

# SYMPLECTIC RESOLUTION OF THE QUOTIENT OF $\mathbb{R}^2$ BY A NON-FINITE SYMPLECTIC GROUP

HICHEM LASSOUED<sup>1</sup>, CAMILLE LAURENT-GENGOUX<sup>1</sup>

ABSTRACT. We construct smooth symplectic resolutions of the quotient of  $\mathbb{R}^2$  under some infinite but discrete sub-group of  $\mathfrak{sl}_2(\mathbb{R})$ . This extends to the setting of smooth real differential geometry the classical Du Val symplectic resolution of  $\mathbb{C}^2$  divided by a finite group in algebraic geometry. The first of these infinite groups is  $G = \mathbb{Z}$ , identified to triangular matrices with spectrum  $\{1\}$ . Smooth functions on the quotient  $\mathbb{R}^2/G$  come with a natural Poisson bracket, and  $\mathbb{R}^2/G$  is for an arbitrary  $k \geq 1$  set-isomorphic to the real Du Val singular variety  $A_{2k} = \{(x, y, z) \in \mathbb{R}^3, x^2 + y^2 = z^{2k}\}$ . We show that each one of the usual minimal resolutions of these Du Val varieties are symplectic resolutions of  $\mathbb{R}^2/G$ . The same holds for  $G = \mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$  (identified to triangular matrices with spectrum  $\{\pm 1\}$ ), with one half of  $D_{2k+1}$  playing the rôle of  $A_{2k}$ .

## CONTENTS

1. Introduction	1
2. Quotient of $\mathbb{R}^2$ by two infinite groups	3
3. Real Du Val symplectic resolutions	7
4. Quotient Poisson structure	10
5. Symplectic resolutions of the quotients $\mathbb{R}^2/G$ and $\mathbb{R}^2/G'$	13
References	16

## 1. INTRODUCTION

This article aims to initiate the study of quotients of smooth Poisson manifolds  $(M, \pi)$  by *infinite* but discrete groups  $G$  of Poisson automorphisms of  $(M, \pi)$ . We call *quotient of  $M$  by  $G$*  and denote by  $M/G$  (with an abuse of notation) the quotient of  $M$  under the closure of the equivalence relation induced by  $G$ , *i.e.*  $x \sim y$  if and only if there exists sequences  $(x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}}$  with limits  $x$  and  $y$  respectively such that  $x_n$  and  $y_n$  are in the same  $G$ -orbit for all  $n \in \mathbb{N}$ . Such quotients are neither varieties nor algebraic varieties in general, but they are always "Poisson sets" [9] in the following sense: a real valued function on  $M/G$  is said to be *smooth* if its pull-back to  $M$  is smooth on  $M$ , *i.e.*  $C^\infty(M/G) \simeq C^\infty(M)^G$ . Since  $G$  preserves structure, the algebra  $C^\infty(M/G)$  comes equipped with a Poisson bracket (see [12], Section 5.4).

**Definition 1.1.** [10] *Let  $(M, \pi_M)$  be a Poisson manifold, and  $C^\infty(M/G) := C^\infty(M)^G$ . A symplectic realization of  $M/G$  is a triplet  $(\Sigma, \pi_\Sigma, \varphi)$  where  $(\Sigma, \pi_\Sigma)$  is a symplectic manifold (maybe with boundary),  $\varphi : \Sigma \rightarrow M/G$  is a map such that*

---

*Date:* ..., 2020.

i)  $\varphi: \Sigma \rightarrow M/G$  is smooth, i.e.  $\varphi^*(F)$  is a smooth function on  $\Sigma$  for all  $F \in C^\infty(M/G)$ ,

ii) the algebra morphism  $\varphi^*: C^\infty(M/G) \rightarrow C^\infty(\Sigma)$  is a Poisson morphism.

Such a symplectic realization  $(\Sigma, \pi_\Sigma, \varphi)$  is called a symplectic resolution of  $M/G$  when:

iii) there exists an open dense  $G$ -saturated subset  $U \in M$  on which the  $G$ -action is discrete, proper and free<sup>1</sup>,

iiii)  $\varphi: \varphi^{-1}(U/G) \rightarrow U/G$  is a diffeomorphism.

Definition 1.1 echoes the nowadays classical notion of symplectic realizations of smooth Poisson manifolds [3]-[4]-[5]-[8]. However, in Definition 1.1, the manifolds  $\Sigma$  and  $M$  have the same dimension while in the latter theory, the dimension is doubled. Definition 1.1 is therefore more similar to that of symplectic resolutions of affine Poisson varieties as in [1]-[6]-[11]: indeed, it is an extension of this notion to the context of real differential geometry (as in [11]). Several examples of symplectic resolutions include quotients of symplectic vector spaces by *finite* subgroup of the symplectic group, see [2]. The most basic examples of such quotients are the very classical Du Val singularities  $(A_n, D_n, E_6, E_7, E_8)$ , see [15]. It is known that the minimal desingularization of these singularities is a symplectic resolution in the sense of algebraic geometry [1]-[7]. This implies that it is also a symplectic resolution in the sense of Definition 1.1.

As previously said, we wish to investigate quotients of Poisson vector spaces under infinite but discrete groups and their symplectic resolutions, if any. Notice that for  $G$  a discrete infinite subgroup of  $GL(V)$ , there may be very few non-constant algebraic  $G$ -invariant functions, while  $G$ -invariant smooth functions  $C^\infty(V/G)$  may be still large enough to be worth of study. As we will see, our purpose can not be addressed within algebraic geometry: We have to work with smooth functions.

But this is certainly not an easy matter. Our more reasonable present purpose is to understand in detail two examples that we think to be generic in some sense. Both will turn to be surprisingly subtle, although they are simply the quotients of  $\mathbb{R}^2$  by the groups

$$(1.2) \quad G = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \mid n \in \mathbb{Z} \right\} \text{ and } G' = \left\{ \begin{pmatrix} 1 & n \\ 0 & \epsilon \end{pmatrix} \mid \epsilon \in \{\pm 1\}, n \in \mathbb{Z} \right\}$$

which are isomorphic to  $\mathbb{Z}$  and  $\mathbb{Z}/2\mathbb{Z} \rtimes \mathbb{Z}$  respectively. These infinite but discrete groups act on  $\mathbb{R}^2$  by Poisson automorphisms for the Poisson structure

$$(1.3) \quad \pi_{\mathbb{R}^2} := \frac{1}{2\pi} q \frac{\partial}{\partial p} \wedge \frac{\partial}{\partial q}$$

where  $(p, q)$  stands for the canonical coordinates on  $\mathbb{R}^2$ .

It was proven in [10] that the previous Poisson structure does not admit symplectic resolutions. In contrast, the main theorems of the present articles state that the quotients  $\mathbb{R}^2/G$  and  $\mathbb{R}^2/G'$  do admit symplectic resolutions.

Let us describe the content of the article. In Section 2, we study the algebras  $C^\infty(\mathbb{R}^2/G) := C^\infty(\mathbb{R}^2)^G$  and  $C^\infty(\mathbb{R}^2/G') := C^\infty(\mathbb{R}^2)^{G'}$ . In the process, we will explain why, unlike in the finite case, real analytic or holomorphic contexts are not relevant here because they admit too little invariant functions. For smooth

<sup>1</sup>This implies that  $U/G$  is a manifold (and makes sense of Condition (iiii)).

invariant functions, which form an algebra large enough to be interesting, we give a useful Fourier series expansion.

Section 3 is dedicated to reminders on Du Val's symplectic resolutions (in the real case) of  $A_{2k}$  and  $D_{2k+1}$ .

In section 4, we show that for all  $k \geq 2$ , the quotient space  $\mathbb{R}^2/G$  is in bijection (as a set) with the quotient singularity of Du Val

$$A_{2k} = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = z^{2k}\}.$$

We then show that this bijection  $\underline{\phi}_k$  behaves well with respect to Poisson structures. By "behaves well", we mean that  $\underline{\phi}_k^* : C^\infty(\mathbb{R}^3) \rightarrow C^{\lfloor k/2 \rfloor}(\mathbb{R}^2/G)$  is a Poisson algebra morphism.

We also show similar results for the second group  $G' \simeq \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ , but it is then the singularities (Du Val) of type  $D_{2k+1} = \{(x, y, z) \in \mathbb{R}^3 \mid zx^2 + y^2 = z^{2k+1}\}$  which intervenes (more precisely,  $D_{2k+1} \cap \{z \geq 0\}$ ).

Section 5 concludes this construction by an extremely surprising result: the (usual) symplectic resolution  $\varphi : Z_k \rightarrow A_{2k}$  of the Du Val singularity  $A_{2k}$  induces a symplectic resolution of the Poisson quotient  $\mathbb{R}^2/G$ , when composed with  $\underline{\phi}_k^{-1} : A_{2k} \rightarrow \mathbb{R}^2/G$ . In other words,  $\phi^{-1} \circ \varphi : Z_k \rightarrow \mathbb{R}^2/G$  satisfies the requirements of Definition 1.1. This is not an easy result, and we have to deploy various techniques to achieve this, including Fourier analysis, and delicate consideration on the Du Val resolution.

We also show similar results for the second group  $G' := \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ . The usual symplectic resolution  $\varphi : Z_k \rightarrow D_{2k+1}$ , restricted to  $\varphi^{-1}(D_{2k+1} \cap \{z \geq 0\})$ , composed with  $\overline{\phi}_k^{-1} : D_{2k+1} \cap \{z \geq 0\} \rightarrow \mathbb{R}^2/G'$  gives a smooth symplectic resolution of the Poisson quotient  $\mathbb{R}^2/G'$  in this case.

It is of course interesting to see that there are infinitely many symplectic resolutions for both quotient spaces. However, it is unexpected to find  $A_{2k}$  in one case and  $D_{2k+1}$  in the other. There must be a general phenomena behind this astonishing result.

## 2. QUOTIENT OF $\mathbb{R}^2$ BY TWO INFINITE GROUPS

We equip  $\mathbb{R}^2$  with the coordinates  $(p, q)$ . Consider the infinite groups  $G \simeq \mathbb{Z}$  and  $G' \simeq \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$  as in (1.2) and their actions on  $\mathbb{R}^2$  by

$$\begin{array}{ccc} \mathbb{Z} \times \mathbb{R}^2 & \rightarrow & \mathbb{R}^2 \\ n \cdot (p, q) & \mapsto & (p + nq, q) \end{array} \quad \text{and} \quad \begin{array}{ccc} (\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) \times \mathbb{R}^2 & \rightarrow & \mathbb{R}^2 \\ (n, \epsilon) \cdot (p, q) & \mapsto & (p + nq, \epsilon q) \end{array}$$

for all  $(n, \epsilon) \in \mathbb{Z} \times \{-1, 1\}$  and  $(p, q) \in \mathbb{R}^2$ .

The closure  $\sim$  of the equivalence relation induced by the action of  $G = \mathbb{Z}$  above is given by

$$(2.1) \quad (p, q) \sim (p + nq, q),$$

for any pair  $(p, q) \in \mathbb{R}^2$ ,  $n \in \mathbb{Z}$  and  $(p, 0) \sim (p', 0)$  for all  $p, p' \in \mathbb{R}$ . This equivalence relation is closed *i.e.*,  $\{(p_1, q_1) \times (p_2, q_2) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid (p_1, q_1) \sim (p_2, q_2)\}$  is a closed set of  $\mathbb{R}^2 \times \mathbb{R}^2$ . For  $G' = \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ , we still have to identify  $(p, q)$  and  $(p, -q)$ . We denote by  $\sim'$  this relation.

**Convention 2.2.** We denote (with a slight abuse of notation) the equivalence classes of the closed relations  $\sim$  and  $\sim'$  by  $\mathbb{R}^2/G$  and  $\mathbb{R}^2/G'$ .

In general, for  $\sim$  be a closed equivalence relation on a manifold  $M$ , we call *smooth function, real analytic function or function of class  $C^k$  on  $M/\sim$*  a function whose pull-back on  $M$  through the natural projection  $M \mapsto M/\sim$  is smooth, respectively real analytic or of class  $C^k$ . Equivalently, functions of one of these given types on  $M/\sim$  are functions on  $M$  constant on the equivalence classes.

When the equivalence relation is the closure of a relation given by the action of a group  $G$ , smooth functions (resp. real analytic or  $C^k$ -functions) on the quotient are exactly  $G$ -invariant smooth functions (resp. real analytic or  $C^k$ -functions) on  $M$ . In the case we are interested with, we therefore have  $\mathcal{F}[\mathbb{R}^2]^G \simeq \mathcal{F}[\mathbb{R}^2/G]$  and  $\mathcal{F}[\mathbb{R}^2]^{G'} \simeq \mathcal{F}[\mathbb{R}^2/G']$  for  $\mathcal{F} = C^\infty$  or  $C^k$ .

Let us study this algebra. finite groups, the algebra of invariant functions  $\mathcal{F}[\mathbb{R}^2]^G$  or  $\mathcal{F}[\mathbb{R}^2]^{G'}$  may not be finitely generated. We take  $G = \mathbb{Z}$  and we start with the algebra  $C^\infty(\mathbb{R}^2/G) = C^\infty(\mathbb{R}^2)^G$ .

**Lemma 2.3.** *Any function  $f \in C^\infty(\mathbb{R}^2)^G$  satisfies for all  $i \geq 1, j \geq 0$*

$$\frac{\partial^{i+j} f}{\partial p^i \partial q^j}(p, 0) = 0, \quad \forall p \in \mathbb{R}.$$

*Proof.* The  $k^{\text{th}}$  derivative of the relation  $f(p, q) = f(p + nq, q)$  with respect to  $q$  gives:

$$\frac{\partial^k f}{\partial q^k}(p, q) = \sum_{i=0}^k \binom{k}{i} n^{k-i} \frac{\partial^k f}{\partial p^i \partial q^{k-i}}(p + nq, q).$$

For  $q = 0$ , we have

$$\sum_{i=0}^k \binom{k}{i} n^i \frac{\partial^k f}{\partial p^i \partial q^{k-i}}(p, 0) = \frac{\partial^k f}{\partial q^k}(p, 0).$$

The right hand term coincides with the  $i = 0$  term on left hand. We therefore obtain for all  $n \in \mathbb{Z}$ :

$$\sum_{i=1}^k \binom{k}{i} n^i \frac{\partial^k f}{\partial p^i \partial q^{k-i}}(p, 0) = 0.$$

The coefficients of a polynomial function that admits infinity many roots are zero. In particular, the coefficients of the polynomial

$$P_k(X) := \sum_{i=1}^k \binom{k}{i} \frac{\partial^k f}{\partial p^i \partial q^{k-i}}(p, 0) X^i$$

are zero for all  $k \in \mathbb{N}^*$  and  $p \in \mathbb{R}$ . This completes the proof.  $\square$

**Proposition 2.4.** *The algebra  $C^\infty(\mathbb{R}^2/G)$  of smooth functions on  $\mathbb{R}^2/G$  breaks down as follows*

$$C^\infty(\mathbb{R}^2/G) = C^\infty(q) + \tilde{C},$$

where  $C^\infty(q)$  is the algebra of smooth functions which depends only on the variable  $q$ , and  $\tilde{C} \subset C^\infty(\mathbb{R}^2/G)$  is the subalgebra of  $G$ -invariant smooth functions vanishing with all their partial derivatives along the line  $q = 0$ .

*Proof.* Consider a  $G$ -invariant smooth function  $f(p, q)$ . Define smooth functions  $g, h$  on  $\mathbb{R}$  and  $\mathbb{R}^2$  respectively by  $h(q) := g(0, q)$  and  $g(p, q) := f(p, q) - h(q)$ . We have  $f = h + g$  by construction. It therefore remains to show that  $g$  is an element of  $\tilde{C}$ .

Since the function  $g$  is obviously  $G$ -invariant, Lemma 2.3 implies that for all  $i \geq 1, j \geq 0$ :

$$\frac{\partial^{i+j}g}{\partial p^i \partial q^j}(p, 0) = 0, \quad \forall p \in \mathbb{R}.$$

By integration,

$$\frac{\partial^j g}{\partial q^j}(p, 0) = \frac{\partial^j g}{\partial q^j}(0, 0) + \int_{t=0}^p \frac{\partial^{1+j}g}{\partial p \partial q^j}(t, 0) dt.$$

Since  $g(0, q) = 0$  by construction, we have  $\frac{\partial^j g}{\partial q^j}(0, 0) = 0$ , and the first term on the right vanishes. Lemma 2.3 implies the vanishing of the second one. This completes the proof.  $\square$

There is a similar result for the  $G'$ -action on  $\mathbb{R}^2$ .

**Proposition 2.5.** *The algebra  $C^\infty(\mathbb{R}^2/G')$  of smooth functions on  $\mathbb{R}^2/G'$  breaks down as follows*

$$C^\infty(\mathbb{R}^2/G) = C^\infty(q^2) + \tilde{C},$$

where  $C^\infty(q^2)$  is the algebra of smooth even functions depending only on the variable  $q$ , and  $\tilde{C} \subset C^\infty(\mathbb{R}^2/G')$  is the subalgebra of  $G'$ -invariant smooth functions vanishing with all their partial derivatives along the line  $q = 0$ .

The following corollary shows that the real analytic case is not interesting: there are too little  $G$ -invariant functions.

**Corollary 2.6.** *Real analytic functions on  $\mathbb{R}^2/G$  (on  $\mathbb{R}^2/G'$ ) are the functions (resp. even function) which depend only on the variable  $q$ .*

*Proof.* A real analytic function on  $\mathbb{R}^2$  which vanishes along a line with all its partial derivatives is zero on  $\mathbb{R}^2$ . The result now follows from Propositions 2.4 and 2.5.  $\square$

This corollary also means that it is not possible to make such a study in the holomorphic context and study  $\mathbb{C}^2/G$  instead. There is not enough  $G$ -invariant holomorphic functions. Let us complete our description of smooth functions on  $\mathbb{R}^2/G$ :

**Proposition 2.7.** *Any function  $f \in C^\infty(\mathbb{R}^2)^G \simeq C^\infty(\mathbb{R}^2/G)$  reads*

$$(2.8) \quad f(p, q) = \alpha_0(q) + \sum_{n \geq 1} \alpha_n(q) \cos\left(2n\pi \frac{p}{q}\right) + \sum_{n \geq 1} \beta_n(q) \sin\left(2n\pi \frac{p}{q}\right),$$

where  $\alpha_0$  is a smooth function on  $\mathbb{R}$  and where  $(\alpha_n(q), \beta_n(q))_{n \geq 1}$  are smooth functions on  $\mathbb{R}$  that vanish together with all their derivatives at  $q = 0$ . Both above series, moreover, are absolutely convergent on any open ball.

*Proof of Proposition 2.7. Step 1.* Let us construct functions  $\alpha_n, \beta_n$  that satisfy (2.8). For  $q \in \mathbb{R}^*$  fixed, the function  $p \mapsto f(p, q)$  is smooth and  $q$ -periodic. Its Fourier transform with respect to the variable  $p$  reads

$$f(p, q) = \alpha_0(q) + \sum_{n \geq 1} \alpha_n(q) \cos\left(2n\pi \frac{p}{q}\right) + \sum_{n \geq 1} \beta_n(q) \sin\left(2n\pi \frac{p}{q}\right)$$

Let us use the change of variables  $u = \frac{p}{q}$  to find a friendly expression of the Fourier coefficients:

$$\begin{aligned}\alpha_0(q) &= \frac{1}{q} \int_{-q/2}^{q/2} f(p, q) dp, \\ &= \int_{-1/2}^{1/2} f(uq, q) du\end{aligned}\quad (*_0),$$

and

$$\begin{aligned}\alpha_n(q) &= \frac{2}{q} \int_{-q/2}^{q/2} f(p, q) \cos\left(2n\pi\frac{p}{q}\right) dp, \\ &= 2 \int_{-1/2}^{1/2} f(uq, q) \cos(2n\pi u) du,\end{aligned}\quad (*_n)$$

and

$$\begin{aligned}\beta_n(q) &= \frac{2}{q} \int_{-q/2}^{q/2} f(p, q) \sin\left(2n\pi\frac{p}{q}\right) dp, \\ &= 2 \int_{-1/2}^{1/2} f(uq, q) \sin(2n\pi u) du.\end{aligned}\quad (**_n)$$

Formulas  $(*_0, *_n, **_n)$  still make sense for  $q = 0$ , so that the functions  $\alpha_n, \beta_n$  are defined on  $\mathbb{R}$ . They are smooth as integrals of smooth functions on a compact interval.

*Step 2.* Let us show that for all  $n \geq 1$ , the Fourier coefficients  $\alpha_n(q), \beta_n(q) \in C^\infty(\mathbb{R}, \mathbb{R})$  are smooth functions which are zero with all their derivatives at  $q = 0$ .

A. If the function  $f(p, q)$  only depends on  $q$ , then  $\alpha_n = \beta_n = 0$  for all  $n \geq 1$ , while  $\alpha_0(q) = f(p, q)$  and the result is obvious.

B. If  $f(p, q)$  belongs to  $\tilde{C}$ , it vanishes with all its partial derivative along the line  $q = 0$ . A simple computation using  $(*_n, **_n)$  then gives the result.

By Proposition 2.4, any function in  $C^\infty(\mathbb{R}^2/G)$  is a sum of a function of the type A with a function of the second type B. The result therefore holds.

*Step 3.* Let us show that the series in (2.8) are absolutely convergent on any relatively compact subset  $\mathcal{U}$  of  $\mathbb{R}^2$ . Let us integrate by part the right hand side of  $(*_n)$  twice. Since  $u \mapsto f(uq, q)$  is periodic of period 1, all boundary term disappear, and we obtain:

$$(2.9) \quad \alpha_n(q) = \frac{-1}{n^2} \frac{q^2}{4\pi^2} \int_{-1/2}^{1/2} \frac{\partial^2 f(uq, q)}{\partial q^2} \cos(2n\pi u) du.$$

Let  $K = \frac{B^2}{2\pi^2} N$ , where  $B, N$  are the greatest possible value of  $|q|$  and  $|\frac{\partial^2 F(p, q)}{\partial p^2}|$  on  $\mathcal{U}$ . We have  $|\alpha_n(q)| < \frac{K}{n^2}$ . The same applies to  $(**_n)$ . Since the series  $\sum 1/n^2$  converges, both series in (2.8) are absolutely convergent.  $\square$

In fact, the inequality (2.9) can be made much stronger - and it will be needed later.

**Lemma 2.10.** *For every  $i \geq 1, k \geq 1$  and  $M > 0$ , there exists a constant  $K$  that does not depend on  $n$  such that:*

$$\left| \frac{\alpha_n^{(i)}(q)}{q^k} \right| \leq \frac{K}{n^k} \quad \text{and} \quad \left| \frac{\beta_n^{(i)}(q)}{q^k} \right| \leq \frac{K}{n^k}$$

for all  $-M < q < M$

*Proof.* Using successive integration by part, we obtain:

$$(2.11) \quad \alpha_n^{(i)}(q) = \frac{1}{n^k} \frac{q^k}{(2\pi)^k} \int_{-1/2}^{1/2} \frac{\partial^{k+i} f(uq, q)}{\partial q^{k+i}} \text{cossin}_k(2n\pi u) du.$$

where  $\text{cossin}_k$  is  $\cos, \sin, -\cos, -\sin$  for  $k \equiv 0, 1, 2, 3$  modulo 4 respectively. Since  $|\text{cossin}_k| \leq 1$ ,

$$K = \frac{1}{(2\pi)^k} \text{Max} \left\{ \left| \frac{\partial^{k+i} f(uq, q)}{\partial q^{k+i}} \right|, (u, q) \in [0, 1] \times [-M, M] \right\}$$

This maximum exists by compactness: This concludes the proof for the functions  $\alpha_n^{(i)}$ . The proof is similar for the functions  $\beta_n^{(i)}$ .  $\square$

Here is an immediate consequence of Lemma 2.10:

**Proposition 2.12.** *Both series in Equation (2.8) are uniformly convergent on any compact set, together with all their partial derivatives.*

For the group action  $G'$ , a similar argument leads to:

**Proposition 2.13.** *A function  $f \in C^\infty(\mathbb{R}^2/G')$  reads*

$$f(p, q) = \alpha_0(q) + \sum_{n \geq 1} \alpha_n(q) \cos\left(2n\pi \frac{p}{q}\right) + \sum_{n \geq 1} \beta_n(q) \sin\left(2n\pi \frac{p}{q}\right),$$

where the functions  $(\alpha_n, \beta_n)_{n \geq 1}$  are smooth and vanish with all their derivatives at  $q = 0$ . Moreover the functions  $\alpha_n$  are even functions and the functions  $\beta_n$  are odd functions for all  $n \in \mathbb{N}$ . The above series, moreover, is absolutely convergent on any relatively compact open set, and so are any of its partial derivatives.

### 3. REAL DU VAL SYMPLECTIC RESOLUTIONS

In this section, we mainly recall from [1] and adapt to the real case the construction of the symplectic resolution of the Du Val singularities  $A_{2k}$  and  $D_{2k+1}$ .

First, we define a Poisson structure on the varieties  $A_{2k}$  and  $D_{2k+1}$ . For any polynomial function  $F(x, y, z)$  on  $\mathbb{R}^3$ , a Poisson bracket on  $\mathbb{R}^3$  is given by [14]:

$$(3.1) \quad \{x, y\} = \frac{\partial F}{\partial z}, \{y, z\} = \frac{\partial F}{\partial x}, \{z, x\} = \frac{\partial F}{\partial y}.$$

This Poisson bracket admits  $F$  as a Casimir, and therefore goes to the quotient to  $\frac{\mathbb{R}[x, y, z]}{(F)}$ . Unlike in the complex case, the latter may not be easily identified to polynomial functions on the set  $F = 0$  (for instance, for  $F = x^2 + y^2 + z^2$ , the latter set is a point). It only makes sense when the quotient  $\frac{\mathbb{R}[x, y, z]}{(F)}$  injects itself into the algebra of real valued functions on  $\{F = 0\}$ . This is the case, in particular, for  $F = x^2 + y^2 - z^{2k}$ : We denote by  $A_{2k}$  the set

$$A_{2k} = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = z^{2k}\}.$$

Formula (3.1) equips  $A_{2k}$  with the Poisson structure:

$$(3.2) \quad \{x, y\} = -2kz^{2k-1}, \quad \{y, z\} = 2x, \quad \{z, x\} = 2y.$$

Let us study the successive blowing up at the origin (the only singular point) of  $A_{2k}$  and the behavior of the above Poisson structure.

Let us first describe the minimal resolution of  $A_{2k}$ .

- (1) The blowup  $\varphi : Z \rightarrow A_{2k}$  at the origin  $(0, 0, 0) \in A_{2k}$  is covered by 3 pieces.
  - (a) In two of them, the strict transform of  $A_{2k}$  is smooth.

- (b) In the last one, the  $z$ -chart, with coordinates  $u, v, z$ , the morphism is given by  $\varphi_1 : Z_1 \rightarrow \mathbb{R}^3$  defined by  $x = uz, y = vz, z = z$ , and the inverse image of  $A_{2k}$  is defined by  $u^2z^2 + v^2z^2 = z^{2k}$  so that the strict transform is  $u^2 + v^2 = z^{2(k-1)}$ . As a consequence, the strict transform on this chart is isomorphic to  $A_{2k-2}$ .

- (2) We then blow-up  $A_{2k-2}$ , and repeat the procedure

⋮

- (k) Therefore, after  $k$ -successive blowups at  $(0, 0, 0)$ :  $\varphi_k : Z_k \rightarrow Z_{k-1}$ , the strict transform is given by  $x^2 + y^2 = 1$  with  $z$  a free variable.

At this point, we obtain a smooth projective variety, called the *(real) Du Val resolution of  $A_{2k}$* .

Let us describe the behaviour of the Poisson structure under the successive blow-up leading to the Du Val resolution.

- (1) The Poisson structure (3.2) pulls-back to a Poisson structure on the strict transform of  $A_{2k}$  under the first blow-up  $A_{2k}$

- (a) In the two first affine charts, the pulled-back Poisson structure is symplectic  
 (b) In the third one, (where the strict transform is  $\simeq A_{2k-2}$ ), the pulled-back Poisson structure is of the form (3.2) with  $k$  being replaced by  $k - 1$ .

- (2) The same applies then to the blow-up of  $A_{2k-2}$

⋮

- (k) Therefore, after  $k$ -successive blowups at  $(0, 0, 0)$ , the pulled-back Poisson structure, which is still well-defined and has no singularity, is given by

$$(3.3) \quad \{x, y\} = 0, \quad \{y, z\} = 2x, \quad \{z, x\} = 2y.$$

which is symplectic.

We have therefore obtained by successive blowing a symplectic resolution of  $A_{2k}$ , which we denote by  $(Z_k, \pi_k)$ . Let us recapitulate.

**Proposition 3.4.** *For all  $k \geq 2$ :*

- (1) *The map  $\phi : Z_k$  is onto, and restricts to a diffeomorphism  $Z_k \setminus \phi^{-1}(\{0\}) \simeq A_{2k} \setminus \{0\}$*   
 (2) *The pull-back of the Poisson structure (3.2) to the Du Val resolution of  $\varphi : Z_k \rightarrow A_{2k}$  is a symplectic structure.*

*Equivalently,  $(Z_k, \varphi, \pi_k)$  is a symplectic resolution of  $A_{2k}$ .*

The next point is easy to verify and will be important. Notice that, now, it is only true on  $\mathbb{R}$ , and not valid on  $\mathbb{C}$ .

**Proposition 3.5.** *The pull back of the rational functions  $\frac{x}{z^k}$  and  $\frac{y}{z^k}$  by the symplectic Du Val resolution  $Z_k \rightarrow A_{2k}$  are smooth functions on  $Z_k$ .*

*Proof.* The unique point where the rational function  $\frac{x}{z^k}$  is not defined is  $x = y = z = 0$ . Let us see the behaviour under the successive blowups that lead to the Du Val resolution:

- (1) For the blowing-up of the singularity  $A_{2k}$  at the origin, the following phenomenon occurs:
- (a) in the two first affine charts (the  $x$ -chart and the  $y$ -chart), the pull-back is a smooth function, because the pull-back of  $z$  can not be equal to 0,
  - (b) in the last one, the restriction to the strict transform (which is isomorphic to  $A_{2k-2}$ ) of the pull back of  $\frac{x}{z^k}$  is the function  $\frac{x}{z^{k-1}}$ .
- We are therefore brought back to the previous function, upon changing  $k$  by  $k - 1$ . We can then repeat the procedure.
- ⋮
- (k) At the last step, the strict transform is given by  $x^2 + y^2 - 1 = 0$  with  $z$  being free, and the back pull of the function  $x/z^k$  is simply the function  $x$ , which is smooth.

This completes the proof. □

Let us define two sequences  $(x_n)_{n \geq 1}$  and  $(y_n)_{n \geq 1}$  of rational functions on  $A_{2k} \subset \mathbb{R}^3$  by  $x_1(x, y, z) := x$ ,  $y_1(x, y, z) := y$  for all  $(x, y, z) \in A_{2k}$  and

$$(3.6) \quad \begin{cases} x_n = \frac{x_1}{z^k} x_{n-1} - \frac{y_1}{z^k} y_{n-1} \\ y_n = \frac{x_1}{z^k} y_{n-1} + \frac{y_1}{z^k} x_{n-1}. \end{cases}$$

From Proposition 3.5, it follows that

**Corollary 3.7.** *For all  $n \in \mathbb{N}$  the pull back  $X_n := \varphi^* x_n$  and  $Y_n = \varphi^* y_n$  of the functions  $x_n, y_n$  defined in (3.6) to the Du Val resolution  $\varphi : Z_k \rightarrow A_{2k}$  are smooth functions on  $Z_k$ .*

By same procedure, we equip the variety defined by

$$D_{2k+1} := \{(x, y, z) \in \mathbb{R}^3 \mid zx^2 + y^2 = z^{2k+1} = 0\}$$

with the Poisson structure defined as in (3.1) with the help of the function  $zx^2 + y^2 - z^{2k+1}$ :

$$(3.8) \quad \{x, y\} = -(2k+1)z^{2k} + x^2, \{y, z\} = 2zx, \{z, x\} = 2y.$$

Let us now describe a symplectic resolution of the latter. The procedure is at first very similar.

- (1) The strict transform of the blowing up at zero is covered by the three natural charts.
- (a) In the  $x$ -chart the strict transform is the variety  $zx - z^{2k+1}x^{2k-1} = -y^2$ , whose unique singular point is  $(0, 0, 0)$ .
    - (i) The strict transform of the pull-back at the origin of the latter variety is smooth, and the pull-back of the Poisson structure (3.8) is symplectic.
  - (b) In the  $y$ -chart the strict transform is smooth and the pull-back of the Poisson structure (3.8) is symplectic.
  - (c) In the  $z$ -chart the strict transform is  $D_{2(k-1)}$ , and the pull-back of the Poisson structure (3.8) is the Poisson structure (3.8), with  $k$  being replaced by  $k - 1$ .

We are therefore brought back to the previous problem in changing  $k$  by  $k - 1$  and then repeat the operation.

⋮

- (k) At the  $k^{\text{th}}$ -step, the situation is quite different from the one of  $A_{2k}$ . The strict transform is the variety:

$$zx^2 + y^2 - z = 0$$

which admits two singular points:  $(1, 0, 0)$  and  $(-1, 0, 0)$ . The pull-back of the Poisson structure 3.8 is given (as in (3.1)) by

$$(3.9) \quad \{x, y\} = x^2 - 1, \{y, z\} = 2xz, \{z, x\} = 2y.$$

- (a) The blow up at  $(1, 0, 0)$  admits a singular point. Blowing-up at this point, we obtain at last a smooth variety, and the pull-back of the Poisson structure (3.9) is symplectic.
- (b) The blow up at  $(-1, 0, 0)$  admits a singular point. Blowing-up at this point, we obtain at last a smooth variety, and the pull-back of the Poisson structure (3.9) is symplectic. We denote by  $(Z_{k+2}, \pi_{k+2})$  this symplectic manifold.

Let  $D_{2k+1}^+ = D_{2k+1} \cap \{z > 0\}$  and  $Z_{k+2}^+ := \phi^{-1}(\{z > 0\}) \subset Z_{k+2}$ .  $Z_{k+2}^+$  is manifold (with boundary). We have:

**Proposition 3.10.** *For all  $k \geq 2$ :*

- (1) *The map  $\varphi : Z_{k+2}^+ \rightarrow D_{2k+1}^+$  is onto, and restricts to a diffeomorphism  $Z_{k+2}^+ \setminus \phi^{-1}(\{0\}) \simeq D_{2k+1}^+ \setminus \{0\}$*
- (2) *The pull-back of the Poisson structure (3.8) to the Du Val resolution of  $\varphi : Z_{k+2}^+ \rightarrow D_{2k+1}^+$  is a symplectic structure.*

*Equivalently,  $(Z_{k+2}^+, \varphi, \pi_k)$  is a symplectic resolution of  $D_{2k+1}^+$ .*

**Proposition 3.11.** *The pull back  $\frac{x}{z^{k+\frac{1}{2}}}$  and  $\frac{y}{z^{k+\frac{1}{2}}}$  by the map  $Z_{k+2} \rightarrow D_{2k+1}^+$  are smooth functions on  $Z_{k+2}$ .*

*Proof.* The proof is similar to that of the Proposition 3.5. We leave it to the reader.  $\square$

We shall consider the following functions, similar to those in (3.6):

$$(3.12) \quad \begin{cases} x_n = \frac{x_1}{z^{k+1/2}} x_{n-1} - \frac{y_1}{z^{k+1/2}} y_{n-1} \\ y_n = \frac{x_1}{z^{k+1/2}} y_{n-1} + \frac{y_1}{z^{k+1/2}} x_{n-1}. \end{cases}$$

It follows from all of the above that:

**Corollary 3.13.** *The pull back  $X_n := \varphi^* x_n$  and  $Y_n := \varphi^* y_n$  of functions  $x_n, y_n$  defined in (3.12), by the Du Val resolution  $\varphi : Z_{k+2} \rightarrow D_{2k+1}^+$  are polynomial functions on  $Z_{k+2}$ .*

#### 4. QUOTIENT POISSON STRUCTURE

We now introduce the Poisson bracket on the quotient space  $\mathbb{R}^2/G$  (resp  $\mathbb{R}^2/G'$ ), with  $G$  and  $G'$  acting on  $M = \mathbb{R}^2$  as in Section 2. We introduce for  $k \geq 2$  the algebra of functions noted  $C^{k,\infty}(\mathbb{R}^2)$  which are:

- a) differentiable  $\lfloor k/2 \rfloor$ -times on  $\mathbb{R}^2$ ,
- b) and smooth outside the straight line  $q = 0$ .

We now equip  $M = \mathbb{R}^2$  with the following Poisson bracket

$$(4.1) \quad \{p, q\} = \frac{q}{2\pi}.$$

The following lemmas are easily verified.

**Lemma 4.2.** *For all  $k \in \{2, 3, \dots, +\infty\}$ , the algebra  $C^{k, \infty}(\mathbb{R}^2)$  is stable under the Poisson bracket.*

and

**Lemma 4.3.** *The Poisson bracket (4.1) is invariant under the actions of  $G = \mathbb{Z}$  and  $G'$ .*

In view of Lemmas 4.2 and 4.3, the following algebras come equipped with the induced Poisson brackets:  $C^\infty(M/G) := C^\infty(M)^G$ ,  $C^\infty(M/G') := C^\infty(M)^{G'}$ ,  $C^{k, \infty}(M/G) := C^{k, \infty}(M)^G$ , and  $C^{k, \infty}(M)^{G'}$  (see Section 2 for notations).

**Remark 4.4.** The symplectic Poisson bracket  $\{p, q\} = 1$  is also invariant under the actions of  $G$  and  $G'$ , but does not satisfy Lemma 4.2. Also, since the singular locus of both group actions (*i.e.* the straight line  $q = 0$ ) projects to a single point in the quotient space, it makes more sense to choose a Poisson structure on  $\mathbb{R}^2$  that vanishes on this straight line.

From now on, we deal with the  $G$ -action only: results for the  $G'$ -action will be stated at the end of the section.

Let  $\mathcal{A}$  be the subalgebra of  $C^{k, \infty}(\mathbb{R}^2/G)$  generated by

$$\left( x(p, q) := q^k \cos\left(2\pi \frac{p}{q}\right), y(p, q) := q^k \sin\left(2\pi \frac{p}{q}\right), z(p, q) := q \right).$$

**Proposition 4.5.** *The algebra  $\mathcal{A}$  is stable under the Poisson bracket (4.1).*

*Proof.* A direct computation with this bracket gives:

$$(4.6) \quad \begin{cases} \{x, y\} = -2k z^{2k-1} \in \mathcal{A} \\ \{y, z\} = 2x \in \mathcal{A} \\ \{z, x\} = 2y \in \mathcal{A}. \end{cases}$$

This proves the result.  $\square$

Let us consider the map  $\mathbb{R}^2 \rightarrow \mathbb{R}^3$  defined by

$$\phi_k : (p, q) \mapsto (x(p, q), y(p, q), z(p, q)).$$

Since  $x, y, z$  are  $G$ -invariant functions, and are equal to 0 on the straight line  $q = 0$ , this map goes to the quotient to define a map:  $\underline{\phi}_k : \mathbb{R}^2/G \rightarrow \mathbb{R}^3$ . Since  $x^2 + y^2 = z^{2k}$ , this map takes values in the Du Val singular surface  $A_{2k}$  (see Section 2).

**Lemma 4.7.** *The map  $\underline{\phi}_k : \mathbb{R}^2/G \rightarrow A_{2k}$  is one-to-one. Its restriction to  $\mathbb{R}^* \times \mathbb{R}/G \simeq A_{2k} \setminus \{0\}$  is a diffeomorphism.*

*Proof.* The relation

$$(4.8) \quad \begin{aligned} x^2 + y^2 &= \left( q^k \cos\left(2\pi \frac{p}{q}\right) \right)^2 + \left( q^k \sin\left(2\pi \frac{p}{q}\right) \right)^2 \\ &= q^{2k} = z^{2k}. \end{aligned}$$

proves that  $\phi_k$  is valued in  $A_{2k}$ . Hence so is  $\underline{\phi}_k$ .

Injectivity follows from the following two points:

- For any two  $(p, q), (p', q')$  with  $q, q' \neq 0$ ,  $x(p, q) = y(p, q)$  if and only if  $q = q'$  and  $p = p' + nq$  for some  $n \in \mathbb{N}$ .
- $x(p, q) = y(p, q) = z(p, q) = 0$  implies  $q = 0$

Hence  $\underline{\phi}_k(p, q) = \underline{\phi}_k(p', q')$  if and only if  $(p, q) \sim (p', q')$ .

Surjectivity follows easily from the surjectivity of  $u \rightarrow (\cos u, \sin u)$  from  $\mathbb{R}$  to  $S^1$ .  $\square$

Both  $\mathbb{R}^2/G$  and  $A_{2k}$  come equipped with Poisson structures. We would like to state that  $\underline{\phi}_k : \mathbb{R}^2/G \simeq A_{2k}$  is a Poisson isomorphism. But we have to be careful because there are several algebras of functions on  $\mathbb{R}^2/G$ : the one that allows to make this statement is  $\mathcal{A}$ , as we now state (here  $\mathcal{F}(A_{2k}) = \frac{\mathbb{R}[x, y, z]}{x^2 + y^2 - z^{2k}}$  stands for the algebra of polynomial functions on  $A_{2k}$ ):

**Proposition 4.9.** *The pull-back map of  $\underline{\phi}_k : \mathbb{R}^2/G \rightarrow A_{2k}$  is a Poisson morphism:*

$$C^\infty(\mathbb{R}^3) \rightarrow C^{\lfloor k/2 \rfloor}(\mathbb{R}^2/G).$$

It restricts to a Poisson algebra isomorphism  $\underline{\phi}_k^* : \mathcal{F}(A_{2k}) \simeq \mathcal{A}$ .

*Proof.* By Equation (4.8), the functions  $x, y, z \in C^{k, \infty}(\mathbb{R}^2)$  satisfy  $x^2 + y^2 = z^{2k}$ . Since there is no other relations between them, save for multiples of this relation, and since the generators  $x, y, z$  of  $\mathcal{F}(A_{2k})$  are also subject to the relation  $x^2 + y^2 = z^{2k}$ , the algebra morphism  $\underline{\phi}_k^* : \mathcal{F}(A_{2k}) \hookrightarrow \mathcal{A}$  is an algebra isomorphism. It is still necessary to show that it is a Poisson morphism.

A comparison of the Poisson bracket on  $\mathcal{F}(A_{2k})$  defined in (3.2) with the Poisson brackets of  $x, y, z \in C^{k, \infty}(M)$  as computed in (4.6) shows that  $\phi_k : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  is a Poisson map. In view of the following commutative diagram

$$(4.10) \quad \begin{array}{ccc} \mathbb{R}^2 & \xrightarrow{\phi_k} & \mathbb{R}^3 \\ \downarrow P & & \uparrow i \\ \mathbb{R}^2/G & \xrightarrow{\underline{\phi}_k} & A_{2k} \end{array} \quad \text{hence} \quad \begin{array}{ccc} \mathbb{R}[x, y, z] & \xrightarrow{\phi_k^*} & C^{\lfloor k/2 \rfloor}(\mathbb{R}^2) \\ \downarrow i^* & & \uparrow P^* \\ \mathcal{F}(A_{2k}) & \xrightarrow{\underline{\phi}_k^*} & C^{\lfloor k/2 \rfloor}(\mathbb{R}^2/G) \end{array}$$

this implies that  $\underline{\phi}_k^*$  is a Poisson morphism.  $\square$

For the quotient space  $\mathbb{R}^2/G'$ , we consider the subalgebra  $\mathcal{A}' \subset C^k(\mathbb{R}^2/G')$  generated by

$$\left( x = q^{2k} \cos\left(2\pi \frac{p}{q}\right), y = q^{2k+1} \sin\left(2\pi \frac{p}{q}\right), z = q^2 \right), \text{ for any integer } k \geq 2.$$

**Proposition 4.11.** *The algebra  $\mathcal{A}'$  is stable under the Poisson bracket (4.1).*

*Proof.* A direct computation with this bracket gives:

$$\begin{cases} \{x, y\} = -(2k+1)z^{2k} + x^2 \in \mathcal{A}' \\ \{y, z\} = 2zx \in \mathcal{A}' \\ \{z, x\} = 2y \in \mathcal{A}'. \end{cases}$$

$\square$

Let us consider the map  $\mathbb{R}^2 \rightarrow \mathbb{R}^3$  defined by

$$\phi_k : (p, q) \mapsto (x(p, q), y(p, q), z(p, q)).$$

Since  $x, y, z$  are  $G'$ -invariant functions, and are equal to 0 on the straight line  $q = 0$ , this map goes to the quotient to define a map:  $\underline{\phi}_k : \mathbb{R}^2/G' \rightarrow \mathbb{R}^3$ . Since  $zx^2 + y^2 = z^{2k+1}$ , this map takes values in the Du Val singular surface  $D_{2k+1}$  (see Section 2). More precisely, it takes values in  $D_{2k+1}^+ = D_{2k+1} \cap \{z \geq 0\}$

**Lemma 4.12.** *The map  $\bar{\phi}_k : \mathbb{R}^2/G' \rightarrow D_{2k+1}^+$  is one-to-one*

*Proof.* The relation

$$(4.13) \quad \begin{aligned} zx^2 + y^2 &= \left( q^{2k+1} \cos \left( 2\pi \frac{p}{q} \right) \right)^2 + \left( q^{2k+1} \sin \left( 2\pi \frac{p}{q} \right) \right)^2 \\ &= q^{4k+2} = z^{2k+1} \end{aligned}$$

proves that  $\phi_k$  is valued in  $D_{2k+1}$ . Since  $q^2 \geq 0$ , it takes indeed values in  $D_{2k+1}^+$ . Hence so is  $\bar{\phi}_k$ . It remains to prove that  $\bar{\phi}_k$  is bijective: The proof is identical to that of the Lemma 4.7.  $\square$

As for  $\mathbb{R}^2/G$  and  $A_{2n}$ , since  $\mathbb{R}^2/G'$  and  $D_{2k+1}^+$  are in bijection, and since they are also Poisson sets, we would like to assert that  $\bar{\phi}_k$  is a Poisson isomorphism. As for  $\mathbb{R}^2/G$  and  $A_{2n}$ , the precise statement will use the subalgebra  $\mathcal{A}' \subset C^k(\mathbb{R}^2/G')$ .

**Proposition 4.14.** *The pull-back map of  $\bar{\phi}_k : \mathbb{R}^2/G' \rightarrow D_{2k+1}^+$  is a Poisson morphism:*

$$C^\infty(\mathbb{R}^3) \rightarrow C^{\lfloor k/2 \rfloor}(\mathbb{R}^2/G).$$

*It restrict to a Poisson algebra isomorphism  $\bar{\phi}_k^* : \mathcal{F}(D_{2k+1}) \simeq \mathcal{A}'$ .*

The proof of this Proposition is identical to that of the Proposition 4.9.

## 5. SYMPLECTIC RESOLUTIONS OF THE QUOTIENTS $\mathbb{R}^2/G$ AND $\mathbb{R}^2/G'$

We deal with the case of  $G$  first, and use the notations of the previous section. The map  $\underline{\phi}_k : \mathbb{R}^2/G \rightarrow A_{2k}$  is bijective by Lemma 4.7. Let us denote by  $\psi_k : A_{2k} \rightarrow \mathbb{R}^2/G$  its inverse map. We note  $\varphi : Z_k \rightarrow A_{2k}$  the usual symplectic resolution of  $A_{2k}$ , see Section 3. Corollary 3.7 will now be used to show the very surprising following result:

**Proposition 5.1.** *The map  $\psi_k \circ \varphi$  is a smooth map from  $(Z_k, \{\cdot, \cdot\}_{Z_k})$  to  $(\mathbb{R}^2/G, \{\cdot, \cdot\})$ , i.e. for every smooth function  $F \in C^\infty(\mathbb{R}^2/G)$ , its pull-back  $(\psi_k \circ \varphi)^* F$  is a smooth function on  $Z_k$ .*

*Proof.* Let  $F \in C^\infty(\mathbb{R}^2/G)$ . By Proposition 2.7, we have

$$F(p, q) = \alpha_0(q) + \sum_{n \geq 1} \alpha_n(q) \cos \left( 2n\pi \frac{p}{q} \right) + \sum_{n \geq 1} \beta_n(q) \sin \left( 2n\pi \frac{p}{q} \right)$$

for some smooth functions  $(\alpha_n)_{n \geq 1}$  and  $(\beta_n)_{n \geq 1}$  on  $\mathbb{R}$  that vanish at 0 together with all their derivatives, and some smooth function  $\alpha_0$  (all unique).

We wish to define a function  $H$  on  $Z_k$  by:

$$(5.2) \quad H := \alpha_0(Z) + \sum_{n \geq 1} \frac{\alpha_n(Z)}{Z^k} X_n + \sum_{n \geq 1} \frac{\beta_n(Z)}{Z^k} Y_n$$

where the functions  $X_n, Y_n \in C^\infty(Z_k)$  are as in Corollary 3.7 and  $Z = \varphi^*z$ . This needs some justification.

- (1) All terms of both series defining  $H$  (see (5.2)) are smooth:
  - (a) Corollary 3.7 states that  $X_n, Y_n$  are smooth functions for all  $n \geq 1$ .
  - (b)  $Z = \varphi^*z$  is also a smooth function.
  - (c) Last, since  $\alpha_n, \beta_n$  vanish at 0 together with all their derivatives, the quotients

$$\alpha_n(z)/z^n, \beta_n(z)/z^n$$

are smooth functions on  $\mathbb{R}$ . In view of (b),  $\alpha_n(Z)/Z^n, \beta_n(Z)/Z^n$  are also smooth functions.

- (2) Both series are convergent, together with all their partial derivatives. We will come back to this point at the end the proof.

Now that the existence and the smoothness of the function  $H$  is well-established, let us check that

$$(5.3) \quad (\psi_k \circ \varphi)^*F = H.$$

It suffices to check that the pull-back through  $(\psi_k \circ \varphi)^*$  of each one of the terms of the series defining  $F$  are the terms of both series defining  $H$ . Consider the following smooth functions on  $\mathbb{R}^3$ :

$$f_n(x, y, z) := \frac{\alpha_n(z)}{z^k}x_n \text{ and } g_n(x, y, z) := \frac{\beta_n(z)}{z^k}y_n$$

where the functions  $x_n, y_n$  are as in (3.6). We have:

$$\begin{cases} \varphi^*f_n &= \frac{\alpha_n(Z)}{Z^n}X_n & \text{by definition of } X_n \text{ (see Cor. 3.7)} \\ \underline{\phi_k}^*f_n &= \alpha_n(q) \cos\left(2n\pi\frac{p}{q}\right) & \text{by definition of } \underline{\phi_k} \end{cases}$$

and

$$\begin{cases} \varphi^*g_n &= \frac{\beta_n(Z)}{Z^n}Y_n & \text{by definition of } Y_n \text{ (see Cor. 3.7)} \\ \underline{\phi_k}^*g_n &= \beta_n(q) \sin\left(2n\pi\frac{p}{q}\right) & \text{by definition of } \underline{\phi_k} \end{cases}$$

Hence

$$\begin{aligned} \varphi^* \circ \psi_k^* \left( \alpha_n(q) \cos\left(2n\pi\frac{p}{q}\right) \right) &= \alpha_n(Z)X_n \\ \varphi^* \circ \psi_k^* \left( \beta_n(q) \sin\left(2n\pi\frac{p}{q}\right) \right) &= \beta_n(Z)Y_n \end{aligned}$$

Summing up these identities, we obtain (5.3).

To complete the proof, we need to prove uniform convergence of the series (5.2) defining  $H$ , and uniform convergence of all the series obtained by applying a given partial differential operator to all its terms.

Let  $C \subset Z_k$  be a compact neighborhood of  $\phi^{-1}(0)$ . Pulling-back the relation:

$$x_n^2 + y_n^2 = z^{2k}$$

we obtain that

$$(5.4) \quad X_n^2 + Y_n^2 = Z^{2k}$$

so that  $X_n, Y_n$  is bounded on  $C$  by some constant  $k = \max_C|Z|$  that does not depend on  $n$ .

Now let us consider  $\varphi_k^{-1}(\phi(C)) \subset \mathbb{R}^2$ . This is a neighborhood of the horizontal line in  $\mathbb{R}^2$ , contained into a band  $\mathcal{B}$  of the form  $-M < q < M$  for some  $M \in \mathbb{R}$ . In

view of Lemma 2.10, there exists a constant  $K$  such that  $|\alpha_n(q)/q^k| < K/n^k$  and  $|\beta_n(q)/q^k| < K/n^k$  for all  $-M < q < M$ , so that

$$\left| \frac{\alpha_n(Z)}{Z^k} \right| < \frac{K}{n^k} \text{ and } \left| \frac{\beta_n(Z)}{Z^k} \right| < \frac{K}{n^k}$$

uniformly on  $C$ . This proves the uniform convergence on  $C$  of the series (5.2) defining  $H$ .

To prove that  $H$  is smooth, it suffices to prove that the series that define  $D(H)$  converge for every partial derivative operator  $D$  on  $Z$ .

For every partial derivative operator  $D$  of order  $d$ , we derive from the recursion relation (3.6) that:

$$\begin{pmatrix} D(X_{n+1}) \\ D(Y_{n+1}) \end{pmatrix} = \begin{pmatrix} X_1 & -Y_1 \\ Y_1 & X_1 \end{pmatrix} \begin{pmatrix} D(X_n) \\ D(Y_n) \end{pmatrix} + \begin{pmatrix} L(X_n, Y_n) \\ M(X_n, Y_n) \end{pmatrix}$$

where  $L(X, Y), M(X, Y)$  depend linearly on a finite number of partial derivatives of order  $\leq d-1$  of  $X, Y$ . Hence

$$U_n^D := (D(X_n))^2 + (D(Y_n))^2$$

satisfies

$$(5.5) \quad U_{n+1}^D = U_n^D + S^D(X_n, Y_n)$$

where  $S^D(X, Y)$  depend linearly on a finite number of partial derivatives of order  $\leq d-1$  of  $X, Y$ .

We claim that for any partial derivative operator  $D$  of degree  $d$ , there exists constants  $A, B$  such that  $U_n^D \leq A + Bn^D$ . For  $d=0$ , Equation (5.4) implies the desired assumption. If this is true for all  $D$  of degree  $\leq d-1$ , then in Equation (5.5), there exists  $A, B$  such that

$$|S^D(X_n, Y_n)| \leq A + Bn^{d-1}$$

Equation (5.5) implies then that there exists constants  $A', B'$  such that  $U_n^D \leq A' + B'n^d$ .

Since the convergence to zero of the  $d$ -first derivatives of  $|\alpha_n(Z)/Z^k|, |\beta_n(Z)/Z^k|$  can be made smaller than  $\frac{K}{n^r}$  for an arbitrary  $r$ , uniformly on  $\phi^{-1}(C)$ . This proves the uniform convergence of the series obtained by applying  $D$  to all the terms in the series in (5.2). This proves the smoothness of its limits, and in turn the smoothness of  $H$ . □

We conclude with the main result of this article:

**Theorem 5.6.** *The map  $\psi_k \circ \varphi : Z_k \rightarrow \mathbb{R}^2/G$  is a symplectic resolution of  $(\mathbb{R}^2/G, \{\cdot, \cdot\})$ .*

*Proof.* Let us check that all four conditions in Definition 1.1 are satisfied. Condition (i) has been established in Proposition 4.9. Condition (ii) follows from Proposition 4.9 and the first item in Proposition 3.4. Conditions (iii) and (iiii) hold in view of Lemma 4.7 and the second item in Proposition 3.4. □

For the quotient space  $\mathbb{R}^2/G'$ , the map  $\bar{\phi}_k : \mathbb{R}^2/G' \rightarrow D_{2k+1}^+$  is bijective by Lemma 4.12. Let us call  $\underline{\psi}_k : D_{2k+1}^+ \rightarrow \mathbb{R}^2/G'$  its inverse map. We note  $\varphi : Z_{k+2} \rightarrow D_{2k+1}^+$  the usual symplectic resolution of  $D_{2k+1}^+$ , see Section 3.

**Proposition 5.7.** *The map  $\underline{\psi}_k \circ \varphi$  is a smooth morphism of  $(Z_{k+2}, \{\cdot, \cdot\}_{Z_{k+2}})$  to  $(\mathbb{R}^2/G', \{\cdot, \cdot\})$ .*

*Proof.* The proof of this result is identical to that of Proposition 5.1. We simply use Proposition 2.13 instead of Proposition 2.7, and Corollary 3.13 instead of Corollary 3.7.  $\square$

**Theorem 5.8.** *The map  $\underline{\psi}_k \circ \varphi : Z_{k+2} \rightarrow \mathbb{R}^2/G'$  is a smooth symplectic resolution of the quotient Poisson  $(\mathbb{R}^2/G', \{\cdot, \cdot\})$ .*

*Proof.* As for  $\mathbb{R}^2/G$  and  $A_{2k}$ , let us check that all four conditions in Definition 1.1 are satisfied. Condition (i) has been established in Proposition 4.14. Condition (ii) follows from Proposition 4.14 and the first item in Proposition 3.10. Conditions (iii) and (iiii) hold in view of Lemma 4.12 and the second item in Proposition 3.10.  $\square$

#### REFERENCES

- [1] Arnaud Beauville, *Symplectic singularities*, Invent. Math., **139**, (2000), 541-549.
- [2] Gwyn Bellamy and Travis Schedler, *A new linear quotient of  $\mathbf{C}^4$  admitting a symplectic resolution*, Math. Z., **273**, (2013), 753-769.
- [3] Damien Broka, Ping Xu, *Symplectic Realizations of Holomorphic Poisson Manifolds*, arXiv, 1512.08847v1. (2015).
- [4] Alain Coste, Pierre Dazord, Alan Weinstein, *Groupoïdes symplectiques*, Publications du Département de Mathématiques de Lyon, **2A**, (1987), 1-62.
- [5] Jean-Paul Dufour, Nguyen Tien Zung, *Poisson Structures and Their Normal Forms*, Springer Birkhauser, (2005).
- [6] Baohua Fu, *A survey on symplectic singularities and resolutions*. Ann. math. Blaise Pascal, cedram, **13**, (2006), 209-236.
- [7] BaoHua Fu, *Symplectic Resolutions for Nilpotent Orbits*, Invent. math., **151**, (2003), 157-186.
- [8] M.V. Karasev, *Analogues of s of the theory of Lie groups for nonlinear Poisson brackets*, Math. USSR-Izv., **28**, (1987), 497-527.
- [9] Hichem Lassoued, *Résolutions symplectiques et de contact de variétés de Poisson et de Jacobi*, PhD thesis in mathematics under the direction of Camille Laurent-Gengoux and Benhammadi Zoubida, Université de Lorraine, (2019).
- [10] Hichem Lassoued, *Dimension preserving resolutions of singular Poisson structures*, Volume **65**, 212-226, August (2019).
- [11] Camille Laurent-Gengoux, *From Lie groupoids to resolutions of singularities. Applications to symplectic and Poisson resolution*, arXiv, 0610288v2. (2007).
- [12] Camille Laurent-Gengoux, Anne Pichereau, Pol Vanhaecke, *Poisson Structures*, Springer, (2013).
- [13] André Lichnerowicz, *Variétés de Poisson et leurs algèbres*, J. Diff. Geom., **12**, (1977), 253-300.
- [14] Anne Pichereau. *Cohomologie de Poisson en dimension trois*, C. R. Acad. Sci. Paris, Ser. I 340 (2005) 151.154.
- [15] Miles Reid, *The Du Val singularities  $A_n, D_n, E_6, E_7, E_8$* , <https://homepages.warwick.ac.uk/~masda/surf/more/DuVal.pdf>, consulted in Nov. 2020.