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Heun operator of Lie type and the modified algebraic Bethe ansatz

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Abstract

The generic Heun operator of Lie type is identified as a certain BC -Gaudin magnet Hamiltonian in a magnetic field. By using the modified algebraic Bethe ansatz introduced to diagonalize such Gaudin models, we obtain the spectrum of the generic Heun operator of Lie type in terms of the Bethe roots of inhomogeneous Bethe equations. We show also that these Bethe roots are intimately associated to the roots of polynomial solutions of the differential Heun equation. We illustrate the use of this approach in two contexts: the representation theory of $O(3)$ and the computation of the entanglement entropy for free Fermions on the Krawtchouk chain.

1 Introduction

In the spirit of [5] where the diagonalization of the Heun–Askey–Wilson operator has been performed by using the modified algebraic Bethe ansatz, we indicate in the present paper how the Heun operator of Lie type can be diagonalized by a similar method. We show that the particular Heun operator introduced in [19] can be identified with the BC -Gaudin magnet Hamiltonian in an external magnetic field with one site in a spin s representation of $\mathfrak{su}(2)$. By a slight generalization of the modified algebraic Bethe ansatz used in [13], we succeed in diagonalizing the generic Heun operator of Lie type. We then look at two mathematical and physical problems where the Heun operator of Lie type appears and the results apply.

The algebraic Heun operators are generalizations, introduced in [21], of the differential Heun operator which is the standardized form of the Fuchsian second order differential equation with four regular singularities. These operators are the most general bilinear combination of the bispectral pair associated to the orthogonal polynomials of the Askey scheme and they share the name of these polynomials. The differential Heun operator is recovered in this framework when we consider the dual pair associated to the Jacobi polynomials [22]. The Heun–Askey–Wilson operator has been studied in [6] whereas the Heun–Racah, the Heun–Bannai–Ito and the Heun–Hahn operators have been examined in [9, 30]. These operators have found applications in different contexts. For example, they give a nice algebraic interpretation of the time-band limiting problem [21] and provide the commuting tridiagonal operator [20] that allows to compute the entanglement entropy for free Fermion models [16, 17]. This paper focuses on the simplest algebraic Heun operators which are those of Lie type studied in [19]. In [5], the Heun–Askey–Wilson operator has been identified in the transfer matrix of the XXZ spin chain which allows to use the methods of quantum integrable models to diagonalize this operator. We prove here a similar result: the algebraic Heun operator of Lie type is associated to the BC -Gaudin model in an external magnetic field. In the spin s representation of $\mathfrak{su}(2)$, this identification allows to use the algebraic Bethe ansatz to diagonalize this Heun operator.

The usual algebraic Bethe ansatz (or quantum inverse scattering method) has been developed in [27] to diagonalize integrable model with periodic boundary conditions. Its generalization to open boundary conditions has been introduced in [26]. The problem of how to apply this method for generic open boundary has remained unsolved for 30 years. In [7], the modified algebraic Bethe ansatz was finally introduced to compute the eigenvectors of the XXX spin chain with generic boundaries associated to the eigenvalues found in [10]. This modified method has also been used to diagonalize the totally asymmetric exclusion process [12] and the XXZ spin chain [8, 2] and it provides the spectrum in terms of inhomogeneous Bethe equations. In this paper, we generalize the result of [13] where the BC -Gaudin model is diagonalized by the modified algebraic Bethe ansatz. An extension is required to take into account the external magnetic field.

The Bethe equations consist in N algebraic relations between N unknowns that need to be solved in order to obtain the eigenvalues and the eigenvectors. These equations usually have multiple solutions and one must demonstrate that all the eigenvalues are obtained in this manner to prove that the ansatz provides the complete spectrum. One way to tackle the problem is to show that obtaining the solutions of Bethe equations is equivalent to finding the polynomial solutions of a differential (or difference) equation. For a second order differential equation, this problem is referred to the Heine–Stieljes problem and is well-studied. This program has already been used with success in [31, 23] for various quantum integrable models. We show in this fashion that the spectrum obtained by the modified algebraic Bethe ansatz of the algebraic Heun operator is complete. The associated differential equation is an inhomogeneous differential Fuchsian equation with 4 regular singularities (*i.e.* an inhomogeneous differential Heun operator). We show moreover, by using the Bargmann realization of the Lie algebra $\mathfrak{su}(2)$, that the eigenvalues problem is also associated to a homogeneous differential Fuchsian equation with 5 regular singularities.

The plan of the paper is as follows. Section 2 offers a review of the Heun operators of Lie type [19] and of the algebraic framework needed to study the BC -Gaudin magnet. Section 3 provides the main steps of the modified algebraic Bethe ansatz for the Gaudin model. In Section 4, we study in more details the inhomogeneous Bethe equations and associate their solutions, the Bethe roots, to the roots of a polynomial solution of the differential Heun equation with an inhomogeneous term. Section 5 focuses on a particular case where the parameters of the algebraic Heun operator satisfy some constraints. It is seen in this case that there exists an alternative to compute the spectrum that leads to homogeneous Bethe equations. The last two sections discuss the use of the Heun operator of Lie type in two different contexts: first, in the representation theory of $O(3)$ (see Section 6) and second, in the computation of the entanglement entropy for free Fermions on a Krawtchouk chain (see Section 7). We conclude with an outlook in Section 8. Useful formulas are gathered in Appendix A.

2 Heun operators and the Gaudin magnet

2.1 Algebraic Heun operator of Lie type

The Lie algebra $\mathfrak{su}(2)$ is generated by J_1, J_2 and J_3 subject to the following defining relations

$$[J_1, J_2] = iJ_3, \quad [J_2, J_3] = iJ_1, \quad [J_3, J_1] = iJ_2. \quad (2.1)$$

We set

$$X = \alpha J_1 + \beta J_2, \quad \text{and} \quad Y = J_1. \quad (2.2)$$

The Heun operator of Lie type is the following element of the universal enveloping algebra $U(\mathfrak{su}(2))$

$$W = r_1[X, Y] + r_2\{X, Y\} + r_3X + r_4Y + r_5, \quad (2.3)$$

where r_i are free parameters and $\{X, Y\} = XY + YX$ is the anti-commutator. This Heun operator can be rewritten as follows in terms of the Lie generators (up to a renormalization)

$$W = \rho_1 J_1 + \rho_2 J_2 + \rho_3 J_3 + \{J_1, J_2\} + \rho_4 J_1^2 + \rho_5 \quad (2.4)$$

where ρ_i ($1 \leq i \leq 5$) are free real parameters. The normalization of W is chosen so that the coefficient in front of the anti-commutator $\{J_1, J_2\}$ be one. The parameters ρ_i have been taken real to ensure that W is Hermitian if J_i are represented by Hermitian matrices. The generators X and W satisfy the Heun algebra of Lie type studied in [19].

2.2 BC-Gaudin magnet in magnetic field

Let us introduce the r -matrix

$$r_{12}(u, v) = \frac{1}{(u-v)(uv-1)} \begin{pmatrix} u(1-v^2) & 0 & 0 & -2(u-v) \\ 0 & -u(1-v^2) & -2v(uv-1) & 0 \\ 0 & -2u(uv-1) & -u(1-v^2) & 0 \\ -2uv(u-v) & 0 & 0 & u(1-v^2) \end{pmatrix} \quad (2.5)$$

solution of the non-standard classical Yang-Baxter equation

$$[r_{13}(u_1, u_3), r_{23}(u_2, u_3)] = [r_{21}(u_2, u_1), r_{13}(u_1, u_3)] + [r_{23}(u_2, u_3), r_{12}(u_1, u_2)], \quad (2.6)$$

where we denote $r_{12}(u) = r(u) \otimes \mathbb{I}$, $r_{23}(u) = \mathbb{I} \otimes r(u)$ and so on. This r -matrix has been used to give the FRT presentations of the Onsager algebra [3] and of the classical Askey–Wilson algebra [4]. Let us recall that the parameters u and v are usually referred to as spectral parameters. This r -matrix can be obtained from the one of the 6-vertex model under a twist by a matrix solution of the classical reflection equation (see [3]).

Let us now define the K -matrix containing as follows the generators of $\mathfrak{su}(2)$

$$K(u) = \frac{2}{(1-au)(u-a)} \begin{pmatrix} u(a^2-1)J_3 & (a^2u-2a+u)J_1 - iu(a^2-1)J_2 \\ -u(a^2-2au+1)J_1 + iu(a^2-1)J_2 & -u(a^2-1)J_3 \end{pmatrix}, \quad (2.7)$$

where a is a free parameter called the inhomogeneity parameter. The K -matrix satisfies the classical reflection equation

$$[K_1(u), K_2(v)] = [r_{21}(v, u), K_1(u)] + [K_2(v), r_{12}(u, v)]. \quad (2.8)$$

In fact, this classical reflection equation is equivalent to the defining relations (2.1) of $\mathfrak{su}(2)$.

We introduce also a scalar matrix

$$c(u) = \left(\frac{2iu(a^2 - 1)\rho_1}{a(u^2 - 1)} + \frac{2(au - 1)(a - u)\rho_2}{a(u^2 - 1)} \right) j_1 + 2i\rho_2 j_2 + 2i\rho_3 j_3 + \rho_3 \mathbb{1}, \quad (2.9)$$

where j_i are the 2 by 2 matrices representing J_i given explicitly by $j_1 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $j_2 = \frac{1}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ and $j_3 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and ρ_i are the parameters entering in the definition (2.4) of the Heun operator. This matrix $c(x)$ is solution of

$$0 = [r_{21}(v, u), c_1(u)] + [c_2(v), r_{12}(u, v)]. \quad (2.10)$$

Such a scalar solution has been used to study the Gaudin model in a magnetic field (see [28]). It is easy to see that $\tilde{K}(x) = K(u) + c(u)$ satisfies also (2.8).

We introduce at this point the transfer matrix

$$t(u) = \text{tr}(\tilde{K}(u)^2). \quad (2.11)$$

The important feature of the transfer matrix is its commutativity property for different spectral parameters:

$$[t(u), t(v)] = 0. \quad (2.12)$$

We are now in a position to give the first result of this paper: amongst the conserved quantities of the Gaudin model (with one site) in a magnetic field, there is the Heun operator of Lie type (2.4). More explicitly

$$\frac{a}{8i(1 - a^2)u} t(u) \Big|_{u=0} = W, \quad (2.13)$$

with $\rho_4 = \frac{2i(a^2 + 1)}{a^2 - 1}$ and $\rho_5 = \frac{\rho_1 \rho_2}{2} + \frac{i\rho_2^2(1 + a^2)}{2(1 - a^2)}$. Since ρ_4 is a real parameter, a must be a pure phase. The parameter ρ_5 is given in terms of the other parameters but we can add the identity operator to recover the generic Heun operator.

The second result of this paper is the diagonalization of the Heun operator of Lie type W with the Bethe ansatz.

3 Modified algebraic Bethe ansatz

In this section, we recall the results of reference [13] and use them to diagonalize the operator W by the algebraic Bethe ansatz.

Gauge transformations. Let us introduce the following matrix

$$M(u) = \begin{pmatrix} 1/2 & 1/u \\ -u/2 & 1 \end{pmatrix}. \quad (3.1)$$

It allows to transform the r -matrix (2.5), the K -matrix (2.7) and the matrix $c(x)$ as follows

$$\begin{aligned} \tilde{r}(u, v) &= M_1(u)^{-1} M_2(v)^{-1} r(u, v) M_1(u) M_2(v) \\ &= \frac{1}{(uv - 1)((u - v))} \begin{pmatrix} -v(u^2 - 1) & 0 & \frac{2(uv - 1)((u - v))}{u} & 0 \\ 0 & v(u^2 - 1) & -2v(v^2 - 1) & -\frac{2(uv - 1)((u - v))}{u} \\ -\frac{u(uv - 1)((u - v))}{2} & -\frac{2u^2(v^2 - 1)}{v} & v(u^2 - 1) & 0 \\ 0 & \frac{u(uv - 1)v((u - v))}{2} & 0 & -v(u^2 - 1) \end{pmatrix}, \end{aligned} \quad (3.3)$$

and

$$\tilde{K}(u) = M(u)^{-1}K(u)M(u), \quad \tilde{c}(u) = M(u)^{-1}c(u)M(u). \quad (3.4)$$

It is easy to show that the matrices with tilde satisfy relations similar to those obeyed by $r(u, v)$, $K(u)$ and $c(u)$. Namely, one gets

$$[\tilde{K}_1(u), \tilde{K}_2(v)] = [\tilde{r}_{21}(v, u), \tilde{K}_1(u)] + [\tilde{K}_2(v), \tilde{r}_{12}(u, v)]. \quad (3.5)$$

Moreover the gauge transformation does not modify the transfer matrix

$$t(u) = \text{tr}((\tilde{K}(u) + \tilde{c}(u))^2). \quad (3.6)$$

We introduce the new $\mathfrak{su}(2)$ generators: $\tilde{J}_1 = -\frac{(a^2-1)(a^2-4)}{8a^2}J_1 + i\frac{(a^2+1)(a^2-4)}{8a^2}J_2 + \frac{a^2+4}{4a}J_3$, $\tilde{J}_2 = i\frac{(a^2-1)(a^2+4)}{8a^2}J_1 + \frac{(a^2+1)(a^2+4)}{8a^2}J_2 - i\frac{a^2-4}{4a}J_3$ and $\tilde{J}_3 = -\frac{a^2-1}{2a}J_1 + i\frac{a^2-1}{2a}J_2$ so that

$$\tilde{K}(u) = \begin{pmatrix} \frac{2a(u^2-1)}{(au-1)(a-u)}\tilde{J}_3 & \frac{4}{u}\tilde{J}_3 + \frac{2(a^2-1)a}{(au-1)(a-u)}\tilde{J}_- \\ -u\tilde{J}_3 + \frac{2u^2(a^2-1)}{a(au-1)(a-u)}\tilde{J}_+ & -\frac{2a(u^2-1)}{(au-1)(a-u)}\tilde{J}_3 \end{pmatrix}, \quad (3.7)$$

where $\tilde{J}_+ = \tilde{J}_1 + i\tilde{J}_2$ and $\tilde{J}_- = \tilde{J}_1 - i\tilde{J}_2$. We thus recover a particular case of the K -matrix used in [13] (see relations (4.1) and (4.2) in [13] for $L \rightarrow 1$, $x \rightarrow u$, $v_1 \rightarrow a$, $\alpha \rightarrow 1$, $\beta \rightarrow 0$ and $\gamma \rightarrow 0$).

Commutation relations. As is familiar in the context of algebraic Bethe ansatz, we define the operators $\tilde{A}(u)$, $\tilde{B}(u)$, $\tilde{C}(u)$ and $\tilde{D}(u)$ as follows

$$\tilde{K}(u) + \tilde{c}(u) = \begin{pmatrix} \tilde{A}(u) & \tilde{B}(u) \\ \tilde{C}(u) & \tilde{D}(u) \end{pmatrix}. \quad (3.8)$$

Only the special case $\tilde{D}(u) = -\tilde{A}(u)$ has been treated in [13]. It does not occur here because of the additional term $\tilde{c}(u)$. We must therefore slightly generalize the results of [13]. The commutation relations of these operators are computed from relation (3.5) and are given by

$$[\tilde{A}(u), \tilde{A}(v)] = [\tilde{D}(u), \tilde{D}(v)] = [\tilde{A}(u), \tilde{D}(v)] = 0, \quad (3.9)$$

$$B(u, n)B(v, n+1) = B(v, n)B(u, n+1), \quad C(u, n)C(v, n-1) = C(v, n)C(u, n-1), \quad (3.10)$$

$$[\tilde{A}(u), B(v, n)] = -[\tilde{D}(u), B(v, n)] = \frac{2(u^2-1)}{(u-v)(vu-1)}(vB(v, n) - uB(u, n)), \quad (3.11)$$

$$[\tilde{A}(u), C(v, n)] = -[\tilde{D}(u), C(v, n)] = \frac{2(u^2-1)v^2}{(u-v)(vu-1)}\left(\frac{1}{u}C(u, n) - \frac{1}{v}C(v, n)\right), \quad (3.12)$$

$$C(u, n)B(v, n) = B(v, n+1)C(u, n+1) + 8n\frac{u}{v} + \frac{2u}{(u-v)(uv-1)v}\left(v(u^2-1)(\tilde{A}(u) - \tilde{D}(u)) - u(v^2-1)(\tilde{A}(v) - \tilde{D}(v))\right), \quad (3.13)$$

where $B(u, n) = \tilde{B}(u) - \frac{2(2n-1)}{u}$ and $C(u, n) = \tilde{C}(u) + \frac{(2n-1)u}{2}$.

Shifted transfer matrix. Following [13], we define the shifted transfer matrix

$$t(u, n) = \tilde{A}(u)^2 + B(u, n+1)C(u, n+1) + C(u, n)B(u, n) + \tilde{D}(u)^2 + 2. \quad (3.14)$$

We can see that $t(u, 0) = t(u)$ is the transfer matrix we are interested in. This shifted transfer matrix has the following commutation relation with $B(v, n)$:

$$t(u, n-1)B(v, n) - B(v, n)t(u, n) = 4B(v, n)\left(\frac{v(u^2-1)}{(u-v)(uv-1)}\left(\tilde{A}(u) - \tilde{D}(u) + 4\frac{u^2+1}{u^2-1}\right) - 2\right) - 4\frac{u}{v}B(u, n)\left(\frac{u(v^2-1)}{(u-v)(uv-1)}\left(\tilde{A}(v) - \tilde{D}(v) + 4\frac{v^2+1}{v^2-1}\right) - 4(n-1)\right). \quad (3.15)$$

Representation of $\mathfrak{su}(2)$. Denote by ω_s (with $s \in \mathbb{N}/2$) the highest weight vector of the spin s representation of $\mathfrak{su}(2)$ which satisfies

$$\tilde{J}_3 \omega_s = s \omega_s, \quad \tilde{J}_+ \omega_s = 0. \quad (3.16)$$

From the explicit form of the operators, one deduces that

$$\tilde{A}(u) \omega_s = \alpha(u) \omega_s = \left(\rho_3 + \frac{2as(u^2 - 1)}{(au - 1)(a - u)} - \frac{i(a^2 - 1)(u^2 + 1)\rho_1}{2a(u^2 - 1)} - \frac{(a - u)^2 + (au - 1)^2}{2a(u^2 - 1)} \rho_2 \right) \omega_s, \quad (3.17)$$

$$\tilde{D}(u) \omega_s = \delta(u) \omega_s = \left(\rho_3 - \frac{2as(u^2 - 1)}{(au - 1)(a - u)} + \frac{i(a^2 - 1)(u^2 + 1)\rho_1}{2a(u^2 - 1)} + \frac{(a - u)^2 + (au - 1)^2}{2a(u^2 - 1)} \rho_2 \right) \omega_s, \quad (3.18)$$

$$C(u, n) \omega_s = u \gamma_n \omega_s = u \left(-s + n - \frac{1}{2} - \frac{i(a^2 - 1)\rho_1}{4a} - \frac{(a^2 + 1)\rho_2}{4a} + \frac{i\rho_3}{2} \right) \omega_s. \quad (3.19)$$

Bethe vectors. We construct as follows the Bethe vectors that depend on the parameters $\mathbf{z} = \{z_1, z_2, \dots, z_M\}$:

$$\mathbb{V}(\mathbf{z}) = B(z_1, 1)B(z_2, 2) \dots B(z_M, M) \omega_s. \quad (3.20)$$

Due to relation (3.10), the entries of the vector $\mathbb{V}(\mathbf{z})$ do not depend on the order of the parameters z_i . After some algebraic manipulations, we can express the action of the transfer matrix on the Bethe vector as follows

$$\begin{aligned} t(u) \mathbb{V}(\mathbf{z}) &= 2u \gamma_{M+1} B(z_1, 1)B(z_2, 2) \dots B(z_M, M)B(u, M+1) \omega_s \\ &\quad + \mathcal{W}(u, \mathbf{z}) \mathbb{V}(\mathbf{z}) \\ &\quad - \sum_{k=1}^M \frac{16u^2(z_k^2 - 1)}{(u - z_k)(uz_k - 1)z_k} \mathcal{U}_k(\mathbf{z}) \mathbb{V}(\mathbf{z}_k, u) \end{aligned} \quad (3.21)$$

where $\mathbb{V}(\mathbf{z}_k, u) = B(z_1, 1) \dots B(z_{k-1}, k-1)B(u, k)B(z_{k+1}, k+1) \dots B(z_M, M) \omega_s$ and

$$\mathcal{W}(u, \mathbf{z}) = \lambda(u) + \sum_{k=1}^M \frac{16z_k(u^2 - 1)}{(u - z_k)(uz_k - 1)} \left(\frac{\alpha(u) - \delta(u)}{4} + \frac{u^2 + 1}{u^2 - 1} + \sum_{\substack{p=1 \\ p \neq k}}^M \frac{(u^2 - 1)z_k z_p}{u(z_k - z_p)(z_k z_p - 1)} \right), \quad (3.22)$$

$$\lambda(u) = \alpha^2(u) + \delta^2(u) + 2 + 2u(\alpha'(u) - \delta'(u)) + 2 \frac{u^2 + 1}{u^2 - 1} (\alpha(u) - \delta(u)), \quad (3.23)$$

$$\mathcal{U}_k(\mathbf{z}) = \frac{\alpha(z_k) - \delta(z_k)}{4} + \frac{z_k^2 + 1}{z_k^2 - 1} + \sum_{\substack{p=1 \\ p \neq k}}^M \frac{z_p(z_k^2 - 1)}{(z_k - z_p)(z_k z_p - 1)}. \quad (3.24)$$

The prime in $\alpha'(u)$ and $\delta'(u)$ stands for the derivative with respect to u . In the usual algebraic Bethe ansatz, the first line in (3.21) is not present.

Modified algebraic Bethe ansatz and inhomogeneous Bethe equations. For generic values of the parameters, one gets $\gamma_{M+1} \neq 0$ for any M (the particular case where there exists a M such that $\gamma_M = 0$ is the object of Section 5). One must compute $B(z_1, 1)B(z_2, 2) \dots B(z_M, M)B(u, M+1) \omega_s$. For a generic M there is no simple formula but for $M = 2s$, one gets (see [13] for a proof of this result):

$$2u B(z_1, 1)B(z_2, 2) \dots B(z_{2s}, 2s)B(u, 2s+1) \omega_s = \overline{\mathcal{W}}(u, \mathbf{z}) \mathbb{V}(\mathbf{z}) - \sum_{k=1}^{2s} \frac{16u^2(z_k^2 - 1)}{(u - z_k)(uz_k - 1)z_k} \overline{\mathcal{U}}_k(\mathbf{z}) \mathbb{V}(\mathbf{z}_k, u) \quad (3.25)$$

where

$$\overline{\mathcal{W}}(u, \mathbf{z}) = -8\gamma_{2s+1}^* \prod_{p=1}^{2s} \frac{(u-a)(au-1)z_p}{a(u-z_p)(uz_p-1)}, \quad (3.26)$$

$$\overline{\mathcal{U}}_k(\mathbf{z}) = -\frac{\gamma_{2s+1}^*}{2(z_k^2-1)} \prod_{\substack{p=1 \\ p \neq k}}^{2s} \frac{(z_k-a)(az_k-1)z_p}{a(z_k-z_p)(z_kz_p-1)}, \quad (3.27)$$

and $*$ is the complex conjugate. Then, the eigenvalues of the transfer matrix are

$$\mathcal{W}(u, \mathbf{z}) + \gamma_{2s+1} \overline{\mathcal{W}}(u, \mathbf{z}), \quad (3.28)$$

where $\mathcal{W}(u, \mathbf{z})$ is given by (3.22) with $M = 2s$ and $\overline{\mathcal{W}}(u, \mathbf{z})$ is given by (3.26) if \mathbf{z} satisfies the inhomogeneous Bethe equations

$$\frac{\alpha(z_k) - \delta(z_k)}{4} + \frac{z_k^2 + 1}{z_k^2 - 1} + \sum_{\substack{p=1 \\ p \neq k}}^{2s} \frac{z_p(z_k^2 - 1)}{(z_k - z_p)(z_k z_p - 1)} = \frac{|\gamma_{2s+1}|^2}{2(z_k^2 - 1)} \prod_{\substack{p=1 \\ p \neq k}}^{2s} \frac{z_p(z_k - a)(az_k - 1)}{a(z_k - z_p)(z_k z_p - 1)}. \quad (3.29)$$

4 Bethe roots and Heun operator

4.1 Inhomogeneous Heun differential equation

In this section, we study in more details the inhomogeneous Bethe equations obtained above.

Finding the Bethe roots of the inhomogeneous Bethe equations given by (3.29) is equivalent to computing the roots of the monic polynomial solution of degree $2s$ of the following differential Heun equation with an inhomogeneous term

$$y''(X) + \left(\frac{a_0}{X} + \frac{a_1}{X-1} + \frac{a_2}{X-A} \right) y'(X) + \frac{a_3(X-\mu)}{X(X-1)(X-A)} y(X) = |\gamma_{2s+1}|^2 \frac{(X-A)^{2s}}{X(X-1)}, \quad (4.1)$$

where

$$\begin{aligned} A &= -\frac{(a-1)^2}{4a}, & a_0 &= 1 - \frac{i(a^2-1)\rho_1}{4a} - \frac{(a-1)^2\rho_2}{4a}, & a_1 &= 1 - \frac{i(a^2-1)\rho_1}{4a} - \frac{(a+1)^2\rho_2}{4a}, \\ a_2 &= -2s, & a_3 &= |\gamma_{2s+1}|^2 - 2s(2s-1+a_0+a_1+a_2). \end{aligned} \quad (4.2)$$

The roots Z_i of the polynomials $y(X)$ and the Bethe roots z_i are linked by $Z_i = \frac{1}{4}(2 - z_i - 1/z_i)$. The parameter μ in the previous relation must be chosen so that there exists a polynomial solution. We come back to this point below.

The proof of this statement is quite standard since it is a generalization of the Heine–Stieljes problem. Let us recall the main steps here. Suppose that for a given μ there exists a polynomial solution $y(x)$ of degree $2s$ and denote by Z_i the $2s$ roots of this polynomial. The Heun differential equation (5.3) at $X = Z_i$ simplifies to

$$y''(Z_i) + \left(\frac{a_0}{Z_i} + \frac{a_1}{Z_i-1} + \frac{a_2}{Z_i-A} \right) y'(Z_i) = |\gamma_{2s+1}|^2 \frac{(Z_i-A)^{2s}}{Z_i(Z_i-1)}. \quad (4.3)$$

Defining z_i through $Z_i = \frac{1}{4}(2 - z_i - 1/z_i)$, one can show that (4.3) implies that z_i satisfy the Bethe equations.

The inverse is also true. For a given solution z_i of the Bethe equations (3.29), we define the following monic polynomial of degree $2s$:

$$y(X) = \prod_{i=1}^{2s} \left(X - \frac{1}{4}(2 - z_i - 1/z_i) \right). \quad (4.4)$$

One can prove that (4.3) holds for this polynomial and one deduces that the following polynomial

$$X(X-1)(X-A)y''(X) + ((X-1)(X-A)a_0 + a_1X(X-A) + a_2X(X-1))y'(X) - |\gamma_{2s+1}|^2(X-A)^{2s+1} \quad (4.5)$$

of degree $2s+1$ vanishes for $X = Z_i = \frac{1}{4}(2 - z_i - 1/z_i)$ which are the roots of $y(X)$. Therefore it is equal to $p(X)y(X)$ for some polynomial $p(x)$ of degree 1. By looking now at the term of degree $2s+1$ in equation (4.5), we conclude that $p(x)$ is of the form $p(x) = a_3(X - \mu)$ (with a_3 given by (4.2)) and that $y(X)$ satisfies the inhomogeneous differential Heun equation (4.1).

The goal is hence to find μ such that (4.1) has a monic polynomial solution of degree $2s$. To arrive at that, we start by putting the polynomial solution

$$y(X) = X^{2s} + \sum_{n=0}^{2s-1} c_n X^n \quad (4.6)$$

in the Heun equation (4.1). We obtain the following constraints for the coefficients c_n , for $0 \leq n \leq 2s+1$,

$$\begin{aligned} & A(n+1)(n+a_0)c_{n+1} - n((1+A)(a_0+n-1) + a_1A + a_2)c_n + |\gamma_n|^2 c_{n-1} \\ & = \mu a_3 c_n + |\gamma_{2s+1}|^2 \binom{2s+1}{n} (-A)^{2s+1-n}, \end{aligned} \quad (4.7)$$

with the conventions $c_{2s} = 1$ and $c_{-1} = c_{2s+1} = c_{2s+2} = 0$. The previous relation is directly satisfied for $n = 2s+1$ given the explicit value of a_3 . Relation (4.7) for $n = 2s$ gives c_{2s-1} as a polynomial of degree 1 with respect to μ . Then, by recurrence, relation (4.7) for $n = 2s-p$ ($p = 1, \dots, 2s-1$) gives c_{2s-p-1} as a polynomial of degree $p+1$ with respect to μ . Finally, relation (4.7) for $n = 0$ implies that the resulting polynomial $P_{2s+1}(\mu)$ of degree $2s+1$ in μ must vanish.

The previous discussion allows to conclude that for each solution μ of the following relation

$$P_{2s+1}(\mu) = 0, \quad (4.8)$$

there exists a polynomial $y_\mu(X)$ solution of the Heun equation (4.1) of degree $2s$. The $2s$ roots of the polynomial $y_\mu(X)$ provide a solution of the Bethe equations (3.29). As $P_{2s+1}(\mu)$ is of degree $2s+1$, there are $2s+1$ different μ and we obtain $2s+1$ different solutions of the Bethe equations. This proves that the spectrum obtained from the modified algebraic Bethe ansatz is complete.

Let us remark that in the discussion above, we have assumed that $P_{2s+1}(\mu)$ has simple roots. We can expect this to be true for generic parameters. As far as we know, there is no closed formula for $P_{2s+1}(\mu)$ but it is easy to compute this polynomial from the relations (4.7).

4.2 Eigenvalues of the algebraic Heun operator

Using the identification (2.13) between the Gaudin model Hamiltonian and the algebraic Heun operator, we can conclude that the eigenvalues of the algebraic Heun W are given by

$$w = \frac{2is|\gamma_{2s}|^2(a-1)}{a+1} - \frac{4ai|\gamma_{2s}|^2}{1-a^2} \sum_{j=1}^{2s} Z_j + \frac{\rho_1\rho_2}{2} + \frac{s(a^2+1)\rho_1 - is(a^2-1)\rho_2}{2a} + \frac{i(a^2+1)(4s^2 - \rho_2^2)}{2(a^2-1)} \quad (4.9)$$

where $\{Z_i\}$ are the roots of a polynomial solution of the inhomogeneous differential Heun equation (4.1) or equivalently $Z_i = \frac{1}{4}(2 - z_i - 1/z_i)$ with $\{z_i\}$ the Bethe roots of (3.29).

In addition, remarking that $-\sum_{i=1}^{2s} Z_i$ is equal to the coefficient c_{2s-1} in (4.6), we can express this sum in terms of μ . Indeed, as explained before, relation (4.7) for $n = 2s$ gives c_{2s-1} as a polynomial in μ of degree 1. Therefore, we can conclude that

$$w = \frac{4aia_3\mu}{1-a^2} + \rho_1\left(\frac{\rho_2}{2} - s\right) - \frac{i(a^2+1)(\rho_2^2 - 2s(s+1))}{2(a^2-1)} + \frac{i(a-1)(\rho_2 - |\gamma_{2s+1}|^2)}{(a+1)} + \frac{2is(s-1)a}{a^2-1}. \quad (4.10)$$

This relation allows to view w , the eigenvalues of the algebraic Heun operator, as the parameters in the differential inhomogeneous Heun operator (4.1) such that this equation has a polynomial solution. Similar results can be obtained directly from the Bargmann representation of $\mathfrak{su}(2)$ as explained in the following subsection.

4.3 Bargmann realization of the algebraic Heun operator of Lie type

It is well-known that the spin s representation of $\mathfrak{su}(2)$ can be realized in terms of differential operators acting on the space of univariate polynomials of order less or equal to $2s$:

$$J_1 = \frac{1-z^2}{2} \frac{d}{dz} + sz, \quad J_2 = \frac{1+z^2}{2i} \frac{d}{dz} - \frac{sz}{i}, \quad J_3 = s - z \frac{d}{dz}. \quad (4.11)$$

Indeed, the actions on the monomials

$$|s, m\rangle = \frac{z^{s-m}}{\sqrt{(s-m)!(s+m)!}}, \quad \text{for } -s \leq m \leq s, \quad (4.12)$$

are given by

$$J_{\pm}|s, m\rangle = (J_1 \pm iJ_2)|s, m\rangle = \sqrt{(s \mp m)(s \pm m + 1)}|s, m \pm 1\rangle, \quad J_3|s, m\rangle = m|s, m\rangle, \quad (4.13)$$

which is the standard spin s representation of $\mathfrak{su}(2)$.

The algebraic Heun operator W in this realization becomes:

$$W = \frac{i(z^2-1)(z^2a^2-1)}{(a^2-1)} \frac{d^2}{dz^2} + \left(\frac{1-2s}{2}(z^3(\rho_4+2i) - \rho_4z) - z^2 \frac{\rho_1+i\rho_2}{2} - z\rho_3 + \frac{\rho_1-i\rho_2}{2} \right) \frac{d}{dz} + z^2 \left(s^2\rho_4 + 2is^2 - \frac{s\rho_4}{2} - is \right) + sz(\rho_1+i\rho_2) + s\rho_3 + \frac{s\rho_4}{2} + \rho_5 \quad (4.14)$$

with $\rho_4 = \frac{2i(a^2+1)}{a^2-1}$ and $\rho_5 = \frac{\rho_1\rho_2}{2} + \frac{i\rho_3^2(1+a^2)}{2(1-a^2)}$. Then, the eigenvalues w of the algebraic Heun operator W can be seen as the eigenvalues of this differential operator

$$W\phi(z) = w\phi(z). \quad (4.15)$$

Thus, w is obtained by asking that there is a polynomial solution $\phi(z)$ of degree less or equal to $2s$ of this differential equation. This operator W leads to a Fuchsian second order differential equation with five regular singularities $\{1, -1, 1/a, -1/a, \infty\}$, but in the special case where $\rho_1 = \rho_2 = 0$, it reduces to the differential Heun operator. Indeed, let us perform the change of variable

$$y = a^2 z^2 \quad \text{and} \quad \psi(y) = \phi(a^2 z^2). \quad (4.16)$$

In this new variable when $\rho_1 = \rho_2 = 0$, (4.15) becomes the Heun differential equation

$$\frac{d^2\psi}{dy^2} + \left(\frac{1}{2y} + \frac{1-2s-i\rho_3}{2(y-1)} + \frac{1-2s+i\rho_3}{2(y-a^2)} \right) \frac{d\psi}{dy} + \frac{2(2s-1)sy + is\rho_3(1-a^2) + s(a^2+1)}{4y(y-1)(y-a^2)} \psi = \frac{iw(1-a^2)}{4y(y-1)(y-a^2)} \psi. \quad (4.17)$$

Comparing this differential Heun operator obtained from the Bargmann realisation and the one in (4.1) obtained from the Bethe ansatz, we see that one singularity of the former is located at a^2 whereas for the latter one singularity is at $A = -\frac{(a-1)^2}{4a}$.

5 Homogeneous case

In the previous sections, we have explained how the modified algebraic Bethe ansatz can be used to diagonalize the transfer matrix of the Gaudin model and the Heun operator, that leads to inhomogeneous Bethe equations. However, for particular values of the parameters, there exist homogeneous Bethe equations which diagonalize also this model. Indeed, let us suppose that s , a , ρ_1 , ρ_2 and ρ_3 are such that there is an integer $0 \leq \mathcal{M} \leq 2s - 1$ satisfying $\gamma_{\mathcal{M}+1} = 0$ *i.e.*

$$\begin{cases} \mathcal{M} = s - \frac{1}{2} + \frac{i(a^2 - 1)\rho_1}{4a} + \frac{(a^2 + 1)\rho_2}{4a} \\ \rho_3 = 0 \end{cases}, \quad (5.1)$$

where the real parameter ρ_3 is set to zero to ensure that the imaginary part of the relation $\gamma_{\mathcal{M}+1} = 0$ is verified. In this case, some eigenvalues of the transfer matrix are given by $\mathcal{W}(u, \mathbf{z})$ (relation (3.22) for $M = \mathcal{M}$), if \mathbf{z} satisfies the Bethe equations $\mathcal{U}_k(\mathbf{z}) = 0$ *i.e.*

$$\frac{1}{4}(\alpha(z_k) - \delta(z_k)) + \frac{z_k^2 + 1}{z_k^2 - 1} + \sum_{\substack{p=1 \\ p \neq k}}^{\mathcal{M}} \frac{z_p(z_k^2 - 1)}{(z_k - z_p)(z_k z_p - 1)} = 0. \quad (5.2)$$

In this case, the Bethe equations are called homogeneous. As before, finding the Bethe roots z_k , solutions of the Bethe equations (5.2) is the same as finding polynomial solutions of a differential equation. More precisely, it amounts to finding polynomial solutions of degree \mathcal{M} of the homogeneous Heun differential equation

$$y''(X) + \left(\frac{a_0}{X} + \frac{a_1}{X-1} + \frac{a_2}{X-A} \right) y'(X) + \mathcal{M}^2 \frac{(X-\mu)}{X(X-1)(X-A)} y(X) = 0, \quad (5.3)$$

for a suitable μ , where

$$A = -\frac{(a-1)^2}{4a}, \quad a_0 = s + \frac{1}{2} - \mathcal{M} + \frac{\rho_2}{2}, \quad a_1 = s + \frac{1}{2} - \mathcal{M} - \frac{\rho_2}{2}, \quad a_2 = -2s. \quad (5.4)$$

If $\{Z_i\}$ are the \mathcal{M} roots of a polynomial solution of the Heun differential equation, then, for each Z_i , take z_i as one solution of $Z_i = \frac{1}{4}(2 - z_i - 1/z_i)$, then these $\{z_i\}$ are solutions of the Bethe equations (5.2).

The parameter μ must be chosen such that (5.3) has a polynomial solution of degree \mathcal{M} . If we take the polynomial solution of the form

$$y(X) = X^{\mathcal{M}} + \sum_{n=0}^{\mathcal{M}-1} c_n X^n \quad (5.5)$$

upon inserting this expression in the Heun equation (5.3), we obtain the following constraints for the coefficients c_n , for $0 \leq n \leq \mathcal{M} + 1$,

$$A(n+1)(n+a_0)c_{n+1} - n((1+A)(a_0+n-1) + a_1A + a_2)c_n + (\mathcal{M}+1-n)^2 c_{n-1} = \mu \mathcal{M}^2 c_n, \quad (5.6)$$

with the conventions $c_{\mathcal{M}} = 1$ and $c_{-1} = c_{\mathcal{M}+1} = c_{\mathcal{M}+2} = 0$. Again, these equations are consistent only if μ is a root of the polynomial $P_{\mathcal{M}+1}(\mu)$ of degree $\mathcal{M} + 1$ in μ .

When the constraints (5.1) are satisfied, the eigenvalues of the algebraic Heun operator W (2.4) can also be deduced from those of the Gaudin transfer matrix thanks to formula (2.13), they are given by

$$w = \frac{i}{a^2 - 1} (s(a^2 + 1) - 2\mathcal{M}a - a\rho_2) + 4 \frac{ia}{a^2 - 1} \sum_{i=1}^{\mathcal{M}} Z_i, \quad (5.7)$$

where $\{Z_i\}$ are the roots of a polynomial solution of the differential Heun equation (5.3) or equivalently $Z_i = \frac{1}{4}(2 - z_i - 1/z_i)$ with $\{z_i\}$ the Bethe roots of (5.2).

Since $-\sum_{i=1}^{\mathcal{M}} Z_i$ is equal to the coefficient $c_{\mathcal{M}-1}$ in (4.6), we can express this sum in terms of μ as follows

$$i \sum_{i=1}^{\mathcal{M}} Z_i = -ic_{\mathcal{M}-1} = -i(\mu\mathcal{M}^2 + \mathcal{M}((1+A)(a_0 + \mathcal{M} - 1) + a_1A + a_2)). \quad (5.8)$$

Then, the eigenvalues of the algebraic Heun operator W (2.4) become

$$w = \frac{4ia\mu\mathcal{M}^2}{1-a^2} + \frac{i}{1-a^2} (((a^2+1)s + a\rho_2)(2\mathcal{M}+1) - \mathcal{M}^2(a-1)^2), \quad (5.9)$$

where μ are the roots of the polynomials $P_{\mathcal{M}+1}(\mu)$.

The previous construction provides $\mathcal{M}+1$ eigenvalues. They belong to the spin range $\{s-\mathcal{M}, s-\mathcal{M}+1, \dots, s\}$. We remark that this spin domain is stabilized by the algebraic Heun operator since (when relations (5.1) hold)

$$W_{\mathcal{M}+1, \mathcal{M}+2} W_{\mathcal{M}+2, \mathcal{M}+1} = 0. \quad (5.10)$$

To obtain the second part of the spectrum, we must start from the Bethe vectors $\bar{\mathbb{V}}(\bar{z})$ constructed from the lowest weight of $\mathfrak{su}(2)$. We give the definition and the useful formulas in Appendix A. We note that for $M = 2s - 1 - \mathcal{M}$, one gets $\beta_{-M} = 0$ and relation (A.7) simplifies since the first line vanishes. Therefore, the usual Bethe ansatz works and one finds that $\bar{\mathbb{W}}(u, \bar{z})$ (see (A.8)) for $M = 2s - 1 - \mathcal{M}$ is an eigenvalue if \bar{z} satisfies the Bethe equations,

$$\frac{\bar{\delta}(z_k) - \bar{\alpha}(z_k)}{4} + \frac{\bar{z}_k^2 + 1}{\bar{z}_k^2 - 1} + \sum_{\substack{p=1 \\ p \neq k}}^{2s-1-\mathcal{M}} \frac{\bar{z}_p(\bar{z}_k^2 - 1)}{(\bar{z}_k - \bar{z}_p)(\bar{z}_k\bar{z}_p - 1)} = 0, \quad (5.11)$$

for $1 \leq k \leq 2s - 1 - \mathcal{M}$. Once more, finding the Bethe roots \bar{z} is equivalent to finding a polynomial solution of a differential Heun equation. In this way, we prove that we obtain $2s - \mathcal{M}$ solutions for spins running from $-s$ to $s - \mathcal{M} - 1$ in unit steps.

From the Bethe vectors $\mathbb{V}(z)$ and $\bar{\mathbb{V}}(\bar{z})$, we obtain the complete spectrum of the Gaudin model or of the algebraic Heun operators when relations (5.1) hold.

6 Representations of the rotation group $O(3)$ and Heun operator

A special and important case of the Heun operator of Lie type arises when $\rho_1 = \rho_2 = \rho_3 = 0$:

$$W|_{\rho_1=\rho_2=\rho_3=0} = \{J_1, J_2\} + \frac{2i(a^2+1)}{a^2-1} J_1^2. \quad (6.1)$$

It corresponds to the situation where there is no external magnetic field for the Gaudin magnet (*i.e.* the scalar matrix $c(u)$ given in (2.9) vanishes). Consider the following rotation of the generators of $\mathfrak{su}(2)$

$$J_1 = \cos(\theta)\bar{J}_1 - \sin(\theta)\bar{J}_2, \quad \text{and} \quad J_2 = \sin(\theta)\bar{J}_1 + \cos(\theta)\bar{J}_2, \quad (6.2)$$

with $a = e^{2i\theta}$. In terms of the generators \bar{J}_1 and \bar{J}_2 , the algebraic Heun operator reads

$$E := \frac{4i(1-a)}{1+a} W|_{\rho_1=\rho_2=\rho_3=0} = 4(\bar{J}_1^2 + r\bar{J}_2^2), \quad r = \left(\frac{1-a}{1+a}\right)^2. \quad (6.3)$$

The operator E occurs in many physical and mathematical contexts. It is seen to be equivalent (up to an affine transformation) to the Hamiltonian of the quantum Euler top [29]. It also appears in the representation theory of the group $O(3)$ and its universal covering group $SU(2)$ as follows. As is very familiar, the standard representation basis is defined by the joint eigenvectors of the Casimir element $C = J_1^2 + J_2^2 + J_3^2$ and of the generator J_3 . It is said to be of subgroup type since it corresponds to the group reduction $O(3) \supset O(2)$ with one subgroup generator, J_3 , diagonalized. Close to 50 years

ago, Patera and Winternitz stressed [25] the existence of a second interesting basis stemming from the classification of second order polynomials in the generators. This second basis is provided by the eigenfunctions of C and $E = 4(J_1^2 + rJ_2^2)$ (and of the discrete operators X and PZ , where X and Z correspond to reflections in the yz and xy planes and P is the parity operator). It is not of subgroup type as E is not the generator of any subgroup of $O(3)$. Ways to obtain the eigenvalues of E are described in [25] but none are providing a closed formula. Our construction via the Bethe ansatz strikingly advances the characterization of the eigenvalues ϵ of E . Indeed, by specializing formulas (3.29) and (4.9), one gets that the eigenvalues of E are given by

$$\epsilon = \frac{2s(4s^2 + 1)(a^2 + 1)}{(a + 1)^2} - \frac{4a(s - \frac{1}{2})^2}{(a + 1)^2} \sum_{p=1}^{2s} (z_p + z_p^{-1}), \quad (6.4)$$

with the Bethe roots satisfying

$$\frac{as(z_k^2 - 1)}{(az_k - 1)(a - z_k)} + \frac{z_k^2 + 1}{z_k^2 - 1} + \sum_{\substack{p=1 \\ p \neq k}}^{2s} \frac{z_p(z_k^2 - 1)}{(z_k - z_p)(z_k z_p - 1)} = \frac{(2s + 1)^2}{8(z_k^2 - 1)} \prod_{\substack{p=1 \\ p \neq k}}^{2s} \frac{z_p(z_k - a)(az_k - 1)}{a(z_k - z_p)(z_k z_p - 1)}. \quad (6.5)$$

Let us also remark that if s is a half-integer, the result of Section 5 can also be used to characterize the spectrum of E . In this case the eigenvalues read

$$\epsilon = \frac{4s(a^2 + 1)}{(a + 1)^2} - \frac{4a}{(a + 1)^2} \sum_{p=1}^{s-\frac{1}{2}} (z_p + z_p^{-1}), \quad (6.6)$$

with the Bethe roots satisfying

$$\frac{as(z_k^2 - 1)}{(az_k - 1)(a - z_k)} + \frac{z_k^2 + 1}{z_k^2 - 1} + \sum_{\substack{p=1 \\ p \neq k}}^{s-1/2} \frac{z_p(z_k^2 - 1)}{(z_k - z_p)(z_k z_p - 1)} = 0. \quad (6.7)$$

The specialization used here of relation (4.17) for the Bargmann realization of the Heun operator reproduces the result of [25]. We should point out that the supplementary condition $0 < r < 1$ is imposed in [25] which implies that $a = \frac{1-\sqrt{r}}{1+\sqrt{r}}$. In the construction above we have required that a be a pure phase but the results can easily be generalized and the relations (6.4)-(6.7) are still valid for any a .

7 Entanglement entropy for the Krawtchouk chain and Heun operator

In this section, we present another interesting problem to which the results of this paper can be applied, namely the determination of the entanglement entropy for free Fermions on Krawtchouk chains. These chains are interesting in many respects. They emerge from the projection of spin systems defined on hypercubes [11] and they are known to allow for perfect state transfer [1]. They are usually introduced in the following way. Let us consider the free Fermion inhomogeneous Hamiltonian with nearest neighbor interactions and local magnetic fields defined by:

$$\hat{\mathcal{H}} = \frac{\beta}{2} \sum_{n=0}^{2s-1} \sqrt{(n+1)(2s-n)} (c_n^\dagger c_{n+1} + c_{n+1}^\dagger c_n) - \alpha \sum_{n=0}^{2s} (n-s) c_n^\dagger c_n, \quad (7.1)$$

where $\{c_m^\dagger, c_n\} = \delta_{mn}$ and where α and β are free parameters. We can choose the normalization of \mathcal{H} such that $\alpha^2 + \beta^2 = 1$ and take $\alpha = \cos(2\theta)$ and $\beta = \sin(2\theta)$. In what follows, it will be more

convenient to rewrite (7.1) in terms of the matrices J_1 and J_3 from an irreducible representation of $\mathfrak{su}(2)$ of dimension $2s + 1$. Indeed, we have that

$$\hat{\mathcal{H}} = (c_0^\dagger \quad \dots \quad c_{2s}^\dagger) \hat{H} \begin{pmatrix} c_0 \\ \vdots \\ c_{2s} \end{pmatrix}, \quad (7.2)$$

where

$$\hat{H} = \cos(2\theta)J_3 + \sin(2\theta)J_1. \quad (7.3)$$

We easily find an orthonormal basis $\{|\omega_k\rangle\}_{k \in \{0, \dots, 2s\}}$ of \mathbb{C}^{2s+1} such that $\hat{H}|\omega_k\rangle = \omega_k|\omega_k\rangle$. The eigenvalues are given by $\omega_k = k - s$ and the diagonalized Hamiltonian is

$$\hat{\mathcal{H}} = \sum_{k=0}^{2s} \omega_k \tilde{c}_k^\dagger \tilde{c}_k,$$

where the $\tilde{c}_k = \sum_{n=0}^{2s} \langle n|\omega_k\rangle c_n$ also respect $\{\tilde{c}_j, \tilde{c}_k^\dagger\} = \delta_{jk}$. The overlap coefficients are given in terms of Krawtchouk polynomials [24]:

$$\langle n|\omega_k\rangle = \sqrt{\binom{2s}{n} \binom{2s}{2s-k}} (\sin\theta)^{2s} |\cot\theta|^{k-n} K_n(2s-k; \sin^2\theta, 2s). \quad (7.4)$$

The ground state $|\Psi_0\rangle\rangle$ is defined by filling up the Fermi sea. If $|0\rangle\rangle$ represents the vacuum state which is annihilated by all the \tilde{c}_k , we have

$$|\Psi_0\rangle\rangle = \tilde{c}_K^\dagger \tilde{c}_{K-1}^\dagger \dots \tilde{c}_1^\dagger \tilde{c}_0^\dagger |0\rangle\rangle. \quad (7.5)$$

K is taken to be the largest integer k such that states associated to a negative energy ω_k are filled. A natural question to ask about such chain is the following: if we split it in two, what is the entanglement between the two parts? This information is contained in the entanglement entropy:

$$S = \text{tr}(\rho \ln(\rho)), \quad (7.6)$$

where ρ is the reduced density matrix associated to one of the two parts. Here, we take the subsystem to be the first $l + 1$ sites of the chain. In particular, the projection operator π_ℓ over the subsystem is taken to be

$$\pi_\ell = \sum_{n=0}^{\ell} |n\rangle \langle n|. \quad (7.7)$$

For a Krawtchouk chain in its ground state $|\Psi_0\rangle\rangle$, it is known that the value of S can also be extracted from the eigenvalues of the chopped correlation matrix [17]. This matrix is obtained by first considering the complete correlation matrix, which is the $(2s + 1) \times (2s + 1)$ matrix having for entries:

$$\hat{C}_{mn} = \langle\langle \Psi_0 | c_m^\dagger c_n | \Psi_0 \rangle\rangle. \quad (7.8)$$

It is useful to note that \hat{C} can be expressed as:

$$\hat{C} = \sum_{k=0}^K |\omega_k\rangle \langle \omega_k|. \quad (7.9)$$

Then, the chopped correlation matrix associated to the first $\ell + 1$ sites in the chain is obtained by considering the submatrix of \hat{C} defined as:

$$C = \pi_\ell \hat{C} \pi_\ell = |\hat{C}_{mn}|_{0 \leq m, n \leq \ell}. \quad (7.10)$$

Given (7.4), we see that

$$C_{mn} = \sum_{k=0}^K \sqrt{\binom{2s}{n} \binom{2s}{m} \binom{2s}{k}} (\sin^2 \theta)^{2s} |\cot \theta|^{2k-n+m} K_m(2s-k) K_n(2s-k), \quad (7.11)$$

where the last two parameters of the Krawtchouk polynomials are kept implicit.

In general, we see that the submatrices of (7.9) have a rather complicated expression and do not prove easy to diagonalize. However, considerations of bispectrality in the context of the time and band limiting problem have shown how to identify a tridiagonal operator T , in fact an algebraic Heun operator [21, 17], that has the property of commuting with the chopped correlation matrix. It is expressed as

$$T = \{\hat{H}, J_3\} + \mu J_3 + \nu \hat{H}, \quad (7.12)$$

where $\mu = -2K - 1 + 2s$ and $\nu = 2\ell + 1 - 2s$ [16]. Thus, one could instead diagonalize T and use the results to extract the spectrum of C . This is where the modified algebraic Bethe ansatz comes into play. If we make the identification

$$-\sin(2\theta)J_1 + \cos(2\theta)J_3 \rightarrow \tilde{J}_1, \quad \cos(2\theta)J_1 + \sin(2\theta)J_3 \rightarrow \tilde{J}_2 \quad (7.13)$$

for the generators and

$$\begin{aligned} \rho_1 &= \cot(2\theta)\mu + \frac{\nu}{\sin(2\theta)}, & \rho_2 &= \mu, & \rho_3 &= 0, \\ \rho_4 &= 2 \cot 2\theta & \text{and} & & \rho_5 &= \frac{\mu\nu}{2 \sin 2\theta} \end{aligned} \quad (7.14)$$

for the parameters, we see that

$$T = \sin 2\theta(W - \rho_5), \quad (7.15)$$

where W is defined in (2.3). We thus see that the spectrum of T is given, up to an affine transformation, by the one of W .

Moreover, it is observed that this W belongs to the homogeneous case (discussed in Section 5). Indeed, one can check that we here have $\mathcal{M} = \ell$ when considering the Bethe vector constructed with the lowest weight. Therefore, the $\ell + 1$ eigenvalues $t_{\bar{z}}$ of T will be given by:

$$t_{\bar{z}} = s \cos(2\theta) + \frac{\mu}{2} - \frac{1}{2} \sum_{i=1}^{\ell} \left(\bar{z}_i + \frac{1}{\bar{z}_i} \right), \quad (7.16)$$

where the \bar{z} are solutions to the Bethe equations given by (5.11):

$$\frac{e^{2i\theta} s(\bar{z}_k^2 - 1)}{(e^{2i\theta} \bar{z}_k - 1)(e^{2i\theta} - \bar{z}_k)} + \frac{(\bar{z}_k^2 + 1)(1 - \frac{\nu}{2}) - \mu \bar{z}_k}{(\bar{z}_k^2 - 1)} + \sum_{\substack{p=1 \\ p \neq k}}^{\ell} \frac{\bar{z}_p(\bar{z}_k^2 - 1)}{(\bar{z}_k - \bar{z}_p)(\bar{z}_k \bar{z}_p - 1)} = 0. \quad (7.17)$$

At this point, it is possible to recover the spectrum of C . Usually, this is done by acting with C on the eigenvectors of T , *i.e.* the Bethe vectors for W [14]. Since these are also eigenvectors of the chopped correlation matrix, we can read the eigenvalues of C from the results. We here want to present an

alternative way of obtaining the spectrum. It amounts to constructing a polynomial P of order ℓ such that

$$C = P(T) = \sum_{j=0}^{\ell} a_j T^j. \quad (7.18)$$

If we have P , then it is easy to see that $P(t_{\mathbf{z}})$ gives the eigenvalue of C associated to \mathbf{z} . To prove the existence of P , it is sufficient to know that $[C, T] = 0$ is verified and to notice that, since T is tridiagonal, the first $\ell + 1$ powers of T are linearly independent. The fact that T is tridiagonal also allows to determine P . Indeed, since we have that

$$\langle 0 | T^r | m \rangle = 0 \quad \text{if } r < m, \quad (7.19)$$

we are led to

$$a_{\ell} = \langle 0 | C | \ell \rangle / \langle 0 | T^{\ell} | \ell \rangle \quad (7.20)$$

and to the recurrence relation:

$$a_{\ell-j} = \frac{\left[\langle 0 | C | \ell - j \rangle - \sum_{r=0}^{j-1} a_{\ell-r} \langle 0 | T^{\ell-r} | \ell - j \rangle \right]}{\langle 0 | T^{\ell-j} | \ell - j \rangle}. \quad (7.21)$$

Given this relation, constructing P is straightforward and obtaining the eigenvalues of C and the entanglement entropy follows.

8 Conclusion and outlook

This paper has woven threads between the algebraic Heun operator of Lie type, the differential Heun operator, a Fuchsian second order differential equation with five regular singularities, the inhomogeneous Bethe equation and the Gaudin magnet. It showed in particular that there exist two equivalent ways to compute the eigenvalues of the algebraic Heun operator of Lie type by studying the polynomial solution of two differential equations, one inhomogeneous with four singularities and the other homogeneous with five singularities. This type of equivalence has already been noticed in a different context [15] and certainly deserves further investigation.

We have examined the Gaudin model with only one site. It is well-known however that the model with N sites remains integrable; the Bethe equations are also given in [13]. In the framework of the Bargmann realization, a differential equation with N variables is obtained in that case. A natural question is to compare this differential equation with the one obtained from the Bethe equations. This would provide a generalization to N variables of the results of this paper.

We showed that the tools of quantum integrable systems and the algebraic Heun operators of Lie type are relevant to the representation theory of $\mathfrak{su}(2)$. We expect that other quantum integrable systems and algebraic Heun operators of different types will find their way in the representation theory of higher rank Lie algebras. For example, algebraic Heun–Hahn or Racah–Hahn operators appear in [18] in the study of the diagonal centralizer of two $\mathfrak{su}(3)$.

Finally, we demonstrated the usefulness of the Bethe ansatz to compute the entanglement entropy of free Fermion chain with couplings given by the recurrence coefficients of the Krawtchouk polynomials. It would be interesting to use this approach for other chains. Algebraic Heun operators of different type which commute with the chopped correlation matrix associated to different inhomogeneous free Fermion chains have been introduced in [16, 17]. The computation of their spectra with the Bethe ansatz could be used to determine the entanglement entropy at least in the thermodynamical limit. We plan on returning to some of these questions.

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A Bethe ansatz from the lowest weight

The commutation relation of the shifted transfer matrix with $C(v, n)$ is given by

$$t(u, n)C(v, n) - C(v, n)t(u, n-1) = 4C(v, n) \left(\frac{v(u^2-1)}{(u-v)(uv-1)} \left(\tilde{D}(u) - \tilde{A}(u) + 4\frac{u^2+1}{u^2-1} \right) - 2 \right) (A.1) \\ - 4\frac{v}{u}C(u, n) \left(\frac{u(v^2-1)}{(u-v)(uv-1)} \left(\tilde{D}(v) - \tilde{A}(v) + 4\frac{v^2+1}{v^2-1} \right) + 4n \right).$$

Let $\bar{\omega}_s$ (with $s \in \mathbb{N}/2$) denote the lowest weight vector of the spin s representation of $\mathfrak{su}(2)$ which satisfies

$$\tilde{J}_3 \bar{\omega}_s = -s \bar{\omega}_s, \quad \tilde{J}_- \bar{\omega}_s = 0. \quad (A.2)$$

From the explicit form of the operators, one deduces that

$$\tilde{A}(u) \bar{\omega}_s = \bar{\alpha}(u) \bar{\omega}_s = \left(\rho_3 - \frac{2as(u^2-1)}{(au-1)(a-u)} - \frac{i(a^2-1)(u^2+1)\rho_1}{2a(u^2-1)} - \frac{(a-u)^2 + (au-1)^2}{2a(u^2-1)} \rho_2 \right) \bar{\omega}_s, \quad (A.3)$$

$$\tilde{D}(u) \bar{\omega}_s = \bar{\delta}(u) \bar{\omega}_s = \left(\rho_3 + \frac{2as(u^2-1)}{(au-1)(a-u)} + \frac{i(a^2-1)(u^2+1)\rho_1}{2a(u^2-1)} + \frac{(a-u)^2 + (au-1)^2}{2a(u^2-1)} \rho_2 \right) \bar{\omega}_s, \quad (A.4)$$

$$B(u, n) \bar{\omega}_s = \frac{1}{u} \beta_n \bar{\omega}_s = \frac{4}{u} \left(-s - n + \frac{1}{2} + \frac{i(a^2-1)\rho_1}{4a} + \frac{(a^2+1)\rho_2}{4a} + \frac{i\rho_3}{2} \right) \bar{\omega}_s. \quad (A.5)$$

We construct as follows the Bethe vectors, that depend on the parameters $\bar{\mathbf{z}} = \{\bar{z}_1, \bar{z}_2, \dots, \bar{z}_M\}$,

$$\bar{\mathbb{V}}(\bar{\mathbf{z}}) = C(\bar{z}_1, 0)C(\bar{z}_2, -1) \dots C(\bar{z}_M, -M+1) \bar{\omega}_s. \quad (A.6)$$

Due to relation (3.10), the entries of the vector $\bar{\mathbb{V}}(\bar{\mathbf{z}})$ do not depend on the order of the parameters \bar{z}_i . The action of the transfer matrix on the Bethe vector $\bar{\mathbb{V}}(\bar{\mathbf{z}})$ is given by

$$t(u) \bar{\mathbb{V}}(\bar{\mathbf{z}}) = \frac{2}{u} \beta_{-M} C(\bar{z}_1, 0)C(\bar{z}_2, -1) \dots C(\bar{z}_M, -M+1) C(u, -M) \bar{\omega}_s \\ + \bar{\mathbb{W}}(u, \bar{\mathbf{z}}) \bar{\mathbb{V}}(\bar{\mathbf{z}}) \\ - \sum_{k=1}^M \frac{16\bar{z}_k(\bar{z}_k^2-1)}{(u-\bar{z}_k)(u\bar{z}_k-1)} \bar{\mathbb{U}}_k(\bar{\mathbf{z}}) \bar{\mathbb{V}}(\bar{\mathbf{z}}, u) \quad (A.7)$$

where $\bar{\mathbb{V}}(\bar{\mathbf{z}}, u) = C(\bar{z}_1, 0) \dots C(\bar{z}_{k-1}, -k+2)C(u, -k+1)C(\bar{z}_{k+1}, -k) \dots C(\bar{z}_M, -M+1) \bar{\omega}_s$ and

$$\bar{\mathbb{W}}(u, \bar{\mathbf{z}}) = \bar{\lambda}(u) + \sum_{k=1}^M \frac{16\bar{z}_k(u^2-1)}{(u-\bar{z}_k)(u\bar{z}_k-1)} \left(\frac{\bar{\delta}(u) - \bar{\alpha}(u)}{4} + \frac{u^2+1}{u^2-1} + \sum_{\substack{p=1 \\ p \neq k}}^M \frac{(u^2-1)\bar{z}_k\bar{z}_p}{u(\bar{z}_k-\bar{z}_p)(\bar{z}_k\bar{z}_p-1)} \right), \quad (A.8)$$

$$\bar{\lambda}(u) = \bar{\alpha}^2(u) + \bar{\delta}^2(u) + 2 - 2u(\bar{\alpha}'(u) - \bar{\delta}'(u)) - 2\frac{u^2+1}{u^2-1}(\bar{\alpha}(u) - \bar{\delta}(u)), \quad (A.9)$$

$$\bar{\mathbb{U}}_k(\bar{\mathbf{z}}) = \frac{\bar{\delta}(z_k) - \bar{\alpha}(z_k)}{4} + \frac{\bar{z}_k^2+1}{\bar{z}_k^2-1} + \sum_{\substack{p=1 \\ p \neq k}}^M \frac{\bar{z}_p(\bar{z}_k^2-1)}{(\bar{z}_k-\bar{z}_p)(\bar{z}_k\bar{z}_p-1)}. \quad (A.10)$$

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