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A Multifamily GLRT for CFAR Detection of Signals in a Union of Subspaces

Francois-Xavier Socheleau, *Senior Member, IEEE*

Abstract—We consider the problem of detecting an unknown signal that lies in a union of subspaces (UoS) and that is observed in additive white Gaussian noise with unknown variance. The main contribution of this paper is the derivation of a detector that can accommodate a union made of nested subspaces. This detector includes the generalized likelihood ratio test (GLRT) as a special case when the subspace dimensions are all identical. It relies on the framework of multifamily likelihood ratio tests (MFLRT) and is shown by numerical examples to achieve better performance than existing detectors.

Index Terms—Multifamily likelihood ratio test, nested models, subspace detection.

I. INTRODUCTION

Testing whether a signal lies within a known subspace is a well-studied problem in the framework of matched-subspace detectors [1]–[6]. However, in practice, signals are often generated by multimodal processes so that there is not one but several possible subspace models. Among the set of possible models, the active subspace that generated the observed signal may be unknown a priori. This kind of signals complies with what is referred to as the UoS model. More precisely, $\mathbf{x} \in \mathbb{R}^N$ is an unknown signal belonging to some union of M known subspaces, i.e., $\mathbf{x} \in \bigcup_{i=1}^M \mathcal{S}_i$, if and only if there exists i_0 such that $\mathbf{x} \in \mathcal{S}_{i_0}$ [7]. In other words, \mathbf{x} belongs to one of the subspace \mathcal{S}_i , but we do not know a priori to which one. UoS examples include signals with unknown spectral support [7], spectral signatures of radar targets [8] or sparse representations [9].

Signal detection under the UoS model has been partly addressed in the literature in the specific case where all subspaces are distinct, i. e., there are no subspaces in the union such that $\mathcal{S}_i \subset \mathcal{S}_\ell$, for $i \neq \ell$ [8]–[10]. All these works resort to the GLRT and estimate the active subspace index i_0 as the one that maximizes the likelihood functions. By assuming distinct subspaces, these methods fail to handle nested models properly. In our UoS context, nesting occurs when the signal can be expressed as the sum of an unknown number of basis vectors so that $\mathcal{S}_i \subset \mathcal{S}_\ell$, for $1 \leq i < \ell \leq M$. Examples include signals of unknown duration, periodic signal with an unknown number of harmonics, signals transmitted through an unknown multipath channel, etc. In that case, the GLRT always chooses the subspace model with the largest dimension leading to possible detection losses. More generally, problems arise with the GLRT when subspaces are of different dimensions and are not pairwise disjoint.

In [11], the MFLRT has been introduced as a general solution to accommodate nested signal models. The idea is to add a penalty term to the GLR statistic to counteract its tendency to increase with the model order. More recently, a new normalizing transformation has been introduced in [12]. It generalizes the results of [11] and relies on the Legendre transform of the cumulant generating function (LT-CGF) of the test statistic. Although effective for various applications [13], [14], these results are not directly applicable to our context since the GLR statistic involved in our problem does not satisfy the required properties.

The main contribution of this letter is to adapt the LT-CGF framework to the detection of an unknown signal in a UoS, with possibly overlapping or nested subspaces. The additive noise is assumed to be white Gaussian with unknown variance. The proposed detector is obtained by establishing a convergence result on the LT-CGF of a Fisher-z random variable. This approach generalizes the standard GLRT under the UoS model and includes it as a special case when the subspace dimensions are all identical. The paper is organized as follows. Sec. II is devoted to the problem formulation. The derivation of the Multifamily CFAR GLRT is presented in Sec. III. Numerical results are provided in Sec. IV, followed by conclusions in Sec. V.

Notation: We denote by $X_n \xrightarrow{d} X$ the convergence in distribution as $n \rightarrow +\infty$. $O(\cdot)$, $O_p(\cdot)$ and $o_p(\cdot)$ designate the usual asymptotic notations for non-random and random variables, respectively [15, Ch. 14]. That is, $a_n = O(b_n)$ if the ratio a_n/b_n is bounded for large n , $X_n = O_p(b_n)$ if the random sequence X_n/b_n is stochastically bounded and $X_n = o_p(1)$ if X_n converges to zero in probability. Finally, $f^{(m)}(x)$ denotes the m -th derivative of f with respect to x .

II. PROBLEM FORMULATION

Let $\mathbf{x} \in \bigcup_{i=1}^M \mathcal{S}_i$, where \mathcal{S}_i are subspaces of \mathbb{R}^N . No particular assumption is made with respect to the intersection of the subspaces and we do not assume all subspaces to have the same dimension but simply that $n_i = \dim(\mathcal{S}_i) < N$, $\forall i$. The detection problem addressed in this paper can be stated as

$$\begin{cases} \mathcal{H}_0 : & \mathbf{y} = \mathbf{w} \\ \mathcal{H}_1 : & \mathbf{y} = \mathbf{x} + \mathbf{w}, \end{cases} \quad (1)$$

where \mathbf{y} is the observation vector and \mathbf{w} is a zero-mean white Gaussian vector with unknown variance. Because of the unknown parameters involved in the problem, no uniformly powerful test exists and the likelihood ratio test cannot be applied. A standard approach is then to resort to GLRT techniques that replace the unknown parameters with their maximum likelihood estimates.

F.-X. Socheleau is with IMT-Atlantique, Lab-STICC, France (e-mail: {fx.socheleau}@imt-atlantique.fr).

Let $\mathbf{H}_i \in \mathbb{R}^{N \times n_i}$ be a basis for the n_i -dimensional subspace \mathcal{S}_i and let $\mathbf{P}_{\mathcal{S}_i} = \mathbf{H}_i (\mathbf{H}_i^T \mathbf{H}_i)^{-1} \mathbf{H}_i^T$ be its projection matrix. The GLRT for Problem (1) is a straightforward extension of the well-studied CFAR matched-subspace detector [1, Ch. 4] and can be written as [10]

$$T_{\text{GLRT}}(\mathbf{y}) = \max_{1 \leq i \leq M} L_i(\mathbf{y}) \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\geq}} \eta_{\text{GLRT}}, \quad (2)$$

where η_{GLRT} is the threshold chosen so as to satisfy some fixed test size. $L_i(\mathbf{y})$ is defined as twice the generalized log-likelihood ratio of \mathbf{y} when \mathbf{x} is assumed to belong to subspace \mathcal{S}_i . This ratio can be expressed as [16, pp. 372]

$$L_i(\mathbf{y}) = N \log \left(\frac{\mathbf{y}^T \mathbf{y}}{\mathbf{y}^T (\mathbf{I}_N - \mathbf{P}_{\mathcal{S}_i}) \mathbf{y}} \right), \quad (3)$$

where \mathbf{I}_N is the $N \times N$ identity matrix. As expressed in (2), the GLRT will always select the subspace containing the greatest amount of signal energy. As long as the subspaces have similar dimensions and/or are pairwise disjoint, this may be fine but in any other cases this is problematic. The worst case happens when the subspaces are nested, i.e., when $\mathcal{S}_i \subset \mathcal{S}_\ell$, for $1 \leq i < \ell \leq M$. In this scenario, the GLRT will always implement the test statistic with the matrix $\mathbf{P}_{\mathcal{S}_M}$ that projects the data in the larger subspace. If the actual signal subspace is smaller than \mathcal{S}_M , the GLRT will include noise-only samples under either hypothesis, degrading the performance of the detector. For instance, this problem is illustrated in [11] for a signal of unknown samples and unknown length corrupted by an additive white Gaussian noise with known variance. For this signal model, but now with an unknown noise variance, the GLRT (2) becomes a CFAR energy detector. More precisely, each basis satisfies

$$\mathbf{H}_i = \begin{pmatrix} \mathbf{I}_{n_i} \\ \mathbf{O}_{(N-n_i) \times n_i} \end{pmatrix} \quad (4)$$

where $\mathbf{O}_{n \times m}$ is the $n \times m$ null matrix and $n_i < n_\ell$ for $i < \ell$. The test statistic in (2) is then a monotonically increasing function of the ratio $\sum_{k=0}^{N-1} |y(k)|^2 / \sum_{k=n_M}^{N-1} |y(k)|^2$, where $y(k)$ is the k -th entry of \mathbf{y} . If the actual signal length is smaller than n_M , the GLRT will not be effective since this ratio will be overestimated. More generally, this problem arises for any signal expressed as the sum of an unknown number of basis vectors.

Improvement of the GLRT can be achieved by using a model order selection criterion in the decision procedure. A well known criterion is the minimum description length (MDL) criterion [17], which leads to the following test¹

$$T_{\text{MDL}}(\mathbf{y}) = \max_{1 \leq i \leq M} \{L_i(\mathbf{y}) - n_i \log N\} \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\geq}} \eta_{\text{MDL}}. \quad (5)$$

In [11], a multifamily likelihood ratio test (MFLRT) has also been proposed to accommodate nested models. It implements

the following decision rule:

$$T_{\text{MFLRT}}(\mathbf{y}) = \max_{1 \leq i \leq M} \left\{ \left[L_i(\mathbf{y}) - n_i \left(\log \left(\frac{L_i(\mathbf{y})}{n_i} \right) + 1 \right) \right] \times u \left(\frac{L_i(\mathbf{y})}{n_i} - 1 \right) \right\} \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\geq}} \eta_{\text{MFLRT}}, \quad (6)$$

where $u(\cdot)$ is the unit step function. As illustrated in Sec. IV, by adding a penalty term, both tests (5) and (6) can provide a better performance than the GLRT. However, by expanding the scope of a new normalizing transformation introduced in [12], we will show that the performance can be further improved.

III. MULTIFAMILY CFAR GLRT

A. Preliminary results

To begin, we give some preliminary definitions and recall the main result of [12]. Let $K_X(t)$ be the cumulant generating function (CGF) of some random variable X , that is,

$$K_X(t) = \log \mathbb{E} (e^{tX}), \quad (7)$$

where \mathbb{E} denotes the mathematical expectation. The Legendre transform (LT) of $K(\cdot)$ is then defined as

$$K_X^*(x) = \sup_{t \in \mathcal{T}} (tx - K_X(t)), \quad (8)$$

where $\mathcal{T} = \{t : |K_X(t)| < +\infty\}$. Next, define

$$g_X(x) = 2K_X^*(x)u(X - \mathbb{E}(X)). \quad (9)$$

Proposition 1. [12] Let $X_n = \sum_{k=1}^n U_k$ where the U_k 's are IID random variables with finite mean and variance. Also, assume that $\mathbb{E}(|U_k - \mathbb{E}(U_k)|^3) < \infty$. Define the random variable Y_n as $Y_n = g_{X_n}(X_n)$. Then, $Y_n \xrightarrow{d} Y$, where the (generalized) probability density function (PDF) of Y satisfies

$$f_Y(y) = \begin{cases} \frac{1}{2\sqrt{2\pi y}} e^{-y/2}, & y > 0 \\ \frac{1}{2} \delta(y), & y = 0 \\ 0, & \text{otherwise.} \end{cases} \quad (10)$$

This proposition states that, as n tends to $+\infty$, Y_n equals zero with probability 1/2 and has a χ_1^2 PDF with probability 1/2. By applying a quadratic transformation to the standardized random variable X_n , a similar result could have been obtained by invoking the central limit-theorem (CLT). However, as illustrated in [12], the asymptotic regime can be reached faster than the CLT in some cases. Prop. 1 is relevant for nested detection problems as it can be used to *equalize* the model-order dependent statistics involved in a GLRT solution. The function g , defined in (9), actually transforms the (asymptotic) GLR statistic to a χ_1^2 random variable independent of the model order n . As shown in [12], if $L_i(\mathbf{y})$ can be expressed as the sum of $n_i \chi_1^2$ random variables under \mathcal{H}_0 , then g takes the form of the expression enclosed in braces in (6). The MFLRT (6) is thus a specific case of a more general theory resulting from the application of Prop. 1. Unfortunately, this proposition is not directly applicable to Problem (1) because the GLR statistic (3) is not expressed as a sum of independent random variables. However, by transforming (3), we will show that a similar result to that of Prop. 1 can be obtained for our specific problem.

¹Other criteria such as AIC or GIC could also be considered [18].

B. Proposed detector

Let $\tilde{L}_i(\mathbf{y})$ be the monotonically increasing function of $L_i(\mathbf{y})$ defined as

$$\tilde{L}_i(\mathbf{y}) = \frac{1}{2} \log \left(\frac{\mathbf{y}^T \mathbf{P}_{S_i} \mathbf{y}}{\mathbf{y}^T (\mathbf{I}_N - \mathbf{P}_{S_i}) \mathbf{y}} \times \frac{N - n_i}{n_i} \right). \quad (11)$$

The proposed detector is based on the derivation of a model-order dependent penalty factor that results from the analysis of the LT-CGF of $\tilde{L}_i(\mathbf{y})$ under \mathcal{H}_0 .

Under \mathcal{H}_0 , $\tilde{L}_i(\mathbf{y})$ is half the logarithm of an F-distribution variate and therefore follows Fisher's z-distribution with n_i and $N - n_i$ degrees of freedom. Let $\rho_i = N/n_i > 1$, for $t < n_i(\rho_i - 1)$, the CGF of $\tilde{L}_i(\mathbf{y})$ under \mathcal{H}_0 satisfies [19]

$$K_{\tilde{L}_i}(t) = \frac{t}{2} \log(\rho_i - 1) - \log \left(\Gamma \left(\frac{n_i(\rho_i - 1)}{2} \right) \Gamma \left(\frac{n_i}{2} \right) \right) + \log \left(\Gamma \left(\frac{n_i + t}{2} \right) \Gamma \left(\frac{n_i(\rho_i - 1) - t}{2} \right) \right), \quad (12)$$

where $\Gamma(z) = \int_0^{+\infty} t^{z-1} e^{-t} dt$. The LT of $K_{\tilde{L}_i}$ is then defined as $K_{\tilde{L}_i}^*(x) = \sup_{t < n_i(\rho_i - 1)} (tx - K_{\tilde{L}_i}(t))$ and

$$g_{\tilde{L}_i}(x) = 2K_{\tilde{L}_i}^*(x)u(\tilde{L}_i(\mathbf{y}) - \mu_i), \quad (13)$$

where $\mu_i = \mathbb{E}(\tilde{L}_i(\mathbf{y})) = \frac{1}{2n_i} \left(\frac{1}{\rho_i - 1} - 1 \right)$.

Proposition 2. Let Y_{n_i} be the random variable defined as $Y_{n_i} = g_{\tilde{L}_i}(\tilde{L}_i(\mathbf{y}))$ and assume that the ratio $\rho_i = N/n_i > 1$ is kept constant as $n_i \rightarrow +\infty$. Under \mathcal{H}_0 , $Y_{n_i} \xrightarrow{d} Y$, where the PDF of Y is defined in Eq. (10).

Proof: See Appendix A. ■

Applying the transformation $g_{\tilde{L}_i}$ to the statistic $\tilde{L}_i(\mathbf{y})$ is therefore a way to equalize the model-order dependent PDFs involved in our problem. Unfortunately, $g_{\tilde{L}_i}(\tilde{L}_i(\mathbf{y}))$ has no closed-form expression. However, based on the asymptotic analysis provided in App. B, we are in position to formulate an explicit approximation of $g_{\tilde{L}_i}(\tilde{L}_i(\mathbf{y}))$, that is

$$g_{\tilde{L}_i}(\tilde{L}_i(\mathbf{y})) \approx 2 \left[\tilde{L}_i(\mathbf{y}) h_i(\tilde{L}_i(\mathbf{y})) - K_{\tilde{L}_i}(h_i(\tilde{L}_i(\mathbf{y}))) \right] \times u(\tilde{L}_i(\mathbf{y}) - \mu_i), \quad (14)$$

where

$$h_i(x) = n_i \left(\frac{\rho_i}{(\rho_i - 1)e^{-2x} + 1} - 1 \right). \quad (15)$$

Finally, to solve Problem (1), we propose the following test, referred to as the MFCLRT,

$$T_{\text{MFCLRT}}(\mathbf{y}) = \max_{1 \leq i \leq M} \left\{ g_{\tilde{L}_i}(\tilde{L}_i(\mathbf{y})) \right\} \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\geq}} \eta_{\text{MFCLRT}}, \quad (16)$$

where the approximation (14) is used for $g_{\tilde{L}_i}$. This test does not depend on unknown parameters and therefore ensures the CFAR property. Moreover, it can be shown that $g_{\tilde{L}_i}(\tilde{L}_i(\mathbf{y}))$ is a decreasing function of n_i so that it counteracts the tendency of the GLR statistic to increase with n_i . Also, note that if the subspace dimensions are all identical ($n_i = n, \forall i$), then the tests (2), (5), (6) and (16) are all equivalent since their statistics are all monotonically increasing with $L_i(\mathbf{y})$.

IV. NUMERICAL RESULTS

To illustrate the performance of the MFCLRT (16), we consider the detection of a known signal \mathbf{s} in an unknown deterministic multipath channel \mathbf{h} . More precisely, $x(k) = \sum_{l=0}^{n_M-1} h(l)s(k-l)$, where n_M denotes the number of channel taps. In the simulation, the channel has an exponential decay power-delay profile, i.e., $h(k) = e^{-k \frac{\log(10)}{2k_0}}$, where $k_0 > 0$ denotes the 10 dB-maximum delay spread.² Without loss of generality, we assume that $s(k) = d(k)$, where $d(k)$ is the unit impulse.³ Since the channel is unknown, the detector is unaware of the number of significant echoes that must be taken into account in the test statistic. The set of all possible echoes therefore constitutes a union of nested subspaces, each of which corresponds to a set of delays. More precisely, since $s(k) = d(k)$, each subspace basis is defined as in Eq. (4) for any $1 \leq n_i \leq n_M$.

We investigate the performance of the proposed detector in terms of probability of detection P_d . The probability of false alarm P_{fa} is set to 10^{-3} . The detection thresholds and the probabilities of detection are obtained with $100/P_{fa}$ and 10^4 independent trials, respectively. The signal length N is set to 100 and $n_M = 50$. The energy-to-noise ratio (ENR) is defined as $\text{ENR} = 10 \log_{10}(\mathbf{x}^T \mathbf{x} / \sigma^2)$, where σ^2 is the noise variance.

Fig. 1 shows the probability of detection as function of the ENR for the four tests (2), (5), (6) and (16). k_0 is set to 15. The MFCLRT clearly outperforms the other tests as it better estimates the order of the channel. The GLRT includes all the echoes in the statistic, even those with insignificant power, and therefore performs the worst. A detailed analysis (not shown here) reveals that the MDL test tends to only consider the first few powerful echoes, whereas the MFLRT often underpenalizes insignificant echoes.

The effect of the model order is analyzed in Fig. 2 by varying the value of the maximum delay spread k_0 . Once again, the MFCLRT outperforms the other tests for any value k_0 . Both the MFCLRT and the MFLRT converge to the performance of the GLRT as k_0 increases. This results from the increase in the number of significant echoes as k_0 increases, leading to a model order close to the maximum n_M . Surprisingly, a collapse of performance rapidly occurs for the MDL test. This can be explained by the penalty term in Eq. (5) that linearly increases with n_i so that the channel tail is too often ignored.

V. CONCLUSION

We have shown that existing methods, such as the GLRT, are not adapted to the detection of signals in a union of possibly nested subspaces. By adapting to our context the CGF-based transformation presented in [12], we have derived a new detector capable of equalizing the model-order dependent statistics. Although the equalization is of an asymptotic nature (as the subspace dimension becomes large), the resulting test turns out to be very efficient, even for low-dimensional subspaces.

²We recall that this is the time delay (in number of samples) after which the multipath power falls 10 dB below the maximum.

³ $s(k)$ can be considered as the output of a correlation-based channel sounder for instance.

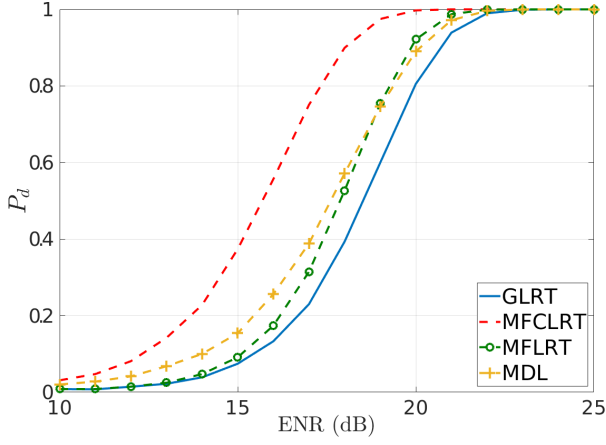


Fig. 1. P_d as a function of ENR for the GLRT, the MDL test, the MFLRT and the MFCLRT. $P_{fa} = 10^{-3}$, $N = 100$, $n_M = 50$, $k_0 = 15$.

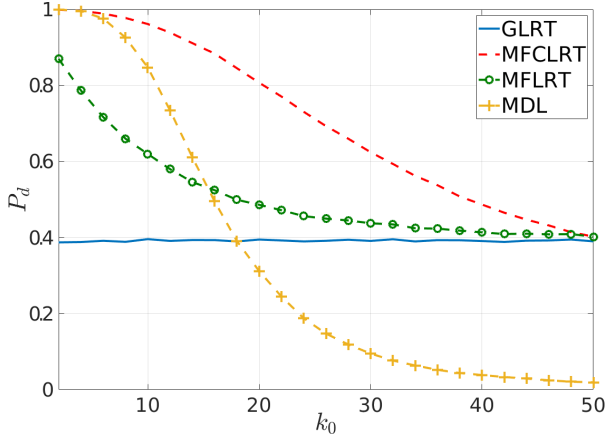


Fig. 2. P_d as a function of k_0 for the GLRT, the MDL test, the MFLRT and the MFCLRT. $P_{fa} = 10^{-3}$, $N = 100$, $n_M = 50$, ENR=18 dB.

APPENDIX A PROOF OF PROP. 2

Let μ_i and σ_i^2 denote the expectation and the variance of the random variable $\tilde{L}_i(\mathbf{y})$ under \mathcal{H}_0 . Since $\tilde{L}_i(\mathbf{y})$ follows Fisher's z-distribution, we have $\mu_i = \frac{1}{2n_i} \left(\frac{1}{\rho_i - 1} - 1 \right)$ and $\sigma_i^2 = \frac{1}{2n_i} \left(\frac{1}{\rho_i - 1} + 1 \right)$. Using a Taylor expansion for the LT-CGF about the expectation μ_i , we have [12, App. B]

$$g_{\tilde{L}_i}(\tilde{L}_i(\mathbf{y})) = u \left(\tilde{L}_i(\mathbf{y}) - \mu_i \right) \times \left[\frac{\left(\tilde{L}_i(\mathbf{y}) - \mu_i \right)^2}{\sigma_i^2} + \frac{1}{3} K_{\tilde{L}_i}^{*(3)}(\xi_i) \sigma_i^3 \frac{\left(\tilde{L}_i(\mathbf{y}) - \mu_i \right)^3}{\sigma_i^3} \right], \quad (17)$$

where $\xi_i = \mu_i + \theta(\tilde{L}_i(\mathbf{y}) - \mu_i)$ and $0 < \theta < 1$. It is known that the distribution of a Fisher's z variable approaches normality when its degrees of freedom tends to infinity [19, Sec. 15]. More precisely, let $V_{n_i} = \frac{\tilde{L}_i(\mathbf{y}) - \mu_i}{\sigma_i}$. For any $\rho_i = N/n_i > 1$, $V_{n_i} \xrightarrow{d} \mathcal{N}(0, 1)$. Therefore, by the continuous mapping theorem, $V_{n_i}^2 \xrightarrow{d} \chi_1^2$. The main step of the proof is then to show that the second term on the r.h.s of (17) is bounded in probability by a decreasing function of n_i .

First, since σ_i^2 is finite, then [15, Th. 14.4-1] $\left(\tilde{L}_i(\mathbf{y}) - \mu_i \right) / \sigma_i^3 = O_p(1)$. In addition, using [12, App. A], we note that the third derivative of the LT-CGF satisfies

$$K_{\tilde{L}_i}^{*(3)}(\xi_i) = - \frac{K_{\tilde{L}_i}^{(3)}(t_i^*(\xi_i))}{\left(K_{\tilde{L}_i}^{(2)}(t_i^*(\xi_i)) \right)^3}, \quad (18)$$

where $t_i^*(x) = \operatorname{argsup}_{t < n_i(\rho_i - 1)} (tx - K_{\tilde{L}_i}(t))$. Based on the expression of the CGF defined in (12), we observe that the ratio (18) will depend on the 3rd and 2nd-order derivative of the logarithm of the gamma function. These derivatives are known as the polygamma functions of order 2 and 1 [20, Ch. 6.4], respectively. Based on the asymptotic expansion given [20, Eq. (6.4.11)], we deduce that the m -th order polygamma function $\Psi^{(m)}(z)$ satisfies $\Psi^{(m)}(z) = O(z^{-m})$ ($z \rightarrow \infty$). Moreover, as shown in App. B, $\forall x > 0$, $t_i^*(x)$ is strictly increasing in n_i . Consequently, $\forall x > 0$,

$$\frac{K_{\tilde{L}_i}^{(3)}(t_i^*(x))}{\left(K_{\tilde{L}_i}^{(2)}(t_i^*(x)) \right)^3} = O \left(\frac{\Psi^{(2)}(n_i)}{\left(\Psi^{(1)}(n_i) \right)^3} \right) = O(n_i). \quad (19)$$

In addition, provided that $\tilde{L}_i(\mathbf{y}) > \mu_i$, ξ_i is strictly positive. Therefore, based on Eqs (18) and (19), we get $K_{\tilde{L}_i}^{*(3)}(\xi_i) = O_p(n_i)$, so that

$$\frac{1}{3} K_{\tilde{L}_i}^{*(3)}(\xi_i) \sigma_i^3 \frac{\left(\tilde{L}_i(\mathbf{y}) - \mu_i \right)^3}{\sigma_i^3} = O_p \left(n_i^{-1/2} \right) = o_p(1). \quad (20)$$

In summary, $g_{\tilde{L}_i}(\tilde{L}_i(\mathbf{y})) = u(V_{n_i}) V_{n_i}^2 + o_p(1)$ with $V_{n_i} \xrightarrow{d} \mathcal{N}(0, 1)$. Therefore, according to the continuous mapping theorem and to [15, Th. 4.4-3], $g_{\tilde{L}_i}(\tilde{L}_i(\mathbf{y}))$ converges in distribution to the random variable Y whose PDF is expressed in Eq. (10).

APPENDIX B ASYMPTOTICS OF $t_i^*(x)$

Let $t_i^*(x)$ be defined as $t_i^*(x) = \operatorname{argsup}_{t < n_i(\rho_i - 1)} G_i(t, x)$ where $G_i(t, x) = (tx - K_{\tilde{L}_i}(t))$. From (12), we deduce that

$$\frac{\partial G_i(t, x)}{\partial t} = \frac{1}{2} \left[2x - \log(\rho_i - 1) - \Psi^{(0)} \left(\frac{1}{2}(n_i + t) \right) + \Psi^{(0)} \left(\frac{1}{2}(n_i(\rho_i - 1) - t) \right) \right], \quad (21)$$

where $\Psi^{(0)}(\cdot)$ is the digamma function. For any real variable z , this function satisfies [20, Eq. (6.3.18)]: $\Psi^{(0)}(z) = \log(z) + o(z^{-1})$. Therefore, as $n_i \rightarrow \infty$, we have

$$\frac{\partial G_i(t, x)}{\partial t} \underset{n_i \rightarrow \infty}{\sim} \frac{1}{2} \left[2x + \log \left(\frac{n_i - t/(\rho_i - 1)}{n_i + t} \right) \right]. \quad (22)$$

By setting the derivative to zero, we can conclude that

$$t_i^*(x) \underset{n_i \rightarrow \infty}{\sim} n_i \left(\frac{\rho_i}{(\rho_i - 1)e^{-2x} + 1} - 1 \right). \quad (23)$$

Note that $\frac{\partial^2 G_i(t, x)}{\partial t^2} \Big|_{t=t_i^*(x)} < 0$, $\forall x$, so that $t_i^*(x)$ is an (asymptotic) supremum. Also, observe that $\forall x > 0$, $t_i^*(x)$ is strictly increasing in n_i .

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