



## On the star forest polytope for trees and cycles

Méziante Aïder, Lamia Aoudia, Mourad Baïou, A. Ridha Mahjoub, Viet Hung Nguyen

### ► To cite this version:

Méziante Aïder, Lamia Aoudia, Mourad Baïou, A. Ridha Mahjoub, Viet Hung Nguyen. On the star forest polytope for trees and cycles. *RAIRO - Operations Research*, 2019, 53 (5), pp.1763-1773. 10.1051/ro/2018076 . hal-03071987

**HAL Id: hal-03071987**

**<https://hal.science/hal-03071987>**

Submitted on 16 Dec 2020

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

## ON THE STAR FOREST POLYTOPE FOR TREES AND CYCLES

MEZIANE AIDER<sup>1</sup>, LAMIA Aoudia<sup>1</sup>, MOURAD BAÏOU<sup>2</sup>,  
A. RIDHA MAHJOUB<sup>3</sup> AND VIET HUNG NGUYEN<sup>4,\*</sup>

**Abstract.** Let  $G = (V, E)$  be an undirected graph where the edges in  $E$  have non-negative weights. A star in  $G$  is either a single node of  $G$  or a subgraph of  $G$  where all the edges share one common end-node. A star forest is a collection of vertex-disjoint stars in  $G$ . The weight of a star forest is the sum of the weights of its edges. This paper deals with the problem of finding a Maximum Weight Spanning Star Forest (MWSFP) in  $G$ . This problem is *NP*-hard but can be solved in polynomial time when  $G$  is a cactus [Nguyen, *Discrete Math. Algorithms App.* **7** (2015) 1550018]. In this paper, we present a polyhedral investigation of the MWSFP. More precisely, we study the facial structure of the star forest polytope, denoted by  $SFP(G)$ , which is the convex hull of the incidence vectors of the star forests of  $G$ . First, we prove several basic properties of  $SFP(G)$  and propose an integer programming formulation for MWSFP. Then, we give a class of facet-defining inequalities, called *M*-tree inequalities, for  $SFP(G)$ . We show that for the case when  $G$  is a tree, the *M*-tree and the nonnegativity inequalities give a complete characterization of  $SFP(G)$ . Finally, based on the description of the dominating set polytope on cycles given by Bouchakour *et al.* [*Eur. J. Combin.* **29** (2008) 652–661], we give a complete linear description of  $SFP(G)$  when  $G$  is a cycle.

**Mathematics Subject Classification.** 90C27, 90C10, 05Cxx.

Received June 1, 2015. Accepted September 4, 2018.

### 1. INTRODUCTION

Given an undirected graph  $G = (V, E)$  where  $n = |V|$  and  $m = |E|$ , a *star* in  $G$  is either a single node of  $G$  or a subgraph of  $G$  where every edge shares one common end-node. The latter is called the *center* of the star when the star is not reduced to a single node. If the star is a single edge, then any of its end-nodes can be designated as the center. A *star forest* is a collection of vertex-disjoint stars in  $G$ . An *edge dominating set* in  $G$  is an edge subset  $F \subseteq E$  such that for any edge  $e$  in  $G$  either  $e \in F$  or  $e$  shares at least one common end-node with some edge in  $F$ . A *dominating set* in  $G$  is a node subset  $S \subseteq V$  such that for any node  $u \in V$  either  $u \in S$  or  $u$  is neighbor with some node in  $S$ . We suppose that the edges in  $G$  have non-negative weights (note that

---

*Keywords.* Combinatorial optimization, polyhedral combinatorics, star forest, facility location, dominating set.

<sup>1</sup> Laromad, Faculty of Mathematics, University of Sciences and Technology, Houari Boumediène, Algiers, Algeria.

<sup>2</sup> LIMOS, CNRS UMR 6158, Clermont-Ferrand, France.

<sup>3</sup> Université Paris-Dauphine, PSL, CNRS UMR 7243 LAMSADE, Place du Maréchal De Lattre de Tassigny, 75775, Paris Cedex 16, France.

<sup>4</sup> Sorbonne Université, CNRS UMR 7606, LIP6 – 4 place Jussieu, Paris, France.

\*Corresponding author: [Hung.Nguyen@lip6.fr](mailto:Hung.Nguyen@lip6.fr)

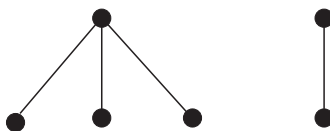


FIGURE 1. A star forest of weight 4 with weights 1 on the edges.

the unweighted case can be seen as a special weighted case when weights are 0 or 1), then the weight of a star forest or an edge dominating set is the sum of the weights of its edges. The *Maximum Weight spanning Star Forest Problem* (MWSFP) is to find a star forest spanning the nodes of  $G$  of maximum weight. The *Minimum Weight Edge Dominating Problem* (MWEDP) is to find an edge dominating set in  $G$  of minimum weight. If the nodes are weighted, the weight of a dominating set is the sum of the weights of its nodes. The *Minimum Weight Dominating Set Problem* (MWDSP) is to find a minimum weight dominating set in  $G$ . The two last problems are well-known to be  $NP$ -hard. They have been the subject of many works in the literature [20], [13]. The MWSFP, however, is a recent problem which has been introduced by Nguyen *et al.* in [17]. It has applications in several areas, especially in computational biology [17] and automobile industry [1]. In [17], the authors show the  $NP$ -hardness of MWSFP by observing that in a *maximal star forest*  $F$  (a maximal star forest is a star forest to which no more edge can be added), the set of the centers of the stars in  $F$  is a dominating set of  $G$ . Conversely, for any dominating set  $S$  we can build a maximal star forest with centers as the nodes belonging to  $S$ . Thus, given a maximal star forest  $F$ , there exists a dominating set  $S$  such that  $|S| = |V| - |F|$  and vice versa. Hence the case of 0/1 weights of the MWSFP is  $NP$ -hard by a reduction from the 0/1 weights case of the MWDSP. In [17], the authors also give a linear time algorithm to solve the MWSFP when  $G$  is a tree and a  $\frac{1}{2}$ -approximation algorithm for the general case. Since then, the MWSFP has been intensively investigated, in particular for the unweighted version. Nguyen *et al.* [17] prove that the problem is APX-hard by presenting an explicit inapproximability bound of 259/260, and present a combinatorial 0.6-approximation algorithm for the unweighted MWSFP. Polynomial-time algorithms are presented for special classes of graphs such as planar graphs and trees in the same paper. Chen *et al.* [12] present a better approximation algorithm with ratio 0.71 for unweighted MWSFP. Later, Athanassopoulos *et al.* [2] improve this approximation ratio to 0.803 by using the fact that the problem is a special case of the complementary set cover problem. Interesting generalizations including node-weighted and edge-weighted versions of the MWSFP have also been considered. In [12, 17] the authors present approximation algorithms and APX-hardness results for these problems as well. Stronger inapproximability results for these problems recently appeared in [11, 14]. For the weighted version, Nguyen [18] has given a linear time algorithm for solving the MWSFP when  $G$  is a cactus.

Let  $SFP(G)$  (respectively  $EDP(G)$ ) be the convex hull of the incidence vectors of the star forests (respectively the edge dominating sets) in  $G$ . Let  $\mathbb{R}^n$  be the real space indexed by the nodes in  $V$ . Let  $D$  be any dominating set in  $G$ , let  $\chi(D) \in \mathbb{R}^n$  be the incidence vector of  $D$ , defined as

$$\chi(D)_v = \begin{cases} 1 & \text{if } v \text{ is a nodes in } V \text{ and } v \in D \\ 0 & \text{otherwise.} \end{cases}$$

Let  $DP(G)$  be the convex hull of the incidence vectors of the dominating sets in  $G$ . To the best of our knowledge, no polyhedral investigation has been done for  $SFP(G)$  and  $EDP(G)$  though some integer formulations have been used in approximation algorithms for the MWSFP and the MWEDP. There are, however, several works on  $DP(G)$ , in particular, Saxena [19] has given a complete description for  $DP(G)$  when  $G$  is a tree, and Bouchakhour *et al.* [8] have given a complete description for  $DP(G)$  when  $G$  is a cycle. In [16], Mahjoub has given a complete description of  $DP(G)$  in threshold graphs. And in [9], Bouchakour and Mahjoub have studied compositions for the polytope  $DP(G)$  in graphs that decompose by one-node cutsets.

In this paper, we present a polyhedral investigation of the MWSFP for trees and cycles. More precisely, we study the facial structure of the star forest polytope. In the first part of the paper, we give a complete characterization of  $SFP(G)$  when  $G$  is a tree, which is obtained by projection of a simple extended formulation issued from the work of Baïou and Barahona [3] on the uncapacitated facility location polytope. Also, we show that the facet-defining inequalities for  $SFP(G)$  when  $G$  is a tree can be generalized to valid inequalities for  $SFP(G)$  when  $G$  is an arbitrary graph. These inequalities define facets for  $SFP(G)$  under certain conditions, and can be separated in polynomial time.

In the second part of the paper, we give a complete description for  $SFP(G)$  when  $G$  is a simple cycle  $C$ . More precisely, we establish the relation between spanning star forests and dominating sets when the graph is a simple cycle  $C$  and give a complete linear description for  $SFP(C)$  based on the one for  $DP(C)$  given by Bouchakhour *et al.* [8].

The paper is organized as follows. In the next section, we describe some properties of  $SFP(G)$  and give an integer programming formulation for the MWSFP. In section 3, we introduce a class of valid inequalities, called the  $M$ -tree inequalities, for  $SFP(G)$  and give a complete description for  $SFP(G)$  when  $G$  is a tree. In Section 4, we present a complete description for  $SFP(G)$  when  $G$  is a cycle.

In the rest of this section, we give some notations that will be used in the paper. For  $x \in \mathbb{R}^m$ , given any  $F \subseteq E$ , we let  $x(F)$  denote  $\sum_{e \in F} x_e$ . For  $x \in \mathbb{R}^n$ , given a set  $S \subseteq V$ , we let  $x(S)$  denote  $\sum_{v \in S} x_v$ . Given a set of vertices  $S$ , we denote by  $E(S)$  the set of edges with both ends belonging to  $S$ . Let  $v \in V$ , the *neighborhood* of  $v$ , denoted by  $N(v)$ , is the vertex set consisting of  $v$  and the nodes which are adjacent to  $v$ . Given any edge subset  $F \subseteq E$ , let  $V(F)$  denote the set of the end-nodes of the edges in  $F$ . We call a *3-path* a simple path having 3 edges in  $G$  and a *3-cycle* a triangle in  $G$ . Let  $\mathcal{P}_4$  (respectively  $\mathcal{C}_3$ ) denote the collection of the 3-paths (resp. 3-cycles) in  $G$ .

## 2. BASIC PROPERTIES OF $SFP(G)$ AND INTEGER PROGRAMMING FORMULATION FOR THE MWSP

### 2.1. Basic properties of $SFP(G)$ .

The following remark is about zero vector  $\mathbf{0} \in \mathbb{R}^m$  which is the incidence vector associated with the single node star forests.

**Remark 2.1.** The zero vector  $\mathbf{0} \in \mathbb{R}^m$  is an extreme point of  $SFP(G)$ .

*Proof.* We can see that  $\mathbf{0}$  is the incidence vector associated with the single node star forests and as for any  $x \in SFP(G)$  and any  $e \in E$ ,  $x_e \geq 0$ ,  $\mathbf{0}$  is an extreme point of  $SFP(G)$ .  $\square$

Hence,  $SFP(G)$  is a polytope pointed at  $\mathbf{0}$ . Moreover, the following theorem shows that  $SFP(G)$  is full dimensional.

**Theorem 2.2.**  $SFP(G)$  is a full dimensional polytope, i.e.  $\dim(SFP(G)) = m$ .

*Proof.* Suppose that the incidence vectors of all the star forests in  $G$  satisfy some equality  $\alpha^t x = \beta$ . As  $\alpha^t \mathbf{0} = \beta$ ,  $\beta = 0$ . As any edge  $e \in E$  is a star forest in  $G$ , we have  $\alpha_e = \beta = 0$  for all  $e \in E$ .  $\square$

**Theorem 2.3.** All the facet-defining inequalities of  $SFP(G)$ , that are different from  $x_e \geq 0$  for some  $e \in E$ , are of the form  $a^t x \leq b$  with  $a \in \mathbb{R}_+^m$  and  $b \geq 0$  scalar.

*Proof.* Let  $a^t x \leq b$  be any facet-defining inequality for  $SFP(G)$  which is not  $x_e \geq 0$  for some  $e \in E$ . As  $a^t \mathbf{0} \leq b$ , we have  $b \geq 0$ . Suppose that for some edge  $e$ ,  $a_e < 0$ . As  $a^t x \leq b$  defines a facet different from  $x_e \geq 0$ , there exists a star forest  $F$  in  $G$  containing  $e$  such that  $a^t \chi^F = b$ . As  $F' = F \setminus \{e\}$  is also a star forest, we have  $a^t \chi^{F'} = a^t \chi^F - a_e = b - a_e > b$ . This contradicts the fact that  $a^t x \leq b$  is valid for  $SFP(G)$ .  $\square$

Given  $a^t x \leq b$  any facet-defining inequality of  $SFP(G)$ , the support graph of  $a^t x \leq b$  is the subgraph  $G_a = (V_a, E_a)$  of  $G$  induced by the edges  $e \in E$  such that  $a_e > 0$ . A tight star forest  $F$  with respect to  $a^t x \leq b$  is a star forest with which the associated incidence vector satisfies  $a^t x \leq b$  at equality. A star forest  $F$  is maximal with respect to an edge subset  $E' \subseteq E$  if for any edge  $e \in E' \setminus F$ ,  $F \cup \{e\}$  is not anymore a star forest.

Theorem 2.3 implies the following corollary.

**Corollary 2.4.** *Given  $a^t x \leq b$  any facet-defining inequality of  $SFP(G)$  with  $G_a = (V_a, E_a)$  its support graph, all the tight star forests with respect to  $a^t x \leq b$  are maximal with respect to  $E_a$ .*

**Lemma 2.5.** *Let  $\bar{G} = (\bar{V}, \bar{E})$  be any induced subgraph of  $G$ , then any facet-defining inequality for  $SFP(\bar{G})$  also defines a facet for  $SFP(G)$ .*

*Proof.* First notice that the theorem trivially holds for the trivial inequalities  $x(e) \leq 1$  for all  $e \in \bar{E}$ , since these inequalities define facets for  $SFP(G)$  for any graph  $G$ . Let  $I$  be any facet-defining inequality for  $SFP(\bar{G})$ , different from  $x(e) \leq 1$  for all  $e \in \bar{E}$ . Then  $I$  defines a facet for  $SFP(G)$  if we are able to show that for any edge  $ij \in E \setminus \bar{E}$ , there always exists a tight star forest  $F$  in  $\bar{G}$  with respect to  $I$  such that  $F \cup \{ij\}$  is also star forest in  $G$ . Suppose that for some edge  $ij \in E \setminus \bar{E}$ , no such  $F$  exists. This implies that for every tight star forest  $F$  in  $\bar{G}$  w.r.t  $I$ ,  $F \cup \{ij\}$  is not a star forest in  $G$ . In this case, exactly one node  $i$  or  $j$ , say  $i$ , belongs to  $\bar{V}$ . Moreover, as  $\bar{G}$  is an induced subgraph of  $G$ , every tight forest  $F$  in  $\bar{G}$  w.r.t  $I$  should contain an edge  $ik$  such that  $i$  is of degree 1 in  $F$  and  $k$  is of degree at least 2 in  $F$ . Otherwise,  $F \cup \{ij\}$  would be a star forest of  $G$ . Then every tight star forest  $F$  in  $\bar{G}$  w.r.t  $I$  also satisfies  $x(\delta_{\bar{G}}(i)) = 1$ . This, together with Theorem 2.2, implies that  $I$  should be  $x(\delta_{\bar{G}}(i)) \leq 1$ . Since  $I$  is different from  $x(e) \leq 1$  for all  $e \in \bar{E}$ ,  $i$  should be of degree at least 2 in  $\bar{G}$ . But then  $x(\delta_{\bar{G}}(i)) \leq 1$  is not valid for  $SFP(\bar{G})$ , a contradiction.  $\square$

## 2.2. Integer programming formulation for the MWSFP

In this subsection, we give an integer programming formulation for MWSFP. First we state the following lemma.

**Lemma 2.6.** *A graph is a star forest iff it does not contains 3-paths and 3-cycles.*

*Proof.* It can be immediately verified from the definition of a star forest.  $\square$

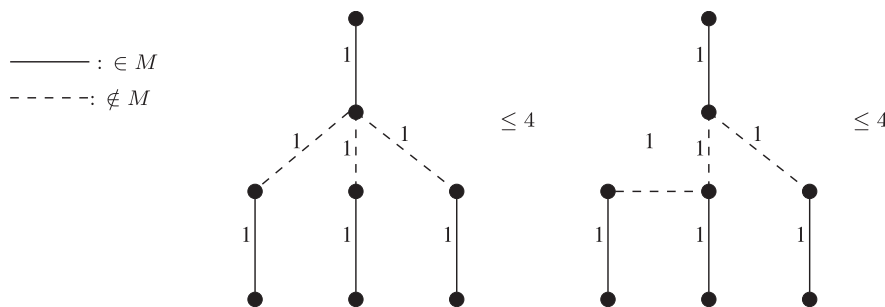
Let us consider the following integer program.

$$\begin{aligned}
 (IP) \quad & \max \quad c^T x \\
 & \text{s.t.} \\
 & x(P) \leq 2 && \text{for all } P \in \mathcal{P}_4 && (2.1) \\
 & x(C) \leq 2 && \text{for all } C \in \mathcal{C}_3 && (2.2) \\
 & 0 \leq x_e \leq 1 && \text{for all } e \in E && (2.3) \\
 & x \text{ integer}
 \end{aligned}$$

Inequalities (2.1), called the *3-path inequalities*, state the fact that a star forest can only take at most 2 edges in a 3-path. Similarly, inequalities (2.2), called the *3-cycle inequalities*, state the fact that a star forest can only take at most 2 edges in a 3-cycle. Inequalities (2.3) are the *trivial inequalities*.

**Theorem 2.7.** *(IP) is equivalent to the MWSFP.*

*Proof.* It is clear that by inequalities (2.1) and (2.2) in a solution of (IP) there is neither 3-paths and nor 3-cycles. By Lemma 2.6, this solution represents a star forest.  $\square$

FIGURE 2. A  $M$ -tree inequalities associated with a  $M$ -matching  $M$  of cardinality 4.

### 3. $M$ -TREE INEQUALITIES AND COMPLETE DESCRIPTION OF $SFP(G)$ IN TREES

In this section, we shall give a complete description of  $SFP(G)$  in trees. For this, let us first describe a class of inequalities which are valid for any graph, not only for trees.

#### 3.1. $M$ -tree inequalities

In this subsection,  $G$  is an arbitrary graph (not necessarily a tree).

**Definition 3.1** ( $M$ -tree). A  $M$ -tree  $\tau$  is a tree in which every non-pendant node is connected to exactly one pendant node (leaf).

Given a  $M$ -tree  $\tau$ , let us call  $M$ -matching of  $\tau$ , the set of the edges incident to the leaves of  $\tau$ .

**Definition 3.2** ( $M$ -tree inequality). The  $M$ -tree inequality associated with a  $M$ -tree  $\tau$  is the inequality  $x(\tau) \leq |M|$  where  $M$  is the  $M$ -matching of  $\tau$ .

We can remark that the  $M$ -tree inequalities generalize inequalities  $x_e \leq 1$  for  $e \in E$  and the 3-path inequalities. These are  $M$ -tree inequalities with  $|M| = 1$  and  $|M| = 2$ , respectively.

**Theorem 3.3.** The  $M$ -tree inequalities define facets for  $SFP(G)$ .

*Proof.* Let  $\tau$  be any  $M$ -tree and  $M$  the  $M$ -matching of  $\tau$ , let us consider the corresponding  $M$ -tree inequality

$$x(\tau) \leq |M|. \quad (3.1)$$

Let us first prove the validity. Let  $F$  be any star forest of  $G$ . We will prove the validity by showing that  $|F \cap \tau| \leq |M|$ . If  $F$  only contains the edges in  $M$ , then  $|F \cap \tau| \leq |M|$ . If  $F$  only contains edges in  $\tau \setminus M$  then  $|F \cap \tau| \leq |\tau \setminus M|$  and, from the definition of a  $M$ -tree,  $|\tau \setminus M| = |M| - 1 < |M|$ . Now suppose that  $F$  contains edges in both  $M$  and  $\tau \setminus M$ . If  $F \cap \tau$  is a matching then  $|F \cap \tau| \leq |M|$  since  $M$  covers all the nodes of  $\tau$ . So suppose that  $F \cap \tau$  is not a matching. Thus,  $F \cap \tau$  should contain a (sub)star  $S$  with, say  $u_2$ , as center, which contains at least two edges in  $F$ : one edge, say  $u_1u_2$ , which belongs to  $M$  and one other, say  $u_2v_2$ , which belongs to  $\tau \setminus M$ . As  $v_2$  should be also covered by  $M$ , there exists an edge  $v_1v_2 \in M$ . As  $u_1u_2$  and  $u_2v_2$  are in  $F$ ,  $v_1v_2 \notin F$ . Moreover, by definition of  $M$ -tree,  $v_1$  should be of degree 1 in  $\tau$ . It follows that each edge in  $F$  which belongs to  $\tau \setminus M$  (e.g.  $u_2v_2$ ) correspond exactly to another edge (e.g.  $v_1v_2$ ) in  $M \setminus F$ . Hence,  $|F \cap (\tau \setminus M)| \leq |M \setminus F|$  which implies  $|(F \cap (\tau \setminus M)) \cup (F \cap M)| \leq |(M \setminus F) \cup (F \cap M)| = |M|$ . As  $M \subset \tau$ ,  $|(F \cap (\tau \setminus M)) \cup (F \cap M)| = |F \cap \tau|$ . Thus,  $|F \cap \tau| \leq |M|$ .

Let us prove now that (3.1) defines a facet for  $SFP(G)$ . Suppose that there exists a facet-defining inequality  $\alpha^t x \leq \beta$  for  $SFP(T)$  such that all the star forests satisfying (3.1) at equality, satisfy also  $\alpha^t x \leq \beta$  at equality.

Since by Theorem 2.2,  $SFP(T)$  is full dimensional, it suffices to show that  $\alpha^t x \leq \beta$  is a positive multiple of (3.1).

Let us remark that  $M$  is a star forest satisfying (3.1) at equality and hence also satisfies  $\alpha^t x \leq \beta$  at equality, i.e.  $\alpha(M) = \beta$ . For any edge  $e \in E \setminus \tau$ , we can see that  $M \cup \{e\}$  is also star forest satisfying (3.1) at equality. Hence,  $\alpha(M \cup \{e\}) = \beta$ . This implies that  $\alpha_e = 0$  for all  $e \in E \setminus \tau$ .

Let  $u_2 v_2$  be any edge in  $\tau \setminus M$  and  $u_1 u_2$  and  $v_1 v_2$  be the edges in  $M$  incident to  $u_2$  and  $v_2$  respectively. We can see that  $M$ ,  $(M \setminus \{u_1 u_2\}) \cup \{u_2 v_2\}$  and  $(M \setminus \{v_1 v_2\}) \cup \{u_2 v_2\}$  are all star forests satisfying (3.1) at equality. This implies that  $\alpha_{u_1 u_2} = \alpha_{u_2 v_2} = \alpha_{v_1 v_2}$ . If we extend this to all the edges in  $\tau \setminus M$ , we obtain that  $\alpha_e = \alpha_{e'}$  for all  $e, e' \in \tau$ . Hence,  $\alpha^t x \leq \beta$  is a positive multiple of (3.1), which ends the proof of the theorem.  $\square$

### 3.2. Complete description of $SFP(G)$ in trees

From now on and throughout this section,  $G$  will be a tree denoted by  $T$ .

**Proposition 3.4.** *All the maximal star forests with respect to a  $M$ -tree  $\tau$  are of cardinality  $|M|$  where  $M$  is the  $M$ -matching of  $\tau$ .*

*Proof.* Let  $S$  be any maximal star forest in  $\tau$ . Let  $m = |M|$  then  $|V(\tau)| = 2m$  and  $|\tau| = 2m - 1$ . By the validity of  $M$ -tree inequalities, we have  $|S| \leq m$ . We will prove that  $|S| \geq m$  by showing that each edge in  $M$  correspond to an edge in  $S$ . Let  $v_1$  be any leaf in  $\tau$  and let  $v_2$  be the non-pendant node such that the edge  $v_1 v_2 \in M$ . We distinguish two cases:

- $v_1 \notin S$ . In this case,  $v_2$  should belong to  $V(S)$  since otherwise we can add  $v_1 v_2$  to  $S$  and  $S$  remains a star forest. Moreover,  $v_2$  should be of degree 1 in  $S$  since otherwise we can also add  $v_1 v_2$  to  $S$  and  $S$  remains a star forest. Thus, the edge  $v_1 v_2$  correspond to the edge incident to  $v_2$  in  $S$ .
- $v_1 \in S$ . Then the edge  $v_1 v_2$  should belong to  $S$  and hence it corresponds to itself.

Hence,  $|S| \leq m$  and  $|S| \geq m$  which implies that  $|S| = m$ .  $\square$

We will prove the following theorem.

**Theorem 3.5.** *The  $M$ -tree and nonnegativity inequalities completely define  $SFP(T)$ .*

*Proof.* Suppose that  $a^t x \leq b$  is any facet-defining inequality for  $SFP(T)$  which is not a  $M$ -tree inequality neither the nonnegativity inequality. Let  $G_a$  be the support graph of  $a^t x \leq b$ . We can suppose without loss of generality that  $G_a$  is a tree. Let  $T_a$  denote this tree. Hence,  $T_a$  is a subtree of  $T$ . We have two possible cases.

- $T_a$  is a  $M$ -tree. Let  $F$  be any tight star forest with respect to  $a^t x \leq b$ . By Corollary 2.4,  $F$  is maximal with respect to  $T_a$ . Consequently, by Proposition 3.4,  $F$  is tight with respect to the  $M$ -tree inequalities. Contradiction to the fact that  $a^t x \leq b$  is a facet defining inequality different from a  $M$ -tree inequality.
- $T_a$  is not a  $M$ -tree. Hence, in  $T_a$  there must be one non-pendant node of one of the two following types:

**Type 1.** A non-pendant node not connected to a leaf of  $T_a$ .

**Type 2.** A non-pendant node connected to at least two leaves of  $T_a$ .

We distinguish two cases:

**Case 1.**  $T_a$  contains only non-pendant nodes of Type 2. In this case, one can obtain a  $M$ -tree  $\tau$  from  $T_a$  by keeping for each non-pendant node, only one leaf connecting to it. Let  $x(\tau) \leq |M_\tau|$  be the  $M$ -tree inequality associated with  $\tau$ . The following remark can be easily proved.

**Remark 3.6.** Given a non-pendant node  $v$  of Type 2, any star forest  $F$  satisfying  $a^t x \leq b$  at equality must be maximal in  $\tau$  and contains either all the leaves connected to  $v$  or no of them.



Let  $F$  be any tight star forest with respect to  $a^t x \leq b$ . We have  $F \cap \tau$  is a maximal star forest with respect to  $\tau$ . Since by Proposition 3.4, every maximal star forest in  $\tau$  satisfies  $x(\tau) \leq |M_\tau|$  at equality, it follows that every star forest satisfying  $a^t x \leq b$  at equality satisfies also the  $M$ -tree inequality associated with  $\tau$  at equality. This contradicts the fact that  $a^t x \leq b$  is a facet-defining inequality.

**Case 2.**  $T_a$  contains at least one non-pendant node of Type 1. We will show the following lemma.

**Lemma 3.7.** *There exists a non-pendant node  $s$  of Type 1 such that all the other non-pendant nodes of Type 1 belong to a same connected component obtained by the removal of  $s$  from  $T_a$ .*

*Proof.* We give a constructive proof.

**Initialization.** Let us choose any non-pendant node  $s_0$  of Type 1 and let  $i = 0$ .

**Iteration  $i$ .** Suppose that  $T_1^i, \dots, T_{p_i}^i$  are the subtrees of  $T_a$  obtained if  $s_i$  is removed from  $T_a$  and suppose without loss of generality that  $T_1^i$  is always the tree which contains  $s_0$ . We have two possible cases.

- If all the other non-pendant nodes of Type 1 belong to a same tree  $T_{k_i}$  ( $1 \leq k_i \leq p_i$ ) then  $s_i$  is a non-pendant node of type 1 satisfying the condition stated in the lemma. STOP.
- If the other non-pendant nodes of Type 1 belong to at least two trees. Suppose without loss of generality that  $T_{p_i}^i$  is one of them. Let us choose  $s_{i+1}$  to be any non-pendant node of Type 1 in  $T_{p_i}^i$  and set  $i \leftarrow i + 1$ . Reiterate.

We have the following remark.

**Remark 3.8.**  $T_1^i$  contains all the node previously chosen  $s_0, \dots, s_{i-1}$  which are all distincts. The set containing these nodes is called the kernel.

Thus, the procedure should be ended by finding a non-pendant node of Type 1 satisfying the condition stated in the lemma after at most  $|V(T_a)|$  iterations since the kernel grows after each iteration.  $\square$

Let  $s$  be a non-pendant node of Type 1 satisfying the condition stated in Lemma 3.7. Let  $T_1$  be the connected component which contains the other non-pendant nodes of Type 1 obtained by removing  $s$  from  $T_a$ . We can observe that  $s$  is connected to  $T_1$  by only one edge and the subgraph  $H$  of  $T_a$  induced by the edges in  $T_a \setminus T_1$  is a tree. Moreover,  $H$  is either a  $M$ -tree or a tree which contains some non-pendant nodes of Type 2 but no one of Type 1. Hence, in all the cases as we have proved in Case 1,  $H$  contains a  $M$ -tree  $\tau$  which contains all the non-pendant nodes of  $H$ . Let  $u$  be the neighbour of  $s$  in  $T_1$ , i.e. the edge  $su \in H$ . Observe that  $su$  is a bridge in  $T_a$  that links  $H$  to  $T_1$ . We can see that any maximal star forest in  $T_a$ , whatever it contains  $su$  or not, contains a maximal star forest in  $H$  (if a maximal star forest in  $T_a$  does not contain  $su$ , it must contain some edge  $sv$  where  $v$  is of degree at least 2 in the star forest). The latter, by Remark 3.6, contains a maximal star forest in  $\tau$ . By Corollary 2.4, any star forest  $F$  satisfying  $a^t x \leq b$  at equality is maximal in  $T_a$ . Hence  $F \cap \tau$  is a maximal star forest in  $\tau$ . By Proposition 3.4,  $F \cap \tau$  satisfies the  $M$ -tree inequality associated with  $\tau$  at equality. This contradicts the fact that  $a^t x \leq b$  is facet-defining.  $\square$

#### 4. POLYHEDRAL RESULTS ON CYCLES

In this section,  $G$  will be a chordless cycle  $C = (V(C), E(C))$  of  $n$  nodes with  $V(C) = \{1, 2, \dots, n\}$  numbered clockwise, i.e. the  $n$  edges in  $E(C)$  will be  $e_i = (i, i+1)$  for  $i = 1, \dots, n-1$ , and the edge  $e_n = (n, 1)$ . We will sometimes use  $|C|$  instead of  $n$ ,  $|V(C)|$  or  $|E(C)|$  which are all equal. The edges in  $C$  are weighted by a vector  $c \in \mathbb{R}^n$  where  $c_i$  is the weight associated with edge  $e_i$  for  $i = 1, \dots, n$ . We also consider  $L(C) = (V^L, E^L)$  the line graph of  $C$  where the nodes correspond to the edges of  $C$ , and two nodes of  $L(C)$  are adjacent if the corresponding edges are adjacent in  $C$ . Note that  $L(C)$  is also a cycle node-weighted by vector  $c$ . For the sake of convenience, given a node  $1 \leq i \leq n$  and an integer  $t > 0$ , let  $i+t$  designate the node  $i+t$  if  $i+t \leq n$  and the node  $(i+t) \bmod n$  if  $(i+t) > n$ . For two nodes  $u$  and  $v$  with  $v = u+t$  for some integer  $t > 0$  in  $C$ , let  $C(u, v)$  denote the path  $(u+1, \dots, u+t-1)$  of  $C$  between  $u+1$  and  $u+t-1$  (note that the path does not contain



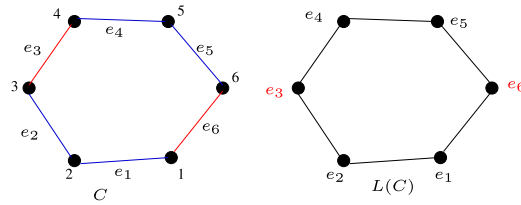


FIGURE 3. A cycle  $C$  and its line graph  $L(C)$ . The edge subset  $\{e_1, e_2, e_4, e_5\}$  is a star forest in  $C$  and its complement  $\{e_3, e_6\}$  is an edge dominating set in  $C$  and a dominating set in  $L(C)$ .

$u$  and  $v$ ). For two edges  $e$  and  $f$  in  $C$  with  $e = e_i$  and  $f = e_{i+t}$  for some integer  $t > 0$ , let  $C(e, f)$  denote the path consisting of the edges  $e_{i+1}, \dots, e_{i+t-1}$  (note that the path does not contain neither  $e_i$  nor  $e_{i+t}$ ). In what follows, we will establish the relations between star forests, dominating sets and edge dominating sets on cycles. Then, using these relations with the polyhedral results in [9] and [8], we will derive a complete description of  $SFP(C)$ . Note that in [18], a linear time algorithm for the MWSFP when  $G$  is a cactus have been presented, hence the MWSFP can be solved in linear time in cycles.

#### 4.1. Star forests, dominating sets and edge dominating sets on cycles

In the context of a cycle, Lemma 2.6 can be restated as follows.

**Lemma 4.1.** *An edge subset  $F \subset C$  is a star forest if and only if  $F$  does not contain any 3-path.*

The following lemma establishes the link between edge dominating sets and star forests in a cycle.

**Lemma 4.2.** *The complement of a star forest  $F$  in  $C$  is an edge dominating set and vice versa.*

*Proof.* ( $\Rightarrow$ ) Let  $F \subseteq E(C)$  be a star forest in  $C$  and let  $\bar{F} = C \setminus F$ . Suppose that  $\bar{F}$  is not an edge dominating set. Then there exists an edge  $e_i \in F$  not adjacent to any edge in  $\bar{F}$ . As  $C$  is a cycle, the neighbors of  $e_i$ ,  $e_{i-1}$  and  $e_{i+1}$  do not belong to  $\bar{F}$ . Hence,  $e_{i-1}, e_i, e_{i+1}$  form a path of length 3 in  $F$ , a contradiction with Lemma 4.1.

( $\Leftarrow$ ) Let  $ED \subseteq C$  be an edge dominating set in  $C$ . Let  $F = C \setminus ED$  and suppose that  $F$  is not a star forest. Then  $F$  contains a 3-path  $(v_1, v_2, v_3, v_4)$ . We can see that the edge  $(v_2, v_3)$  is not dominated by  $ED$  implying that  $ED$  is not an edge dominating set, a contradiction.  $\square$

By the one-to-one correspondence between the nodes of  $L(C)$  and the edges of  $C$ , we have the following result.

**Lemma 4.3.** *Any edge dominating set in  $C$  is a dominating set in  $L(C)$  and vice versa.*

The following lemma reformulates these relations in polyhedral terms for the polytopes  $SFP(C)$ ,  $EDP(C)$  and  $DP(L(C))$ .

**Lemma 4.4.** *The following statements are equivalent:*

- (i)  $\alpha^t y \geq \beta$  with  $y \in \mathbb{R}^n$  defines a facet for  $DP(L(C))$ ,
- (ii)  $\alpha^t x \geq \beta$  with  $x \in \mathbb{R}^n$  defines a facet for  $EDP(C)$ ,
- (iii)  $\alpha^t x \leq \sum_{e \in E(C)} \alpha(e) - \beta$  with  $x \in \mathbb{R}^n$  defines a facet for  $SFP(C)$ .

Hence, the polytopes  $SFP(C)$ ,  $DP(L(C))$  and  $EDP(C)$  are equivalent in the sense that there is a one-to-one correspondence between their facets.

*Proof.* Note that these polytopes are all defined in  $\mathbb{R}^n$  and are full dimensional. The lemma follows from the relations described in Lemmas 4.2 and 4.3 as they are all preserved under affine transformations.  $\square$

As in [8], a complete linear description for  $DP(L(C))$  is given, by Lemma 4.4, we can also derive complete descriptions for  $SFP(C)$  and  $EDP(C)$ . We will explicit these complete descriptions in the following section.

## 4.2. Complete description of $SFP(C)$

Let us consider the graph  $L(C)$  and the polytope  $DP(L(C))$  in  $\mathbb{R}^n$  whose component indexed by the nodes in  $V^L$ . In [9] and [8], Bouchakour *et al.* give the following integer formulation for MWDSP,

$$\begin{aligned} \min c^t x \\ 0 \leq x_v \leq 1 \end{aligned} \quad \text{for all } v \in V^L \quad (4.1)$$

$$\begin{aligned} x(N(v)) \geq 1 \\ x_v \text{ integer} \end{aligned} \quad \text{for all } v \in V^L \quad (4.2)$$

They have also characterized two classes of facet-defining inequalities for  $DP(L(C))$ .

**Theorem 4.5.** [9] *The inequality*

$$x(V^L) \geq \left\lceil \frac{|C|}{3} \right\rceil \quad (4.3)$$

*defines a facet for  $DP(L(C))$  if and only if either  $|C| = 3$  or  $|C| \geq 4$  and  $|C|$  is not a multiple of 3.*

**Theorem 4.6.** [8] *Let  $W = \{v_1, \dots, v_p\}$  be a subset of  $p \geq 3$  nodes in  $V^L$  satisfying the following conditions:*

**C1:**  *$p$  is odd and  $v_1 < v_2 < \dots < v_p$ ,*

**C2:**  *$|C(v_i, v_{i+1})| = 3k_i$ ,  $k_i \geq 1$ , for  $i = 1, \dots, p$  with  $v_{p+1} = v_1$ .*

*Then the constraint*

$$2 \sum_{v \in W} x_v + \sum_{v \in V^L \setminus W} x_v \geq \sum_{i=1}^p k_i + \left\lceil \frac{p}{2} \right\rceil \quad (4.4)$$

*defines a facet for  $DP(L(C))$ .*

Let us apply Lemma 4.4 to derive facet-defining inequalities for  $SFP(C)$ . It is clear that applying Lemma 4.4 to inequalities (4.1) yields the trivial inequalities

$$0 \leq x_e \leq 1 \text{ for all } e \in E(C),$$

for  $SFP(C)$ .

By applying Lemma 4.4 to inequalities (4.2), we get the 3-path inequalities

$$x(P) \leq 2 \text{ for all path of length 3 in } C,$$

for  $SFP(C)$  which have been described in Section 2. The following proposition can be obtained by applying Lemma 4.4 to inequalities (4.3).

**Proposition 4.7.** *The cycle inequality*

$$x(E(C)) \leq \left\lfloor \frac{2|C|}{3} \right\rfloor \quad (4.5)$$

*defines a facet for  $SFP(C)$  when, either  $|C| = 3$  or  $|C| \geq 4$  and  $|C|$  is not multiple of 3.*

Let  $W = \{v_1, \dots, v_p\} \subset V^L$  be a subset of nodes of  $V^L$  as defined in Theorem 4.6, let  $f_i$  denote the edge in  $C$  corresponding to the node  $v_i$  in  $L(C)$  and let  $M = \{f_1, \dots, f_p\}$ . For  $i = 1, \dots, p$ , we define  $C(f_i, f_{i+1})$  with  $f_{p+1} = f_1$  to be the path between  $f_i$  and  $f_{i+1}$  in  $C$  which does not contain any edge in  $M$ . The conditions C1 and C2 on the set  $W$  can be transformed into conditions M1 and M2 on the set  $M$  as follows:

**M1:**  *$M$  is a matching of odd cardinality,*

**M2:**  *$|C(f_i, f_{i+1})| = 3k_i$ ,  $k_i \geq 1$ , for  $i = 1, \dots, p$  with  $f_{p+1} = f_1$ .*

We then deduce the following result by applying Lemma 4.4 to inequalities (4.4).

**Proposition 4.8.** *The matching-cycle inequalities*

$$2x(M) + x(E(C) \setminus M) + \leq 2 \sum_{i=1}^p k_i + \left\lfloor \frac{3p}{2} \right\rfloor \text{ for all } M \subset E(C) \text{ satisfying conditions M1 and M2.} \quad (4.6)$$

define facets for  $SFP(C)$ .

*Proof.* Given a matching  $M$  of  $C$  satisfying conditions M1 and M2, suppose that  $v_1, \dots, v_p$  are the nodes in  $L(C)$  corresponding respectively to  $f_1, \dots, f_p$ . It is easy to see that  $v_1, \dots, v_p$  satisfy conditions C1 and C2 of Theorem 4.6, and hence, we have that

$$2 \sum_{v \in W} x_v + \sum_{v \in V^L \setminus W} x_v \geq \sum_{i=1}^p k_i + \left\lceil \frac{p}{2} \right\rceil$$

defines a facet for  $DP(L(C))$ . The result thus follows from Lemma 4.4.  $\square$

In [8], Bouchakour *et al.* have shown the following theorem.

**Theorem 4.9.** [8] *A complete linear description for  $DP(L(C))$  is given by inequalities (4.1), (4.2), (4.3), (4.4).*

As a direct consequence, we have the following result.

**Corollary 4.10.** *When  $G$  is a cycle,  $SFP(G)$  is completely described by the trivial inequalities, the 3-path inequalities, the cycle inequality (4.5) and the matching-cycle inequalities (4.6).*

## 5. CONCLUSIONS

In this paper, we have presented an IP formulation for MWSFP. We have also given two complete linear descriptions for  $SFP(G)$ , the star forests polytope for the cases where  $G$  is a tree and  $G$  is a cycle. An interesting direction for future works would be to exploit these results to derive a complete linear description for  $SFP(G)$  when  $G$  is a cactus. Our complete description for  $SFP(G)$  when  $G$  is a tree could be helpful to find an exact solution for the MWSFP in general graphs. A star forest is always a subgraph of a spanning tree in  $G$  and a complete linear description of the spanning tree polytope is known.

*Acknowledgements.* We would like to thank one anonymous referee for her/his helpful comments which greatly improved the presentation of the paper.

## REFERENCES

- [1] A. Agra, D. Cardoso, O. Cerfeira and E. Rocha, A spanning star forest model for the diversity problem in automobile industry. In: *ECCO XVIII, Minsk* (2005).
- [2] S. Athanassopoulos, I. Caragiannis, C. Kaklamanis and M. Kyropoulou, An improved approximation bound for spanning star forest and color saving. In: *MFCs*. Springer, Berlin, Heidelberg (2009) 90–101.
- [3] M. Baïou and F. Barahona, On the integrality of some facility location polytopes. *SIAM J. Discrete Math.* **23** (2009) 665–679.
- [4] M. Baïou and F. Barahona, Simple extended formulation for the dominating set polytope via facility location. Tech. Rep. RC25325, IBM Research (2012).
- [5] M. Baïou and F. Barahona, Algorithms for minimum weighted dominating sets in cycles and cacti. Tech. Rep. RC25488, IBM Research (2014).
- [6] M. Baïou and F. Barahona, The dominating set polytope via facility location. Combinatorial Optimization. *ISCO 2014*. In Vol. 8596 of *Lecture Note Computer Sciences* (2014) 38–49.
- [7] V. Berry, S. Guillemot, F. Nicholas, and C. Paul, On the approximation of computing evolutionary trees. In: *Proc. of the Eleventh Annual International Computing and Combinatorics Conference*. Springer, Berlin, Heidelberg (2005) 115–123.
- [8] M. Bouchakour, T.M. Contenza, C.W. Lee, A.R. Mahjoub, On the dominating set polytope. *Eur. J. Combin.* **29** (2008) 652–661.

- [9] M. Bouchakour and A.R. Mahjoub, One-node cutsets and the dominating set polytope. *Discrete Math.* **165/166** (1997) 101–123.
- [10] B.-M. Bui-Xuan, J.A. Telle and M. Vatshelle, Boolean-width of graphs. *Theoret. Comput. Sci.* **412** (2011) 5187–5204.
- [11] D. Chakrabarty and G. Goel, On the approximability of budgeted allocations and improved lower bounds for submodular welfare maximization and GAP. In: *FOCS '08. IEEE 49th Annual IEEE Symposium on Foundations of Computer Science*, 1975 (2008) 687–696.
- [12] N. Chen, R. Engelberg, C.T. Nguyen, P. Raghavendra, A. Rudra and G. Singh, Improved approximation algorithms for the spanning star forest problem. In: *APPROX/RANDOM* In Vol. 4627 of *Lecture Notes in Computer Science book series* (2007) 44–58.
- [13] T.W. Haynes, P.J. Slater and S.T. Hedetniemi, Fundamentals of Domination in Graphs. CRC Press, Boca Raton, FL (1998).
- [14] J. He and H. Liang, On variants of the spanning star forest problem. In: *Proc. of FAW-AAIM* (2011) 70–81.
- [15] T. Ito, N. Kakimura, N. Kamiyama, Y. Kobayashi and Y. Okamoto, Minimum-cost  $b$ -edge dominating sets on trees. In Vol. 8880 of *Lecture Notes Computer Sciences* (2014) 195–207.
- [16] A.R. Mahjoub, Polytope des absorbants dans une classe de graphes seuil. *Annal. Discrete Math.* **17** (1983) 443–452.
- [17] C.T. Nguyen, J. Shen, M. Hou, L. Sheng, W. Miller and L. Zhang, Approximating the spanning star forest problem and its applications to genomic sequence alignment. In: *Proc. of SODA* (2007) 645–654.
- [18] V.H. Nguyen, The maximum weight spanning star forest problem on cactus graphs. *Discrete Math. Algorithms App.* **7** (2015) 1550018.
- [19] A. Saxena, Some results on the dominating set polytope of a cycle. Technical Report CMU (2004).
- [20] M. Yannakakis and F. Gavril, Edge dominating sets in graphs. *SIAM J. Appl Math.* **38** (1980) 364–372.