^{2\mu}\right) \alpha + (-2\mu + 1)\gamma \left(\frac{\eta}{2}\right)^{2\mu} \right] \\ &\leq -n \left(\frac{\eta}{4} + \frac{(2\mu - 1)\gamma}{\alpha} \left(\frac{\eta}{2}\right)^{2\mu} \right), \end{aligned}$$

and this time we obtain

$$\left| \frac{1}{2\pi \mathbf{i}} \int_{\Gamma_{\pm}^{in}} e^{n\tau} \mathbf{G}_{\tau}(j) d\tau \right| \leq C e^{-n\left(\frac{\eta}{4} + \frac{(-2\mu+1)\gamma}{\alpha} \left(\frac{\eta}{2}\right)^{2\mu}\right)}.$$

As a summary, we have proved the following result.

Lemma 9. *There exist constants $C > 0$ and $M > 0$ such that for $n \geq 1$ and $n\delta \leq j \leq nr$ the following estimate holds:*

$$|\mathcal{G}_j^n| \leq \frac{C}{n^{\frac{1}{2\mu}}} \exp\left(-\frac{1}{M} \left(\frac{|j - \alpha n|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right).$$

Proof. One only needs to check that the purely exponentially decaying in n contributions obtained when $\rho\left(\frac{\zeta}{\gamma}\right) > \epsilon_0$ or $\rho\left(\frac{\zeta}{\gamma}\right) < -\frac{\eta}{2}$ can be subsumed into Gaussian estimates. For example, in the case $\rho\left(\frac{\zeta}{\gamma}\right) > \epsilon_0$, there exists some large constant $M > 0$ such that

$$-n \leq -\frac{1}{M} \left(\frac{|j - \alpha n|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}},$$

as

$$\frac{\beta}{2\alpha^{2\mu-1}} \leq \gamma \leq \frac{\beta}{\alpha^{2\mu}} r,$$

and

$$\frac{j}{n\alpha} - 1 = \frac{2\mu\zeta}{\alpha} > \frac{2\mu}{\alpha} \gamma \epsilon_0^{2\mu-1} \geq \frac{\mu\beta}{\alpha^{2\mu}} \epsilon_0^{2\mu-1}.$$

All other cases can be dealt with in a similar way. \square

Combining Lemma 7, Lemma 8 and Lemma 9 proves our main Theorem 1 in the explicit case with $K = 1$.

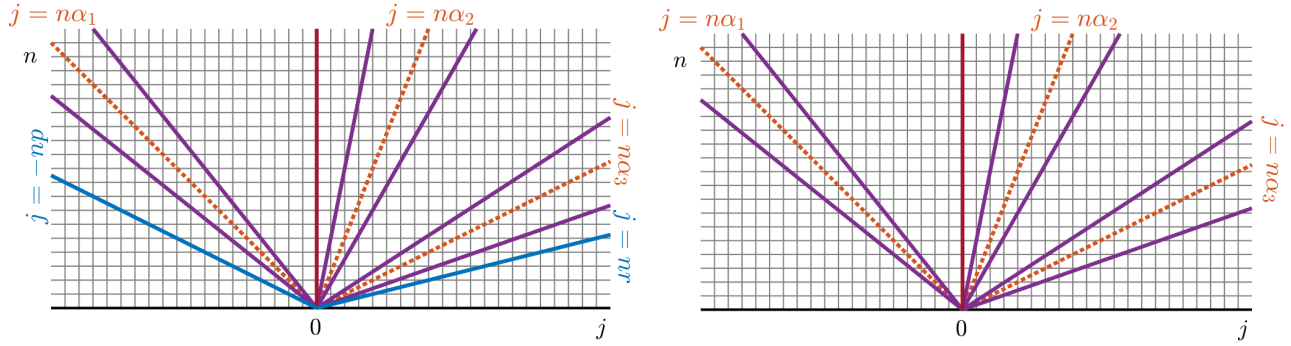


Figure 2: Illustration for the explicit (left) and implicit (right) cases of the different domains in the (j, n) plane where generalized Gaussian estimates are obtained; here $K = 3$ with $\alpha_1 < 0$ and $\alpha_2 \neq \alpha_3 > 0$. Generalized Gaussian estimates are derived near each $j \approx n \alpha_k$, $k = \{1, 2, 3\}$ in the interior of the sectors delimited by the magenta lines. In the explicit case below the lines $j = -np$ and $j = nr$ (blue) the Green's function \mathcal{G}_j^n vanishes.

3.2 The implicit case with $K = 1$

The main difference compared with the explicit case is that now it is no longer true that \mathcal{G}_j^n vanishes for $j > nr$ or $j < -pn$. First, our analysis in cases (a) and (b) are unchanged, and Lemma 7, Lemma 8 and Lemma 9 still hold true. Actually, the proofs of Lemma 7, Lemma 8 and Lemma 9 naturally extends to $-nL \leq j \leq 0$ for case (a) and $1 \leq j \leq nL$ for case (b) for any large constant $L \geq \max(p, r)$ that is fixed a priori. As a consequence, one only needs to consider the case $n \geq 1$ and $|j| > nL$ for some large constant $L > 0$. To obtain the desired estimate in that case, we use the bound at infinity obtained in Lemma 5. More precisely, there exists $R \geq \pi/2$ and two constants $C > 0$, $c > 0$ such that there holds

$$\forall |\tau| \geq R, \quad \forall j \in \mathbb{Z}, \quad |\mathbf{G}_\tau(j)| \leq C \exp(-c|j|).$$

We have the following result.

Lemma 10. *Let $L \geq \max(p, r) > 0$ be large enough such that $L > 2R/c$. Then, there exists $C > 0$, such that for $n \geq 1$ and $|j| > nL$, we have*

$$|\mathcal{G}_j^n| \leq C \exp\left(-\frac{c}{2}|j|\right),$$

where $c > 0$ is the constant from Lemma 5.

Proof. In (26), we now use the contour $\Gamma = \left\{ \frac{c|j|}{2n} + i\ell \mid \ell \in [-\pi, \pi] \right\}$. With our choice of L , we have that for all $\tau \in \Gamma$, $|\tau| \geq \frac{c|j|}{2n} \geq R$ and so

$$\left| \frac{1}{2\pi i} \int_{\Gamma} e^{n\tau} \mathbf{G}_\tau(j) d\tau \right| \leq C e^{-\frac{c}{2}|j|}.$$

□

Finally, we notice that

$$-\frac{c}{2}|j| \leq -n \frac{cL}{4} - \frac{c}{4}|j|, \quad \text{for } |j| > nL,$$

such that the bound obtained in the previous Lemma can be subsumed into the generalized Gaussian estimate of Lemma 9. This concludes the proof of Theorem 1 in the implicit case for $K = 1$.

3.3 The explicit and implicit cases with $K > 1$

We now briefly explain how to handle the general case with $K > 1$ and refer to Figures 2-3 for illustrations. From Assumption 2, we have the existence of K tangency points $\underline{\kappa}_k$ with associated nonzero real numbers α_k . We will distinguish two cases:

- A. All α_k are distinct from one and another.
- B. There exist two or more α_k which are equal.

We only discuss the explicit case here, as the implicit case does not distinguish between Cases A and B.

Case A. This is precisely the case depicted in Figure 2 with $K = 3$. Without loss of generality we label the α_k by increasing order such that

$$\alpha_1 < \cdots < \alpha_k < \cdots < \alpha_K.$$

For each $k = 1, \dots, K$ we define two real numbers $\underline{\delta}_k < \bar{\delta}_k$ such that we have the ordering

$$-p < \underline{\delta}_1 < \alpha_1 < \bar{\delta}_1 < \cdots < \underline{\delta}_k < \alpha_k < \bar{\delta}_k < \cdots < \underline{\delta}_K < \alpha_K < \bar{\delta}_K < r,$$

with $\text{sgn}(\underline{\delta}_k) = \text{sgn}(\bar{\delta}_k) = \text{sgn}(\alpha_k)$. For each $k = 1, \dots, K$, we define the following sectors in the (j, n) -plane:

$$\mathcal{D}_k := \{(j, n) \in \mathbb{Z} \times \mathbb{N}^* \mid n \underline{\delta}_k \leq j \leq n \bar{\delta}_k\},$$

together with

$$\mathcal{D}_* := \{(j, n) \in \mathbb{Z} \times \mathbb{N}^* \mid -np \leq j \leq nr\} \setminus \bigcup_{k=1}^K \mathcal{D}_k.$$

Finally, we let be $\underline{\tau}_k = \mathbf{i}\theta_k := \log(z_k)$ for $\theta_k \in [-\pi, \pi]$ for each $k = 1, \dots, K$. Our first Lemma pertains at obtaining exponential bounds in the region \mathcal{D}_* .

Lemma 11. *There exist $C > 0$ and $\delta_* > 0$ such that for each $(j, n) \in \mathcal{D}_*$ the following estimate holds:*

$$|\mathcal{G}_j^n| \leq C e^{-n\delta_*}.$$

Proof. We only sketch the proof as it is almost identical to the proofs of Lemma 7 and Lemma 8. We select the contour $\Gamma = \{-\eta + \mathbf{i}\ell \mid \ell \in [-\pi, \pi]\}$, with $0 < \eta < \eta_\varepsilon$ and we denote by Γ_k^{in} the portion of Γ which lie within $B_\varepsilon(\tau_k)$ and Γ^{out} the union of the remaining portions. As a consequence, we have $\Gamma = \Gamma_1^{in} \cup \cdots \cup \Gamma_K^{in} \cup \Gamma^{out}$, and we get

$$\left| \frac{1}{2\pi\mathbf{i}} \int_{\Gamma} e^{n\tau} \mathbf{G}_\tau(j) d\tau \right| \leq \sum_{k=1}^K \left| \frac{1}{2\pi\mathbf{i}} \int_{\Gamma_k^{in}} e^{n\tau} \mathbf{G}_\tau(j) d\tau \right| + \left| \frac{1}{2\pi\mathbf{i}} \int_{\Gamma^{out}} e^{n\tau} \mathbf{G}_\tau(j) d\tau \right|.$$

Our objective is to bound each above term separately. Along Γ^{out} , we get an estimate of the form

$$\left| \frac{1}{2\pi\mathbf{i}} \int_{\Gamma^{out}} e^{n\tau} \mathbf{G}_\tau(j) d\tau \right| \leq C e^{-n\eta - c|j|}, \quad \forall (j, n) \in \mathcal{D}_*,$$

as in Γ^{out} the Green's function $\mathbf{G}_\tau(j)$ enjoys pointwise exponential bound from Corollary 1.

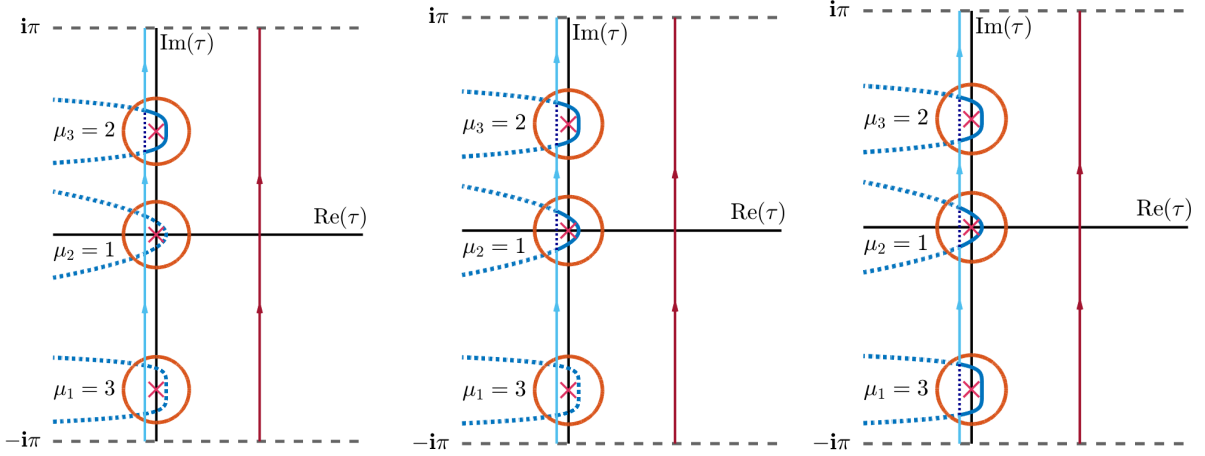


Figure 3: Left: typical contour used when $(j, n) \in \mathcal{D}_3$ in the case depicted in Figure 2 where all α_k are distinct $\alpha_1 < \alpha_2 < \alpha_3$ (case **A**). Middle: typical contour used when $(j, n) \in \mathcal{D}_2 = \mathcal{D}_3$ in the case $\alpha_1 < \alpha_2 = \alpha_3$ (case **B**). Right: typical contour used when $(j, n) \in \mathcal{D}_1 = \mathcal{D}_2 = \mathcal{D}_3$ in the case $\alpha_1 = \alpha_2 = \alpha_3$ (case **B**). Here $K = 3$.

We now derive pointwise bound for each contour integral along Γ_k^{in} , $k = 1, \dots, K$. We first handle the case where $\mathcal{S}_k = \{k\}$, and assume without loss of generality that $\alpha_k > 0$. Then Lemma 3 translates into

$$\forall \tau \in B_\varepsilon(\tau_k), \quad \forall j \in \mathbb{Z}, \quad |\mathbf{G}_\tau(j)| \leq \begin{cases} C \exp(-c|j|), & \text{if } j \leq 0, \\ C e^{j \operatorname{Re}(\varpi_k(\tau))}, & \text{if } j \geq 1, \end{cases} \quad (31)$$

with

$$\varpi_k(\tau) = -\frac{1}{\alpha_k}(\tau - \tau_k) + (-1)^{\mu_k+1} \frac{\beta_k}{\alpha_k^{2\mu_k+1}} (\tau - \tau_k)^{2\mu_k} + O(|\tau - \tau_k|^{2\mu_k+1}), \quad \forall \tau \in B_\varepsilon(\tau_k).$$

If $(j, n) \in \mathcal{D}_*$ is such that $j \leq 0$, then we directly get

$$\left| \frac{1}{2\pi \mathbf{i}} \int_{\Gamma_k^{in}} e^{n\tau} \mathbf{G}_\tau(j) d\tau \right| \leq C e^{-n\eta - c|j|}.$$

For now on, we consider $(j, n) \in \mathcal{D}_*$ with $j \geq 1$. In that case, we observe that

$$\operatorname{Re}(\varpi_k(\tau)) = -\frac{1}{\alpha_k} \operatorname{Re}(\tau) - \frac{\beta_k}{\alpha_k^{2\mu_k+1}} \sum_{m=0}^{\mu_k} (-1)^m \binom{2\mu_k}{2m} \operatorname{Re}(\tau)^{2m} (\operatorname{Im}(\tau) - \theta_k)^{2(\mu_k-m)} + O(|\tau|^{2\mu_k+1}),$$

for all $\tau \in B_\varepsilon(\tau_k)$. As in the proof of Lemma 8, we split each Γ_k^{in} into two parts $\Gamma_k^{in} = \Gamma_{k,<0}^{in} \cup \Gamma_{k,\geq 0}^{in}$, where

$$\Gamma_{k,<0}^{in} := \{\tau = -\eta + \mathbf{i}\ell \mid \operatorname{Re}(\varpi_k(\tau)) < 0\} \quad \text{and} \quad \Gamma_{k,\geq 0}^{in} := \{\tau = -\eta + \mathbf{i}\ell \mid \operatorname{Re}(\varpi_k(\tau)) \geq 0\}.$$

Trivially, we get

$$\left| \frac{1}{2\pi \mathbf{i}} \int_{\Gamma_{k,<0}^{in}} e^{n\tau} \mathbf{G}_\tau(j) d\tau \right| \leq C e^{-n\eta}.$$

Then, from the expansion of $\varpi_k(\tau)$, we have the existence of $c_k > 0$ such that

$$|\operatorname{Im}(\tau) - \theta_k| \leq c_k \eta^{\frac{1}{2\mu_k}}, \quad \forall \tau \in \Gamma_{k, \geq 0}^{\text{in}},$$

from which we deduce that

$$\operatorname{Re}(\varpi_k(\tau)) \leq \frac{\eta}{\alpha_k} + \tilde{c}_k \eta^{1+\frac{1}{2\mu_k}},$$

for some $\tilde{c}_k > 0$ and all $\tau \in \Gamma_{k, \geq 0}^{\text{in}}$. As a consequence, for all $(j, n) \in \mathcal{D}_*$ with $1 \leq j \leq n\bar{\delta}_k$, we have

$$-n\eta + j \operatorname{Re}(\varpi_k(\tau)) \leq n\eta \left(-1 + \frac{j}{n\alpha_k} + \frac{j}{n} \tilde{c}_k \eta^{1+\frac{1}{2\mu_k}} \right) \leq -\frac{n\eta}{2} \left(1 - \frac{\bar{\delta}_k}{\alpha_k} \right),$$

provided that η is chosen small enough, which is always possible. For the remaining cases $(j, n) \in \mathcal{D}_*$ and $n\bar{\delta}_k \leq j \leq nr$, we use a different contour near the ball $B_\varepsilon(\tau_k)$. We deform locally the ray $\{-\eta + \mathbf{i}\ell \mid \ell \in [\theta_k - 2\varepsilon, \theta_k + 2\varepsilon]\}$ into $\tilde{\Gamma}_k = \{\operatorname{Re}(\tau) = \tau_m - \xi \mid \operatorname{Im}(\tau) - \theta_k \mid -\eta \leq \operatorname{Re}(\tau) \leq \tau_m\}$ with $\tau_m = \frac{c}{2} \frac{j}{n}$ and $\xi = \frac{\tau_m + \eta}{2\varepsilon} > 0$. We refer to Figure 4 for an illustration. Note that $\tilde{\Gamma}_k$ intersects the ray Γ outside $B_\varepsilon(\tau_k)$ at $-\eta + \mathbf{i}(\theta_k \pm 2\varepsilon)$. As $\tau_m = \frac{c}{2} \frac{j}{n} \geq \frac{c}{2} \bar{\delta}_k > 0$, it is always possible to choose $\varepsilon > 0$ small enough such that $\tilde{\Gamma}_k$ lies outside the ball $B_\varepsilon(\tau_k)$, and in that case we get

$$\left| \frac{1}{2\pi \mathbf{i}} \int_{\tilde{\Gamma}_k} e^{n\tau} \mathbf{G}_\tau(j) d\tau \right| \leq C e^{-\frac{c}{2}j},$$

which gives the desired estimate. Let finally comment on the case where $\mathcal{S}_k = \{k_1, k_2\}$ with $\alpha_{k_1} < 0 < \alpha_{k_2}$. This time (28) reads

$$\forall \tau \in B_\varepsilon(\tau_k), \quad \forall j \in \mathbb{Z}, \quad |\mathbf{G}_\tau(j)| \leq \begin{cases} C e^{j \operatorname{Re}(\varpi_{k_1}(\tau))}, & \text{if } j \leq 0, \\ C e^{j \operatorname{Re}(\varpi_{k_2}(\tau))}, & \text{if } j \geq 1, \end{cases}$$

The analysis for $(j, n) \in \mathcal{D}_*$ and $j \geq 1$ is unchanged, and we apply the same strategy for $(j, n) \in \mathcal{D}_*$ and $j \leq 0$ without any difficulty.

As a conclusion, we have obtained that there exist $C > 0$ and $\delta_* > 0$ such that for each $k = 1, \dots, K$ we have

$$\left| \frac{1}{2\pi \mathbf{i}} \int_{\tilde{\Gamma}_k} e^{n\tau} \mathbf{G}_\tau(j) d\tau \right| \leq C e^{-n\delta_*}, \quad \forall (j, n) \in \mathcal{D}_*,$$

which ends the proof. \square

We prove in the next Lemma that we obtain generalized Gaussian estimates in each sector \mathcal{D}_k , $k = 1, \dots, K$.

Lemma 12. *There exist constants $C > 0$ and $M > 0$ such that for $(j, n) \in \mathcal{D}_k$ the following estimate holds:*

$$|\mathcal{G}_j^n| \leq \frac{C}{n^{\frac{1}{2\mu_k}}} \exp \left(-\frac{1}{M} \left(\frac{|j - \alpha_k n|}{n^{\frac{1}{2\mu_k}}} \right)^{\frac{2\mu_k}{2\mu_k - 1}} \right).$$

Proof. Let $(j, n) \in \mathcal{D}_k$, that is $n \geq 1$ and $n\bar{\delta}_k \leq j \leq n\bar{\delta}_k$. Assume without loss of generality that $\alpha_k > 0$. We consider first the case where $\mathcal{S}_k = \{k\}$. We use a contour Γ_k which consists of Γ_p^k in $B_\varepsilon(\tau_k)$ and the

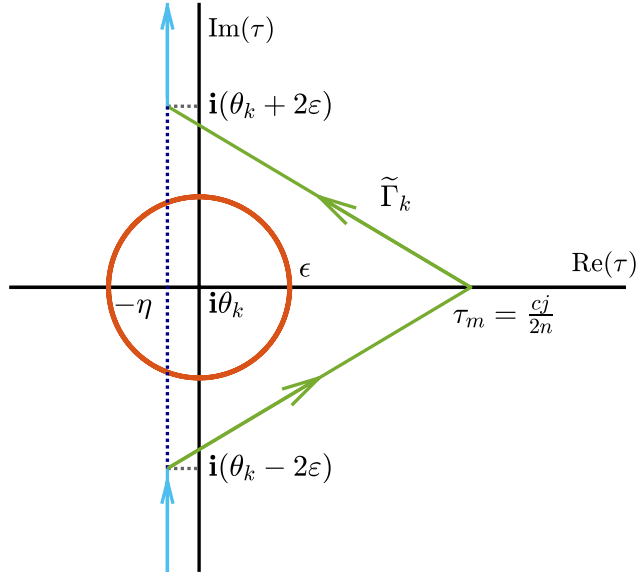


Figure 4: Illustration of the contour $\tilde{\Gamma}_k$ used in the proof of Lemma 11 and Lemma 12.

ray $\{-\eta + \mathbf{i}\ell \mid \ell \in [-\pi, \pi]\}$ otherwise (see Figure 3, left panel for an illustration in the case $K = 3$). Note that Γ_p^k is defined as

$$\Gamma_p^k := \left\{ \operatorname{Re}(\tau) + \frac{\beta_k}{\alpha_k^{2\mu_k}} \sum_{m=0}^{\mu_k} (-1)^m \binom{2\mu_k}{2m} \operatorname{Re}(\tau)^{2m} (\operatorname{Im}(\tau) - \theta_k)^{2(\mu_k - m)} = \Psi_k(\tau_{p,k}) \mid -\eta \leq \operatorname{Re}(\tau) \leq \tau_{p,k} \right\}$$

with $\Psi_k(\tau_{p,k}) := \tau_{p,k} + (-1)^{\mu_k} \frac{\beta}{\alpha_k^{2\mu_k}} \tau_p^{2\mu_k}$. Furthermore, the curve Γ_p^k intersects the real axis at $\tau_{p,k}$ which we define as

$$\tau_{p,k} = \begin{cases} \rho_k \left(\frac{\zeta_k}{\gamma_k} \right) & \text{if } -\frac{\eta}{2} \leq \rho_k \left(\frac{\zeta_k}{\gamma_k} \right) \leq \epsilon_{0,k}, \\ \epsilon_{0,k} & \text{if } \rho_k \left(\frac{\zeta_k}{\gamma_k} \right) > \epsilon_{0,k}, \\ -\frac{\eta}{2} & \text{if } \rho_k \left(\frac{\zeta_k}{\gamma_k} \right) < -\frac{\eta}{2}, \end{cases}$$

where $0 < \epsilon_{0,k} < \epsilon$ is chosen such that Γ_p^k with $\tau_{p,k} = \epsilon_{0,k}$ intersects the segment $\{-\eta + \mathbf{i}\ell \mid \ell \in [-\pi, \pi]\}$ precisely on the boundary of $B_\epsilon(\tau_k)$. There only remains to define ζ_k , γ_k and the function ρ_k . As in the proof of Lemma 9, we let

$$\zeta_k := \frac{j - n\alpha_k}{2\mu_k n}, \quad \text{and} \quad \gamma_k := \frac{j}{n} \frac{\beta_k}{\alpha_k^{2\mu_k}} > 0,$$

and $\rho_k \left(\frac{\zeta_k}{\gamma_k} \right)$ is given by

$$\rho_k \left(\frac{\zeta_k}{\gamma_k} \right) := \operatorname{sgn}(\zeta_k) \left(\frac{|\zeta_k|}{\gamma_k} \right)^{\frac{1}{2\mu_k - 1}}.$$

Depending on the ratio $\frac{\zeta_k}{\gamma_k}$, there exists (or not) a portion of the ray $\{-\eta + \mathbf{i}\ell \mid \ell \in [-\pi, \pi]\}$ within the ball $B_\epsilon(\tau_k)$ that we denote Γ_k^{in} . Note that when $\rho_k\left(\frac{\zeta_k}{\gamma_k}\right) > \epsilon_{0,k}$ we have $\Gamma_k^{in} = \emptyset$. The analysis along $\Gamma_p^k \cup \Gamma_k^{in}$ is exactly the same as the one conducted in the proof of Lemma 9 and we get

$$\left| \frac{1}{2\pi \mathbf{i}} \int_{\Gamma_p^k \cup \Gamma_k^{in}} e^{n\tau} \mathbf{G}_\tau(j) d\tau \right| \leq \frac{C}{n^{\frac{1}{2\mu_k}}} \exp\left(-\frac{1}{M} \left(\frac{|j - \alpha_k n|}{n^{\frac{1}{2\mu_k}}}\right)^{\frac{2\mu_k}{2\mu_k-1}}\right), \quad (j, n) \in \mathcal{D}_k.$$

Along the ray $\{-\eta + \mathbf{i}\ell \mid \ell \in [-\pi, \pi]\}$, we denote by Γ^{out} all portions that lie outside the balls $B_\epsilon(\tau_\nu)$ with $\nu \neq k$, and we get

$$\left| \frac{1}{2\pi \mathbf{i}} \int_{\Gamma^{out}} e^{n\tau} \mathbf{G}_\tau(j) d\tau \right| \leq e^{-n\eta - c|j|}, \quad (j, n) \in \mathcal{D}_k.$$

Thus, it only remains to estimate the contour integral along the ray $\{-\eta + \mathbf{i}\ell \mid \ell \in [-\pi, \pi]\}$ within a ball $B_\epsilon(\tau_\nu)$ with $\nu \neq k$, that we denote Γ_ν^{in} . Let assume first that $\mathcal{T}_\nu = \{\nu\}$. We split the analysis in two cases.

(i) If $\alpha_\nu < 0$, then we have the estimate

$$|\mathbf{G}_\tau(j)| \leq C e^{-c j}, \quad \tau \in \Gamma_\nu^{in},$$

as $0 < n \underline{\delta}_k \leq j \leq n \bar{\delta}_k$, and we obtain

$$\left| \frac{1}{2\pi \mathbf{i}} \int_{\Gamma_\nu^{in}} e^{n\tau} \mathbf{G}_\tau(j) d\tau \right| \leq C e^{-n\eta - c j}, \quad (j, n) \in \mathcal{D}_k.$$

(ii) If $\alpha_\nu > 0$, then we have the estimate

$$|\mathbf{G}_\tau(j)| \leq C e^{j \operatorname{Re}(\varpi_\nu(\tau))}, \quad \tau \in \Gamma_\nu^{in}.$$

As in the proof of Lemma 8, we split each Γ_ν^{in} into two parts $\Gamma_\nu^{in} = \Gamma_{\nu, < 0}^{in} \cup \Gamma_{\nu, \geq 0}^{in}$, where

$$\Gamma_{\nu, < 0}^{in} := \{\tau = -\eta + \mathbf{i}\ell \mid \operatorname{Re}(\varpi_\nu(\tau)) < 0\} \quad \text{and} \quad \Gamma_{\nu, \geq 0}^{in} := \{\tau = -\eta + \mathbf{i}\ell \mid \operatorname{Re}(\varpi_\nu(\tau)) \geq 0\}.$$

Trivially we get

$$\left| \frac{1}{2\pi \mathbf{i}} \int_{\Gamma_{\nu, < 0}^{in}} e^{n\tau} \mathbf{G}_\tau(j) d\tau \right| \leq C e^{-n\eta}.$$

Then, from the expansion of $\varpi_\nu(\tau)$, we have the existence of $c_{\nu,0} > 0$ such that

$$|\operatorname{Im}(\tau) - \theta_\nu| \leq c_{\nu,0} \eta^{\frac{1}{2\mu_\nu}}, \quad \forall \tau \in \Gamma_{\nu, \geq 0}^{in},$$

from which we deduce again that

$$\operatorname{Re}(\varpi_\nu(\tau)) \leq \frac{\eta}{\alpha_\nu} + c_{\nu,1} \eta^{1+\frac{1}{2\mu_\nu}},$$

for some $c_{\nu,1} > 0$ and all $\tau \in \Gamma_{\nu, \geq 0}^{in}$. As a consequence, we have

$$-n\eta + j \operatorname{Re}(\varpi_\nu(\tau)) \leq n\eta \left(-1 + \frac{j}{n\alpha_\nu} + \frac{j}{n} c_{\nu,1} \eta^{1+\frac{1}{2\mu_\nu}}\right).$$

Thus if $\alpha_\nu > \alpha_k$, we get that

$$-n\eta + j \operatorname{Re}(\varpi_\nu(\tau)) \leq -n \underbrace{\frac{\eta}{2} \left(1 - \frac{\bar{\delta}_k}{\alpha_\nu}\right)}_{>0},$$

which gives

$$\left| \frac{1}{2\pi \mathbf{i}} \int_{\Gamma_{\nu, < 0}^{in}} e^{n\tau} \mathbf{G}_\tau(j) d\tau \right| \leq C e^{-n \frac{\eta}{2} \left(1 - \frac{\bar{\delta}_k}{\alpha_\nu}\right)}.$$

Finally, if $\alpha_\nu < \alpha_k$ we use a different contour near the ball $B_\varepsilon(\tau_\nu)$. Namely, we deform locally the ray $\{-\eta + \mathbf{i}\ell \mid \ell \in [\theta_\nu - 2\varepsilon, \theta_\nu + 2\varepsilon]\}$ into $\Gamma_\nu = \{\operatorname{Re}(\tau) = \tau_m - \xi \mid |\operatorname{Im}(\tau) - \theta_\nu| \mid -\eta \leq \operatorname{Re}(\tau) \leq \tau_m\}$ with $\tau_m := \frac{c}{2} \frac{j}{n}$ and $\xi := \frac{\tau_m + \eta}{2\varepsilon} > 0$. Note that Γ_ν intersects the ray Γ outside $B_\varepsilon(\tau_\nu)$ at $-\eta + \mathbf{i}(\theta_\nu \pm 2\varepsilon)$. As $\tau_m = \frac{c}{2} \frac{j}{n} \geq \frac{c}{2} \delta_k > 0$, it is always possible to chose $\varepsilon > 0$ small enough such that Γ_ν lies outside the ball $B_\varepsilon(\tau_\nu)$, in that case we get

$$\left| \frac{1}{2\pi \mathbf{i}} \int_{\Gamma_k} e^{n\tau} \mathbf{G}_\tau(j) d\tau \right| \leq C e^{-\frac{c}{2} j},$$

which gives the desired estimate.

If now $\mathcal{J}_\nu = \{\nu_1, \nu_2\}$, then we have $\alpha_{\nu_1} < 0 < \alpha_{\nu_2}$ and for $0 < n\delta_k \leq j \leq n\bar{\delta}_k$, we get

$$|\mathbf{G}_\tau(j)| \leq C e^{j \operatorname{Re}(\varpi_{\nu_2}(\tau))}, \quad \tau \in \Gamma_\nu^{in},$$

such that the analysis is similar to the above case (ii). Finally, when $\mathcal{J}_k = \{k_1, k_2\}$, we necessarily have that $\alpha_{k_1} < 0 < \alpha_{k_2} = \alpha_k$ and the analysis remains unchanged. \square

Case B. In that case two or more α_k are equal. Note that for $(j, n) \in \mathcal{D}_*$ the analysis remains unchanged and Lemma 11 still holds true in that case. Let us assume for simplicity that $\alpha_{\nu_1} = \alpha_{\nu_2}$ and all other α_k are distinct. The estimate from Lemma 12 is still valid for $(j, n) \in \mathcal{D}_k$ for each k with $\alpha_k \neq \alpha_{\nu_1} = \alpha_{\nu_2}$. For $(j, n) \in \mathcal{D}_{\nu_1} = \mathcal{D}_{\nu_2}$, in the ball $B_\varepsilon(\tau_{\nu_1})$ we use the contour $\Gamma_p^{\nu_1}$ and in the ball $B_\varepsilon(\tau_{\nu_2})$ we use the contour $\Gamma_p^{\nu_2}$. And we refer to Figure 3 for an illustration of such contours. Reproducing the analysis of Lemma 12, we obtain the existence of $C > 0$ and $M > 0$ such that for $(j, n) \in \mathcal{D}_{\nu_1} = \mathcal{D}_{\nu_2}$ the following estimate holds:

$$|\mathcal{G}_j^n| \leq C \sum_{\nu \in \{\nu_1, \nu_2\}} \frac{1}{n^{\frac{1}{2\mu_\nu}}} \exp\left(-\frac{1}{M} \left(\frac{|j - \alpha_\nu n|}{n^{\frac{1}{2\mu_\nu}}}\right)^{\frac{2\mu_\nu}{2\mu_\nu - 1}}\right).$$

Finally, we remark that Lemma 10 naturally extends to the case $K > 1$ in the implicit setting. This concludes the proof of Theorem 1.

4 Examples and extensions

We first give several examples of operators (2) that fit into the framework of Theorem 1, and that arise when discretizing the transport equation:

$$\partial_t u + \partial_x u = 0, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}, \quad (32)$$

with Cauchy data at $t = 0$. We refer to [GKO95, Gus08] for a detailed analysis and more examples of finite difference schemes in that context.

4.1 Example 1: the Lax-Friedrichs scheme

The Lax-Friedrichs scheme is an explicit finite difference approximation of (32), which corresponds to the operators:

$$Q_1 := I, \quad Q_0 := \frac{1 + \lambda}{2} \mathbf{S}^{-1} + \frac{1 - \lambda}{2} \mathbf{S}, \quad (33)$$

where here and below, λ is a real parameter⁷ and \mathbf{S} denotes the so-called shift operator defined by:

$$\mathbf{S} : (u_j)_{j \in \mathbb{Z}} \mapsto (u_{j+1})_{j \in \mathbb{Z}}.$$

We now restrict to $\lambda \in (0, 1)$ so that both coefficients in the definition (33) are positive and they sum to 1. In probability theory, this corresponds to a random walk with probability $(1 + \lambda)/2$ to jump of $+1$ and probability $(1 - \lambda)/2$ to jump of -1 at each time iteration (recall our convention on the coefficients a_ℓ which differs from the standard convolution product).

In the notation of (2), we have $r = p = 1$. Since we are dealing here with an explicit scheme, Assumptions 1 and 4 are trivially satisfied. The definition (5) reduces here to:

$$F(e^{i\xi}) = \cos \xi - i \lambda \sin \xi.$$

Computing:

$$\left| F(e^{i\xi}) \right|^2 = \cos^2 \xi + \lambda^2 \sin^2 \xi,$$

we find that $F(\kappa)$ belongs to $\overline{\mathbb{D}}$ for all $\kappa \in \mathbb{S}^1$, and $F(\kappa)$ belongs to \mathbb{S}^1 for such κ if and only if $\kappa = \pm 1$. We thus have (6) with $\underline{\kappa}_1 := 1$ and $\underline{\kappa}_2 := -1$, and the reader can check that (7) is satisfied with:

$$\alpha_1 = \alpha_2 := \lambda, \quad \beta_1 = \beta_2 := \frac{1 - \lambda^2}{2},$$

which means that Assumption 2 is satisfied. Since the modulus of $F(\kappa)$ attains its maximum at two points of \mathbb{S}^1 , we cannot apply the uniform gaussian bound from [DSC14] but only its sub-gaussian analogue. The improvement of Theorem 1 is thus meaningful here. The spectral curve $F(\mathbb{S}^1)$ is illustrated in Figure 5 in the case $\lambda = 1/2$.

We now turn to Assumption 3 and compute (see the general definition (8)):

$$\mathbb{A}_{-1}(z) = -\frac{1 + \lambda}{2}, \quad \mathbb{A}_1(z) = -\frac{1 - \lambda}{2}.$$

Hence Assumption 3 is satisfied too.

At last, Assumption 5 is satisfied since we have here $K = 2$, $z_1 = 1$ and $z_2 = -1 \neq z_1$, which means that both sets \mathcal{S}_1 and \mathcal{S}_2 in (10) are singletons. Overall, the conclusion of Theorem 1 for the Lax-Friedrichs scheme in (33) is the uniform bound⁸:

$$\mathcal{G}_j^n \leq \frac{C}{\sqrt{n}} \exp\left(-c \frac{(j - \lambda n)^2}{n}\right).$$

This behavior is illustrated in Figure 5 in the case $\lambda = 1/2$.

⁷In the theory of finite difference schemes, it is referred to as the Courant-Friedrichs-Lewy parameter [CFL28].

⁸We do not use the absolute value here since all coefficients \mathcal{G}_j^n are nonnegative.

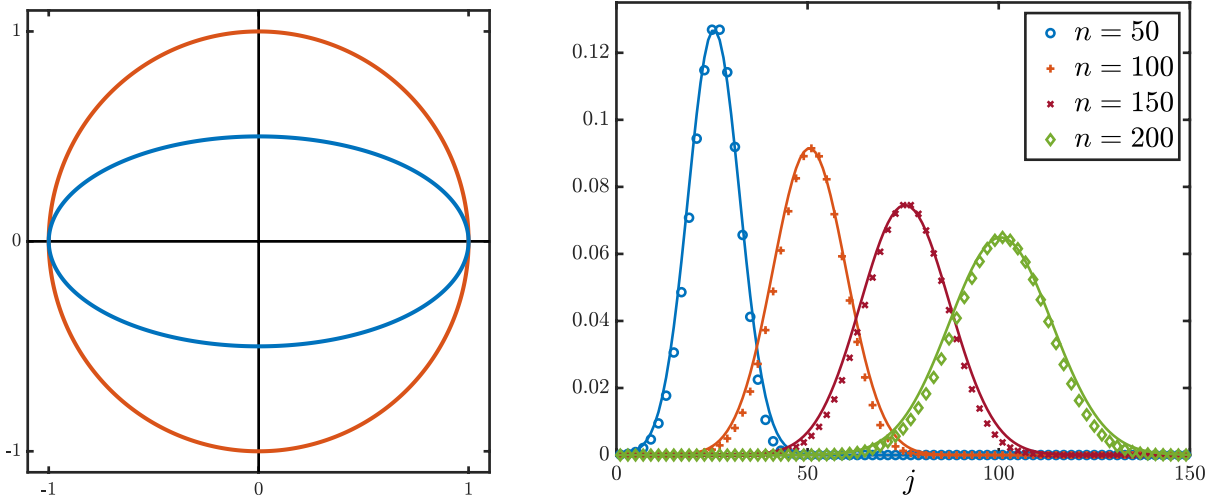


Figure 5: Left: Spectrum (blue curve) $\sigma(\mathcal{L}) = F(\mathbb{S}^1)$ for the Lax-Friedrichs scheme (33). Right: The Green's function at different time iterations for the Lax-Friedrichs scheme (33). Here, we have set $\lambda = 1/2$.

For readers who are familiar with the theory of the transport equation (see [GKO95] otherwise), the parameter λ stands for the ratio $\Delta t/\Delta x$ of the time and space steps. Hence the bound of Theorem 1 equivalently reads (with new constants that are still denoted C and c):

$$\mathcal{G}_j^n \leq \frac{C}{\sqrt{n}} \exp\left(-c \frac{(j \Delta x - n \Delta t)^2}{\Delta x (n \Delta t)}\right),$$

which corresponds to the heat kernel at point $j \Delta x$, time $n \Delta t$ with a diffusion coefficient proportional to Δx .

4.2 Example 2: an implicit scheme

Our second example is based on the so-called method of lines for discretizing (32) (see [GKO95, Gus08] for a detailed exposition of the method and its outcome). Here we first apply the centered finite difference for the spatial derivative and we then apply the implicit Euler scheme for the time integration. As in the case of the Lax-Friedrichs scheme (33), we introduce a positive parameter $\lambda > 0$ (which plays the role of the ratio $\Delta t/\Delta x$ but its origin is meaningless here), and we use the operators:

$$Q_1 := I + \frac{\lambda}{2} (\mathbf{S} - \mathbf{S}^{-1}), \quad Q_0 := I. \quad (34)$$

In the notation of (2), this corresponds again to $r = p = 1$, but the scheme is now implicit because Q_1 is not the identity. We compute:

$$\widehat{Q}_1(e^{i\xi}) = 1 + i \lambda \sin \xi \neq 0,$$

which means that Q_1 is an isomorphism on $\ell^2(\mathbb{Z}; \mathbb{C})$. The index condition (4) is also satisfied since the complex number $\widehat{Q}_1(\kappa)$ has positive real part for all $\kappa \in \mathbb{S}^1$ so we can write:

$$\widehat{Q}_1(\kappa) = \exp q(\kappa),$$

thanks to the standard determination of the logarithm (which implies the validity of (4)). The operator \mathcal{L} is given by:

$$\mathcal{L} = \frac{1}{\sqrt{1+\lambda^2}} \left\{ \sum_{\ell \geq 0} x^\ell \mathbf{S}^{-\ell} + \sum_{\ell \geq 1} (-1)^\ell x^\ell \mathbf{S}^\ell \right\},$$

where $x \in (0, 1)$ is given by:

$$x := \frac{\sqrt{1+\lambda^2} - 1}{\lambda}.$$

We are thus dealing with a convolution operator with infinite support.

We compute:

$$F(e^{i\xi}) = \frac{1}{1 + i\lambda \sin \xi},$$

which means that $F(\kappa)$ belongs to $\overline{\mathbb{D}}$ for all $\kappa \in \mathbb{S}^1$ and, again, $F(\kappa)$ belongs to \mathbb{S}^1 if and only if $\kappa = \pm 1$ ($K = 2$ in the notation of Assumption 2). Setting $\underline{\kappa}_1 := 1$ and $\underline{\kappa}_2 := -1$, we find that the relation (7) is satisfied with:

$$\alpha_1 := \lambda, \quad \beta_1 = \beta_2 := \frac{\lambda^2}{2}, \quad \alpha_2 := -\lambda.$$

Assumption 2 is thus satisfied but we now have $z_1 = z_2 = 1$, and we immediately see that Assumption 5 is also satisfied: both sets \mathcal{S}_1 and \mathcal{S}_2 equal $\{1, 2\}$ and $\alpha_1 \alpha_2 = -\lambda^2 < 0$. The spectral curve $F(\mathbb{S}^1)$ is illustrated in Figure 6 in the case $\lambda = 1/2$.

As far as Assumption 3 is concerned, we compute:

$$\mathbb{A}_{-1}(z) = -\frac{\lambda}{2}z, \quad \mathbb{A}_1(z) = \frac{\lambda}{2}z,$$

so Assumption 3 is satisfied again. We also note that Assumption 4 is satisfied since we have $a_{-1,1} = -\lambda/2$ and $a_{1,1} = \lambda/2$. We can therefore apply Theorem 1 which, in the case of (34), yields the uniform Gaussian bound:

$$|\mathcal{G}_j^n| \leq \frac{C}{\sqrt{n}} \left(\exp\left(-c \frac{(j + \lambda n)^2}{n}\right) + \exp\left(-c \frac{(j - \lambda n)^2}{n}\right) \right).$$

This behavior is illustrated in Figure 6 in the case $\lambda = 1/2$.

4.3 Further extensions

For the sake of clarity, we have focused here on the case of scalar iterations, but the techniques developed in this article apply to multistep iterations of the form

$$\begin{cases} Q_{s+1} u^{n+s+1} = Q_s u^{n+s} + \dots + Q_0 u^n, & n \in \mathbb{N}, \\ u^0, \dots, u^s \in \ell^2(\mathbb{Z}), \end{cases}$$

where $s \in \mathbb{N}$ is a given fixed integer and there are now $s + 2$ convolution operators with finite support involved. Of course, the statement of the assumptions should be suitably modified (for instance, Assumption 1 now bears on Q_{s+1} and not on Q_1).

We have focused here on numerical schemes for which the modulus of the amplification factor F is not constant on \mathbb{S}^1 and such that the local behavior of F near a point where its modulus attains its maximum is dictated as in [Tho65]. We recall that for operators of the form (2), the main result in [Tho65]

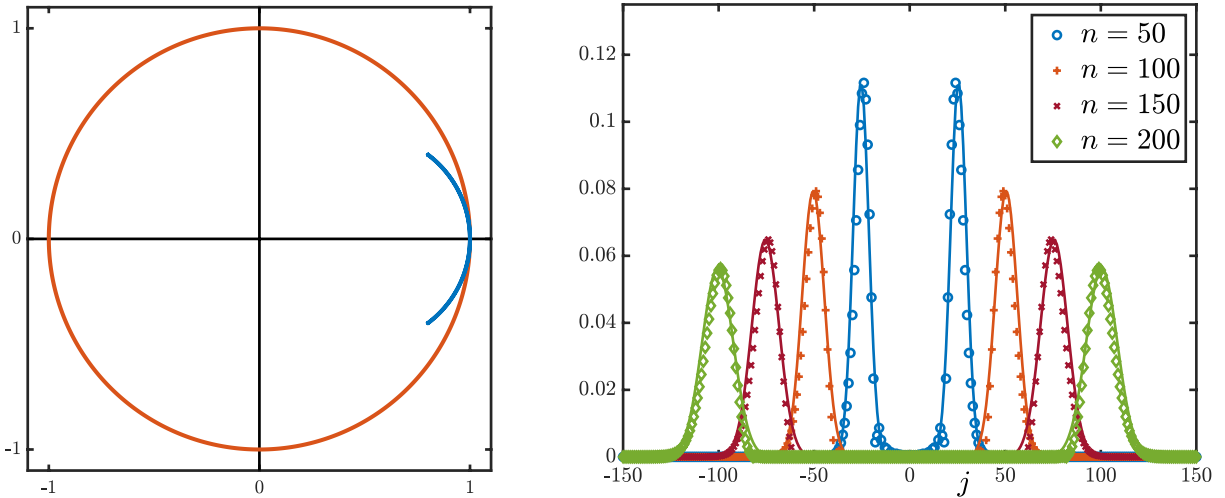


Figure 6: Left: Spectrum (blue curve) $\sigma(\mathcal{L}) = F(\mathbb{S}^1)$ for the the implicit scheme (34). Right: The absolute value of the Green's function at different time iterations for the implicit scheme (34).

shows that Assumption 2 is necessary and sufficient for $Q_1^{-1}Q_0$ to be power bounded from $\ell^1(\mathbb{Z}; \mathbb{C})$ to $\ell^1(\mathbb{Z}; \mathbb{C})$ (or equivalently from $\ell^\infty(\mathbb{Z}; \mathbb{C})$ to $\ell^\infty(\mathbb{Z}; \mathbb{C})$). We intend to pursue the above strategy for dealing with discretized boundary value problems under suitable uniform or degenerate versions of the Kreiss-Lopatinskii condition [GKS72, KW93]. We also expect that the combination of the above approach with the refined tools from [RSC15] (in one space dimension) or from [RSC17] (in several space dimensions) might yield interesting results.

We also intend to study the case where α_k vanishes in a near future. A typical situation where (the analogue of the function) F has constant modulus and the drift α may vanish is provided by the leap-frog scheme [Tre82, Tre84].

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