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# SYMMETRY GROUP OF THE EQUIANGULAR CUBED SPHERE

JEAN-BAPTISTE BELLET

**ABSTRACT.** The equiangular cubed sphere is a spherical grid, widely used in computational physics. This paper deals with the symmetry group of this grid, *i.e.* the group of the orthogonal transformations that leave the cubed sphere invariant. The main result is that it coincides with the full symmetry group of the cube. In the proof, metric properties of the cubed sphere are investigated. Indeed, we identify the shortest arcs of great circles on the cubed sphere. In particular, we match the minimal arcs with the vertices of a cuboctahedron.

## 1. INTRODUCTION

**1.1. Cubed sphere.** Various fields in computational physics, for instance climatology modelling [27], involve numerical computations on the sphere. This includes the use of spherical grids [28]. The grids obtained by radial projection of a circumscribed cube on the sphere are among the most employed. These *cubed sphere grids* have been originally introduced in [24], and further studied, for example in [12, 16–18, 20–22]. A wide variety of numerical methods have been successfully adapted to cubed sphere grids, *e.g.* in [2–9, 11, 13, 19, 23, 25, 26] and the references therein.

This paper deals with mathematical properties of a cubed sphere structured by equiangular great circles: the *equiangular cubed sphere*  $CS_N$  [2, 18], depicted in Figure 1 and in Appendix A. As can be observed, the cubed sphere is quasi-uniform, is not polarized along a specific axis, and is shaped by the cube, including discontinuities accross “edges” (radial projection of the edges of the circumscribed cube). Furthermore, some symmetry properties, such as invariance under permutation of the cartesian coordinates, can be noticed.

The symmetry of the cubed sphere has been previously used in various contexts, for instance in [2, 14, 15, 22]. However, to the best of our knowledge, a systematic study of the isometries of  $CS_N$  is not available so far. This paper aims at filling a gap in this direction by identifying the full set of symmetries of  $CS_N$ .

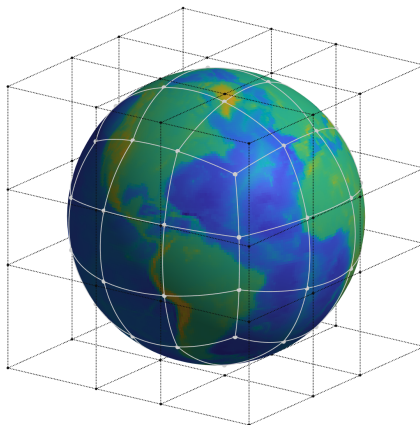


FIGURE 1. Cubed sphere grid  $CS_N$  ( $N = 3$ ). A cartesian grid  $C_N$  (black dots) is defined on the faces of the cube  $[-1, 1]^3$ , by means of the two-dimensional grid  $T_N \times T_N$ , where  $T_N = \{\tan \frac{k\pi}{2N}, -\frac{N}{2} \leq k \leq \frac{N}{2}\}$  is an equiangular grid. The cubed sphere grid  $CS_N$  (white dots) is defined as the radial projection of  $C_N$  on the unit sphere:  $CS_N = \rho(C_N)$ , with  $\rho : x \in \mathbb{R}^3 \setminus \{0\} \mapsto \frac{x}{\|x\|} \in \mathbb{S}^2$ . On the faces of the cube, the grid  $C_N$  defines a mesh (dotted straight lines). The projection  $\rho$  maps this mesh onto a mesh of the sphere: the (dotted) straight lines are mapped onto equiangular arcs of great circles (white curves).

**1.2. Main results.** We study the symmetries of the cubed sphere, by means of its symmetry group [1]:

**Definition 1.** The symmetry group of  $\text{CS}_N$  is the set  $\mathcal{G}_N$  of the orthogonal matrices that leave  $\text{CS}_N$  invariant:

$$\mathcal{G}_N = \{Q \in \mathbb{R}^{3 \times 3} : Q^\top Q = QQ^\top = I_3 \text{ and } Q\text{CS}_N = \{Qu, u \in \text{CS}_N\} = \text{CS}_N\}.$$

For  $N = 1$ ,  $\text{CS}_1 = \{-1/\sqrt{3}, 1/\sqrt{3}\}^3$  is a scaling of  $\{-1, 1\}^3$ . Hence the group  $\mathcal{G}_1$  is exactly the symmetry group  $\mathcal{O}$  of the cube  $\{-1, 1\}^3$ :  $\mathcal{G}_1 = \mathcal{O}$ . This group is well known: it is isomorphic to the group  $\mathfrak{S}_4 \times \mathbb{Z}/2\mathbb{Z}$  [1, pp. 37, 38, 55]; any symmetry of the cube is indeed identified with a permutation of the four principal diagonals, combined with a toggle for inversion of the cube, or not. We refer to Appendix B for a matricial representation of  $\mathcal{O}$ .

The main result of this paper is that  $\mathcal{G}_N = \mathcal{O}$ , for every  $N \geq 1$ :

**Theorem 2** (Symmetry group of the equiangular cubed sphere). *Let  $N \geq 1$ . The symmetry group  $\mathcal{G}_N$  of the cubed sphere  $\text{CS}_N$  coincides with the symmetry group  $\mathcal{O}$  of the cube  $\{-1, 1\}^3$ . In other words, an orthogonal matrix  $Q$  leaves  $\text{CS}_N$  invariant if, and only if, it leaves  $\{-1, 1\}^3$  invariant.*

The most difficult inclusion to be proved is  $\mathcal{G}_N \subset \mathcal{O}$ . One possible approach deduces this result from  $\mathcal{O} \subset \mathcal{G}_N$ , using group theory as Appendix C. Instead of this theoretical approach, this paper focuses on a direct proof which investigates metric properties of  $\text{CS}_N$ . This direct approach has the advantage of deepening the knowledge of  $\text{CS}_N$ . More precisely, we solve the problem of the shortest arcs between distinct points on  $\text{CS}_N$ :

$$\min\{\theta(u, v) := \arccos u \cdot v; u \in \text{CS}_N, v \in \text{CS}_N, u \neq v\}. \quad (\text{P})$$

One original feature of problem (P) is that  $u$  and  $v$  are allowed to belong to distinct grid lines, or even distinct panels; in particular, any spherical diagonal of the mesh is realizable. The main difficulty of problem (P) comes from this specificity. This is somehow the lock of Theorem 2. Any solution of (P) is given by the following theorem.

**Theorem 3** (Shortest arcs on the equiangular cubed sphere). *Let  $u, v \in \text{CS}_N$  with  $u \neq v$ . The arc  $\{u, v\}$  is optimal for (P) if, and only if, there exists an isometry of the cube,  $Q \in \mathcal{O}$ , such that:*

$$\begin{cases} \{u, v\} = \{Q\rho(1, -\tan \frac{\pi}{4N}, 1), Q\rho(1, \tan \frac{\pi}{4N}, 1)\}, & \text{if } N \text{ is odd,} \\ \{u, v\} = \{Q\rho(1, 0, 1), Q\rho(1, \tan \frac{\pi}{2N}, 1)\}, & \text{if } N \text{ is even.} \end{cases}$$

In other words, the minimal arcs are described by the orbit of the arc  $\{\rho(1, -\tan \frac{\pi}{4N}, 1), \rho(1, \tan \frac{\pi}{4N}, 1)\}$ , or  $\{\rho(1, 0, 1), \rho(1, \tan \frac{\pi}{2N}, 1)\}$ , for the group action  $(Q, \{u, v\}) \mapsto \{Qu, Qv\}$ ,  $Q \in \mathcal{O}$ ,  $\{u, v\} \subset \text{CS}_N$ . The matricial representation (11) of  $\mathcal{O}$  shows immediately the following corollary:

**Corollary 4.** (i) *If  $N$  is odd, there are precisely 12 minimal arcs on  $\text{CS}_N$  (one per edge):*

$$\{\rho(-\delta, \epsilon_1, \epsilon_2), \rho(\delta, \epsilon_1, \epsilon_2)\}, \{\rho(\epsilon_1, -\delta, \epsilon_2), \rho(\epsilon_1, \delta, \epsilon_2)\}, \{\rho(\epsilon_1, \epsilon_2, -\delta), \rho(\epsilon_1, \epsilon_2, \delta)\}, \delta = \tan \frac{\pi}{4N}, \epsilon_1 = \pm 1, \epsilon_2 = \pm 1.$$

(ii) *If  $N$  is even, there are precisely 24 minimal arcs on  $\text{CS}_N$  (two per edge):*

$$\{\rho(0, \epsilon_1, \epsilon_2), \rho(\delta, \epsilon_1, \epsilon_2)\}, \{\rho(\epsilon_1, 0, \epsilon_2), \rho(\epsilon_1, \delta, \epsilon_2)\}, \{\rho(\epsilon_1, \epsilon_2, 0), \rho(\epsilon_1, \epsilon_2, \delta)\}, \delta = \pm \tan \frac{\pi}{2N}, \epsilon_1 = \pm 1, \epsilon_2 = \pm 1.$$

Roughly speaking, the minimal arcs are “short arcs around the midpoints on the edges”, as displayed in Figure 2. These midpoints match with the vertices of the *cuboctahedron* (12). The proof of Theorem 2 is mainly based on the invariance of this cuboctahedron under  $\mathcal{G}_N$ .

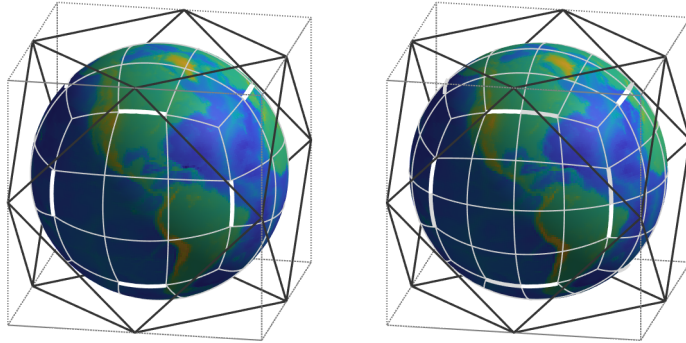


FIGURE 2. The bold arcs, around the midpoints on the edges, are minimal arcs on  $\text{CS}_N$ . Left:  $N$  is odd ( $N = 3$ ); right:  $N$  is even ( $N = 4$ ). These arcs match with the vertices of a cuboctahedron (black line).

**1.3. Organization of the paper.** The paper is organized as follows. In Section 2, we prove Theorem 2. In Section 3, we prove Theorem 3. The proof is based on a series of lemmas which are proved in Section 4. In Appendix A, we recall the definition of  $\text{CS}_N$ . Some geometrical and matricial properties of  $\mathcal{O}$  are given in Appendix B. In Appendix C, we give a proof of Theorem 2, based on group theory.

## 2. PROOF OF THE MAIN THEOREM

**Lemma 5.** *Any symmetry of  $\{-1, 1\}^3$  leaves  $\text{CS}_N$  invariant, i.e.  $\mathcal{O} \subset \mathcal{G}_N$ .*

*Proof.* Using  $(u_1, u_2, u_3) = |u_j| \left( \frac{u_1}{|u_j|}, \frac{u_2}{|u_j|}, \frac{u_3}{|u_j|} \right)$ , the cubed sphere satisfies:

$$\text{CS}_N = \{u = (u_1, u_2, u_3) \in \mathbb{S}^2 : \exists j \in \{1, 2, 3\}, u_j \neq 0 \text{ and } \forall i \in \{1, 2, 3\} \setminus \{j\}, \frac{u_i}{|u_j|} \in \text{T}_N\}.$$

Let  $Q \in \mathcal{O}$ . Let  $u \in \text{CS}_N$ . There exists  $1 \leq j \leq 3$  such that  $u_j \neq 0$ , and  $\forall 1 \leq i \leq 3, \frac{u_i}{|u_j|} \in \text{T}_N$  (this is also true if  $i = j$ ). We show that  $Qu \in \text{CS}_N$ . Let  $\sigma \in \mathfrak{S}_3$  and  $\epsilon \in \{-1, 1\}^3$ , such that  $Q = [\epsilon_1 e_{\sigma_1} \epsilon_2 e_{\sigma_2} \epsilon_3 e_{\sigma_3}]$ . Since  $Q$  is an isometry,  $\|Qu\| = \|u\| = 1$ :  $Qu \in \mathbb{S}^2$ . The component  $\sigma_j$  of  $Qu$  is  $e_{\sigma_j} \cdot Qu = \epsilon_j u_j \neq 0$ . To finish with, the component  $\sigma_k$  of  $Qu$  is  $e_{\sigma_k} \cdot Qu = \epsilon_k u_k$ , thus  $\frac{e_{\sigma_k} \cdot Qu}{|e_{\sigma_j} \cdot Qu|} = \frac{\epsilon_k u_k}{|\epsilon_j| |u_j|}$ , with  $\frac{\epsilon_k}{|\epsilon_j|} = \pm 1$ , and  $\frac{u_k}{|u_j|} \in \text{T}_N$ . Since  $\text{T}_N$  is symmetric with respect to 0,  $\frac{e_{\sigma_k} \cdot Qu}{|e_{\sigma_j} \cdot Qu|} \in \text{T}_N$ . We have proved that  $Q\text{CS}_N \subset \text{CS}_N$ . Since  $Q$  is bijective,  $Q\text{CS}_N = \text{CS}_N$ .  $\square$

*Remark 6.* The result can be extended for other cubed sphere grids: the equiangular property of the grid  $\text{T}_N$  does not matter; the key property is the symmetry of  $\text{T}_N$ .

**Theorem 7.** *Let  $\Omega$  be the cuboctahedron (12). Let  $Q \in \mathcal{G}_N$ . The isometry  $Q$  leaves  $\Omega$  invariant.*

*Proof. Case 1:  $N$  is odd.* We introduce the midpoints of the minimal arcs:

$$\Omega' = \left\{ \frac{1}{2}(u + v), \{u, v\} \text{ is minimal for (P)} \right\}.$$

By Corollary 4,  $\Omega'$  is a scaling of  $\Omega$ :

$$\Omega' = \{(2 + \tan^2 \frac{\pi}{4N})^{-1/2} x, x \in \Omega\}.$$

We prove that  $\Omega'$  is left invariant by  $Q$ . Fix  $w = \frac{u+v}{2} \in \Omega'$  with  $\{u, v\}$  minimal. Then  $\{Qu, Qv\}$  is minimal due to  $Q \in \mathcal{G}_N$ . As a result  $Qw = \frac{Qu+Qv}{2} \in \Omega'$ . Thus  $Q\Omega' \subset \Omega'$ . Finally  $Q\Omega = \Omega$ .

*Case 2:  $N$  is even.* We prove that  $\rho(\Omega) = \{2^{-1/2}x, x \in \Omega\}$  is left invariant by  $Q$ . By Corollary 4,

$$\rho(\Omega) = \{u \in \text{CS}_N : n(u) = 2\},$$

where  $n(u)$  denotes the number of  $v$  such that  $\{u, v\}$  is minimal:

$$n(u) = \#\{v \in \text{CS}_N : \{u, v\} \text{ is minimal for (P)}\}, \quad u \in \text{CS}_N.$$

By the way, the arc  $\{u, v\}$  is minimal if, and only if,  $\{Qu, Qv\}$  is minimal (due to  $Q \in \mathcal{G}_N$ ). At the end, for every  $u \in \text{CS}_N$ ,  $n(u) = n(Qu)$ , and

$$u \in \rho(\Omega) \Leftrightarrow n(u) = 2 \Leftrightarrow n(Qu) = 2 \Leftrightarrow Qu \in \rho(\Omega).$$

Then  $Q\rho(\Omega) = \rho(\Omega)$  and  $Q\Omega = \Omega$ .  $\square$

*Proof of Theorem 2.* We have proved  $\mathcal{O} \subset \mathcal{G}_N$  in Lemma 5. The converse is a corollary to Theorems 7 and 21: any  $Q \in \mathcal{G}_N$  leaves the cuboctahedron  $\Omega$  invariant, so it leaves the cube  $\{-1, 1\}^3$  invariant.  $\square$

## 3. PROOF OF THEOREM 3

In this section we minimize the great circle distance  $\theta(u, v) = \arccos u \cdot v$  between  $u \neq v \in \text{CS}_N$ . Briefly, any minimal arc on  $\text{CS}_N$  is minimal on the edge  $(I) \cap (V)$ , up to an isometry of the cube.

**Notation 8.** On the edge  $(I) \cap (V) = \{\rho(1, \tan \frac{k\pi}{2N}, 1), |k| \leq \frac{N}{2}\}$ , the shortest arc-length is denoted by

$$m^* := \min\{\theta(u, v), u \neq v \in (I) \cap (V)\}. \quad (1)$$

**Theorem 9.** *Let  $u \neq v \in \text{CS}_N$ . If  $\{u, v\}$  is optimal for (P), then there exists  $Q \in \mathcal{O}$  such that  $\{Qu, Qv\}$  is optimal for (1), on the edge  $(I) \cap (V)$ .*

*Proof of Theorem 9.* By Lemma 5, for every  $Q \in \mathcal{O}$ ,  $\{Qu, Qv\}$  is optimal:  $\theta(Qu, Qv) = \theta(u, v)$ . One or both of the following cases applies.

*Case 1:  $u, v$  belong to the same panel.* We define an isometry  $Q \in \mathcal{O}$  such that  $Qu, Qv \in (I)$ . Then  $\{Qu, Qv\}$  is optimal for (3), on the panel  $(I)$ . By Lemma 12,  $\{Qu, Qv\}$  is optimal for (2), on the grid lines of the panel  $(I)$ . By Lemma 11, there exists  $P \in \mathcal{O}$  such that  $\{PQu, PQv\}$  is optimal for (1), on the edge  $(I) \cap (V)$ .

*Case 2:  $u, v$  belong to adjacent panels.* We set an isometry  $Q \in \mathcal{O}$  such that  $Qu \in (I)$  and  $Qv \in (V)$ . Then  $\{Qu, Qv\}$  is optimal for (4), on the adjacent panels  $(I)$  and  $(V)$ ; Lemma 13 shows that  $Qu$  and  $Qv$  belong to the same panel. We conclude using Case 1 for  $\{Qu, Qv\}$ .

*Case 3:  $u, v$  belong to opposite panels.* We set an isometry  $Q \in \mathcal{O}$  such that  $Qu \in (I)$  and  $Qv \in (III)$ . We apply Lemma 14:  $\{Qu, Qv\}$  is optimal for (5), on the opposite panels  $(I)$  and  $(III)$ , with  $Qu \cdot Qv = \frac{1}{3}$ . If  $N = 1$ ,  $\text{CS}_N = \{-1/\sqrt{3}, 1/\sqrt{3}\}^3$ , then  $Qu$  and  $Qv$  belong to the same edge. So we conclude with Case 1 for  $\{Qu, Qv\}$ . If  $N \geq 2$ , this case is impossible. Otherwise,  $\theta(u, v) = \theta(Qu, Qv) > m^*$  which contradicts the optimality of  $\{u, v\}$ .  $\square$

*Proof of Theorem 3.* Problem (P) is an optimization problem, over a finite set. Therefore there exists an optimal arc:  $\{u, v\} \subset \text{CS}_N$ , with  $u \neq v$ . Then, by Theorem 9, there exists  $Q \in \mathcal{O}$  such that  $\{Qu, Qv\} \subset (I) \cap (V)$  is optimal for (1), on the edge  $(I) \cap (V)$ . This arc is one of the arcs of Lemma 10. Furthermore the minimal value (P) is  $\theta(u, v) = \theta(Qu, Qv) = m^*$ . This shows that  $\{Qu, Qv\}$  is optimal for the full problem (P). Then, by Lemma 5, for every  $P \in \mathcal{O}$ ,  $\{PQu, PQv\}$  is optimal for (P):  $\theta(PQu, PQv) = \theta(u, v)$ .  $\square$

We formulate hereafter the lemmas that are used in the proofs of Theorems 3 and 9. They are proved in Section 4.

**Lemma 10** (Shortest arc on an edge). *(i) If  $N$  is odd, there exists a unique minimal arc for (1):*

$$\{u, v\} = \{\rho(1, -\tan \frac{\pi}{4N}, 1), \rho(1, \tan \frac{\pi}{4N}, 1)\}.$$

*(ii) If  $N$  is even, there are precisely two minimal arcs for (1):*

$$\{Qu, Qv\} = \{\rho(1, 0, 1), \rho(1, \pm \tan \frac{\pi}{2N}, 1)\}, \text{ with } Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in \mathcal{O}.$$

**Lemma 11** (Shortest arc on the grid lines). *The shortest arc-length on the grid lines of the panel  $(I)$  satisfies:*

$$\min \{\theta(u, v); u = \rho(1, X, Y), v = \rho(1, Z, T), X = Z \text{ or } Y = T, (X, Y) \neq (Z, T) \in \mathbb{T}_N^2\} = m^*. \quad (2)$$

*Moreover, the arc  $\{u, v\}$  is optimal for (2), if, and only if,  $\{Qu, Qv\}$  is optimal for (1), where  $Q$  is one of the following symmetry of the cube*

$$I_3, Q_1 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, Q_2 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}, Q_3 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \in \mathcal{O}.$$

**Lemma 12** (Shortest arc on a panel). *The shortest arc-length between distinct points on the panel  $(I)$  satisfies:*

$$\min \{\theta(u, v); u \in (I), v \in (I), u \neq v\} = m^*; \quad (3)$$

*moreover, the arc  $\{u, v\}$  is optimal for (3) if, and only if,  $\{u, v\}$  is optimal for (2).*

The shortest arcs on the panel  $(I)$  are displayed in Figure 3.

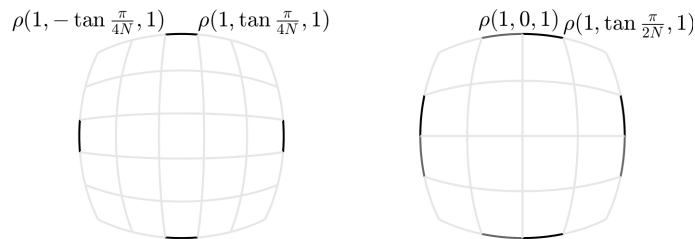


FIGURE 3. Shortest arcs between distinct points in  $(I)$  (in dark). If  $N$  is odd (left), there are 4 minimal arcs, obtained by rotation of  $\{\rho(1, -\tan \frac{\pi}{4N}, 1), \rho(1, \tan \frac{\pi}{4N}, 1)\} \subset (I) \cap (V)$ . If  $N$  is even (right), there are 8 minimal arcs, obtained by symmetry of  $\{\rho(1, 0, 1), \rho(1, \tan \frac{\pi}{2N}, 1)\} \subset (I) \cap (V)$ .

**Lemma 13** (Shortest arc between adjacent panels). *Let  $u \in (I) \setminus (V)$  and  $v \in (V) \setminus (I)$ . The arc  $(u, v)$  is not a minimal arc between  $(I)$  and  $(V)$ :*

$$\theta(u, v) > \min\{\theta(u', v'); u' \in (I), v' \in (V), u' \neq v'\}. \quad (4)$$

**Lemma 14** (Shortest arc between opposite panels). *The shortest arc-length between the panels  $(I)$  and  $(III)$  satisfies:*

$$\min\{\theta(u, v); u \in (I), v \in (III), u \neq v\} = \theta(\rho(1, -1, 1), \rho(1, 1, 1)) \geq m^*, \quad (5)$$

with equality if, and only if,  $N = 1$ .

Lemmas 13–14 are illustrated in Figure 4.

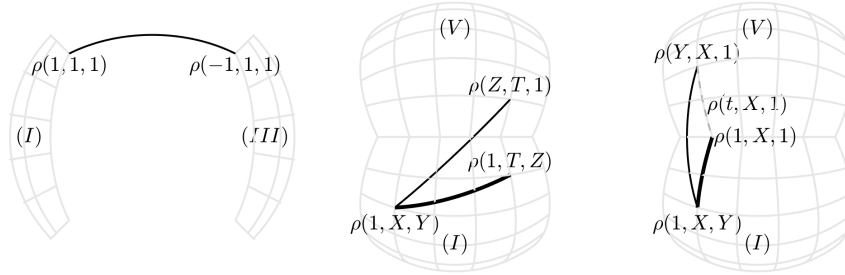


FIGURE 4. The distance between points on distinct panels is dominated by the distance between some points in a panel. Left: the edge between the corners  $\rho(1, 1, 1)$  and  $\rho(-1, 1, 1)$  is a minimal arc between the opposite panels  $(I)$  and  $(III)$ . Middle:  $\rho(1, T, Z) \in (I)$  is closer to  $\rho(1, X, Y) \in (I)$  than the symmetric  $\rho(Z, T, 1) \in (V)$ . Right: the distance between  $\rho(1, X, Y)$  and  $\rho(t, X, 1)$  (on the dashed line) decreases when  $t$  increases from  $Y$  to  $1$ ;  $\rho(1, X, 1) \in (I) \cap (V)$  is closer to  $\rho(1, X, Y)$  than  $\rho(Y, X, 1) \in (V)$ .

#### 4. PROOF OF LEMMAS 10–14

In this section, we minimize the arc-length  $\theta(u, v)$  over various subsets of the cubed sphere, or, equivalently, we maximize the cos-angle  $\cos \theta(u, v) = u \cdot v$  ( $\cos$  is decreasing on  $[0, \pi]$ ). Most of the proofs are based on Lemma 15.

**4.1. Technical lemma.** This subsection is about the continuous face containing the first panel:

$$\rho(\{1\} \times [-1, 1]^2) = \{\rho(1, X, Y), (X, Y) \in [-1, 1]^2\} \supset (I).$$

We investigate various strategies that increase the cos-angle

$$F(X, Y, Z, T) := \rho(1, X, Y) \cdot \rho(1, Z, T) = \frac{1 + XZ + YT}{(1 + X^2 + Y^2)^{1/2}(1 + Z^2 + T^2)^{1/2}}, \quad X, Y, Z, T \in [-1, 1].$$

Firstly, the cos-angle satisfies the following symmetries:

$$F(X, Y, Z, T) = F(Y, X, T, Z) = F(Z, T, X, Y) = F(T, Z, Y, X) \quad (6)$$

$$= F(-X, Y, -Z, T) = F(X, -Y, Z, -T) = F(-X, -Y, -Z, -T). \quad (7)$$

Secondly, the following inequalities are easily checked:

$$XZ \leq 0 \Rightarrow F(X, Y, Z, T) = F(-|X|, Y, |Z|, T) \leq F(|X|, Y, |Z|, T), \quad (8)$$

$$YT \leq 0 \Rightarrow F(X, Y, Z, T) = F(X, -|Y|, Z, |T|) \leq F(X, |Y|, Z, |T|), \quad (9)$$

$$F(|X|, |Y|, -|Z|, -|T|) \leq F(X, Y, Z, T) \leq F(|X|, |Y|, |Z|, |T|). \quad (10)$$

Lastly, we summarize various monotonicity properties:

**Lemma 15.** *Let  $F(X, Y, Z, T) = \rho(1, X, Y) \cdot \rho(1, Z, T)$ ,  $X, Y, Z, T \in [-1, 1]$ .*

- (i) *Let  $-1 \leq X < Z \leq 1$ . The function  $Y \in [0, 1] \mapsto F(X, Y, Z, Y)$  is increasing.*
- (ii) *Let  $0 < \delta \leq \frac{\pi}{4}$ , and  $0 \leq Y \leq 1$ . The function  $x \in [-\frac{\delta}{2}, \frac{\pi}{4} - \delta] \mapsto F(\tan x, Y, \tan(x + \delta), Y)$  is decreasing if  $Y > 0$ , and constant if  $Y = 0$ .*
- (iii) *The function  $X \in [0, 1] \mapsto F(-X, 1, X, 1)$  is decreasing.*
- (iv) *Let  $-1 < Z \leq 1$  and  $-1 \leq Y \leq 1$ . The function  $X \in [-1, Z] \mapsto F(X, Y, Z, Y)$  is increasing.*
- (v) *Let  $-1 \leq X < 1$  and  $-1 \leq Y \leq 1$ . The function  $Z \in [X, 1] \mapsto F(X, Y, Z, Y)$  is decreasing.*
- (vi) *Let  $0 \leq X < Z \leq 1$  and  $0 < Y \leq 1$ . The function  $T \in [0, Y] \mapsto F(X, Y, Z, T)$  is increasing.*

- (vii) Let  $0 \leq X < Z \leq 1$  and  $0 < T \leq 1$ . The maximum of the function  $Y \in [0, T] \mapsto F(X, Y, Z, T)$  is reached only for  $Y = Y^*(X, Z, T) = \frac{T(1+X^2)}{1+ZX} \in (0, T]$ .
- (viii) Let  $0 < \delta \leq \frac{\pi}{4}$  and  $0 < T \leq 1$ . The function  $x \in [0, \frac{\pi}{4} - \delta] \mapsto F(\tan x, Y^*(\tan x, \tan(x+\delta), T), \tan(x+\delta), T)$  is decreasing.
- (ix) Let  $0 \leq Y \leq 1, -1 \leq Z, T \leq 0$ . The function  $X \in [0, 1] \mapsto F(X, Y, Z, T)$  is non-increasing.
- (x) Let  $0 \leq X, Y \leq 1, -1 \leq T \leq 0$ . The function  $Z \in [-1, 0] \mapsto F(X, Y, Z, T)$  is non-decreasing.

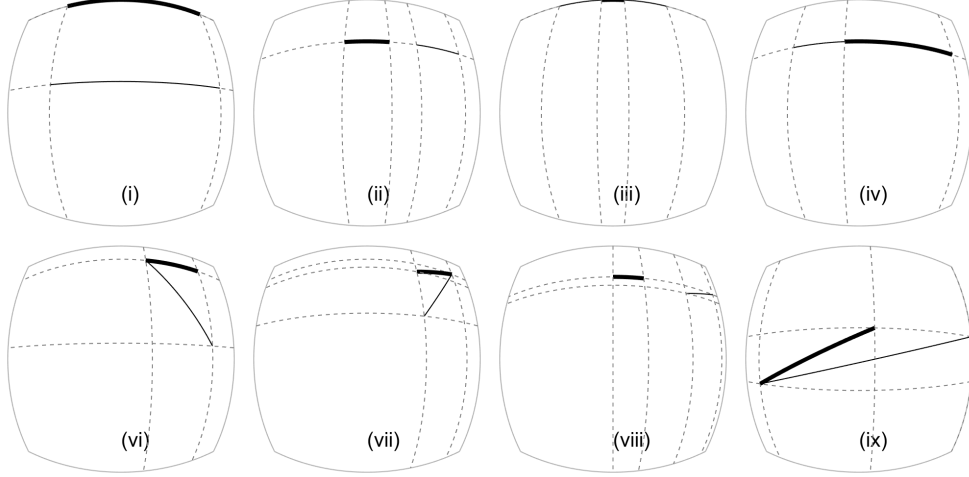


FIGURE 5. Illustrations of Lemma 15. We plot two arcs of great circle associated with two values of  $F$ ; the bold arc is shorter ( $F$  is larger). The dashed arcs are iso- $X$  or iso- $Y$ , in  $\{\rho(1, X, Y), (X, Y) \in [-1, 1]^2\}$ .

- (i)  $F(X, Y, Z, Y) < F(X, 1, Z, 1)$ ,  $X < Z$ ,  $0 \leq Y < 1$ .
- (ii)  $F(\tan x, Y, \tan(x+\delta), Y) < F(-\tan \frac{\delta}{2}, Y, \tan \frac{\delta}{2}, Y)$ ,  $0 < \delta$ ,  $0 < Y$ ,  $-\frac{\delta}{2} < x$ .
- (iii)  $F(-X_1, 1, X_1, 1) < F(-X_2, 1, X_2, 1)$ ,  $0 \leq X_2 < X_1$ .
- (iv)  $F(X_1, Y, Z, Y) < F(X_2, Y, Z, Y)$ ,  $-1 \leq X_1 < X_2 \leq Z$ .
- (vi)  $F(X, Y, Z, T) < F(X, Y, Z, Y)$ ,  $0 \leq X < Z$ ,  $0 \leq T < Y$ .
- (vii)  $F(X, Y, Z, T) < F(X, Y^*(X, Z, T), Z, T)$ ,  $0 \leq X < Z$ ,  $0 < T \leq 1$ ,  $0 \leq Y \leq T$ ,  $Y \neq Y^*(X, Z, T) := \frac{T(1+X^2)}{1+ZX}$ .
- (viii)  $F(\tan x, Y^*(\tan x, \tan(x+\delta), T), \tan(x+\delta), T) < F(0, T, \tan \delta, T)$ ,  $0 < \delta, T, x$ .
- (ix)  $F(1, Y, Z, T) \leq F(X, Y, Z, T)$ ,  $X, Y \geq 0$ ,  $Z, T \leq 0$ .

Before the proof, we comment the results from a geometrical point of view; see Figure 5.

- (i) The length of an iso- $Y$  arc between two fixed meridians (iso- $X$ ) decreases from the equator  $Y = 0$  to the edge  $Y = 1$ .
- (ii) In the Northern Hemisphere, the length of an iso- $Y$  arc between two meridians, separated from a longitude  $\delta$ , decreases when the meridians moves towards the central position (longitudes  $\pm\delta/2$ ). Along the equator, it is constant.
- (iii) Along the edge  $Y = 1$ , the arc-length between the symmetric meridians  $X = -X_1$  and  $X = X_1$  decreases with  $X_1$ .
- (iv)-(v) The length of an iso- $Y$  arc between two meridians decreases when the longitudes of the meridians get closer.
- (vi) In the eastern part of the Northern Hemisphere, we consider a spherical quadrangle delimited by two meridians and two iso- $Y$  arcs. The two meridians and the northernmost iso- $Y$  arc are fixed. The length of the NorthWest-SouthEast diagonal decreases when the southernmost iso- $Y$  arc moves towards the northernmost one.
- (vii) With the same pattern than (vi), the length of the SouthWest-NorthEast diagonal reaches its minimum value for some position of the southernmost iso- $Y$  arc ( $Y = Y^*$ ).
- (viii) The minimal SouthWest-NorthEast diagonal of (vii) decreases when the westernmost meridian moves towards the central meridian  $X = 0$ , while keeping a fixed longitude  $\delta$  between the meridians.
- (ix-x) We consider the arc between a point in the eastern part of the Northern Hemisphere and a point in the western part of the Southern Hemisphere. The arc-length does not decrease when the easternmost point moves towards the East (along the iso- $Y$  arc), or when the westernmost point moves towards the West.

*Proof of Lemma 15.* For each case, we denote by  $f$  the considered function. We give an expression that contains the sign of the derivative  $f'$ ; these expressions have been checked by symbolic computation. By the chain-rule, the derivative  $f'$  always involves the gradient  $\nabla F$ , whose components are analogous expressions to:

$$\frac{\partial F(X, Y, Z, T)}{\partial X} = \frac{(1+Z^2+T^2)(Z(1+Y^2)-X(1+TY))}{(1+X^2+Y^2)^{3/2}(1+Z^2+T^2)^{3/2}}.$$

- (i) The derivative of  $f(Y) = F(X, Y, Z, Y)$ ,  $0 \leq Y \leq 1$ , is  $f'(Y) = \frac{\partial F}{\partial Y} + \frac{\partial F}{\partial T}$ . Thus the sign of  $f'(Y)$  is the sign of:  $Y(Z-X)^2(1+Y^2-ZX)$ . Then  $f'(0) = 0$  and  $f'(Y) > 0$  if  $Y > 0$ .
- (ii) We set  $X = \tan x$  and  $Z = \tan(x + \delta)$ . The derivative of  $f(x) = F(X, Y, Z, Y)$ ,  $x \in [-\frac{\delta}{2}, \frac{\pi}{4} - \delta]$ , is  $f'(x) = \frac{\partial F}{\partial X} \frac{dX}{dx} + \frac{\partial F}{\partial Z} \frac{dZ}{dx}$ , with  $\frac{dX}{dx} = 1+X^2$  and  $\frac{dZ}{dx} = 1+Z^2$ . The sign of  $f'(x)$  is the sign of:  $-(1+Y^2)(Z-X)^2Y^2(X+Z)$ . If  $Y = 0$ , then  $f'(x) = 0$ . Otherwise, if  $Y > 0$ , the sign of  $f'(x)$  is the sign of  $-(X+Z)$  ( $Z \neq X$ ). Since  $-(X+Z)$  is a decreasing function of  $x$  (derivative:  $-2-X^2-Z^2 < 0$ ) that takes the value 0 for  $x = -\frac{\delta}{2}$ , then  $f'(-\frac{\delta}{2}) = 0$ , and  $f'(x) < 0$  if  $x > -\frac{\delta}{2}$ .
- (iii) For  $f(X) = F(-X, 1, X, 1)$ ,  $0 \leq X \leq 1$ , the sign of the derivative  $f'(X) = -\frac{\partial F}{\partial X} + \frac{\partial F}{\partial Z}$  is the sign of:  $-8X(2+X^2)$ .
- (iv) The sign of the derivative of  $f(X) = F(X, 1, Z, 1)$ ,  $-1 \leq X \leq Z$ , is the sign of:  $(1+Y^2)(Z-X)$ .
- (v) The sign of the derivative of  $f(Z) = F(X, Y, Z, Y)$ ,  $X \leq Z \leq 1$ , is the sign of:  $(1+Y^2)(X-Z)$ .
- (vi) The sign of the derivative of  $f(T) = F(X, Y, Z, T)$ ,  $0 \leq T \leq Y$ , is the sign of:  $Y(1+Z^2) - T(1+ZX)$ . Then  $f'(T) > 0$  if  $T < Y \frac{1+Z^2}{1+ZX}$ , whereas  $Y < Y \frac{1+Z^2}{1+ZX}$ .
- (vii) The sign of the derivative of  $f(Y) = F(X, Y, Z, T)$ ,  $0 \leq Y \leq T$ , is the sign of:  $T(1+X^2) - Y(1+ZX)$ . For  $Y^* = \frac{T(1+X^2)}{1+ZX} \in (0, T]$ ,  $f'(Y) > 0$  if  $Y < Y^*$ ,  $f'(Y^*) = 0$ , and  $f'(Y) < 0$  if  $Y > Y^*$ .
- (viii) We set  $X = \tan x$ ,  $Z = \tan(x + \delta)$ ,  $Y^* = Y^*(X, Z, T)$ ,  $f(x) = F(X, Y^*, Z, T)$ ,  $x \in [0, \frac{\pi}{4} - \delta]$ . Then

$$f'(x) = \frac{\partial F}{\partial X}(X, Y^*, Z, T)(1+X^2) + \frac{\partial F}{\partial Y}(X, Y^*, Z, T) \frac{dY^*}{dx} + \frac{\partial F}{\partial Z}(X, Y^*, Z, T)(1+Z^2).$$

By construction of  $Y^*$ ,  $\frac{\partial F}{\partial Y}(X, Y^*, Z, T) = 0$ . Then, the sign of  $(1+ZX)^3 f'(x)$  is the sign of:

$$-(1+X^2)(Z-X)^2 Z T^2 [(1+ZX)^2 + T^2(1+X^2)] < 0.$$

- (ix) The sign of the derivative of  $f(X) = F(X, Y, Z, T)$ ,  $0 \leq X \leq 1$ , is the sign of:  $Z(1+Y^2) - X(1+TY) \leq 0$ .
- (x) The derivative of  $f(Z) = F(X, Y, Z, T)$ ,  $-1 \leq Z \leq 0$  has the sign of:  $X(1+T^2) - Z(1+YT) \geq 0$ .  $\square$

#### 4.2. Proof of Lemmas 10–14.

*Proof of Lemma 10.* We solve:  $\max\{F(X, 1, Z, 1), X \neq Z \in T_N\}$ . Let  $X \neq Z \in T_N$ .

*Case 1:  $N$  is even*

*Case 1.a:  $XZ < 0$ .* Combination of (8) and Lemma 15,(iv) shows:

$$F(X, 1, Z, 1) = F(-|X|, 1, |Z|, 1) < F(0, 1, |Z|, 1), \quad 0 \neq |Z| \in T_N.$$

*Case 1.b:  $XZ \geq 0$ .* By (6)-(7),

$$F(X, 1, Z, 1) = F(|X|, 1, |Z|, 1) = F(|Z|, 1, |X|, 1) = F(X', 1, Z', 1),$$

with  $0 \leq X' = \min(|X|, |Z|) < Z' = \max(|X|, |Z|); X', Z' \in T_N$ .

Write  $X' = \tan x$ ,  $Z' = \tan(x + \delta)$ , with  $0 \leq x < z = x + \delta \leq \frac{\pi}{4}$ , and  $\delta \geq \frac{\pi}{2N}$ . By Lemma 15, (ii) and (v),

$$F(X, 1, Z, 1) = F(X', 1, Z', 1) \leq F(0, 1, \tan \delta, 1) \leq F(0, 1, \tan \frac{\pi}{2N}, 1),$$

with equalities if, and only if,  $x = 0$  and  $\delta = \frac{\pi}{2N}$ , i.e.  $\{|X|, |Z|\} = \{X', Z'\} = \{0, \tan \frac{\pi}{2N}\}$ . This is the maximum.

*Case 2:  $N$  is odd.*

*Case 2.a:  $XZ \geq 0$ .* As Case 1.b,

$$F(X, 1, Z, 1) = F(\tan x, 1, \tan(x + \delta), 1), \quad 0 \leq x < z = x + \delta \leq \frac{\pi}{4}, \delta \geq \frac{\pi}{2N}.$$

Then  $\tan \frac{\delta}{2} \geq \tan \frac{\pi}{4N}$  and Lemma 15,(ii)-(iii), shows that

$$F(X, 1, Z, 1) < F(-\tan \frac{\delta}{2}, 1, \tan \frac{\delta}{2}, 1) \leq F(-\tan \frac{\pi}{4N}, 1, \tan \frac{\pi}{4N}, 1).$$

*Case 2.b:  $XZ < 0$ .* Lemma 15,(iv)-(v), combined with (8), shows:

$$F(X, 1, Z, 1) = F(-|X|, 1, |Z|, 1) \leq F(-\tan \frac{\pi}{4N}, 1, Z, 1) \leq F(-\tan \frac{\pi}{4N}, 1, \tan \frac{\pi}{4N}, 1),$$

with equality if, and only if,  $|X| = |Z| = \tan \frac{\pi}{4N}$ . This is the maximum.  $\square$



*Proof of Lemma 11.* We maximize  $F(X, Y, Z, T)$ ,  $(X, Y) \neq (Z, T) \in \mathbb{T}_N^2$ , and,  $X = Z$  or  $Y = T$ . Let  $(X, Y, Z, T)$  be an optimal solution. Equivalently,  $\{u = \rho(1, X, Y), v = \rho(1, Z, T)\}$  is minimal for (2). We prove that there exists  $Q \in \{I, Q_1, Q_2, Q_3\}$  such that  $\{Qu, Qv\}$  is minimal for (1); this implies  $\theta(u, v) = \theta(Qu, Qv) = m^*$ , because  $Q \in \mathcal{O}$ .

*Case 1:  $X \neq Z$  and  $Y = T \geq 0$ .* Combination of (6) and Lemma 15,(i) shows that

$$F(X, Y, Z, T) = F(Z, T, X, Y) \leq F(X, 1, Z, 1) = F(Z, 1, X, 1),$$

with equality if, and only if,  $Y = 1$ . By optimality of  $(X, Y, Z, T)$ , we obtain  $Y = 1$ . Then  $\{u, v\} \subset (I) \cap (V)$  is optimal for (1), on the edge  $(I) \cap (V)$ .

*Case 2:  $X \neq Z$  and  $Y = T < 0$ .* The invariance (7) shows that  $F(X, -Y, Z, -T) = F(X, Y, Z, T)$ , where  $(X, -Y) \neq (Z, -T) \in \mathbb{T}_N^2$  with  $-Y = -T \geq 0$ . Then  $(X, -Y, Z, -T)$  is optimal. We deduce from Case 1 that  $\{Q_3u, Q_3v\} = \{\rho(1, X, -Y), \rho(1, Z, -T)\}$  is optimal for (1).

*Case 3:  $X = Z \geq 0$  and  $Y \neq T$ .* The invariance (6) shows  $F(X, Y, Z, T) = F(Y, X, T, Z)$ , where  $(Y, X) \neq (T, Z) \in \mathbb{T}_N^2$  with  $X = Z \geq 0$ . Then  $(Y, X, T, Z)$  is optimal. By Case 1,  $\{Q_1u, Q_1v\} = \{\rho(1, Y, X), \rho(1, T, Z)\}$  is optimal for (1).

*Case 4:  $X = Z < 0$  and  $Y \neq T$ .* The invariances (6)-(7) show that  $F(X, Y, Z, T) = F(-Y, -X, -T, -Z)$ , where  $(-Y, -X) \neq (-T, -Z) \in \mathbb{T}_N^2$  with  $-X = -Z \geq 0$ . Then  $(-Y, -X, -T, -Z)$  is optimal. We conclude with Case 1:  $\{Q_2u, Q_2v\} = \{\rho(1, -Y, -X), \rho(1, -T, -Z)\}$  is optimal for (1).

Conversely, for all  $Q \in \{I, Q_1, Q_2, Q_3\}$  and  $\{u^*, v^*\}$  minimal for (1),  $\{u, v\} = \{Q^\top u^*, Q^\top v^*\}$  is realizable for (2), and  $\theta(u, v) = \theta(Q^\top u^*, Q^\top v^*) = m^*$  is the minimal value of (2), so  $\{u, v\}$  is minimal for (2).  $\square$

*Proof of Lemma 12.* Equivalently, we maximize  $F(X, Y, Z, T)$  on the first panel:  $(X, Y) \neq (Z, T) \in \mathbb{T}_N^2$ . Let  $X \neq Z$  and  $Y \neq T$  in  $\mathbb{T}_N$ . We prove that  $F(X, Y, Z, T)$  is strictly dominated by a value on the grid lines:  $F(X', Y', Z', T')$ , with  $(X', Y') \neq (Z', T') \in \mathbb{T}_N^2$  and,  $X' = Z'$  or  $Y' = T'$ .

*Case 1:  $XZ \geq 0$  and  $YT \geq 0$ .*

*Case 1.a:  $0 \leq X < Z \leq 1$  and  $0 \leq T < Y \leq 1$ .* By Lemma 15,(vi),  $F(X, Y, Z, T) < F(X, Y, Z, Y)$ .

*Case 1.b:  $0 \leq X < Z \leq 1$  and  $0 \leq Y < T \leq 1$ .* According to Lemma 15,(vii)-(viii),

$$F(X, Y, Z, T) \leq F(X, Y^*(X, Z, T), Z, T) \leq F(0, Y^*(0, \tan \delta, T), \tan \delta, T), \quad \delta = \arctan Z - \arctan X,$$

with equalities if, and only if,  $Y = Y^*(X, Z, T)$  and  $X = 0$ . If  $X \neq 0$ , the last inequality is strict. Otherwise,  $Y^*(X, Z, T) = Y^*(0, \tan \delta, T) = T > Y$ , and the first inequality is strict. In both cases:

$$F(X, Y, Z, T) < F(0, T, \tan \delta, T), \quad \tan \delta \geq \tan \frac{\pi}{2N}.$$

Application of Lemma 15,(v) and (ii), shows:

$$F(0, T, \tan \delta, T) \leq F(0, T, \tan \frac{\pi}{2N}, T) \leq F(-\tan \frac{\pi}{4N}, T, \tan \frac{\pi}{4N}, T).$$

If  $N$  is even, resp. odd, we conclude with the first, resp. second, inequality.

*Case 1.c:  $0 \leq Z < X \leq 1$ .* By relation (6),  $F(X, Y, Z, T) = F(Z, T, X, Y)$ . The right member is dominated using Case 1.a or Case 1.b.

*Case 2:  $XZ < 0$  or  $YT < 0$ .*

*Case 2.a:  $X = -Z$  or  $Y = -T$ .* Similarly to (8)-(9),

$$F(X, Y, Z, T) < \begin{cases} F(|X|, Y, |Z|, T) = F(|X|, Y, |X|, T), & \text{if } X = -Z, \\ F(X, |Y|, Z, |T|) = F(X, |Y|, Z, |Y|), & \text{if } Y = -T. \end{cases}$$

*Case 2.b:  $X \neq -Z$  and  $Y \neq -T$ .* By (10):

$$F(X, Y, Z, T) \leq F(|X|, |Y|, |Z|, |T|).$$

The right member is dominated using Case 1.  $\square$

*Proof of Lemma 13.* Let  $u \in (I) \setminus (V)$  and  $v \in (V) \setminus (I)$ :  $u = \rho(1, X, Y)$ , and  $v = \rho(Z, T, 1)$ , with  $(X, Y), (Z, T) \in \mathbb{T}_N^2$ , such that  $Y, Z < 1$ . We prove that  $u \cdot v < \cos m^*$ .

*Case 1:  $(T, Z) \neq (X, Y)$ , as in the middle of Figure 4.* Let

$$Q = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \in \mathcal{O}, \quad w = Qv = \rho(1, T, Z) \in (I).$$

It is clear that  $0 < (1 - Z)(1 - Y)$ , so  $Z + XT + Y < 1 + XT + YZ$ , i.e.  $u \cdot v < F(X, Y, T, Z) = u \cdot w$ . By the way,  $u \neq w \in (I)$  so  $u \cdot w \leq \cos m^*$ .

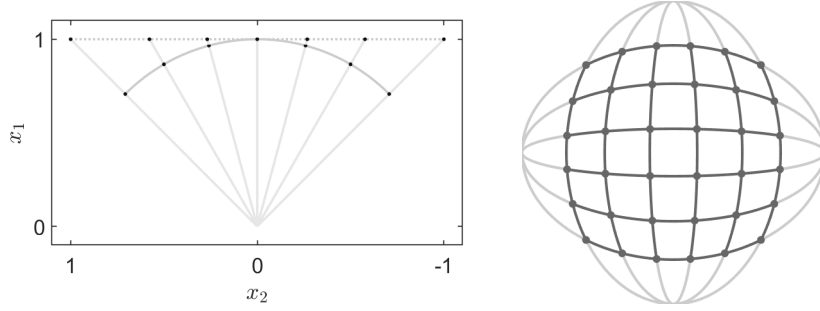


FIGURE 6. Left: construction of the equiangular grid  $T_N = \{\tan \frac{k\pi}{2N}, -\frac{N}{2} \leq k \leq \frac{N}{2}\}$  ( $N = 6$ ). On the unit circle  $x_1^2 + x_2^2 = 1$ , we consider  $N + 1$  equidistributed angles:  $\frac{k\pi}{2N}, -\frac{N}{2} \leq k \leq \frac{N}{2}$ . The origin and the point  $(1, \tan \frac{k\pi}{2N}) \in \{1\} \times T_N$  (black dot on the dotted line) are aligned with  $(\cos \frac{k\pi}{2N}, \sin \frac{k\pi}{2N})$  (black dot on the arc of circle). Right: construction of a panel of the cubed sphere by intersection of two equiangular families of great circles ( $N = 5$ ).

*Case 2:*  $(T, Z) = (X, Y)$ , as in the right of Figure 4. We prove that  $u \cdot v < F(X, Y, X, 1)$  using the auxiliary function

$$f(t) = \rho(1, X, Y) \cdot \rho(t, X, 1), t \in [Y, 1];$$

$f(t)$  represents the cos-angle between  $\rho(1, X, Y)$  and  $\rho(t, X, 1)$ . The sign of  $f'(t)$  is the sign of  $(1 + X^2 + t^2) - (X^2 + Y + t)t = X^2(1 - t) + 1 - Yt > 0$ . Thus  $f$  is increasing and  $f(Y) < f(1)$ , i.e.  $u \cdot v < F(X, Y, X, 1)$ . Due to  $\rho(1, X, Y) \neq \rho(1, X, 1) \in (I)$ ,  $u \cdot v < F(X, Y, X, 1) \leq \cos m^*$ .  $\square$

*Proof of Lemma 14.* Since  $v \in (III) \Leftrightarrow -v \in (I)$ , we solve  $m := \min\{\theta(u, -w); u \neq w \in (I)\}$ , or, equivalently:

$$\cos m = \max\{-u \cdot w; u \neq w \in (I)\} = -\min\{F(X, Y, Z, T); (X, Y) \neq (Z, T) \in T_N^2\}.$$

Fix  $X, Y, Z, T \in T_N$ . We apply: (10), Lemma 15, (ix-x), (6), and Lemma 15, (ix-x) again:

$$\begin{aligned} F(X, Y, Z, T) &\geq F(|X|, |Y|, -|Z|, -|T|) \\ &\geq F(1, |Y|, -|Z|, -|T|) \\ &\geq F(1, |Y|, -1, -|T|) = F(|Y|, 1, -|T|, -1) \\ &\geq F(1, 1, -|T|, -1) \\ &\geq F(1, 1, -1, -1). \end{aligned}$$

Thus the minimal value of  $F$  is reached on a diagonal:  $F(1, 1, -1, -1) = \rho(1, 1, 1) \cdot \rho(1, -1, -1)$ . Its opposite value represents the length of an edge:  $\cos m = -F(1, 1, -1, -1) = \rho(1, 1, 1) \cdot \rho(1, -1, 1)$ . By Lemma 10,  $m > m^*$  if  $N \geq 2$ , and  $m = m^*$  if  $N = 1$ .  $\square$

## 5. CONCLUSION

In this paper we have identified the symmetry group of the equiangular cubed sphere: it is the symmetry group of the cube. This result emphasizes that the cubed sphere behaves like a cube. It provides some theoretical foundation for numerical computation on the cubed sphere, when the symmetries play an important role. For instance, the suitable symmetries for the coefficients of a quadrature formula on the cubed sphere are expected to be the symmetries of the cube. This consolidates the use of the duplicated pattern of [14].

## APPENDIX A. EQUIANGULAR CUBED SPHERE

In this section we recall the definition of the equiangular cubed sphere. Let  $N \geq 1$  be a fixed integer, and the following one-dimensional grid:

$$T_N = \{\tan \frac{k\pi}{2N}, -\frac{N}{2} \leq k \leq \frac{N}{2}\},$$

represented in Figure 6. The index  $k$ , resp. the angle  $\frac{k\pi}{2N}$ , takes the  $N + 1$  equidistributed values from  $-\frac{N}{2}$  to  $\frac{N}{2}$ , resp. from  $-\frac{\pi}{4}$  to  $\frac{\pi}{4}$ . Note that  $T_N$  is an irregular grid on  $[-1, 1]$ . Then we define a cartesian grid on the faces of the cube  $[-1, 1]^3$ , based on the two-dimensional grid  $T_N \times T_N$ , as in Figure 1:

$$C_N = \{(1, u, v), (u, 1, v), (-1, u, v), (u, -1, v), (u, v, 1), (u, v, -1); u, v \in T_N\}.$$

To finish with, we project  $C_N$  on the unit sphere  $\mathbb{S}^2 = \{x \in \mathbb{R}^3 : \|x\| = 1\}$ :

**Definition 16.** The equiangular cubed sphere grid  $\text{CS}_N$  is the radial projection of  $\text{C}_N$  on the sphere:  $\text{CS}_N = \rho(\text{C}_N)$ , with  $\rho : x \in \mathbb{R}^3 \setminus \{0\} \mapsto \frac{x}{\|x\|} \in \mathbb{S}^2$ .

It is easily seen that the radial projection  $\rho$  is injective on the faces of the cube  $[-1, 1]^3$ . Therefore, there is a one-to-one correspondence between  $\text{CS}_N$  and  $\text{C}_N$ ;  $(u_1, u_2, u_3) \in \text{C}_N$  are the so-called *gnomonic* coordinates of the point  $\rho(u_1, u_2, u_3) \in \text{CS}_N$ . By construction, the geometry of the cubed sphere is shaped by the geometry of the cube. In particular, one can define twelve *edges*, as the projection of the edges of the cube, and six *panels*, as the projection of the grid of the faces:

$$\begin{aligned} \text{CS}_N &= \rho(\text{C}_N) = (I) \cup (II) \cup (III) \cup (IV) \cup (V) \cup (VI), \\ (I) &= \rho(\{1\} \times \text{T}_N \times \text{T}_N), \quad (II) = \rho(\text{T}_N \times \{1\} \times \text{T}_N), \quad (III) = \rho(\{-1\} \times \text{T}_N \times \text{T}_N), \\ (IV) &= \rho(\text{T}_N \times \{-1\} \times \text{T}_N), \quad (V) = \rho(\text{T}_N \times \text{T}_N \times \{1\}), \quad (VI) = \rho(\text{T}_N \times \text{T}_N \times \{-1\}). \end{aligned}$$

Two panels are said to be *adjacent*, resp. *opposite*, if they are the projection of adjacent, resp. opposite, faces.

As in Figures 1 and 6, the cubed sphere meshes the sphere with equiangular arcs of great circles: the radial projection maps the cartesian straight lines of the cube onto arcs of great circles. For instance, in panel (I),

$$\begin{aligned} \rho(\{(1, \tan \frac{k\pi}{2N}, z), |z| \leq 1\}) &\subset \{u \in \mathbb{S}^2 : u_1 \sin \frac{k\pi}{2N} - u_2 \cos \frac{k\pi}{2N} = 0\}, \quad |k| \leq \frac{N}{2}; \\ \rho(\{(1, y, \tan \frac{j\pi}{2N}), |y| \leq 1\}) &\subset \{u \in \mathbb{S}^2 : u_1 \sin \frac{j\pi}{2N} - u_3 \cos \frac{j\pi}{2N} = 0\}, \quad |j| \leq \frac{N}{2}. \end{aligned}$$

The panel (I) is obtained by intersection of two equiangular families of great circles:

$$\rho(1, \tan \frac{k\pi}{2N}, \tan \frac{j\pi}{2N}) \in \{u \in \mathbb{S}^2 : u_1 \sin \frac{k\pi}{2N} - u_2 \cos \frac{k\pi}{2N} = 0\} \cap \{u \in \mathbb{S}^2 : u_1 \sin \frac{j\pi}{2N} - u_3 \cos \frac{j\pi}{2N} = 0\};$$

the longitudinal angle  $\frac{k\pi}{2N}$  and the latitudinal angle  $\frac{j\pi}{2N}$  both scan a uniform grid on  $[-\frac{\pi}{4}, \frac{\pi}{4}]$ . Analogous properties can be derived for the other panels.

## APPENDIX B. SYMMETRY GROUP OF THE CUBE

In this section we recall properties about the symmetry group of the cube. We give short direct proofs for completeness.

**Theorem 17.** Let  $e_1 = (1, 0, 0)$ ,  $e_2 = (0, 1, 0)$  and  $e_3 = (0, 0, 1)$ . The symmetry group  $\mathcal{O}$  of the cube  $\{-1, 1\}^3$  is:

$$\mathcal{O} = \{[\epsilon_1 e_{\sigma_1} \quad \epsilon_2 e_{\sigma_2} \quad \epsilon_3 e_{\sigma_3}], \sigma \in \mathfrak{S}_3, \epsilon \in \{-1, 1\}^3\}. \quad (11)$$

*Proof.*  $\subset$ : Fix an orthogonal matrix  $Q$  such that  $\text{C}_1 = \{-1, 1\}^3$  is left invariant by  $Q$ . We remark that for  $x_1, x_2 \in \text{C}_1$ ,  $\|x_1 - x_2\| = 2 \Leftrightarrow \exists \epsilon \in \{-1, 1\}, \exists 1 \leq \sigma \leq 3, x_1 - x_2 = 2\epsilon e_\sigma$  ( $x_1$  and  $x_2$  are the vertices of an edge); this is due to the following identity:  $\|x_1 - x_2\|^2 = \sum_{j=1}^3 |e_j \cdot x_1 - e_j \cdot x_2|^2$ , with  $|e_j \cdot x_1 - e_j \cdot x_2| \in \{0, 2\}$ . Then, the matrix  $Q$  maps  $x_0 = (1, 1, 1) \in \text{C}_1$  onto  $Qx_0 \in \text{C}_1$ . For  $1 \leq i \leq 3$ , the distance between  $x_0, x_0 - 2e_i \in \text{C}_1$  is 2. Then the distance between  $Qx_0, Qx_0 - 2Qe_i \in \text{C}_1$  is also 2, *i.e.* there exist  $\sigma_i \in \{1, 2, 3\}$  and  $\epsilon_i \in \{-1, 1\}$  such that  $Qe_i = \epsilon_i e_{\sigma_i}$ . To finish with, the indices  $\sigma_1, \sigma_2, \sigma_3$  define a permutation; otherwise  $Q$  would contain at least two collinear columns.

$\supset$ : Conversely, let  $Q = [\epsilon_1 e_{\sigma_1} \quad \epsilon_2 e_{\sigma_2} \quad \epsilon_3 e_{\sigma_3}]$ , with  $\sigma \in \mathfrak{S}_3$  and  $\epsilon \in \text{C}_1$ . It is clear that  $Q^T Q = \text{I}_3$ . Let  $x \in \text{C}_1$ . Then  $Qx$  is a permutation of  $[\epsilon_1 x_1 \quad \epsilon_2 x_2 \quad \epsilon_3 x_3]^T \in \text{C}_1$ :  $Qx = [e_{\sigma_1} \quad e_{\sigma_2} \quad e_{\sigma_3}][\epsilon_1 x_1 \quad \epsilon_2 x_2 \quad \epsilon_3 x_3]^T$ ; so  $Qx \in \text{C}_1$ . Then  $QC_1 \subset \text{C}_1$ . But  $Q$  is bijective; so the finite sets  $QC_1$  and  $\text{C}_1$  have the same cardinal number, and  $QC_1 = \text{C}_1$ .  $\square$

*Remark 18.* The group  $\mathcal{O}$  contains  $3! \cdot 2^3 = 48$  symmetries, determined by a permutation of  $\mathfrak{S}_3$  and three signs.

From a geometrical point of view, any symmetry of the cube permutes the centers of the faces:

**Corollary 19.** The symmetry group  $\mathcal{O}$  of the cube  $\{-1, 1\}^3$  coincides with the symmetry group of the following octahedron:  $\{(1, 0, 0), (0, 1, 0), (0, 0, 1), (-1, 0, 0), (0, -1, 0), (0, 0, -1)\}$ .

**Definition 20.** The centers of the edges of the cube  $[-1, 1]^3$  define the cuboctahedron

$$\Omega := \{(0, \epsilon, \eta), (\epsilon, 0, \eta), (\epsilon, \eta, 0), \epsilon = \pm 1, \eta = \pm 1\}. \quad (12)$$

**Theorem 21.** The symmetry group of the cuboctahedron  $\Omega$  coincides with the symmetry group  $\mathcal{O}$  of the cube  $\{-1, 1\}^3$ .

*Proof.*  $\subset$ : Let  $Q$  be in the symmetry group of  $\Omega$ :  $Q$  is orthogonal and leaves  $\Omega$  invariant. Then  $Q$  maps  $(1, 0, 1)$  onto  $x = (x_1, x_2, x_3) \in \Omega$ ; let  $j$  be such that  $x_j = 0$ , and  $x_i = \pm 1, i \neq j$ . The vector  $(1, 0, 1)$  and  $(1, 0, -1)$  are orthogonal, so  $Q$  maps  $(1, 0, -1)$  onto  $\bar{x} = (\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \Omega$  such that  $x \cdot \bar{x} = 0$ . Thus  $\bar{x}_j = 0$ , and there are  $i \neq j$ , such that  $\bar{x}_i = x_i$ , and  $\bar{x}_k = -x_k$ , for  $k \neq i, j$ . Then, the vector  $e_1 = \frac{1}{2}((1, 0, 1) + (1, 0, -1))$  is mapped onto  $Qe_1 = \frac{1}{2}(x + \bar{x}) = x_i e_i$ . This shows that  $Q$  maps the center of face  $e_1$  onto the center of face  $Qe_1 = \epsilon_1 e_{\sigma_1}$ , with  $\epsilon_1 = x_i \in \{-1, 1\}$ , and  $\sigma_1 = i \in \{1, 2, 3\}$ . A similar procedure shows that  $Qe_2 = \epsilon_2 e_{\sigma_2}$ ,  $Qe_3 = \epsilon_3 e_{\sigma_3}$ , with  $\epsilon_2, \epsilon_3 \in \{-1, 1\}$ , and  $\sigma_2, \sigma_3 \in \{1, 2, 3\}$ . To finish with, the vectors  $Qe_i$  are orthogonal so  $(\sigma_1, \sigma_2, \sigma_3) \in \mathfrak{S}_3$ .

$\supset$ : Let  $Q \in \mathcal{O}$ :  $Q = [\epsilon_1 e_{\sigma_1} \ \epsilon_2 e_{\sigma_2} \ \epsilon_3 e_{\sigma_3}]$ , with  $\sigma \in \mathfrak{S}_3$ ,  $\epsilon \in C_1$ . Fix  $x \in \Omega$ :  $x = \eta_1 e_{i_1} + \eta_2 e_{i_2}$ , with  $\eta_1, \eta_2 = \pm 1$ ,  $1 \leq i_1 \neq i_2 \leq 3$ . It is clear that  $Qx = \eta_1 \epsilon_{i_1} e_{\sigma_{i_1}} + \eta_2 \epsilon_{i_2} e_{\sigma_{i_2}} \in \Omega$ . Then  $Q\Omega \subset \Omega$ .  $\square$

### APPENDIX C. PROOF OF THE MAIN THEOREM USING GROUP THEORY

In this section, we prove shortly Theorem 2 using classification of subgroups of orthogonal groups. We refer to [1, Theorems 11.1, 19.1 and 19.2] for more details.

**Theorem 22** (Lagrange's Theorem). *The order of a subgroup of a finite group is always a divisor of the order of the group.*

**Theorem 23.** *A finite subgroup of  $O_2 = \{Q \in \mathbb{R}^{2 \times 2} : Q^T Q = I_2\}$  is either cyclic or dihedral.*

**Theorem 24.** *A finite subgroup of  $SO_3 = \{Q \in \mathbb{R}^{3 \times 3} : Q^T Q = I_3, \det Q = 1\}$  is isomorphic either to a cyclic group, a dihedral group, or the rotation group of a Platonic solid (tetrahedron, cube/octahedron, dodecahedron/icosahedron).*

*Proof of Theorem 2 using group theory.* Lemma 5 proves  $\mathcal{O} \subset \mathcal{G}_N$ . We deduce  $\mathcal{O} = \mathcal{G}_N$  from Theorem 24.

Firstly, we focus on the rotation subgroups,  $\mathcal{O}^+ = \mathcal{O} \cap SO_3$  and  $\mathcal{G}_N^+ = \mathcal{G}_N \cap SO_3$ . Indeed,  $\mathcal{O} = \mathcal{G}_N \Leftrightarrow \mathcal{O}^+ = \mathcal{G}_N^+$ , because  $-I_3 \in \mathcal{O} \subset \mathcal{G}_N$ , so  $\mathcal{O} = \mathcal{O}^+ \cup \{-Q, Q \in \mathcal{O}^+\}$ , and  $\mathcal{G}_N = \mathcal{G}_N^+ \cup \{-Q, Q \in \mathcal{G}_N^+\}$ . Secondly,  $\mathcal{G}_N^+$  is a finite subgroup of  $SO_3$ , by injection into a finite permutation group. Indeed,  $CS_N = \{x_j, 1 \leq j \leq M\}$  is a finite set which contains three linearly independent vectors. So  $\sigma : Q \in \mathcal{G}_N^+ \mapsto \sigma[Q] \in \mathfrak{S}_M$  such that  $Qx_j = x_{\sigma[Q](j)}, 1 \leq j \leq M$ , is injective.

By Theorem 24,  $\mathcal{G}_N^+$  is isomorphic either to a cyclic group, a dihedral group, or the rotation group of a Platonic solid. Since  $\mathcal{O}^+ \subset \mathcal{G}_N^+$ , we eliminate all the candidates, except the rotation group of the cube. By Lagrange's Theorem, the order of  $\mathcal{O}^+$  (24) is a divisor of the order of  $\mathcal{G}_N^+$ . Then  $\mathcal{G}_N^+$  is not isomorphic to the group of the tetrahedron (order 12), neither the group of the dodecahedron/icosahedron (order 60). By the way  $\mathcal{G}_N^+$  is not cyclic, because  $\mathcal{O}^+$  is not commutative. And to finish with,  $\mathcal{G}_N^+$  is not dihedral. Indeed, a dihedral group is isomorphic to a finite subgroup of  $O_2$  ([10, p. 152]). So, by Theorem 23, the subgroups of a dihedral group are either cyclic or dihedral. But the rotation group  $\mathcal{O}^+$  of the cube is neither cyclic nor dihedral, by Theorem 24.  $\square$

*Remark 25.* By Remark 6, the proof is still valid if  $T_N$  is replaced by any symmetric grid. Therefore, the group of any cubed sphere (with  $T_N$  symmetric) coincides with the group of the cube.

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