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# Asymptotic normality of wavelet covariances and of multivariate wavelet Whittle estimators.

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## Abstract

Multivariate processes with long-range memory properties can be encountered in many applications fields. Two fundamentals characteristics in such frameworks are the long-memory parameters and the correlation between time series. We consider multivariate linear processes, not necessarily Gaussian, presenting long-memory dependence. We show that the covariances between the wavelet coefficients in this setting are asymptotically Gaussian. We also study the asymptotic distributions of the estimators of the long-memory parameter and of the long-run covariance by a wavelet-based Whittle procedure. We prove the asymptotic normality of the estimators and we give an explicit expression for the asymptotic covariances. An empirical illustration of this result is proposed on a real dataset of a rat brain connectivity.

**Keywords.** Multivariate processes, long-range memory, covariance, wavelets, asymptotic normality, cerebral connectivity

Multivariate processes with long-range dependence (LRD) properties can be found in a large scope of applications, such as geoscience ([Whitcher and Jensen, 2000](#)), finance ([Gençay et al, 2001](#)) or neuroscience ([Achard and Gannaz, 2016a](#)). Extensions to multivariate settings was initiated by [Robinson \(1995\)](#) and this topic has encountered much interest over the last several decades. Several models have been proposed, like multivariate ARFIMA models ([Lobato, 1997](#); [Sela and Hurvich, 2008](#)). [Kechagias and Pipiras \(2014\)](#) give properties in time and spectral domains of linear representations of multivariate LRD processes. A nonlinear example of multivariate long-range memory processes is also proposed by [Didier et al \(2011\)](#), where a multivariate Brownian motion is defined.

We focus on semi-parametric estimators, which are more robust to model misspecification ([Robinson, 1995](#)). A common estimation procedure in this framework is Whittle estimation, which is based on a Fourier representation of the processes ([Lobato, 1999](#); [Nielsen, 2011](#); [Shimotsu, 2007](#)). The authors establish the consistence and the asymptotic distribution of their estimators. More recently, asymptotic normality of estimators was provided by [Baek et al \(2020\)](#), in prolongation

of [Robinson \(2008\)](#), in a multivariate framework where components can be co-integrated. An estimation with a Lasso penalty was also proposed in this setting by [Pipiras et al \(2017\)](#) and [Düker and Pipiras \(2019\)](#) established the asymptotic normality of this procedure.

As an alternative to Fourier, wavelet-based estimators can be used. Wavelet transforms are interesting especially because the wavelet analysis performs an implicit differentiation, which offers the possibility to consider non stationary processes. Wavelet-based Whittle estimation was introduced in [Moulines et al \(2008\)](#) for univariate LRD time series. It was generalized to multivariate setting by [Achard and Gannaz \(2016a\)](#). Estimators are consistent and they have theoretical rates comparable to Fourier-based estimators. The numerical performances of wavelet-based and Fourier-based estimators are also similar, as illustrated in [Achard and Gannaz \(2019\)](#).

This paper considers linear processes, not necessarily Gaussian, with long-memory properties. The two main characteristics we are interested in are the long-memory parameters, which measure the LRD behavior of the processes, and the long-run covariance, which captures the dependence structure between the components. Long-memory parameters and long-run covariance are estimated jointly with the procedure described in [Achard and Gannaz \(2016a\)](#). Our goal is to establish the asymptotic normality of these estimators. [Roueff and Taqqu \(2009a\)](#) prove that this asymptotic normality is acquired in univariate setting, but no result was provided for wavelet-based estimation in multivariate setting. We first state that the covariances between the wavelet coefficients at a given scale are asymptotically Gaussian. Asymptotic distributions of the estimators of long-memory parameters and long-run covariance are then obtained.

The paper is organized as follows. Section 1 introduces the specific framework of our study. The long-memory properties of the processes are described and assumptions on a linear representation of the time series are given. The wavelet representation of the processes is also synthesized. The asymptotic behavior of the covariance between wavelet coefficients is given in Section 2. Wavelet-based Whittle estimation is considered in Section 3. Asymptotic normality of the estimators is established. Section 4 illustrates the asymptotic normality of the estimators on a real data example, with the study of functional Magnetic Resonance images (fMRI) on a dead rat.

## 1 The semiparametric multivariate long-memory framework

Let  $\mathbf{X} = \{X_\ell(k), k \in \mathbb{Z}, \ell = 1, \dots, p\}$  be a multivariate stochastic process. We consider a long memory process  $\mathbf{X}$  with memory parameters  $\mathbf{d} = (d_1, d_2, \dots, d_p)$ . Let  $\Delta$  denote the difference operator,  $(\Delta \mathbf{X})_t = \mathbf{X}_{t+1} - \mathbf{X}_t$ . The  $k$ -th difference operator,  $\Delta^k$ ,  $k \in \mathbb{N}$ , is defined by  $k$  recursive applications of  $\Delta$ . For any  $\mathbf{D} > \mathbf{d} - 1/2$ , we suppose that the multivariate process  $\mathbf{Z} = \text{diag}(\Delta^{D_\ell}, \ell = 1, \dots, p)\mathbf{X}$  is covariance stationary with a spectral density matrix given by:

$$\text{for all } (\ell, m), f_{\ell, m}^{(D_\ell, D_m)}(\lambda) = \frac{1}{2\pi} \Omega_{\ell, m} (1 - e^{-i\lambda})^{-d_\ell^s} (1 - e^{i\lambda})^{-d_m^s} f_{\ell, m}^S(\lambda), \quad \lambda \in [-\pi, \pi],$$

where the long memory parameters are given by  $d_m^S = d_m - D_m$  for all  $m$ . The functions  $f_{\ell, m}^S(\cdot)$  correspond to the short memory behavior of the process. The generalized cross-spectral density of

processes  $X_\ell$  and  $X_m$  can be written:

$$f_{\ell,m}(\lambda) = \frac{1}{2\pi} \Omega_{\ell,m} (1 - e^{-i\lambda})^{-d_\ell} (1 - e^{i\lambda})^{-d_m} f_{\ell,m}^S(\lambda), \quad \lambda \in [-\pi, \pi].$$

The spectral density of the multivariate process  $\mathbf{X}$  is thus,

$$f(\lambda) = \mathbf{\Omega} \circ (\mathbf{\Lambda}^0(\mathbf{d}) f^S(\lambda) \overline{\mathbf{\Lambda}^0(\mathbf{d})}), \quad \lambda \in [-\pi, \pi], \quad \text{with } \mathbf{\Lambda}^0(\mathbf{d}) = \text{diag}((1 - e^{-i\lambda})^{-\mathbf{d}})$$

where  $\mathbf{d} = \mathbf{D} + \mathbf{d}^s$ . The overline is the conjugate operator and  $\circ$  denotes the Hadamard product.

The spectral density  $f^S(\cdot)$  corresponds to the short-memory behaviour. The matrix  $\mathbf{\Omega}$  is called *fractal connectivity* by [Achard et al \(2008\)](#) or *long-run covariance* matrix by [Robinson \(2005\)](#). Similarly to [Achard and Gannaz \(2016a\)](#); [Moulines et al \(2007\)](#) we assume that  $f^S(\cdot) \in \mathcal{H}(\beta, L)$  with  $0 < \beta \leq 2$  and  $0 < L < \infty$ . The space  $\mathcal{H}(\beta, L)$  is defined as the class of non-negative symmetric functions  $\mathbf{g}(\cdot)$  on  $[\pi, \pi]$  such that for all  $\lambda \in (-\pi, \pi)$ ,  $\|\mathbf{g}(\lambda) - \mathbf{g}(0)\|_\infty \leq L \|\mathbf{g}(0)\|_\infty |\lambda|^\beta$ .

When  $\lambda$  tends to 0, the spectral density matrix can be approximated at the first order by:

$$f(\lambda) \sim \tilde{\mathbf{\Lambda}}(\mathbf{d}) \mathbf{\Omega} \overline{\tilde{\mathbf{\Lambda}}(\mathbf{d})}, \quad \lambda \in (0, \pi], \quad \text{with } \tilde{\mathbf{\Lambda}}(\mathbf{d}) = \text{diag}(|\lambda|^{-\mathbf{d}} e^{-i\pi \mathbf{d}/2}). \quad (1)$$

[Lobato \(1999\)](#) used  $\tilde{\mathbf{\Lambda}}(\mathbf{d}) = \text{diag}(\lambda^{-\mathbf{d}})$  as an approximation of  $f(\cdot)$ . Whereas [Shimotsu \(2007\)](#) chose to approximate  $f(\cdot)$  using  $\tilde{\mathbf{\Lambda}}(\mathbf{d}) = \text{diag}(\lambda^{-\mathbf{d}} e^{-i(\pi-\lambda)\mathbf{d}/2})$ , which corresponds to a second order approximation due to the remaining term  $\lambda$  in the exponential.

We refer to Section 2.1 of [Achard and Gannaz \(2016a\)](#) and references therein for examples of processes which satisfy approximation (1).

## 1.1 Linear decomposition

We suppose hereafter that the multivariate process admits a linear representation.

Suppose that there exists a sequence  $\{\mathbf{A}(u)\}_{u \in \mathbb{Z}}$  in  $\mathbb{R}^{p \times p}$  such that  $\sum_u \|\mathbf{A}(u)\|_\infty^2 < \infty$  and

$$\forall t, \quad \Delta^D \mathbf{X}(t) = \boldsymbol{\mu} + \sum_{u \in \mathbb{Z}} \mathbf{A}(t+u) \boldsymbol{\varepsilon}(u)$$

with  $\boldsymbol{\varepsilon}(t)$  weak white noise process, in  $\mathbb{R}^p$ . Let  $\mathcal{F}_{t-1}$  denote the  $\sigma$ -field of events generated by  $\{\boldsymbol{\varepsilon}(s), s \leq t-1\}$ . Assume that  $\boldsymbol{\varepsilon}$  satisfies  $\mathbb{E}[\boldsymbol{\varepsilon}(t) | \mathcal{F}_{t-1}] = 0$ ,  $\mathbb{E}[\varepsilon_a(t) \varepsilon_b(t) | \mathcal{F}_{t-1}] = \mathbb{1}_{a=b}$  and  $\mathbb{E}[\varepsilon_a(t) \varepsilon_b(t) \varepsilon_c(t) \varepsilon_d(t) | \mathcal{F}_{t-1}] = \mu_{a,b,c,d}$  with  $|\mu_{a,b,c,d}| \leq \mu_\infty < \infty$ , for all  $a, b, c, d = 1, \dots, p$ .

Define for all  $\lambda \in \mathbb{R}$ ,  $\mathbf{A}^*(\lambda) = \sum_{t \in \mathbb{Z}} \mathbf{A}(t) e^{i\lambda t}$  the Fourier series associated to  $\{\mathbf{A}(u)\}_{u \in \mathbb{Z}}$ . We add an additional assumption:

**(A)** For all  $(a, b) \in \{1, \dots, p\}^2$ , for all  $\lambda \in \mathbb{R}$  the sequence  $(2^{-j d_a} |A_{a,b}^*(2^{-j} \lambda)|)_{j \geq 0}$  is convergent as  $j$  goes to infinity.

This assumption is necessary for technical reasons. It does not seem restrictive. Indeed remark that for all  $(a, b) \in \{1, \dots, p\}^2$ , for all  $\lambda \in \mathbb{R}$ , the sequence  $\left( \frac{\sum_{a'=1}^p (2^{-2j d_a} A_{a,a'}^* (2^{-j\lambda}))}{\sum_{b'=1}^p (2^{-2j d_b} A_{b,b'}^* (2^{-j\lambda}))} \right)_{j \geq 0}$  converges (see (12)).

An example of process which satisfies these assumptions is the causal multivariate linear representations with trigonometric power law coefficients proposed in [Kechagias and Pipiras \(2014\)](#).

## 1.2 Wavelet representation

We introduce a pair of wavelet filters  $(\phi, \psi)$  satisfying:

**(W1)** The functions  $\phi(\cdot)$  and  $\psi(\cdot)$  are integrable, have compact supports,  $\int_{\mathbb{R}} \phi(t) dt = 1$  and  $\int \psi^2(t) dt = 1$ ;

**(W2)** There exists  $\alpha > 1$  such that  $\sup_{\lambda \in \mathbb{R}} |\hat{\psi}(\lambda)| (1 + |\lambda|)^\alpha < \infty$ , i.e. the wavelet is  $\alpha$ -regular;

**(W3)** The mother wavelet  $\psi(\cdot)$  has  $M > 1$  vanishing moments.

**(W4)** The function  $\sum_{k \in \mathbb{Z}} k^\ell \phi(\cdot - k)$  is polynomial with degree  $\ell$  for all  $\ell = 1, \dots, M - 1$ .

**(W5)** For all  $i = 1, \dots, p$ ,  $(1 + \beta)/2 - \alpha < d_i \leq M$ .

At a given resolution  $j \geq 0$ , for  $k \in \mathbb{Z}$ , we define the dilated and translated functions  $\phi_{j,k}(\cdot) = 2^{-j/2} \phi(2^{-j} \cdot - k)$  and  $\psi_{j,k}(\cdot) = 2^{-j/2} \psi(2^{-j} \cdot - k)$ . The wavelet coefficients of the process  $\mathbf{X}$  are defined by

$$\mathbf{W}_{j,k} = \int_{\mathbb{R}} \tilde{\mathbf{X}}(t) \psi_{j,k}(t) dt \quad j \geq 0, k \in \mathbb{Z},$$

where  $\tilde{\mathbf{X}}(t) = \sum_{k \in \mathbb{Z}} \mathbf{X}(k) \phi(t - k)$ . For given  $j \geq 0$  and  $k \in \mathbb{Z}$ ,  $\mathbf{W}_{j,k}$  is a  $p$ -dimensional vector  $\mathbf{W}_{j,k} = (W_{j,k}(1) \ W_{j,k}(2) \ \dots \ W_{j,k}(p))$  where  $W_{j,k}(\ell) = \int_{\mathbb{R}} \tilde{X}_\ell(t) \psi_{j,k}(t) dt$ .

As the wavelets have a compact support, only a finite number  $n_j$  of coefficients are non null at each scale  $j$ . Suppose without loss of generality that the support of  $\psi(\cdot)$  is included in  $[0, T_\psi]$  with  $T_\psi \geq 1$ . For every  $j \geq 0$ , define

$$n_j := \max(0, 2^{-j}(N_X - T_\psi + 1) - T_\psi + 1).$$

At each scale  $j$ , the non-zero coefficients belong to  $\{W_{j,k}, k = 0, \dots, n_j\}$ .

Let  $j_0$  be the minimal scale and  $j_1 = j_0 + \Delta$  the maximal scale which are considered in the estimation procedure. In the following,  $n$  will denote the number of wavelet coefficients used for estimation and  $\langle \mathcal{J} \rangle$  the mean of scales, that is,

$$n = \sum_{j=j_0}^{j_1} n_j \quad \text{and} \quad \langle \mathcal{J} \rangle = \frac{1}{n} \sum_{j=j_0}^{j_1} n_j j.$$

Define also

$$\eta_\Delta := \sum_{u=0}^{\Delta} u \frac{2^{-u}}{2 - 2^{-\Delta}} \quad \text{and} \quad \kappa_\Delta := \sum_{u=0}^{\Delta} (u - \eta_\Delta)^2 \frac{2^{-u}}{2 - 2^{-\Delta}}.$$

These sequences converge respectively to 1 and to 2 when  $\Delta$  goes to infinity.

Let

$$\begin{aligned} D_{u,\tau}(\lambda; \delta) &= \sum_{t \in \mathbb{Z}} |\lambda + 2t\pi|^{-\delta} \overline{\widehat{\psi}(\lambda + 2t\pi)} 2^{u/2} \widehat{\psi}(2^u(\lambda + 2t\pi)) e^{-i2^u\tau(\lambda + 2t\pi)}, \\ \widetilde{D}_{u,\infty}(\lambda; \delta) &= \sum_{\tau=0}^{2^u-1} \widetilde{D}_{u,\tau}(\lambda; \delta). \end{aligned} \quad (2)$$

[Moulines et al \(2007\)](#) establish that under assumptions **(W1)-(W5)**  $D_{u,\tau}(\lambda; \delta)$  is an approximation of the cross spectral density between  $\{W_{j,k}, k \in \mathbb{Z}\}$  and  $\{W_{j+u, 2^u + \tau}, \tau = 0, \dots, 2^u - 1, k \in \mathbb{Z}\}$ . Function  $\widetilde{D}_{u,\infty}(\lambda; \delta)$  will appear naturally when taking into account the between-scales dependence.

## 2 Asymptotic normality of wavelet covariances and correlations

Define  $\widehat{\sigma}_{a,b}(j)$  the empirical covariance of the wavelet coefficients at a given scale  $j$ , between components  $a$  and  $b$ , and let  $\sigma_{a,b}(j)$  denote the theoretical covariance,

$$\begin{aligned} \widehat{\sigma}_{a,b}(j) &= \frac{1}{n_j} \sum_{k=0}^{n_j-1} W_{j,k}(a) W_{j,k}(b), \\ \sigma_{a,b}(j) &= \mathbb{E}[W_{j,k}(a) W_{j,k}(b)]. \end{aligned}$$

Proposition 2 of [Achard and Gannaz \(2016a\)](#) proposes an approximation of the wavelet covariance at a given scale. It states that for all  $j \geq 0$ , for all  $\lambda \in (-\pi, \pi)$ ,

$$|\sigma_{a,b}(j) - 2^{-j(d_a+d_b)} G_{a,b}| \leq CL 2^{(d_a+d_b-\beta)j}. \quad (3)$$

with constant  $C$  depending on  $\beta, d, \max_{\ell,m} |\Omega_{\ell,m}|, \phi$  and  $\psi$  and

$$\begin{aligned} G_{a,b} &= \Omega_{a,b} \cos(\pi(d_a - d_b)/2) K(d_a + d_b) \\ K(\delta) &= \int_{-\infty}^{\infty} |\lambda|^{-\delta} |\widehat{\psi}(\lambda)|^2 d\lambda, \end{aligned}$$

$K(\delta)$  being defined for all  $\delta \in (-\alpha, M)$ .

The asymptotic behavior of the covariance process is given in the following theorem.

**Theorem 2.1.** *For all  $j_0 \geq 0, u \geq 0$ , define*

$$\begin{aligned} \widehat{T}(j_0 + u) &= \text{vect} \left( 2^{-j_0(d_a+d_b)} \widehat{\sigma}_{a,b}(j_0 + u), a, b = 1, \dots, p \right) \\ T(j_0 + u) &= \text{vect} \left( 2^{-j_0(d_a+d_b)} \sigma_{a,b}(j_0 + u), a, b = 1, \dots, p \right) \end{aligned}$$

where  $\text{vect}(M)$  denotes the operation which transforms a matrix  $M \in \mathbb{R}^{p_1 \times p_2}$  in a vector of  $\mathbb{R}^{p_1 p_2}$ . Suppose assumptions **(A)**, **(W1)** to **(W5)** hold. Let  $2^{-j_0 \beta} \rightarrow 0$  and  $N_X^{-1} 2^{j_0} \rightarrow 0$ . Then

$$\left\{ \sqrt{n_{j_0+u}} \left( \widehat{T}(j_0+u) - T(j_0+u) \right), u = 0, \dots, \Delta \right\} \xrightarrow{j_0 \rightarrow \infty} \mathcal{L} \{ Q_u^{(d)}, u = 0, \dots, \Delta \}$$

where  $Q^{(d)}$  is a centered Gaussian process with covariance function  $V_{u,u'}^{(d)}((a,b), (a',b')) = \text{Cov}(Q_u^{(d)}(a,b), Q_{u'}^{(d)}(a',b'))$  equal to

$$V_{u,u'}^{(d)}((a,b), (a',b')) = 2\pi 2^{(d_a+d_b+d_{a'}+d_{b'})u \vee u' - |u-u'|/2} \left( G_{a,a'} G_{b,b'} \frac{I_{|u-u'|}(d_a+d_{a'}, d_b+d_{b'})}{K(d_a+d_{a'})K(d_b+d_{b'})} + G_{a,b'} G_{b,a'} \frac{I_{|u-u'|}(d_a+d_{b'}, d_b+d_{a'})}{K(d_a+d_{b'})K(d_b+d_{a'})} \right)$$

with

$$I_u(\delta_1, \delta_2) = \int_{-\pi}^{\pi} \overline{\widetilde{D}_{u,\infty}(\lambda; \delta_1)} \widetilde{D}_{u,\infty}(\lambda; \delta_2) d\lambda,$$

and  $\widetilde{D}_{u,\infty}(\lambda; \delta_2)$  defined in (2).

The proof is given in Section A. It is very similar to the one of the univariate setting given in Theorem 5 of [Roueff and Taqqu \(2009a\)](#). It relies on decimated processes and limit theorems developed in [Roueff and Taqqu \(2009b\)](#).

To simplify the expression of the variance, let define  $\widetilde{I}_u(\delta_1, \delta_2)$  as

$$\widetilde{I}_u(\delta_1, \delta_2) = \frac{I_u(\delta_1, \delta_2)}{K(\delta_1)K(\delta_2)}.$$

Then the variance can be written as

$$V_{u,u'}^{(d)}((a,b), (a',b')) = 2\pi 2^{(d_a+d_b+d_{a'}+d_{b'})u \vee u' - |u-u'|/2} \left( G_{a,a'} G_{b,b'} \widetilde{I}_{|u-u'|}(d_a+d_{a'}, d_b+d_{b'}) + G_{a,b'} G_{b,a'} \widetilde{I}_{|u-u'|}(d_a+d_{b'}, d_b+d_{a'}) \right). \quad (4)$$

Remark that

$$\widetilde{I}_0(\delta_1, \delta_2) = \frac{\int_{-\pi}^{\pi} g_{\psi}(\lambda; \delta_1) g_{\psi}(\lambda; \delta_2) d\lambda}{\left( \int_{-\pi}^{\pi} g_{\psi}(\lambda; \delta_1) d\lambda \right) \left( \int_{-\pi}^{\pi} g_{\psi}(\lambda; \delta_2) d\lambda \right)}$$

where  $g_{\psi}(\lambda; \delta) = \sum_{t \in \mathbb{Z}} |\lambda + 2t\pi|^{-\delta} |\widehat{\psi}(\lambda + 2t\pi)|^2$ . It is straightforward that  $\widetilde{I}_0(\delta_1, \delta_2) \leq 1$ .

In the univariate setting, we obtain the same result as [Roueff and Taqqu \(2009a\)](#). Remark that they use a different normalization, by  $\sqrt{N_X 2^{-j_0}}$  rather than by  $\sqrt{n_{j_0+u}}$ , and that  $\sqrt{n_{j_0+u}} \sim \sqrt{N_X 2^{-j_0-u}}$ .

[Whitcher et al \(2000\)](#) give asymptotic normality for wavelet correlations of bivariate multivariate time series with long-range dependence. They prove their result under the assumption that the wavelet coefficients are a stationary bivariate gaussian process. As stated by [Moulines et al \(2007\)](#), wavelet coefficients are not Gaussian in the general case and the assumption of [Whitcher et al \(2000\)](#) seems restrictive. Consequently, our result is more reliable.

As already pointed out by [Roueff and Taqqu \(2009a\)](#), the covariance of the wavelet coefficients involves between scales correlations which do not vanish when the sample size goes to infinity. This fact contrasts with the behavior of Fourier periodogram. In the variance formulation (4), these correlations appear through quantities  $\{\tilde{I}_u(\delta_1, \delta_2), u \geq 0, (a, b) \in \{1, \dots, p\}^2\}$ .

Using approximation of  $\{\sigma_{a,b}(j), j_0 \leq j \leq j_1\}$  given in (3), we obtain the following corollary.

**Corollary 2.2.** *Under conditions of Theorem 2.1,*

$$\left\{ \sqrt{n_{j_0+u}} \left( \hat{T}(j_0 + u) - \text{vect} \left( \Omega_{a,b} \cos\left(\frac{\pi}{2}(d_a - d_b)\right) K(d_a + d_b), a, b = 1 \dots, p \right) \right), u = 0, \dots, \Delta \right\} \\ \xrightarrow{j_0 \rightarrow \infty} \{Q_u^{(d)}, u = 0, \dots, \Delta\}.$$

We can deduce asymptotic normality for wavelet correlations by Delta method. We do not present here the multivariate result for the sake of concision. We only give the pointwise result to highlight the specificity of our setting.

**Corollary 2.3.** *Let  $(a, b) \in \{1, \dots, p\}^2, a \neq b$ , and  $j \geq j_0 \geq 0$ . Define*

$$\hat{\rho}_{a,b}(j) = \frac{\hat{\sigma}_{a,b}(j)}{\sqrt{\hat{\sigma}_{a,b}(j)\hat{\sigma}_{b,b}(j)}} \text{ and } \rho_{a,b}(j) = \frac{\sigma_{a,b}(j)}{\sqrt{\sigma_{a,b}(j)\sigma_{b,b}(j)}}.$$

*Then, under conditions of Theorem 2.1,*

$$\sqrt{n_j} (\hat{\rho}_{a,b}(j) - \rho_{a,b}(j)) \xrightarrow{j \rightarrow \infty} \mathcal{N} \left( 0, \tilde{V}_{a,b}(j) \right)$$

*with*

$$\tilde{V}_{a,b}(j) = 2\pi \left( \tilde{I}_0(2d_a, 2d_b) + \tilde{I}_0(d_a + d_b, d_a + d_b) (\rho_{a,b}(j)^2 + \rho_{a,b}(j)^4) \right. \\ \left. - (\tilde{I}_0(2d_a, 2d_b) + \tilde{I}_0(2d_b, d_a + d_b)) 2\rho_{a,b}(j)^2 - (\tilde{I}_0(2d_a, 2d_a) + \tilde{I}_0(2d_b, 2d_b)) \rho_{a,b}(j)^2 / 2 \right).$$

*In particular, when  $d_a = d_b$ ,*

$$\sqrt{n_j} 2^{-j d_a} (\hat{\rho}_{a,b}(j) - \rho_{a,b}(j)) \xrightarrow{j \rightarrow \infty} \mathcal{N} \left( 0, 2\pi \tilde{I}_0(2d_a, 2d_a) (1 - \rho_{a,b}^2)^2 \right).$$

The proof does not present major difficulty and it is thus omitted. Approximation (3) has been used to simplify expressions. Remark that when parameter  $d_a$  and  $d_b$  are equal, we recover a traditional form of the variance for a correlation.

[Whitcher et al \(2000\)](#) used the assumption of the convergence of the Fisher transform of  $\hat{\rho}_{a,b}(j)$  to a standard Gaussian distribution at a rate  $\sqrt{n_j}$ , when the correlation  $\rho_{a,b}(j)$  is equal to zero. The authors suppose that wavelet coefficients are independent, which is asymptotically satisfied when the regularity of the wavelet goes to infinity ([McCoy and Walden, 1996](#)). Corollary 2.3 illustrates that an additional normalization by  $\left( 2\pi \tilde{I}_0(2d_a, 2d_b) \right)^{-1/2}$  of  $\hat{\rho}_{a,b}(j)$  is necessary.



Wavelet correlation at a given scale is also known as wavelet coherence. It is used in some applications, as in environmental studies by [Whitcher et al \(2000\)](#), or in neuroscience in [Achard et al \(2019\)](#). In such real data applications, the crucial point is the access to test procedures. In particular, the test of the nullity of the correlations at a given scale is essential. Yet Corollary 2.3 shows that the asymptotic distribution depends on the long-memory parameters  $\mathbf{d}$ . An additional step is thus necessary to provide test statistics, *e.g.* by plugging estimations of parameters  $\mathbf{d}$ .

### 3 Asymptotic normality of the parameters estimates

For a better understanding, the true parameters are denoted with an exponent 0 in this part.

Let  $\widehat{d}$  and  $\widehat{\Omega}$  be the Wavelet Whittle estimators as defined by [Achard and Gannaz \(2016a\)](#). They maximize the criterion

$$\mathcal{L}(\mathbf{G}(\mathbf{d}), \mathbf{d}) = \frac{1}{n} \sum_{j=j_0}^{j_1} \left[ n_j \log \det (\Lambda_j(\mathbf{d}) \mathbf{G}(\mathbf{d}) \Lambda_j(\mathbf{d})) + \sum_k \mathbf{W}_{j,k}^T (\Lambda_j(\mathbf{d}) \mathbf{G}(\mathbf{d}) \Lambda_j(\mathbf{d}))^{-1} \mathbf{W}_{j,k} \right],$$

where the superscript  $T$  denotes the transpose operator and  $\Lambda_j(\mathbf{d})$  and the matrix  $\mathbf{G}(\mathbf{d})$  are given by:

$$\begin{aligned} \Lambda_j(\mathbf{d}) &= \text{diag} \left( 2^{j\mathbf{d}} \right) \\ G_{\ell,m}(\mathbf{d}) &= \Omega_{\ell,m} K(d_\ell + d_m) \cos(\pi(d_\ell - d_m)/2). \end{aligned} \quad (5)$$

The estimation of the vector of long-memory parameters  $\mathbf{d}$  satisfies  $\widehat{\mathbf{d}} = \underset{\mathbf{d}}{\text{argmin}} R(\mathbf{d})$ , with

$$R(\mathbf{d}) = \log \det(\widehat{\mathbf{G}}(\mathbf{d})) + 2 \log(2) \left( \frac{1}{n} \sum_{j=j_0}^{j_1} j n_j \right) \left( \sum_{\ell=1}^p d_\ell \right).$$

The covariance matrix  $\Omega$  is estimated by

$$\begin{aligned} \widehat{\Omega}_{\ell,m} &= \widehat{G}_{\ell,m}(\widehat{\mathbf{d}}) / (\cos(\pi(\widehat{d}_\ell - \widehat{d}_m)/2) K(\widehat{d}_\ell + \widehat{d}_m)), \\ \text{where } \widehat{\mathbf{G}}(\mathbf{d}) &= \frac{1}{n} \sum_{j=j_0}^{j_1} n_j \Lambda_j(\mathbf{d})^{-1} \widehat{\Sigma}(j) \Lambda_j(\mathbf{d})^{-1}, \end{aligned} \quad (6)$$

with  $\widehat{\Sigma}(j) = (\widehat{\sigma}_{a,b}(j))_{a,b=1,\dots,p}$ .

The following theorem gives the asymptotic normality of the estimation of the long-memory parameters.

**Theorem 3.1.** *Suppose assumptions (A), (W1) to (W5) hold. Let*

$$j_1 - j_0 \rightarrow \Delta \in \{1, \dots, \infty\}, \quad j_0 (N_X 2^{-j_0})^{-1/8} \rightarrow 0 \text{ and } N_X 2^{-j_0(1+2\beta)} \rightarrow 0.$$

Then  $\sqrt{n}(\widehat{\mathbf{d}} - \mathbf{d}^0)$  converges in distribution to a centered Gaussian distribution with a variance equal to

$$\frac{\pi}{\log(2)^2} (\mathbf{G}^{0-1} \circ \mathbf{G}^0 + \mathbf{I}_p)^{-1} \mathbf{W}^{(\Delta)} (\mathbf{G}^{0-1} \circ \mathbf{G}^0 + \mathbf{I}_p)^{-1},$$

with component  $(a, a')$  of  $\mathbf{W}^{(\Delta)}$ , for  $(a, a') \in \{1, \dots, p\}$ , given by

$$W_{a,a'}^{(\Delta)} = \sum_{b,b'} G_{a,b}^{0(-1)} G_{a',b'}^{0(-1)} (G_{a,a'}^0 G_{b,b'}^0 \mathcal{I}_\Delta(d_a^0 + d_{a'}^0, d_b^0 + d_{b'}^0) + G_{a,b'}^0 G_{a',b}^0 \mathcal{I}_\Delta(d_a^0 + d_{b'}^0, d_{a'}^0 + d_b^0)) \quad (7)$$

where quantities  $\mathcal{I}_\Delta(\delta_1, \delta_2)$  are defined by

$$\begin{aligned} \mathcal{I}_\Delta(\delta_1, \delta_2) &= \frac{2}{\kappa_\Delta} \widetilde{I}_0(\delta_1, \delta_2) \\ &+ \frac{2}{\kappa_\Delta^2} \sum_{u=1}^{\Delta} (2^{u\delta_1} + 2^{u\delta_2}) 2^{-u} \frac{2 - 2^{-\Delta+u}}{2 - 2^{-\Delta}} (u - \eta_\Delta)(u - \eta_\Delta - \eta_{\Delta-u}) \widetilde{I}_u(\delta_1, \delta_2) \quad \text{if } \Delta < \infty, \\ \mathcal{I}_\infty(\delta_1, \delta_2) &= \widetilde{I}_0(\delta_1, \delta_2) + \sum_{u=1}^{\infty} (2^{u\delta_1} + 2^{u\delta_2}) 2^{-u} \widetilde{I}_u(\delta_1, \delta_2), \quad \text{if } \Delta = \infty. \end{aligned} \quad (8)$$

Quantities  $(G_{a,b}^{0(-1)})$  denote the  $(a, b)$ -th elements of  $\mathbf{G}^{0-1}$ .

The proof is given in Section B.

In univariate setting, we recover Theorem 5 of [Roueff and Taqqu \(2009a\)](#), using the equality

$$\sum_{v=0}^{\Delta-u} \frac{2^{-v}}{2 - 2^{-\Delta}} (u - \eta_\Delta)(u - v - \eta_\Delta) = \frac{2 - 2^{-\Delta+u}}{2 - 2^{-\Delta}} (u - \eta_\Delta)(u - \eta_\Delta - \eta_{\Delta-u}),$$

in (8). Remark that we also normalized by  $\sqrt{n}$  rather than  $\sqrt{N_X 2^{-j_0}}$ .

If the vector  $\mathbf{d}$  is constant equal to  $d$ , the resulting covariance is

$$\frac{\pi}{2 \log(2)^2} \mathcal{I}_\Delta(d, d, d, d) (\mathbf{G}^{0-1} \circ \mathbf{G}^0 + \mathbf{I}_p)^{-1}.$$

We recognize a usual form of the variance, similar for example to the ones given by [Lobato \(1999\)](#), [Shimotsu \(2007\)](#) and [Düker and Pipiras \(2019\)](#) with Fourier-based Whittle estimators. Remark that they used a different approximation of the spectral density at the zero frequency. [Lobato \(1999\)](#) and [Shimotsu \(2007\)](#) considered respectively  $G_{a,b}^0 = \Omega_{a,b} e^{i\pi(d_a - d_b)/2}$  and  $\mathbf{G}^0 = \mathbf{\Omega}$ . [Düker and Pipiras \(2019\)](#)'s model is more general and does not suppose a linear representation of the time series. Their result is valid for a general form of matrix  $\mathbf{\Omega}$ . Remark also that [Baek et al \(2020\)](#) establish asymptotic normality of estimators in a bivariate model with possible co-integration.

Consider the bivariate setting with  $\mathbf{\Omega} = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$  and  $d_1 = d_2 = d$ . The variance matrix is equal to

$$(\mathbf{G}^{-1} \circ \mathbf{G} + \mathbf{I}_p)^{-1} 2 \sigma^2(d, j_1 - j_0) = \begin{pmatrix} 1 - \rho^2/2 & \rho^2/2 \\ \rho^2/2 & 1 - \rho^2/2 \end{pmatrix} \sigma^2(d, j_1 - j_0),$$

where  $\sigma^2(d, j_1 - j_0)$  denotes the variance of univariate components. This result proves that we reduce the variance when we do multivariate estimation rather than univariate estimation. A similar conclusion was provided for Fourier-based estimation by [Lobato \(1999\)](#) and [Nielsen \(2011\)](#). [Achard and Gannaz \(2019\)](#) support this assertion on simulated data. In real data application, [Achard and Gannaz \(2016b\)](#) also establish that the multivariate approach performs better than the univariate one, comparing their application on fMRI data where subjects were scanned twice.

Even if the form of the variance in Theorem 3.1 seems complicated, it would be easily computable once the quantities  $\mathbf{\Omega}$  is given. Yet it prevents from building directly a test procedure on parameters  $\mathbf{d}$ .

We now study the asymptotic behavior of the estimation of the long-run covariance. We give a result on the estimation of  $\mathbf{G}^0 = \mathbf{G}(\mathbf{d}^0)$  by  $\widehat{\mathbf{G}}(\widehat{\mathbf{d}})$ , defined respectively in (5) and in (6).

**Theorem 3.2.** *Suppose assumptions (A), (W1) to (W5) hold. Let*

$$j_1 - j_0 \rightarrow \Delta \in \{1, \dots, \infty\}, \quad j_0(N_X 2^{-j_0})^{-1/8} \rightarrow 0 \text{ and } N_X 2^{-j_0(1+2\beta)} \rightarrow 0.$$

*Then  $\text{vect} \left( \sqrt{n} \left( \widehat{\mathbf{G}}_{a,b}(\widehat{\mathbf{d}}) - \mathbf{G}_{a,b}^0 \right), a, b = 1, \dots, n \right)$  converges in distribution to a centered Gaussian distribution with a variance equal to  $\mathbf{W}^{\mathbf{G}(\Delta)}$ , with*

$$W_{(a,b),(a',b')}^{\mathbf{G}(\Delta)} = 2\pi \left( \mathbf{G}_{a,a'}^0 \mathbf{G}_{b,b'}^0 \mathcal{I}_{\Delta}^{\mathbf{G}}(d_a + d_{a'}, d_b + d_{b'}) + \mathbf{G}_{a,b'}^0 \mathbf{G}_{b,a'}^0 \mathcal{I}_{\Delta}^{\mathbf{G}}(d_a + d_{b'}, d_b + d_{a'}) \right)$$

where

$$\begin{aligned} \mathcal{I}_{\Delta}^{\mathbf{G}}(\delta_1, \delta_2) &= \widetilde{I}_0(\delta_1, \delta_2) + \sum_{u=1}^{\Delta} (2^{u\delta_1} + 2^{u\delta_2}) 2^{-u} \frac{2 - 2^{-\Delta-u}}{2 - 2^{-\Delta}} \widetilde{I}_u(\delta_1, \delta_2) && \text{if } \Delta < \infty, \\ \mathcal{I}_{\infty}^{\mathbf{G}}(\delta_1, \delta_2) &= \widetilde{I}_0(\delta_1, \delta_2) + \sum_{u=1}^{\infty} (2^{u\delta_1} + 2^{u\delta_2}) 2^{-u} \widetilde{I}_u(\delta_1, \delta_2) && \text{if } \Delta = \infty. \end{aligned}$$

The proof is given in Section C.

As previously, we can state a similar result for correlations.

**Corollary 3.3.** *Let  $(a, b) \in \{1, \dots, p\}^2, a \neq b$ . Define*

$$\widehat{r}_{a,b} = \frac{\widehat{\mathbf{G}}_{a,b}(\widehat{\mathbf{d}})}{\sqrt{\widehat{\mathbf{G}}_{a,b}(\widehat{\mathbf{d}}) \widehat{\mathbf{G}}_{b,b}(\widehat{\mathbf{d}})}} \text{ and } r_{a,b} = \frac{\mathbf{G}_{a,b}^0}{\sqrt{\mathbf{G}_{a,b}^0 \mathbf{G}_{b,b}^0}}.$$

*Then, under conditions of Theorem 3.2,*

$$\sqrt{n} (\widehat{r}_{a,b} - r_{a,b}) \xrightarrow{j \rightarrow \infty} \mathcal{N} \left( 0, \widetilde{W}_{a,b}^{\mathbf{G}(\Delta)}(j) \right)$$

with

$$\begin{aligned} \widetilde{W}_{a,b}^{\mathbf{G}(\Delta)} &= 2\pi \left( \mathcal{I}_{\Delta}(2d_a, 2d_b) + \mathcal{I}_{\Delta}(d_a + d_b, d_a + d_b) (r_{a,b}^2 + r_{a,b}^4) \right. \\ &\quad \left. - (\mathcal{I}_{\Delta}(2d_a, 2d_b) + \mathcal{I}_{\Delta}(2d_b, d_a + d_b)) 2r_{a,b}^2 - (\mathcal{I}_{\Delta}(2d_a, 2d_a) + \mathcal{I}_{\Delta}(2d_b, 2d_b)) r_{a,b}^2 / 2 \right). \end{aligned}$$

In particular, when  $d_a = d_b$ ,

$$\sqrt{n}(\widehat{r}_{a,b} - r_{a,b}) \xrightarrow[j \rightarrow \infty]{\mathcal{L}} \mathcal{N}(0, 2\pi \mathcal{I}_{\Delta}(2d_a, 2d_a)(1 - r_{a,b}^2)^2).$$

The proof does not present difficulties and it is omitted. The covariance structure of  $\text{vect}(\widehat{r}_{a,b}, a, b = 1, \dots, p)$  can also be deduced from Theorem 3.2 but it is not displayed here.

The result is very similar than the one presented in Corollary 2.3. Remark that, for all  $(a, b) \in \{1, \dots, p\}^2$ , the sequence  $(\rho_{a,b}(j))_{j \geq 0}$  converges to  $r_{a,b}$  as  $j$  goes to infinity. The strength of Corollary 3.3 is that all the scales are used to estimate  $r_{a,b}$ , which reduces the variance.

Similarly to Corollary 2.3, the variance depends on parameters  $\mathbf{d}$ . Hence a test procedure cannot be directly derived. [Düker and Pipiras \(2019\)](#) propose a global test for non-connectivity. A similar test would consider  $(H_0) \forall a \neq b, r_{a,b} = 0$  against  $(H_1) \exists a \neq b, r_{a,b} = 0$ . In our setting, the variance of  $r_{a,b}$  under the null hypothesis is equal to  $2\pi \mathcal{I}_{\Delta}(2d_a, 2d_b)$ , which still depends of parameters  $d_a$  and  $d_b$ .

We can go further than Theorem 3.1 and Theorem 3.2 by giving the joint distribution of estimates  $\widehat{\mathbf{d}}$  and  $\widehat{\mathbf{G}}$ .

**Proposition 3.4.** *Suppose assumptions of Theorem 3.2 hold.*

Let  $\mathbf{T} = \left( \widehat{\mathbf{d}} - \mathbf{d}^0, \text{vect} \left( \left( \widehat{\mathbf{G}}_{a,b}(\mathbf{d}^0) - \mathbf{G}_{a,b}^0 \right), a, b = 1, \dots, n \right) \right)$ .

Then  $\sqrt{n} \mathbf{T}$  converges in distribution to a centered Gaussian distribution.

A proof is given in Section D. An explicit form of the covariance term is given in (36)-(37). It is not displayed here to gain in clarity.

[Baek et al \(2020\)](#) and [Düker and Pipiras \(2019\)](#) also obtained that the estimates of long-memory and of long-run covariance converges jointly to a Gaussian distribution in a Fourier-based Whittle estimation framework. As stated before, they consider a more general model, allowing for a general form of the matrix  $\Omega$ .

## 4 Illustration on real data

We choose here to illustrate the asymptotic Gaussian behavior on real data either than on simulations. We consider fMRI recordings on a dead rat. The dataset is freely available at [10.5281/zenodo.2452871](https://zenodo.org/record/2452871) ([Becq et al, 2020a,b](#)). The duration of the scanning is 30 minutes with a time repetition of 0.5 second so that 3,600 time points are available at the end of experience. After preprocessing as described in [Pawela et al \(2008\)](#), we extracted 51 time series, each time series being associated with a brain region of the rat. We expect that the long-memory dependence and the long-run covariance are closely linked to the brain activity and not to recording artifacts. We would like to check this assertion on the dataset. This means that we expect  $\mathbf{d}^0 = 0$  and a diagonal matrix  $\mathbf{G}^0$ .

Suppose  $\mathbf{d}^0 = \mathbf{0}$  and  $\mathbf{G}^0$  is diagonal. Theorem 3.1 and Corollary 3.3, with  $\Delta = \infty$ , give

$$\sqrt{n} \hat{\mathbf{d}} \xrightarrow{\mathcal{L}} \mathcal{N}\left(\mathbf{0}, \frac{\pi}{2 \log(2)^2} \mathcal{I}_\Delta(0,0) \mathbf{I}_p\right),$$

$$\forall a, b = 1, \dots, p, a \neq b, \sqrt{n} \hat{r}_{a,b} \xrightarrow{\mathcal{L}} \mathcal{N}(0, 2\pi \mathcal{I}_\Delta^G(0,0)).$$

Figure 1 shows that the estimations of  $\hat{\mathbf{d}}$  and the estimated correlations ( $\hat{r}_{a,b}$ ) are closed from Gaussian distributions. Observed and theoretical standard deviations are given in Table 1. Theoretical standard deviations have the same order of magnitude than empirical observations. Theory slightly overestimates the empirical standard deviations of the estimations of long-run correlation. These observations tend to confirm that long-memory and long-run covariance result from brain activity.

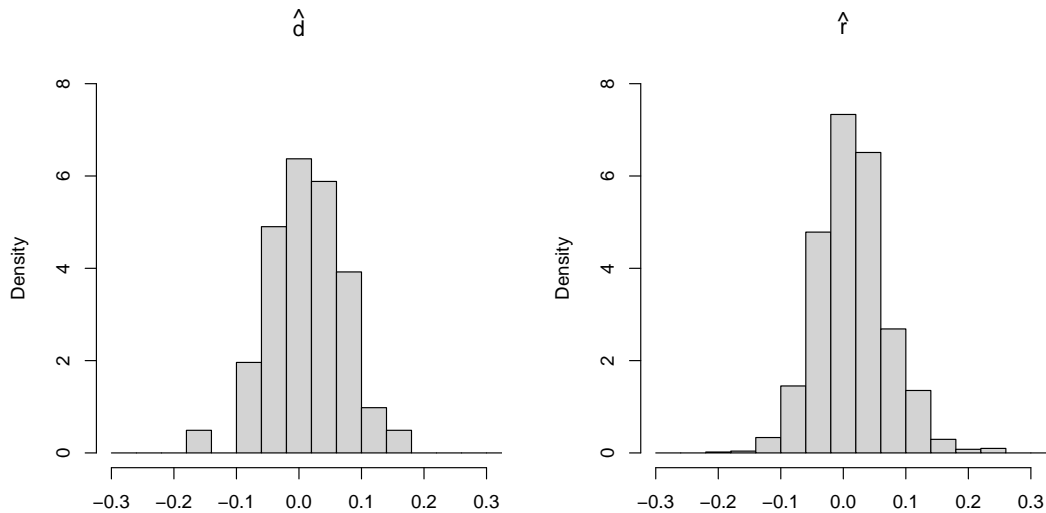


Figure 1: Histograms of long-memory estimators  $\hat{\mathbf{d}}$  (left) and of long-run correlation estimation  $\{\hat{r}_{a,b}, a, b = 1, \dots, p, a \neq b\}$  (right).

Parameter	Observed standard deviation	Theoretical with finite $\Delta$	Theoretical with $\Delta = \infty$
Long-memory $\hat{\mathbf{d}}$	0.061	0.054	0.061
Correlation $(\hat{r}_{a,b})_{a,b}$	0.057	0.083	0.084

Table 1: Standard deviations of the estimators of the parameters on the rat dataset. Estimation was conducted with  $j_0 = 4$ ,  $\Delta = 4$  and  $n = 405$  wavelet coefficients.

## Conclusion

In this paper, we consider a multivariate process with long-memory properties, with a linear representation. We first establish that the covariance between wavelet coefficients is asymptotically

Gaussian. The variance is explicitly given, and the convergence is established under mild assumptions on the wavelet transform and on the process. Asymptotic normality for the wavelet-based Whittle estimators defined in [Achard and Gannaz \(2016a\)](#) is also provided.

An essential point for real data application is the development of statistical tests, on the long-memory parameters and on the long-run covariance. For example, as illustrated here on a real data example, these characteristics are intrinsically related to the brain activity recordings in neuroscience. Some works highlighted that their distributions can be modified by pathologies (see e.g. [Maxim et al \(2005\)](#) and [Achard et al \(2012\)](#)). Statistical tests may be useful to assess rigorously such observations. The variances obtained in the present work show that the estimations are highly correlated. Additional work is necessary to propose test procedures.

## A Proof of Theorem 2.1

As the wavelet  $\psi$  admits  $M$  vanishing moments, at each scale  $j \geq 0$ , the associated filter  $\mathbb{H}_j$  is factorized as  $\mathbb{H}_j(\lambda) = (1 - e^{i\lambda})^M \tilde{\mathbb{H}}_j(\lambda)$ , with  $\tilde{\mathbb{H}}_j$  trigonometric polynomial,  $\tilde{\mathbb{H}}_j(\lambda) = \sum_{t \in \mathbb{Z}} \tilde{h}_{j,t} e^{it\lambda}$ .

Since  $M \geq D$ , the wavelet coefficients can be written as

$$W_{j,k}(a) = \sum_{t \in \mathbb{Z}} \tilde{h}_{j,2^j k - t} (\Delta^D X_a)(t) = \sum_{t \in \mathbb{Z}} \underbrace{\mathbf{B}_a(j, 2^j k - t)}_{1 \times p} \underbrace{\varepsilon(t)}_{p \times 1},$$

where  $\mathbf{B}_\ell(j, 2^j k - t) = \tilde{h}_{j,2^j k - t} (\Delta^{M-D} \mathbf{A}_a)(t)$ . For all  $a = 1, \dots, p$ , the sequence  $\{\mathbf{B}_a(j, u)\}_{u \in \mathbb{Z}}$  belongs to  $\ell^2(\mathbb{Z})$ . Define

$$A_{a,m}^*(\lambda) = (2\pi)^{-1/2} \sum_{t \in \mathbb{Z}} A_{a,m}(t) e^{-i\lambda t}, \quad \lambda \in \mathbb{R}.$$

Let  $j \geq 0$  and  $k \in \{0, \dots, n_j - 1\}$ . Following [Roueff and Taqqu \(2009a\)](#), we introduce a new indexing. This indexing is motivated by the study of the asymptotic normality for the vector  $(\hat{\gamma}_{j_0}(a, b), \dots, \hat{\gamma}_{j_1}(a, b))$  and it would not be necessary if we were interested only in  $\hat{\gamma}_j(a, b)$ . Let  $u = j - j_0$ ,  $u \in \{0, \dots, \Delta\}$ , and define  $(i, s)$  such that  $k = 2^{\Delta-u}(s-1) + i$ , with  $i \in \{2^{\Delta-u}, \dots, 2^{\Delta-u+1} - 1\}$  and  $s \in \mathbb{Z}$ . We have  $2^j k = 2^{j_1}(s-1 + 2^{u-\Delta}i)$ . Index  $i$  varies from 1 to  $N = 2^{\Delta+1} - 1$  and each couple  $(j, k)$  corresponds to a unique couple  $(i, s)$ . We can express wavelet coefficients in an uniform way:

$$2^{-j_0 d_a} W_{j,k}(a) = \sum_{t \in \mathbb{Z}} \sum_{a'=1}^p v_{i,j_0}^{(a,a')} (2^{j_1} s - t) \varepsilon_{a'}(t)$$

with

$$v_{i,j_0}^{(a,a')} (t) = 2^{-j_0 d_a} B_{a,a'}(j, 2^j(i - 2^{j_1-j}) + t), \quad j = j_1 - \lfloor \log_2(i) \rfloor,$$

where  $\lfloor \log_2(i) \rfloor = \Delta - u$  denotes the integer part of  $\log_2(i)$ . Denote

$$Z_{i,s,j_0}(a) = \sum_{t \in \mathbb{Z}} \sum_{a'=1}^p v_{i,j_0}^{(a,a')} (2^{j_1} s - t) \varepsilon_{a'}(t) = 2^{-j_0 d_a} W_{j, 2^{-\lfloor \log_2(i) \rfloor} (s-1) + i}(a).$$

For all  $u = 0, \dots, \Delta$ , denoting  $j = j_0 + u$ , the empirical variance satisfies

$$\begin{aligned} 2^{-j_0(d_a+d_b)}\widehat{\sigma}_{a,b}(j) &= \frac{1}{n_j} \sum_{k=0}^{n_j-1} 2^{-j_0 d_a} W_{j,k}(a) 2^{-j_0 d_b} W_{j,k}(b) \\ &= \frac{\sqrt{n_{j_1}}}{n_j} \sum_{i=2^{\Delta-u}}^{2^{\Delta-u+1}-1} \underbrace{n_{j_1}^{-1/2} \sum_{s=0}^{n_{j_1}-1} Y_{i,s,j_0}(a,b)}_{S_{i,j_0}(a,b)} + n_j^{-1} \underbrace{\sum_{i=2^{\Delta-u}}^{T_\psi(2^{\Delta-u}-1)} Y_{i,n_{j_1},j_0}(a,b)}_{R_j(a,b)}, \end{aligned}$$

with

$$Y_{i,s,j_0}(a,b) = Z_{i,s,j_0}(a)Z_{i,s,j_0}(b).$$

Indeed, when  $s \in \{0, \dots, n_{j_1} - 1\}$  and  $i \in \{2^{\Delta-u}, \dots, 2^{\Delta-u+1} - 1\}$ , then  $k = 2^{\Delta-u}(s-1) + i$  varies in  $\{0, \dots, 2^{\Delta-u}n_{j_1} - 1\}$ , and when  $s = n_{j_1}$  and  $i \in \{2^{\Delta-u}, \dots, T_\psi(2^{\Delta-u}-1)\}$ , then  $k$  varies from  $2^{\Delta-u}n_{j_1}$  to  $2^{\Delta-u}(n_{j_1}-1) + T_\psi(2^{\Delta-u}-1) = 2^{-j}(N_X - T_\psi + 1) - T_\psi = n_j - 1$ .

The vector of empirical correlations at different scales satisfies

$$2^{-j_0(d_a+d_b)} \begin{pmatrix} \widehat{\sigma}_{a,b}(j_0) \\ \vdots \\ \widehat{\sigma}_{a,b}(j_1) \end{pmatrix} = \sqrt{n_{j_1}} \mathbf{A}_n \begin{pmatrix} S_{1,j_0}(a,b) \\ \vdots \\ S_{N,j_0}(a,b) \end{pmatrix} + \begin{pmatrix} R_{j_0}(a,b) \\ \vdots \\ R_{j_1}(a,b) \end{pmatrix}$$

with

$$\mathbf{A}_n = \begin{pmatrix} 0 \dots & \dots & \dots & 0 & \overbrace{n_{j_0}^{-1} \dots n_{j_0}^{-1}}^{2^\Delta \text{ times}} \\ 0 \dots & \dots & 0 & \overbrace{n_{j_0+1}^{-1} \dots n_{j_0+1}^{-1}}^{2^{\Delta-1} \text{ times}} & 0 \dots 0 \\ \vdots & & & \vdots & \vdots \\ n_{j_1}^{-1} 0 \dots & \dots & \dots & \dots & \dots 0 \end{pmatrix}$$

Let  $\{u_{a,b}, a, b = 1, \dots, p\}$  be a given real vector. We want to establish that  $\sum_{a,b=1,\dots,p} u_{a,b} \widehat{\sigma}_{a,b}(j)$  is asymptotically Gaussian. We will establish that the vector  $(\sum_{a,b=1,\dots,p} u_{a,b} S_{i,j_0}(a,b))_{i=1,\dots,N}$  is asymptotically Gaussian when  $j_0$  goes to infinity, while the terms  $(\sum_{a,b=1,\dots,p} u_{a,b} R_j(a,b))$  are negligible.

## A.1 Preliminary results on $(v_{i,s,j_0}^{(a,a')})$

Introduce

$$v_{i,j_0}^{(a,a')*}(\lambda) = (2\pi)^{-1/2} \sum_{t \in \mathbb{Z}} v_{i,j_0}^{(a,a')}(t) e^{-i\lambda t}, \quad \lambda \in \mathbb{R}.$$

We have  $[v_{i,j_0}^{(a,a')}(2^{j_1}s - t)]^*(\lambda) = \overline{v_{i,j_0}^{(a,a')*}(\lambda)} e^{-i2^{j_1}s}$ . We first give some results on the behavior of  $(v_{i,j_0}^{(a,a')*})$ .

**Lemma A.1.** Define for all  $i, i' = 1, \dots, N$ ,

$$w_{i,i'}^{(a,b)*}(\lambda) = \Omega_{a,b} e^{-i \operatorname{sign}(\lambda) \pi(d_a - d_b)/2} 2^{(u-\Delta)/2 + (u'-\Delta)/2 + \Delta(d_a + d_b)} \overline{\widehat{\psi}(2^{u-\Delta}\lambda)} \widehat{\psi}(2^{u'-\Delta}\lambda) |\lambda|^{-d_a - d_b} e^{i(2^{u-\Delta}(i-1) - 2^{u'-\Delta}(i'-1))\lambda},$$

with  $u = \Delta - \lfloor \log_2(i) \rfloor$ ,  $u' = \Delta - \lfloor \log_2(i') \rfloor$ .

For all  $i = 1, \dots, N$ , for all  $\lambda \in \mathbb{R}$ ,

$$\exists \delta_v > 1/2, \forall j_0 \geq 0, \forall a, b = 1, \dots, p, \sup_{|\lambda| < \pi} 2^{-j/2} \left| v_{i,j_0}^{(a,b)*}(\lambda) \right| (1 + 2^j |\lambda|)^{\delta_v} < \infty \quad (10)$$

$$\exists \Phi_{i,j_0}^{(a,b)}(\lambda), \exists v_{i,\infty}^{(a,b)*}(\lambda), 2^{-j_1/2} v_{i,j_0}^{(a,b)*}(\lambda) (2^{-j_1} \lambda) e^{-i \Phi_{i,j_0}^{(a,b)}(2^{-j_1} \lambda)} \xrightarrow{j_0 \rightarrow 0} v_{i,\infty}^{(a,b)*}(\lambda) \quad (11)$$

$$2^{-j_1} \sum_{a'=1}^p \overline{v_{i,j_0}^{(a,a')*}(2^{-j_1} \lambda)} v_{i',j_0}^{(b,a')*}(2^{-j_1} \lambda) \xrightarrow{j_0 \rightarrow \infty} w_{i,i'}^{(a,b)*}(\lambda). \quad (12)$$

*Proof.* Let us give some preliminary results. Remark first that

$$\overline{v_{i,j_0}^{(a,a')*}(\lambda)} v_{i,j_0}^{(b,a')*}(\lambda) = 2^{-j_0(d_a + d_b)} (2\pi)^{-1} |\mathbb{H}_j(\lambda)|^2 \overline{A_{a,a'}^*(\lambda)} A_{b,a'}^*(\lambda)$$

and

$$\left| \sum_{a'=1}^p \overline{A_{a,a'}^*(\lambda)} A_{b,a'}^*(\lambda) \right| = |f_{a,b}(\lambda)| \leq C |\lambda|^{-d_a - d_b}.$$

The last inequality results from the fact that  $|f_{a,b}(\lambda)| \leq L(1 + \pi^\beta) \max_{\ell,m} |\Omega_{\ell,m}| |1 - e^{i\lambda}|^{-d_a - d_b}$  and  $|f_{a,b}^S(\lambda)| \leq L(1 + \pi^\beta)$ .

Moreover, [Moulines et al \(2007\)](#) established that

$$(2\pi)^{-1} |\mathbb{H}_j(\lambda)| \leq C 2^{j/2} \left| 2^j \lambda \right|^M (1 + 2^j |\lambda|)^{-\alpha - M} \quad (13)$$

where  $\alpha$  denotes the regularity of the wavelet function  $\psi$ . Thus, since  $d_a + d_b < 2M$ ,

$$\left| \sum_{a'=1}^p \overline{v_{i,j_0}^{(a,a')*}(\lambda)} v_{i,j_0}^{(b,a')*}(\lambda) \right| \leq C 2^{(j-j_0)(d_a + d_b)} 2^j (1 + 2^j |\lambda|)^{-2\alpha - d_a - d_b}.$$

When  $j - j_0$  is bounded,

$$\left| \sum_{a'=1}^p \overline{v_{i,j_0}^{(a,a')*}(\lambda)} v_{i,j_0}^{(b,a')*}(\lambda) \right| \leq C 2^j (1 + 2^j |\lambda|)^{-2\alpha - d_a - d_b} \quad (14)$$

Now, consider the proof of Lemma A.1. For all  $a' = 1, \dots, p$ ,

$$v_{i,j_0}^{(a,a')*}(\lambda) = 2^{-j_0 d_a} \mathbb{H}_{j_0+u}(\lambda) A_{a,a'}^*(\lambda) e^{i(i-2^{\Delta-u})2^{j_0+u}\lambda}.$$



Using equation (14),

$$\sup_{a'=1,\dots,p} 2^{-j} \left| v_{i,j_0}^{(a,a')*}(\lambda) \right|^2 \leq \sum_{b=1}^p 2^{-j} \left| v_{i,j_0}^{(a,a')*}(\lambda) \right|^2 \leq C(1+2^j|\lambda|)^{-2\alpha-2d_a}.$$

Inequality (10) follows as we have  $\alpha + d_a > 1/2$ .

Proposition 3 of [Moulines et al \(2007\)](#) states that that for all  $j \geq 0$ , for all  $\lambda$ ,

$$\left| \mathbb{H}_j(\lambda) - 2^{j/2} \widehat{\phi}(\lambda) \overline{\widehat{\psi}(2^j \lambda)} \right| \leq C 2^{j/2-j\alpha} |\lambda|^M. \quad (15)$$

Let  $\Phi_{i,j_0}(\lambda)$  be the argument of  $v_{i,j_0}^{(a,b)*}(\lambda)$ . We have

$$\begin{aligned} & \left| 2^{-j_1/2} v_{i,j_0}^{(a,b)*}(2^{-j_1} \lambda) e^{-i\Phi_{i,j_0}^{(a,b)}(2^{-j_1} \lambda)} - 2^{-\Delta/2+\Delta d_a} \left| \widehat{\phi}(2^{-j_1} \lambda) \widehat{\psi}(2^{u-\Delta} \lambda) 2^{-j_1 d_a} A_{a,b}^*(2^{-j_1} \lambda) \right| \right| \\ & \leq C 2^{j(1/2-\alpha-M)+(u-\Delta)M} |\lambda|^M 2^{-\Delta/2+\Delta d_a} \left| 2^{-j_1 d_a} A_{a,b}^*(2^{-j_1} \lambda) \right|. \quad (16) \end{aligned}$$

Using the inequality

$$\left| 2^{-j_1 d_a} A_{a,b}^*(2^{-j_1} \lambda) \right|^2 \leq 2^{-2j_1 d_a} \sum_{a'=1}^p \left| A_{a,a'}^*(2^{-j_1} \lambda) \right|^2 = 2^{-2j_1 d_a} |f_{aa}(2^{-j_1} \lambda)| \leq C |\lambda|^{-2d_a},$$

and the fact that  $1/2 - \alpha - M < 0$ , we obtain that the bound in (16) goes to 0 when  $j_0$  goes to infinity. By continuity,  $|\widehat{\phi}(2^{-j_1} \lambda)|$  tends to  $|\widehat{\phi}(0)| = 1$  when  $j_0$  goes to infinity. We conclude (11) by Assumption **(A)**.

Inequalities (15) and (13) lead to:

$$\begin{aligned} & \left| \overline{\mathbb{H}_j(\lambda)} \mathbb{H}_{j'}(\lambda) - 2^{j/2+j'/2} |\widehat{\phi}(\lambda)| \widehat{\psi}(2^j \lambda) \overline{\widehat{\psi}(2^{j'} \lambda)} \right| \\ & \leq |\mathbb{H}_j(\lambda)| \left| \mathbb{H}_{j'}(\lambda) - 2^{j'/2} \widehat{\phi}(\lambda) \overline{\widehat{\psi}(2^{j'} \lambda)} \right| + |\mathbb{H}_{j'}(\lambda)| \left| \mathbb{H}_j(\lambda) - 2^{j/2} \widehat{\phi}(\lambda) \overline{\widehat{\psi}(2^j \lambda)} \right| \\ & \leq C 2^{j/2+j'/2} 2^{-j\alpha-j'\alpha} |\lambda|^{2M}. \end{aligned}$$

Using the equality  $\sum_{a'=1}^p \overline{A_{a,a'}^*(\lambda)} A_{b,a'}^*(\lambda) = f_{a,b}(\lambda)$ , we get:

$$\begin{aligned} & \left| 2^{-j_1} \sum_{a'=1}^p \overline{v_{i,j_0}^{(a,a')*}(2^{-j_1} \lambda)} v_{i',j_0}^{(b,a')*}(2^{-j_1} \lambda) \right. \\ & \quad \left. - 2^{-j_1-j_0 d_a-j_0 d_b+j/2+j'/2} |\widehat{\phi}(2^{-j_1} \lambda)| \widehat{\psi}(2^{j-j_1} \lambda) \overline{\widehat{\psi}(2^{j'-j_1} \lambda)} f_{a,b}(2^{-j_1} \lambda) e^{i(2^{u-\Delta} i - 2^{u'-\Delta} i') \lambda} \right| \\ & \leq 2^{-j_1} 2^{-j_0 d_a-j_0 d_b} \left| \overline{\mathbb{H}_j(2^{-j_1} \lambda)} \mathbb{H}_{j'}(2^{-j_1} \lambda) - 2^{j/2+j'/2} |\widehat{\phi}(2^{-j_1} \lambda)| \widehat{\psi}(2^{j-j_1} \lambda) \overline{\widehat{\psi}(2^{j'-j_1} \lambda)} \right| |f_{a,b}(2^{-j_1} \lambda)| \\ & \leq C 2^{-j_1} 2^{-j_0 d_a-j_0 d_b} 2^{(j+j')(1/2-\alpha-M)} 2^{(u+u'-2\Delta)M} |\lambda|^{2M} |2^{-j_1} \lambda|^{-d_a-d_b} \\ & \leq C' 2^{-(j+j')(\alpha+M)} 2^{(u+u'-2\Delta)(M+1/2)} 2^{\Delta(d_a+d_b)} |\lambda|^{2M-d_a-d_b}. \end{aligned}$$

The right-hand side goes to 0 when  $j_0$  goes to infinity uniformly for  $\lambda \in (-\pi, \pi)$ . As  $|\widehat{\phi}(2^{-j_1} \lambda)| \xrightarrow{j_0 \rightarrow \infty} |\widehat{\phi}(0)| = 1$  and as  $f_{a,b}(2^{-j_1} \lambda)$  satisfies approximation (1), we obtain convergence (12).  $\square$

## A.2 Asymptotic normality of $S_{i,j}(a, b)$

The steps of the proof of the asymptotic normality of  $\sum_{a,b} u_{a,b} S_{i,j}(a, b)$  are the following:

- We approximate  $\{Y_{i,j}(a, b), i = 1, \dots, N\}$  by a  $m$ -dependent process  $\{Y_{i,j}^{(m)}(a, b), i = 1, \dots, N\}$
- We establish that  $\{Y_{i,j}^{(m)}(a, b), i = 1, \dots, N\}$  is asymptotically normal when  $j$  goes to infinity, using Proposition 2 of Roueff and Taqqu (2009b).
- We obtain the asymptotic normality of  $\sum_{a,b} u_{a,b} S_{i,j_0}^{(m)}(a, b) = n_j^{-1} \sum_{s=0}^{n_j-1} Y_{i,s,j_0}(a, b)$ , thanks to Proposition 3 of Roueff and Taqqu (2009b).
- The asymptotic normality for  $\sum_{a,b} u_{a,b} S_{j_0}(a, b)$  is obtained by letting  $m$  goes to infinity.

### A.2.1 First step, approximation by a $m$ -dependent process

Following Roueff and Taqqu (2009b), we introduce a non-negative infinitely differentiable function  $H$  defined on  $\mathbb{R}$  such that  $H(0) = 1$  and  $H(t) = 0$  if  $|t| > 1/2$ . Denote by  $\widehat{H}$  its Fourier transform. There exists  $c_H > 0$  such that  $|\widehat{H}(\lambda)| \leq c_H |\lambda|^{-\delta_v-1}$  for all  $u \geq 1$  and  $\frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{H}(\lambda) d\lambda = H(0) = 1$ .

Let define

$$\begin{aligned} v_{i,j_0}^{(a,a',m)}(t) &= v_{i,j}^{(a,a')}(t) H(2^{-j_1} t/m), \quad t \in \mathbb{R}, \\ v_{i,j_0}^{(a,a',m)*}(\lambda) &= (2\pi)^{-1/2} \sum_{t \in \mathbb{Z}} v_{i,j_0}^{(a,m)}(t) e^{-i\lambda t} = \frac{m}{2\pi} \int_{-\infty}^{\infty} \widehat{H}(m\xi) v_{i,j}^{(a,a')*}(\lambda - 2^{-j_1} \xi) d\xi, \quad \lambda \in \mathbb{R}, \\ Z_{i,s,j_0}^{(m)}(a) &= \sum_{t \in \mathbb{Z}} \sum_{a'=1}^p v_{i,j}^{(a,a',m)}(2^{j_1} s - t) \varepsilon_{a'}(t). \end{aligned}$$

Then vectors  $\mathbb{Z}_{s,j_0}(a, b) = \left( Z_{1,s,j_0}^{(m)}(a), \dots, Z_{N,s,j_0}^{(m)}(a), Z_{1,s,j_0}^{(m)}(b), \dots, Z_{N,s,j_0}^{(m)}(b) \right)^T$  are  $m$ -dependent relatively to index  $s$ . That is, for all  $q \geq 1$ , for all  $(s_1, \dots, s_q)$  such that  $s_{r+1} \geq s_r + m$ , vectors  $\mathbb{Z}_{s_1,j_0}^{(m)}(a, b), \dots, \mathbb{Z}_{s_q,j_0}^{(m)}(a, b)$  are independent.

We shall first study  $(Y_{i,s,j_0}^{(m)}(a, b))_{i=0, \dots, N_u-1}$ , with  $Y_{i,s,j_0}^{(m)}(a, b) = Z_{i,s,j_0}^{(m)}(a) Z_{i,s,j_0}^{(m)}(b)$ , for  $s \geq 0$ .

### A.2.2 Second step, asymptotic normality for variables $Y^{(m)}$

We first give some results on the behavior of  $(v_{i,j_0}^{(a,a',m)})$  and  $(v_{i,j_0}^{(a,a',m)*})$ , similarly to Lemma A.1.

**Lemma A.2.** For all  $i = 1, \dots, N$ , for all  $m \geq 1$ , sequences  $\{v_{i,j_0}^{(a,b,m)}, a, b = 1, \dots, p, j_0 \geq 0\}$  verify the

following properties:

$$\forall j_0 \geq 0, \forall a = 1, \dots, p, \sum_{t \in \mathbb{Z}} \sum_{a'=1}^p v_{i,j_0}^{(a,a',m)} (2^{j_1} s - t)^2 < \infty \quad (17)$$

$$\sup_{a=1, \dots, p} \sup_{a'=1 \dots p} \sup_{t \in \mathbb{Z}} \left| v_{i,j_0}^{(a,a',m)} (2^{j_1} s - t) \right| \xrightarrow{j_0 \rightarrow \infty} 0 \quad (18)$$

And for all  $m \geq 1$ , for all  $(a, b) \in \{1, \dots, p\}^2$ , for all  $(i, i') \in \{1, \dots, N\}^2$  and  $(s, s') \in \{0, \dots, n_{j_1} - 1\}^2$ , there exists  $\Sigma_{(i,s),(i',s')}^{(m)}(a, b)$  such that

$$\sum_{t \in \mathbb{Z}} \sum_{a'=1}^p v_{i,j_0}^{(a,a',m)} (2^{j_1} s - t) v_{i',j_0}^{(b,a',m)} (2^{j_1} s' - t) \xrightarrow{j_0 \rightarrow \infty} \Sigma_{(i,s),(i',s')}^{(m)}(a, b). \quad (19)$$

*Proof.* Inequality (10) and Lemma 5 of [Roueff and Taqqu \(2009b\)](#) entail that

$$\left| v_{i,j_0}^{(a,a',m)*}(\lambda) \right| \leq C 2^{j/2} (1 + 2^j |\lambda|)^{-\delta_v}$$

with  $C$  not depending of  $\lambda$ ,  $m$ , nor  $j$ . By Parseval's theorem and a change of variable,

$$\sum_{t \in \mathbb{Z}} v_{i,j_0}^{(a,a',m)} (2^{j_1} s - t)^2 = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} |v_{i,j_0}^{(a,a',m)*}(\lambda)|^2 d\lambda < \frac{C}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (1 + |\lambda|)^{-2\delta_v} d\lambda,$$

which implies (17). Similarly, (18) is obtained with inequalities

$$\begin{aligned} \sup_{t \in \mathbb{Z}} \left| v_{i,j_0}^{(a,a',m)} (2^{j_1} s - t) \right| &\leq \sum_{t \in \mathbb{Z}} \left| v_{i,j_0}^{(a,a',m)} (2^{j_1} s - t) \right| \leq \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} |v_{i,j_0}^{(a,a',m)*}(\lambda)| d\lambda \\ &\leq \frac{C}{\sqrt{2\pi}} 2^{-j/2} \int_{-\infty}^{\infty} (1 + |\lambda|)^{-\delta_v} d\lambda. \end{aligned}$$

Recall that

$$v_{i,j_0}^{(a,a',m)*} (2^{-j_1} \lambda) = \frac{m}{2\pi} \int_{-\infty}^{\infty} \widehat{H}(m\tilde{\xi}) v_{i,j_0}^{(a,a')*} (2^{-j_1} (\lambda - \tilde{\xi})) d\tilde{\xi}.$$

By (11),  $v_{i,j_0}^{(a,a')*} (2^{-j_1} (\lambda - \tilde{\xi}))$  converges to  $v_{i,\infty}^{(a,a')*} (\lambda - \tilde{\xi})$ . We can apply dominated convergence and we obtain that  $v_{i,j_0}^{(a,a',m)*} (2^{-j_1} \lambda)$  converges to

$$v_{i,\infty}^{(a,a',m)*} (2^{-j_1} \lambda) = \frac{m}{2\pi} \int_{-\infty}^{\infty} \widehat{H}(m\tilde{\xi}) v_{i,\infty}^{(a,a')*} (\lambda - \tilde{\xi}) d\tilde{\xi}$$

when  $j_0$  goes to infinity. Denote

$$w_{i,i'}^{(a,b,m)*}(\lambda) = \lim_{j_0 \rightarrow \infty} 2^{-j_1} \sum_{a'=1}^p \frac{v_{i,j_0}^{(a,a',m)*} (2^{-j_1} \lambda) v_{i',j_0}^{(b,a',m)*} (2^{-j_1} \lambda)}{v_{i,j_0}^{(a,a',m)*} (2^{-j_1} \lambda)}. \quad (20)$$

As stated above,  $w_{i,i'}^{(a,b,m)*}(\lambda) = \sum_{a'=1}^p v_{i,\infty}^{(a,a',m)*}(\lambda) v_{i',\infty}^{(b,a',m)*}(\lambda)$ .

For all  $a, b = 1, \dots, p$ , using Parseval's inequality,

$$\begin{aligned} & \sum_{t \in \mathbb{Z}} \sum_{a'=1}^p v_{i,j_0}^{(a,a',m)}(2^{j_1} s - t) v_{i',j_0}^{(b,a',m)}(2^{j_1} s' - t) \\ &= \int_{-2^{j_1} \pi}^{2^{j_1} \pi} 2^{-j_1} \sum_{a'=1}^p v_{i,j_0}^{(a,a',m)*}(2^{-j_1} \lambda) v_{i',j_0}^{(b,a',m)*}(2^{-j_1} \lambda) e^{i(s-s')\lambda} d\lambda. \end{aligned}$$

Convergence (19) follows with

$$\Sigma_{(i,s),(i',s')}^{(m)}(a,b) = \int_{-\infty}^{\infty} w_{i,i'}^{(a,b,m)*}(\lambda) e^{-i(s-s')\lambda} d\lambda.$$

Convergence under the integral can indeed be applied thanks to dominated convergence.  $\square$

For all  $a = 1, \dots, p$ , let

$$\mathbb{Z}_{j_0}^{(m)}(a) = \left( \mathbb{Z}_{1,s,j_0}^{(m)}(a), \mathbb{Z}_{2,s,j_0}^{(m)}(a) \dots \mathbb{Z}_{N,s,j_0}^{(m)}(a) \right)_{s=0, \dots, n_{j_1-1}}^T.$$

By Proposition 2 of Roueff and Taqqu (2009b), we get that  $\left( \text{vect} \left( \mathbb{Z}_{j_0}^{(m)}(a), a = 1, \dots, p \right) \text{ vect} \left( \mathbb{Z}_{j_0}^{(m)}(a), a = 1, \dots, p \right) \right)$  converges in distribution to  $\left( \text{vect} \left( \tilde{\mathbb{Z}}_{j_0}^{(m)}(a), a = 1, \dots, p \right) \text{ vect} \left( \tilde{\mathbb{Z}}_{j_0}^{(m)}(a), a = 1, \dots, p \right) \right)$ , which follows a centered Gaussian distribution with

$$\text{Cov}(\tilde{\mathbb{Z}}_{i,s,j_0}^{(m)}(a), \tilde{\mathbb{Z}}_{i',s',j_0}^{(m)}(b)) = \tilde{\Sigma}_{(i,s),(i',s')}^{(m)}(a,b).$$

For  $s = 0, \dots, n_{j_1} - 1$ , denote

$$\mathbf{Y}_{s,j_0}^{(m)}(a,b) = \left( Y_{1,s,j_0}^{(m)}(a,b), \dots, Y_{N,s,j_0}^{(m)}(a,b) \right)^T$$

and  $\mathbf{Y}_{s,j_0}^{(m)} = \sum_{a,b=1, \dots, p} u_{a,b} \mathbf{Y}_{s,j_0}^{(m)}(a,b)$ . The continuous mapping theorem implies that  $\mathbf{Y}_{j_0}^{(m)} = (\mathbf{Y}_{s,j_0}^{(m)})_{s=0, \dots, n_{j_1}-1}$  converges in distribution to  $\tilde{\mathbf{Y}}^{(m)} = \left( \sum_{a,b=1, \dots, p} u_{a,b} \tilde{\mathbf{Y}}_{s,j_0}^{(m)}(a,b) \right)_{s=0, \dots, n_{j_1}-1}$  with  $\tilde{\mathbf{Y}}_{s,j_0}^{(m)}(a,b) = \left( \tilde{\mathbb{Z}}_{i,s}^{(m)}(a) \tilde{\mathbb{Z}}_{i,s}^{(m)}(b) \right)_{i=1, \dots, N}$ .

We now prove that conditions of Proposition 3 of Roueff and Taqqu (2009b) are satisfied.

**Lemma A.3.** For all  $m \geq 1$ ,

$$\sup_{i=1, \dots, N} \sup_{s \geq 0} \sup_{j_0 \geq 0} \mathbb{E}[Y_{i,s,j_0}^{(m)}] < \infty. \quad (21)$$

$$\forall s, s' \geq 0, \lim_{j_0 \rightarrow \infty} \text{Cov}(\mathbf{Y}_{s,j_0}^{(m)}, \mathbf{Y}_{s',j_0}^{(m)}) = \text{Cov}(\tilde{\mathbf{Y}}_s^{(m)}, \tilde{\mathbf{Y}}_{s'}^{(m)}) \quad (22)$$

$$\lim_{\ell \rightarrow \infty} \lim_{j_0 \rightarrow \infty} \text{Cov}(\ell^{-1/2} \sum_{s=0}^{\ell-1} \mathbf{Y}_{s,0}^{(m)}) = \Gamma^{(m)} \quad (23)$$

with  $\Gamma^{(m)} = \sum_{a,b=1,\dots,p} \sum_{a',b'=1,\dots,p} u_{a,b} u_{a',b'} \Gamma^{(m)}((a,b), (a',b'))$  and

$$\Gamma_{i,i'}^{(m)}((a,b), (a',b')) = 2\pi \int_{-\pi}^{\pi} \overline{\tilde{w}_{i,i'}^{(a,a',m)*}(\lambda)} \tilde{w}_{i,i'}^{(b,b',m)*}(\lambda) d\lambda + 2\pi \int_{-\pi}^{\pi} \overline{\tilde{w}_{i,i'}^{(a,b',m)*}(\lambda)} \tilde{w}_{i,i'}^{(b,a',m)*}(\lambda) d\lambda, \quad (24)$$

where  $\tilde{w}_{i,i'}^{(a,b,m)*}(\lambda) = \sum_{t \in \mathbb{Z}} w_{i,i'}^{(a,b,m)*}(\lambda + 2t\pi)$ , function  $w_{i,i'}^{(a,b,m)*}$  being defined in (20).

*Proof.* First, (21) follows from the fact that  $\mathbb{E}[Y_{i,s,j_0}^{(m)}] = \sum_{a,b=1,\dots,p} u_{a,b} \Sigma_{(i,s),(i,s)}^{(m)}(a,b)$ .

Vector  $(\tilde{Z}_s^{(m)}(a) \ \tilde{Z}_s^{(m)}(b) \ \tilde{Z}_{s'}^{(m)}(a') \ \tilde{Z}_{s'}^{(m)}(b'))^T$  follows a centered Gaussian distribution, with covariance given by  $\Sigma^{(m)}$ . We can therefore use Isserlis's theorem. We get:

$$\begin{aligned} & \mathbb{E}(\tilde{Y}_{i,s}^{(m)} \tilde{Y}_{i,s'}^{(m)}) \\ &= \sum_{a,b,a',b'} u_{a,b} u_{a',b'} \mathbb{E}(\tilde{Z}_{i,s}^{(m)}(a) \tilde{Z}_{i,s}^{(m)}(b) \tilde{Z}_{i,s'}^{(m)}(a') \tilde{Z}_{i,s'}^{(m)}(b')) \\ &= \sum_{a,b,a',b'} u_{a,b} u_{a',b'} \left[ \Sigma_{(i,s),(i,s)}^{(m)}(a,b) \Sigma_{(i',s'),(i',s')}^{(m)}(a',b') + \Sigma_{(i,s),(i',s')}^{(m)}(a,a') \Sigma_{(i,s),(i',s')}^{(m)}(b,b') \right. \\ & \quad \left. + \Sigma_{(i,s),(i',s')}^{(m)}(a,b') \Sigma_{(i,s),(i',s')}^{(m)}(b,a') \right]. \end{aligned}$$

It results that:

$$\begin{aligned} \text{Cov}(\tilde{Y}_{i,s}^{(m)}, \tilde{Y}_{i',s'}^{(m)}) &= \sum_{a,b,a',b'} u_{a,b} u_{a',b'} \left[ \Sigma_{(i,s),(i',s')}^{(m)}(a,a') \Sigma_{(i,s),(i',s')}^{(m)}(b,b') \right. \\ & \quad \left. + \Sigma_{(i,s),(i',s')}^{(m)}(a,b') \Sigma_{(i,s),(i',s')}^{(m)}(a',b') \right]. \end{aligned}$$

Additionally,

$$\begin{aligned} \mathbb{E}(Y_{i,s,j_0}^{(m)}) &= \sum_{a,b} u_{a,b} \left[ \sum_{a'} \sum_t v_{i,j_0}^{(a,a',m)}(2j_1s - t) v_{i,j_0}^{(b,a',m)}(2j_1s - t) \right] \\ &\xrightarrow{j_0 \rightarrow \infty} \sum_{a,b} u_{a,b} \Sigma_{(i,s),(i,s)}^{(m)}(a,b). \end{aligned}$$

We deduce that it is sufficient to prove that, when  $j_0$  goes to infinity,  $\mathbb{E}(Y_{i,s,j_0}^{(m)} Y_{i',s',j_0}^{(m)})$  converges to

$$\begin{aligned} \sum_{a,b,a',b'} u_{a,b} u_{a',b'} \left[ \Sigma_{(i,s),(i',s')}^{(m)}(a,a') \Sigma_{(i,s),(i',s')}^{(m)}(b,b') + \Sigma_{(i,s),(i',s')}^{(m)}(a,b') \Sigma_{(i,s),(i',s')}^{(m)}(a,b') \right. \\ \left. + \Sigma_{(i,s),(i,s)}^{(m)}(a,b) \Sigma_{(i',s'),(i',s')}^{(m)}(a',b') \right] \end{aligned}$$

to obtain equality (22).

We have

$$\begin{aligned}
& \mathbb{E}(Y_{i,s,j_0}^{(m)} Y_{i',s',j_0}^{(m)}) \\
&= \sum_{a,b,a',b'} u_{a,b} u_{a',b'} \sum_{a_1,a_2,a_3,a_4} \sum_{t_1,t_2,t_3,t_4} v_{i,j_0}^{(a,a_1,m)}(2^{j_1}s - t_1) v_{i,j_0}^{(b,a_2,m)}(2^{j_1}s - t_1) v_{i',j_0}^{(a',a_3,m)}(2^{j_1}s' - t_3) \\
&\quad v_{i',j_0}^{(b',a_4,m)}(2^{j_1}s' - t_4) \mathbb{E}[\varepsilon_{a_1}(t_1) \varepsilon_{a_2}(t_2) \varepsilon_{a_3}(t_3) \varepsilon_{a_4}(t_4)] \\
&= \sum_{a,b,a',b'} u_{a,b} u_{a',b'} \\
&\quad \left[ \left( \sum_{a_1} \sum_{t_1} v_{i,j_0}^{(a,a_1,m)}(2^{j_1}s - t_1) v_{i,j_0}^{(b,a_1,m)}(2^{j_1}s - t_1) \right) \left( \sum_{a_2} \sum_{t_2} v_{i',j_0}^{(a',a_2,m)}(2^{j_1}s' - t_2) v_{i',j_0}^{(b',a_2,m)}(2^{j_1}s' - t_2) \right) \right. \\
&\quad + \left( \sum_{a_1} \sum_{t_1} v_{i,j_0}^{(a,a_1,m)}(2^{j_1}s - t_1) v_{i',j_0}^{(a',a_1,m)}(2^{j_1}s' - t_1) \right) \left( \sum_{a_2} \sum_{t_2} v_{i,j_0}^{(b',a_2,m)}(2^{j_1}s' - t_2) v_{i',j_0}^{(b',a_2,m)}(2^{j_1}s' - t_2) \right) \\
&\quad + \left( \sum_{a_1} \sum_{t_1} v_{i,j_0}^{(a,a_1,m)}(2^{j_1}s - t_1) v_{i',j_0}^{(b',a_1,m)}(2^{j_1}s' - t_1) \right) \left( \sum_{a_2} \sum_{t_2} v_{i,j_0}^{(a',a_2,m)}(2^{j_1}s' - t_2) v_{i',j_0}^{(b',a_2,m)}(2^{j_1}s' - t_2) \right) \\
&\quad \left. - \tilde{R}_{j_0} \right]
\end{aligned}$$

with

$$\begin{aligned}
\tilde{R}_{j_0} &= \sum_{a_1,a_2,a_3,a_4} \sum_t v_{i,j_0}^{(a,a_1,m)}(2^{j_1}s - t) v_{i,j_0}^{(b,a_2,m)}(2^{j_1}s - t) v_{i',j_0}^{(a',a_3,m)}(2^{j_1}s' - t) v_{i',j_0}^{(b',a_4,m)}(2^{j_1}s' - t) \mu_{a_1,a_2,a_3,a_4} \\
&\quad - \sum_{a_1,a_2} \sum_t v_{i,j_0}^{(a,a_1,m)}(2^{j_1}s - t) v_{i,j_0}^{(b,a_1,m)}(2^{j_1}s - t) v_{i',j_0}^{(a',a_2,m)}(2^{j_1}s' - t) v_{i',j_0}^{(b',a_2,m)}(2^{j_1}s' - t) \\
&\quad - \sum_{a_1,a_2} \sum_t v_{i,j_0}^{(a,a_1,m)}(2^{j_1}s - t) v_{i,j_0}^{(b,a_2,m)}(2^{j_1}s - t) v_{i',j_0}^{(a',a_1,m)}(2^{j_1}s' - t) v_{i',j_0}^{(b',a_2,m)}(2^{j_1}s' - t) \\
&\quad - \sum_{a_1,a_2} \sum_t v_{i,j_0}^{(a,a_1,m)}(2^{j_1}s - t) v_{i,j_0}^{(b,a_2,m)}(2^{j_1}s - t) v_{i',j_0}^{(a',a_2,m)}(2^{j_1}s' - t) v_{i',j_0}^{(b',a_1,m)}(2^{j_1}s' - t).
\end{aligned}$$

Thus it is sufficient to prove that  $\tilde{R}_{j_0} \xrightarrow{j_0 \rightarrow \infty} 0$ . Cauchy-Schwarz's inequality yields to:

$$\begin{aligned}
|\tilde{R}_{j_0}| &\leq (\mu_\infty + 3) \sum_{a_1,a_2,a_3,a_4} \sum_t \left| v_{i,j_0}^{(a,a_1,m)}(2^{j_1}s - t) v_{i,j_0}^{(b,a_2,m)}(2^{j_1}s - t) v_{i',j_0}^{(a',a_3,m)}(2^{j_1}s' - t) v_{i',j_0}^{(b',a_4,m)}(2^{j_1}s' - t) \right| \\
&\leq (\mu_\infty + 3) \sup_{b_1} \sup_{b_2} \sup_{t \in \mathbb{Z}} \left| v_{i',j_0}^{(b_1,b_2,m)}(2^{j_1}s - t) \right|^2 \\
&\quad p \left( \sum_t \sum_{a_1} v_{i,j_0}^{(a,a_1,m)}(2^{j_1}s - t)^2 \right)^{1/2} \left( \sum_t \sum_{a_1} v_{i,j_0}^{(b,a_2,m)}(2^{j_1}s - t)^2 \right)^{1/2}.
\end{aligned}$$

Conclusion follows from (17) and (18).

It remains to study (23). We have

$$\mathbb{E} \left( \left( \sum_{s=0}^{\ell-1} Y_{i,s,j_0}^{(m)} \right) \left( \sum_{s'=0}^{\ell-1} Y_{i',s',j_0}^{(m)} \right) \right) = \sum_{s,s'=0,\dots,\ell-1} \mathbb{E} \left( Y_{i,s,j_0}^{(m)} Y_{i',s',j_0}^{(m)} \right).$$

When  $j_0$  goes to infinity, it converges to

$$\sum_{a,b,a',b'} u_{a,b} u_{a',b'} \sum_{s,s'=0,\dots,\ell-1} \left( \Sigma_{s,s'}^{(m)}(a,a') \Sigma_{s,s'}^{(m)}(b,b') + \Sigma_{s,s'}^{(m)}(a,b') \Sigma_{s,s'}^{(m)}(a,b) + \Sigma_{s,s}^{(m)}(a,b) \Sigma_{s',s'}^{(m)}(a',b') \right).$$

Thus,

$$\begin{aligned} & \text{Cov} \left( \sum_{s=0}^{\ell-1} Y_{i,s,j_0}^{(m)}, \sum_{s'=0}^{\ell-1} Y_{i',s',j_0}^{(m)} \right) \\ & \xrightarrow{j_0 \rightarrow \infty} \sum_{a,b,a',b'} u_{a,b} u_{a',b'} \sum_{s,s'=0,\dots,\ell-1} \left( \Sigma_{(i,s),(i',s')}^{(m)}(a,a') \Sigma_{(i,s),(i',s')}^{(m)}(b,b') + \Sigma_{(i,s),(i',s')}^{(m)}(a,b') \Sigma_{(i,s),(i',s')}^{(m)}(a,b) \right). \end{aligned}$$

Quantity  $\Sigma_{(i,s),(i',s')}^{(m)}(a,b)$  can be written as

$$\Sigma_{(i,s),(i',s')}^{(m)}(a,b) = \int_{-\pi}^{\pi} \tilde{w}_{i,i'}^{(a,b,m)*}(\lambda) e^{-i(s-s')\lambda} d\lambda.$$

Hence, setting  $v = s - s'$ ,

$$\begin{aligned} & \ell^{-1} \sum_{s,s'=0,\dots,\ell-1} \Sigma_{(i,s),(i',s')}^{(m)}(a,a') \Sigma_{(i,s),(i',s')}^{(m)}(b,b') \\ & = \sum_{v \in \mathbb{Z}} \ell^{-1} (\ell - |v|)_+ \left( \int_{-\pi}^{\pi} \tilde{w}_{i,i'}^{(a,a',m)*}(\lambda) e^{-i v \lambda} d\lambda \right) \left( \int_{-\pi}^{\pi} \tilde{w}_{i,i'}^{(b,b',m)*}(\lambda) e^{-i v \lambda} d\lambda \right), \end{aligned}$$

with  $(\ell - |v|)_+ = \ell - |v|$  if  $\ell - |v| \geq 0$  and 0 otherwise. Using Lemma A.5 it results that when  $\ell$  goes to infinity we have the convergence to

$$2\pi \int_{-\pi}^{\pi} \overline{\tilde{w}_{i,i'}^{(a,a',m)*}(\lambda)} \tilde{w}_{i,i'}^{(b,b',m)*}(\lambda) d\lambda.$$

Using similar arguments for the term  $\ell^{-1} \sum_{s,s'=0,\dots,\ell-1} \Sigma_{(i,s),(i',s')}^{(m)}(a,b) \Sigma_{(i,s),(i',s')}^{(m)}(a,b)$ , we obtain (23) and (24).  $\square$

With results of Lemma A.3, we can apply Proposition 3 of [Roueff and Taqqu \(2009b\)](#). We deduce that  $n^{-1/2} \mathbf{Y}_{j_0}^{(m)}$  converges in distribution to a  $\mathcal{N}(0, \Gamma^{(m)})$  with  $\Gamma^{(m)}$  defined in (23) and (24).

### A.2.3 Third step: asymptotic normality for the initial framework

For  $i = 0, \dots, N-1$ , when  $j_0$  goes to infinity,  $S_{i,j_0}^{(m)}(a,b) = n_j^{-1/2} \sum_{s=0}^{n_j-1} Y_{i,s,j_0}^{(m)}(a,b)$  converges to a  $\mathcal{N}(0, \Gamma^{(m)})$  distribution. We want to deduce a similar result for variables  $(S_{i,j_0}(a,b))$ .

Denote

$$\begin{aligned} w_{i,i'}^{(a,b)*}(\lambda) &= 2^{-j_1} \sum_{a'=1}^p v_{i,\infty}^{(a,a')*}(2^{-j_1}\lambda) v_{i',\infty}^{(b,a')*}(2^{-j_1}\lambda) \\ \Gamma_{i,i'}((a,b), (a',b')) &= 2\pi \int_{-\pi}^{\pi} \overline{\tilde{w}_{i,i'}^{((a,a'))*}(\lambda)} \tilde{w}_{i,i'}^{(b,b')*}(\lambda) d\lambda + 2\pi \int_{-\pi}^{\pi} \overline{\tilde{w}_{i,i'}^{(a,b')*}(\lambda)} \tilde{w}_{i,i'}^{(b,a')*}(\lambda) d\lambda, \end{aligned}$$

where  $\tilde{w}_{i,i'}^{(a,b)*}(\lambda) = \sum_{t \in \mathbb{Z}} w_{i,i'}^{(a,b)*}(\lambda + 2t\pi)$ . In order to apply Theorem 3.2 of Billingsley (1999), we prove the following properties.

**Lemma A.4.** For all  $i, i' = 0, \dots, N-1$ ,

$$\lim_{m \rightarrow \infty} \Gamma_{i,i'}^{(m)}((a, b), (a', b')) = \Gamma_{i,i'}((a, b), (a', b')). \quad (25)$$

$$\lim_{m \rightarrow \infty} \lim_{j_0 \rightarrow \infty} \text{Var} \left( n_{j_1}^{-1/2} \sum_{s=0}^{n_{j_1}-1} Y_{i,s,j_0}^{(m)} - n_{j_1}^{-1/2} \sum_{s=0}^{n_{j_1}-1} Y_{i,s,j_0} \right) = 0. \quad (26)$$

With properties of Lemma A.4, Theorem 3.2 of Billingsley (1999) entails that  $(n_{j_1}^{-1/2} \sum_{s=0}^{n_{j_1}-1} Y_{i,s,j_0})_{i=1, \dots, N}$  converges in distribution to  $\mathcal{N}(0, \Gamma)$  when  $j_0$  tends to infinity.

*Proof.* To study the limit of  $\Gamma^{(m)}(a, b)$  when  $m$  goes to infinity, we will inverse the limits, that is, prove that  $\Gamma((a, b), (a', b'))$  in (25) satisfies  $\Gamma_{i,i'}((a, b), (a', b')) = \lim_{j_0 \rightarrow \infty} \lim_{m \rightarrow \infty} \Gamma_{i,i',j_0}^{(m)}((a, b), (a', b'))$ , where

$$\Gamma_{i,i',j_0}^{(m)}((a, b), (a', b')) = 2\pi \int_{-\pi}^{\pi} \overline{\tilde{w}_{i,i',j_0}^{((a,a',m)*}(\lambda)}} \tilde{w}_{i,i',j_0}^{(b,b',m)*}(\lambda) d\lambda + 2\pi \int_{-\pi}^{\pi} \overline{\tilde{w}_{i,i',j_0}^{(a,b',m)*}(\lambda)}} \tilde{w}_{i,i',j_0}^{(b,a',m)*}(\lambda) d\lambda,$$

where  $\tilde{w}_{i,i',j_0}^{(a,b,m)*}(\lambda) = \sum_{t \in \mathbb{Z}} w_{i,i',j_0}^{(a,b,m)*}(\lambda + 2t\pi)$  and

$$w_{i,i',j_0}^{(a,b,m)*}(\lambda) = 2^{-j_1} \sum_{a'=1}^p v_{i,j_0}^{(a,a',m)*}(2^{-j_1}\lambda) v_{i',j_0}^{(b,a',m)*}(2^{-j_1}\lambda).$$

Remark that, as stated in equation (23),  $\lim_{j_0 \rightarrow \infty} \Gamma_{i,i',j_0}^{(m)}((a, b), (a', b')) = \Gamma_{i,i'}^{(m)}((a, b), (a', b'))$ .

Let

$$\begin{aligned} w_{i,i',j_0}^{(a,b)*}(\lambda) &= 2^{-j_1} \sum_{a'=1}^p v_{i,j_0}^{(a,a')*}(2^{-j_1}\lambda) v_{i',j_0}^{(b,a')*}(2^{-j_1}\lambda) \\ \Gamma_{i,i',j_0}((a, b), (a', b')) &= 2\pi \int_{-\pi}^{\pi} \overline{\tilde{w}_{i,i',j_0}^{(a,a')*}(\lambda)}} \tilde{w}_{i,i',j_0}^{(b,b')*}(\lambda) d\lambda + 2\pi \int_{-\pi}^{\pi} \overline{\tilde{w}_{i,i',j_0}^{(a,b')*}(\lambda)}} \tilde{w}_{i,i',j_0}^{(b,a')*}(\lambda) d\lambda, \end{aligned}$$

where  $\tilde{w}_{i,i',j_0}^{(a,b)*}(\lambda) = \sum_{t \in \mathbb{Z}} w_{i,i',j_0}^{(a,b)*}(\lambda + 2t\pi)$ .

Since  $v_{i,s,j_0}^{(a,b)*}$  is continuous, we can apply a convergence under the integral,

$$v_{i,j_0}^{(a,b,m)*}(\lambda) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{H}(u) v_{i,j_0}^{(a,b)*}(\lambda - u/m) du \xrightarrow{m \rightarrow \infty} v_{i,j_0}^{(a,b)*}(\lambda) \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{H}(u) du = v_{i,j_0}^{(a,b)*}(\lambda).$$

Remark additionally that

$$\forall j_0 \geq 0, \forall a, b = 1, \dots, p, \sup_{m \geq 1} \sup_{|\lambda| < \pi} 2^{-j/2} \left| v_{i,s,j_0}^{(a,b,m)*}(\lambda) \right| (1 + 2^j |\lambda|)^{\delta_v} < \infty.$$



Consequently  $\Gamma_{i,i',j_0}^{(m)}((a,b),(a',b'))$  converges uniformly in  $m$  to  $\Gamma_{i,i',j_0}((a,b),(a',b'))$ .

Moreover,  $\Gamma_{i,i',j_0}^{(m)}((a,b),(a',b'))$  converges to  $\Gamma_{i,i'}^{(m)}((a,b),(a',b'))$  when  $j_0$  goes to infinity, the convergence under the integral being obtained by continuity.

It results that

$$\lim_{m \rightarrow \infty} \lim_{j_0 \rightarrow \infty} \Gamma_{i,i',j_0}^{(m)}((a,b),(a',b')) = \lim_{j_0 \rightarrow \infty} \lim_{m \rightarrow \infty} \Gamma_{i,i',j_0}^{(m)}((a,b),(a',b')).$$

Convergence (25) follows.

To obtain convergence (26), we can establish that  $\text{Cov}(Y_{i,s,j_0}^{(m)}(a,b), Y_{i',s',j_0}(a',b'))$  converges as  $j_0$  goes to infinity to

$$2\pi \int_{-\pi}^{\pi} \overline{\tilde{w}_{i,i'}^{(a,a',m)*}(\lambda)} \tilde{w}_{i,i'}^{(b,b',m)*}(\lambda) d\lambda + 2\pi \int_{-\pi}^{\pi} \overline{\tilde{w}_{i,i'}^{(a,b',m)*}(\lambda)} \tilde{w}_{i,i'}^{(b,a',m)*}(\lambda) d\lambda,$$

with  $\tilde{w}_{i,i'}^{(a,b')*(m)*}(\lambda) = \sum_{t \in \mathbb{Z}} w_{i,i'}^{(a,b')*(m)*}(\lambda + 2t\pi)$  and  $w_{i,i'}^{(a,b')*(m)*}(\lambda)$  defined as

$$w_{i,i',j_0}^{(a,b)(m)*}(\lambda) = 2^{-j_1} \sum_{a'=1}^p v_{i,j_0}^{(a,a',m)*}(2^{-j_1}\lambda) v_{i',j_0}^{(b,a')*}(2^{-j_1}\lambda).$$

The proof is very similar to the one carried out for proving (23) and it is thus omitted.

Next we can study the limit as  $m$  goes to infinity and state that  $\text{Cov}(Y_{i,s,j_0}^{(m)}(a,b), Y_{i',s',j_0}(a',b'))$  converges uniformly to  $\Gamma_{i,i',j_0}((a,b),(a',b'))$  when  $m$  goes to infinity. As above we can inverse the limits, so

$$\begin{aligned} \lim_{m \rightarrow \infty} \lim_{j_0 \rightarrow \infty} \text{Cov}(Y_{i,j_0}^{(m)}(a,b), Y_{i',j_0}(a',b')) &= \lim_{j_0 \rightarrow \infty} \lim_{m \rightarrow \infty} \text{Cov}(Y_{i,j_0}^{(m)}(a,b), Y_{i',j_0}(a',b')) \\ &= \Gamma_{i,i'}((a,b),(a',b')). \end{aligned}$$

This concludes the proof. □

### A.3 Study of $R_{j'}(a,b)$

We want to prove that  $\frac{n_j^{1/2}}{(T_\psi-1)(N_u-1)} R_{j_0}(a,b) = n_j^{-1/2} \frac{1}{(T_\psi-1)(N-1)} \sum_{i=0}^{(T_\psi-1)(N-1)-1} Y_{i,n_j,j}(a,b)$  goes to zero as  $j$  goes to infinity. We have

$$\mathbb{E}[|Y_{i,n_j,j_0}(a,b)|] \leq \sum_{a'=1}^p \left( \sum_{t \in \mathbb{Z}} v_{i,j_0}^{(a,a')} (2^{j_1} n_{j_1} - t)^2 \right)^{1/2} \left( \sum_{t \in \mathbb{Z}} v_{i,j_0}^{(b,a')} (2^{j_1} n_{j_1} - t)^2 \right)^{1/2}.$$

Using Parseval's equality and (10),

$$\begin{aligned} \mathbb{E}[|Y_{i,n_j,j_0}(a,b)|] &\leq \sum_{a'=1}^p \left( \int_{-\pi}^{\pi} |v_{i,j_0}^{(a,a')*}(\lambda)|^2 d\lambda \right)^{1/2} \left( \int_{-\pi}^{\pi} |v_{i,j_0}^{(b,a')*}(\lambda)|^2 d\lambda \right)^{1/2} \\ &\leq C \int_{\mathbb{R}} (1 + |\xi|)^{-2\delta_v} d\xi. \end{aligned}$$

Consequently,  $\mathbb{E}[|Y_{i,n_{j_1},j_0}|]$  is finite. Thus  $\frac{n_j^{1/2}}{(T_\psi-1)(N_u-1)}R_{j_0}(a,b) = O_{\mathbb{P}}(n_{j_1}^{-1/2})$  using Markov's inequality.

#### A.4 Expression of the covariance

To express the variance, we use equality A. The variance matrix  $V_{u,u'}^{(d)}((a,b),(a',b'))$  satisfies, for all  $0 \leq u' \leq u \leq \Delta$ ,

$$\begin{aligned} n_{j_0+u}^{1/2} n_{j_0+u'}^{1/2} V_{u,u'}^{(d)}((a,b),(a',b')) &= n_{j_1} n_{j_0+u}^{-1/2} n_{j_0+u'}^{-1/2} (\mathbf{A}_n \boldsymbol{\Gamma}((a,b),(a',b')) \mathbf{A}_n^T)_{u,u'} \\ &= n^{-1} n_{j_1} n_{j_0+u}^{-1} n_{j_0+u'}^{-1} \sum_{i=2^{\Delta-u}}^{2^{\Delta-u+1}-1} \sum_{i'=2^{\Delta-u'}}^{2^{\Delta-u'+1}-1} \Gamma_{i,i'}((a,b),(a',b')), \end{aligned} \quad (27)$$

with

$$\Gamma_{i,i'}((a,b),(a',b')) = \lim_{\ell \rightarrow \infty} \ell^{-1} \sum_{s=0}^{\ell} \sum_{s'=0}^{\ell} \Sigma_{(i,s),(i',s')}(a,a') \Sigma_{(i,s),(i',s')}(b,b') + \Sigma_{(i,s),(i',s')}(a,b') \Sigma_{(i,s),(i',s')}(b,a'). \quad (28)$$

Using (12) and dominated convergence,

$$\begin{aligned} \Sigma_{(i,s),(i',s')}(a,b) &= \int_{-\infty}^{\infty} w_{i,i'}^{(a,b)}(\lambda) e^{-i(s-s')\lambda} d\lambda \\ &= \Omega_{a,b} 2 \cos(\pi(d_a - d_b)/2) 2^{(u-\Delta)/2+(u'-\Delta)/2+\Delta} (d_a+d_b) \Xi_{(u,s),(u',s')}(a,b) \end{aligned} \quad (29)$$

with

$$\begin{aligned} \Xi_{(u,s),(u',s')}(a,b) &= \int_{-\infty}^{\infty} \overline{\widehat{\psi}(2^{u-\Delta}\lambda)} \widehat{\psi}(2^{u'-\Delta}\lambda) |\lambda|^{-d_a-d_b} e^{-i(2^{u-\Delta}(i-1)-2^{u'-\Delta}(i'-1))\lambda} e^{-i(s-s')\lambda} d\lambda, \\ &= 2^{(\Delta-u)(1-d_a-d_b)} \int_{-\infty}^{\infty} \overline{\widehat{\psi}(\lambda)} \widehat{\psi}(2^{u'-u}\lambda) |\lambda|^{-d_a-d_b} e^{-i(i+2^{\Delta-u}s)\lambda+i(2^{\Delta-u'}s'+i')2^{u'-u}\lambda} d\lambda \\ &= 2^{(\Delta-u)(1-d_a-d_b)} \int_{-\infty}^{\infty} g_{u'-u}(\lambda; d_a+d_b) e^{-i(k-2^{u'-u}k')\lambda} d\lambda \end{aligned}$$

with  $g_{u'-u}(\xi; \delta) = \overline{\widehat{\psi}(\lambda)} \widehat{\psi}(2^{u'-u}\lambda) |\lambda|^{-\delta}$ , and  $k = i + 2^{\Delta-u}(s-1)$ ,  $k' = i' + 2^{\Delta-u'}(s'-1)$ .

To get all values in  $\mathbb{Z}$  from  $k - 2^{u'-u}k'$ , we introduce  $\tau \in \{0, \dots, 2^{-(u'-u)} - 1\}$ . Then, when  $i$ ,  $i'$ ,  $s$  and  $s'$  vary respectively in  $\{2^{\Delta-u}, \dots, 2^{\Delta-u+1} - 1\}$ ,  $\{2^{\Delta-u'}, \dots, 2^{\Delta-u'+1} - 1\}$ ,  $\{0, \dots, \ell - 1\}$  and  $\{0, \dots, \ell - 1\}$ , quantity  $q = k - 2^{u'-u}k' + 2^{u'-u}\tau$  takes all relative integers values in  $\{-Q, \dots, Q\}$ , with  $Q = 2^{\Delta-u}(\ell - 1)$ .

We have

$$\begin{aligned}
& 2^{\Delta-u+1-1} 2^{\Delta-u'+1-1} \sum_{i=2^{\Delta-u}} \sum_{i'=2^{\Delta-u'}} \ell^{-1} \sum_{s=0}^{\ell-1} \sum_{s'=0}^{\ell-1} \Xi_{(u,s),(u',s')}(a,b) \Xi_{(u,s),(u',s')}(a',b') \\
&= \frac{Q}{\ell} \sum_{-Q \leq q \leq Q} \left(1 - \frac{q}{Q}\right)_+ 2^{(\Delta-u)(2-d_a-d_b-d_{a'}-d_{b'})} \left( \int_{-\infty}^{\infty} \sum_{\tau=0}^{2^{-(u'-u)}-1} g(\lambda; d_a + d_b) e^{-i(q-2^{u'-u}\tau)\lambda} d\lambda \right) \\
&\quad \left( \int_{-\infty}^{\infty} \sum_{\tau=0}^{2^{-(u'-u)}-1} g(\lambda; d_{a'} + d_{b'}) e^{-i(q-2^{u'-u}\tau)\lambda} d\lambda \right) \\
&= 2^{\Delta-u} \sum_{-Q \leq q \leq Q} \left(1 - \frac{q}{Q}\right)_+ 2^{(\Delta-u)(2-d_a-d_b-d_{a'}-d_{b'})} \left( \int_{-\pi}^{\pi} 2^{-(u'-u)/2} \widetilde{D}_{u'-u,\infty}(\lambda; d_a + d_b) e^{iq\lambda} d\lambda \right) \\
&\quad \left( \int_{-\pi}^{\pi} 2^{-(u'-u)/2} \widetilde{D}_{u'-u,\infty}(\lambda; d_{a'} + d_{b'}) e^{iq\lambda} d\lambda \right)
\end{aligned}$$

since  $\widetilde{D}_{u'-u,\infty}(\lambda; d_a + d_b) = \sum_{v=0}^{2^{u'-u}-1} 2^{(u'-u)/2} \sum_{t \in \mathbb{Z}} g_{u'-u}(\lambda + 2t\pi) e^{i2^{u'-u}\tau(\lambda+2t\pi)}$ . Applying Lemma A.5, we get:

$$\begin{aligned}
& \lim_{\ell \rightarrow \infty} \sum_{i=2^{\Delta-u}}^{2^{\Delta-u+1}-1} \sum_{i'=2^{\Delta-u'}}^{2^{\Delta-u'+1}-1} \ell^{-1} \sum_{s=0}^{\ell} \sum_{s'=0}^{\ell} \Xi_{(u,s),(u',s')}(a,b) \Xi_{(u,s),(u',s')}(a',b') \\
&= 2^{\Delta-u} (2\pi) 2^{(\Delta-u)(2-d_a-d_b-d_{a'}-d_{b'})+u-u'} \int_{-\pi}^{\pi} \overline{\widetilde{D}_{u'-u,\infty}(\lambda; d_a + d_b)} \widetilde{D}_{u'-u,\infty}(\lambda; d_{a'} + d_{b'}) d\lambda. \quad (30)
\end{aligned}$$

Since  $n_{j_1} n_{j_0+u}^{-1/2} n_{j_0+u'}^{-1/2} \sim 2^{-\Delta+u/2+u'/2}$ , we deduce from (27), (28), (29) and (30) that

$$\begin{aligned}
V_{u,u'}^{(d)}((a,b), (a',b')) &= 2\pi 2^{-(u-u')/2} 2^{u(d_a+d_b+d_{a'}+d_{b'})} \\
&\quad \left( G_{a,a'} G_{b,b'} \frac{\int_{-\pi}^{\pi} \overline{\widetilde{D}_{u,\infty}(\lambda; d_a + d_{a'})} \widetilde{D}_{u,\infty}(\lambda; d_b + d_{b'}) d\lambda}{K(d_a + d_{a'}) K(d_b + d_{b'})} \right. \\
&\quad \left. + G_{a,b'} G_{b,a'} \frac{\int_{-\pi}^{\pi} \overline{\widetilde{D}_{u,\infty}(\lambda; d_a + d_{b'})} \widetilde{D}_{u,\infty}(\lambda; d_b + d_{a'}) d\lambda}{K(d_a + d_{b'}) K(d_b + d_{a'})} \right)
\end{aligned}$$

with  $G_{a,b} = \Omega_{a,b} \cos(\pi(d_a - d_b)/2) K(d_a + d_b)$ .

We conclude using the symmetry of the matrix  $\mathbf{V}^{(d)}((a,b), (a',b'))$ .

## A.5 Technical lemma

The following lemma states the convergence of a series of bivariate Fourier coefficients. It is used for the convergence of the variance terms in the above proof.

**Lemma A.5.** Suppose  $\{w_1^*(\lambda), \lambda \in (-\pi, \pi)\}$  and  $\{w_2^*(\lambda), \lambda \in (-\pi, \pi)\}$  are  $\mathbb{C}$ -valued functions of  $L^2((-\pi, \pi))$ . Then

$$\sum_{q \in \mathbb{Z}} (1 - |q|/Q)_+ \left( \int_{-\pi}^{\pi} w_1^*(\lambda) e^{-iq\lambda} d\lambda \right) \left( \int_{-\pi}^{\pi} w_2^*(\lambda) e^{-iq\lambda} d\lambda \right) \xrightarrow{Q \rightarrow \infty} 2\pi \int_{-\pi}^{\pi} \overline{w_1^*(\lambda)} w_2^*(\lambda) d\lambda.$$

*Proof.* Remark that

$$\sum_{q \in \mathbb{Z}} (1 - |q|/Q)_+ \left( \int_{-\pi}^{\pi} w_1^*(\lambda) e^{-iq\lambda} d\lambda \right) \left( \int_{-\pi}^{\pi} w_2^*(\lambda) e^{-iq\lambda} d\lambda \right) = \sum_{q \in \mathbb{Z}} (1 - |q|/Q)_+ c_{1q} c_{2q},$$

with  $c_{1q}$  and  $c_{2q}$  the  $q^{\text{th}}$  Fourier coefficient respectively of functions  $w_1^*$  and  $w_2^*$ .

The sequence  $(1 - |q|/Q)_+ c_{1q} c_{2q}$  converges to  $c_{1q} c_{2q}$  when  $Q$  goes to infinity. By Cauchy-schwarz's inequality and Parseval's inequality,

$$\sum_{|q| \leq Q} |c_{1q} c_{2q}| \leq 2\pi \left( \int_{-\pi}^{\pi} |w_1^*(\lambda)|^2 d\lambda \right)^{1/2} \left( \int_{-\pi}^{\pi} |w_2^*(\lambda)|^2 d\lambda \right)^{1/2} < \infty.$$

Thus dominated convergence entails that the series  $\sum_{q \in \mathbb{Z}} (1 - |q|/Q)_+ c_{1q} c_{2q}$  converges to  $\sum_{q \in \mathbb{Z}} c_{1q} c_{2q}$ . By Parseval's inequality,

$$\sum_{q \in \mathbb{Z}} c_{1q} c_{2q} = 2\pi \int_{-\pi}^{\pi} \overline{w_1^*(\lambda)} w_2^*(\lambda) d\lambda,$$

which concludes the proof. □

## B Proof of Theorem 3.1

For a better understanding, the true parameters are denoted with an exponent 0 in this part.

The criterion  $R$  is equal to  $R(\mathbf{d}) = \log \det \left( \Lambda_{\langle \mathcal{J} \rangle}(\mathbf{d}) \widehat{\mathbf{G}}(\mathbf{d}) \Lambda_{\langle \mathcal{J} \rangle}(\mathbf{d}) \right) - 1$ . It is straightforward that  $\widehat{\mathbf{d}} = \underset{\mathbf{d}}{\operatorname{argmin}} R(\mathbf{d})$  satisfies

$$\begin{aligned} \widehat{\mathbf{d}} &= \underset{\mathbf{d}}{\operatorname{argmin}} \overline{R}(\mathbf{d}) \quad \text{with} \quad \overline{R}(\mathbf{d}) = \log \det \overline{\mathbf{G}}(\mathbf{d}) \\ &\quad \text{and} \quad \overline{\mathbf{G}}(\mathbf{d}) = \Lambda_{\langle \mathcal{J} \rangle}(\mathbf{d} - \mathbf{d}^0) \widehat{\mathbf{G}}(\mathbf{d}) \Lambda_{\langle \mathcal{J} \rangle}(\mathbf{d} - \mathbf{d}^0) \end{aligned}$$

The Taylor expansion of  $\frac{\partial \overline{R}(\mathbf{d})}{\partial \mathbf{d}}$  at  $\widehat{\mathbf{d}}$  at the neighbourhood of  $\mathbf{d}^0$  gives

$$\sqrt{n}(\widehat{\mathbf{d}} - \mathbf{d}^0) = \left( \frac{\partial^2 \overline{R}}{\partial \mathbf{d} \partial \mathbf{d}^T} \Big|_{\widehat{\mathbf{d}}} \right)^{-1} \sqrt{n} \frac{\partial \overline{R}}{\partial \mathbf{d}} \Big|_{\mathbf{d}^0} \quad (33)$$

where  $\bar{\mathbf{d}}$  is such that  $\|\bar{\mathbf{d}} - \mathbf{d}^0\| \leq \|\hat{\mathbf{d}} - \mathbf{d}^0\|$ . The matrix  $\left( \kappa_{j_1-j_0}^{-1} \frac{\partial^2 R}{\partial \mathbf{d} \partial \mathbf{d}^T} \Big|_{\bar{\mathbf{d}}} \right)$  is indeed invertible for sufficiently high  $N_X$  when  $2^{-j_0\beta} + N_X^{-1/2} 2^{j_0/2} \rightarrow 0$

The derivatives of the criterion  $\bar{R}(\mathbf{d})$  are equal to

$$\begin{aligned} \frac{\partial \bar{R}(\mathbf{d})}{\partial d_a} &= \text{trace} \left( \bar{\mathbf{G}}(\mathbf{d})^{-1} \frac{\partial \bar{\mathbf{G}}(\mathbf{d})}{\partial d_a} \right), \\ \frac{\partial^2 \bar{R}(\mathbf{d})}{\partial d_a \partial d_b} &= -\text{trace} \left( \bar{\mathbf{G}}(\mathbf{d})^{-1} \frac{\partial \bar{\mathbf{G}}(\mathbf{d})}{\partial d_b} \bar{\mathbf{G}}(\mathbf{d})^{-1} \frac{\partial \bar{\mathbf{G}}(\mathbf{d})}{\partial d_a} \right) + \text{trace} \left( \bar{\mathbf{G}}(\mathbf{d})^{-1} \frac{\partial^2 \bar{\mathbf{G}}(\mathbf{d})}{\partial d_a \partial d_b} \right). \end{aligned}$$

when  $\bar{\mathbf{G}}(\mathbf{d})^{-1}$  exists. .

It has already been established in Section E.2.5. of [Achard and Gannaz \(2016a\)](#) that

$$\frac{\partial^2 \bar{R}(\mathbf{d})}{\partial \mathbf{d} \partial \mathbf{d}^T} \Big|_{\bar{\mathbf{d}}} \xrightarrow{\mathbb{P}} \kappa_{j_1-j_0} \log(2)^2 2(\mathbf{G}^{0-1} \circ \mathbf{G}^0 + \mathbf{I}_p). \quad (34)$$

We now study the asymptotic convergence of  $\frac{\partial \bar{R}}{\partial \mathbf{d}} \Big|_{\mathbf{d}^0}$ . More precisely, for any vector  $\mathbf{v} = (v_a)_{a=1, \dots, p}$ , we want to prove that  $\mathbf{v}^T \frac{\partial \bar{R}}{\partial \mathbf{d}} \Big|_{\mathbf{d}^0}$  converges to a Gaussian distribution with variance  $\mathbf{v}^T (2\kappa_{j_1-j_0} \mathbf{W}) \mathbf{v}$  where  $\mathbf{W}$  is defined in equation (7). Once this result established, Theorem 3.1 will follow with (33) and (34).

As expressed on page 36 of [Achard and Gannaz \(2016a\)](#),  $\bar{\mathbf{G}}(\mathbf{d})$  can be written as:

$$\bar{\mathbf{G}}_{a,b}(\mathbf{d}^0) = G_{a,b}^0 + S_{a,b}(1/n),$$

with  $S_{a,b}(1/n)$  defined as

$$S_{a,b}(1/n) = \sum_{j,k} \frac{1}{n} \left( \frac{W_{j,k}(a)W_{j,k}(b)}{2^{j(d_a^0+d_b^0)}} - G_{a,b}^0 \right) = \sum_{j=j_0}^{j_1} \frac{1}{n} \left( \frac{\hat{\sigma}_{a,b}(j)}{2^{j(d_a^0+d_b^0)}} - n_j G_{a,b}^0 \right).$$

Applying Proposition 8 of [Achard and Gannaz \(2016a\)](#), we get:

$$\bar{\mathbf{G}}_{a,b}(\mathbf{d}^0) = G_{a,b}^0 + O_{\mathbb{P}}(2^{-j_0\beta} + N^{-1/2} 2^{-j_1/2}).$$

We deduce that

$$\sqrt{n} \mathbf{v}^T \frac{\partial \bar{R}(\mathbf{d})}{\partial \mathbf{d}} \Big|_{\mathbf{d}^0} - S_n \xrightarrow{\mathbb{P}} 0,$$

with

$$S_n = \sqrt{n} \sum_a v_a \text{trace} \left( \mathbf{G}^{0-1} \frac{\partial \bar{\mathbf{G}}}{\partial d_a} \Big|_{\mathbf{d}^0} \right).$$

It is then sufficient to demonstrate that  $S_n$  converges in distribution to  $\mathcal{N}(0, \mathbf{v}^T \mathbf{W} \mathbf{v})$ .

For any  $a = 1, \dots, p$ , let  $\mathbf{i}_a$  be a  $p \times p$  matrix whose  $a$ -th diagonal element is one and all other elements are zero. Let  $a$  and  $b$  be two indexes in  $1, \dots, p$ . The first derivative of  $\overline{\mathbf{G}}(\mathbf{d})$  with respect to  $d_a$ ,  $\frac{\partial \overline{\mathbf{G}}(\mathbf{d})}{\partial d_a}$ , is equal to

$$-\log(2) \frac{1}{n} \sum_{j=j_0}^{j_1} n_j (j - \langle \mathcal{J} \rangle) \mathbf{\Lambda}_{\langle \mathcal{J} \rangle} (\mathbf{d} - \mathbf{d}^0) \mathbf{\Lambda}_j(\mathbf{d})^{-1} (\mathbf{i}_a \mathbf{\Sigma}(j) + \mathbf{\Sigma}(j) \mathbf{i}_a) \mathbf{\Lambda}_j(\mathbf{d})^{-1} \mathbf{\Lambda}_{\langle \mathcal{J} \rangle} (\mathbf{d} - \mathbf{d}^0),$$

with  $\widehat{\mathbf{\Sigma}}(j) = (\widehat{\sigma}_{a',b'}(j))_{a',b'=1,\dots,p}$ . Thus

$$\left. \frac{\partial \overline{\mathbf{G}}}{\partial d_a} \right|_{\mathbf{d}^0} = -\frac{\log(2)}{n} \sum_{j=j_0}^{j_1} n_j (j - \langle \mathcal{J} \rangle) \mathbf{\Lambda}_j(\mathbf{d})^{-1} (\mathbf{i}_a \mathbf{\Sigma}(j) + \mathbf{\Sigma}(j) \mathbf{i}_a) \mathbf{\Lambda}_j(\mathbf{d})^{-1}. \quad (35)$$

Consequently,

$$\mathbb{E}(S_n) = -\frac{2\log(2)}{\sqrt{n}} \sum_{j=j_0}^{j_1} n_j (j - \langle \mathcal{J} \rangle) \sum_{a,b=1,\dots,p} v_a G_{a,b}^{0(-1)} 2^{-j(d_a^0+d_b^0)} \sigma_{a,b}(j).$$

Using inequality (3),

$$\begin{aligned} \mathbb{E}(S_n) &= -\frac{2\log(2)}{\sqrt{n}} \sum_{j=j_0}^{j_1} n_j (j - \langle \mathcal{J} \rangle) \sum_{a,b=1,\dots,p} v_a G_{a,b}^{0(-1)} \left( G_{a,b} + O\left(2^{j_0\beta}\right) \right) \\ &= O\left(j_0 2^{j_0\beta} \sqrt{n}\right) \end{aligned}$$

since  $\sum_{j=j_0}^{j_1} n_j (j - \langle \mathcal{J} \rangle) = 0$ . Thus  $\mathbb{E}(S_n)$  converges to zero when  $j_0^2 N_X 2^{-j_0(1+2\beta)} \rightarrow 0$ .

We can write  $S_n - \mathbb{E}(S_n)$  as

$$S_n - \mathbb{E}(S_n) = \sum_{a,b=1,\dots,p} S_n(a,b)$$

$$\text{with } S_n(a,b) = -n^{1/2} 2^{-j_0(d_a^0+d_b^0)} 2\log(2) \sum_{u=0}^{j_1-j_0} w_u(a,b) (\widehat{\sigma}_{a,b}(j) - \sigma_{a,b}(j))$$

$$\text{and } w_u(a,b) = \frac{n_{j_0+u}}{n} (j_0 + u - \langle \mathcal{J} \rangle) 2^{-u(d_a^0+d_b^0)} v_a G_{a,b}^{0(-1)}.$$

Using the same notations as in Section A,  $2^{-j_0(d_a+d_b)} \widehat{\sigma}_{a,b}(j) = \frac{1}{n_j} \sum_{k=0}^{n_j} Z_{i,s,j_0}(a) Z_{i,s,j_0}(b)$ , with  $k = 2^{j_1}(s-1+2^{j-j_0}i)$ . Following the proof of Section A.2, it can be established that  $n_j^{-1/2} \sum_{k=0}^{n_j} Z_{i,s,j_0}(a) Z_{i,s,j_0}(b)$  converges in distribution. It results that its variance is finite. Hence, there exists a constant  $C$  such that

$$\text{Var}(\widehat{\sigma}_{a,b}(j)) \leq C \frac{2^{j_0(d_a+d_b)}}{\sqrt{n_j}}.$$

The right-hand side is equivalent, up to a constant, to  $2^{j(d_a+d_b+1)} N_X^{-1}$ . We can therefore apply the same proof as in Theorem 3 of [Roueff and Taqqu \(2009a\)](#). As vectors  $(\widehat{\sigma}_{a,b}(j))_{a,b=1,\dots,p}$  are asymptotically Gaussian, the sum  $\sum_{a,b=1,\dots,p} n^{-1/2} 2^{-j_0(d_a^0+d_b^0)} \sum_{u=0}^{\Delta} w_u(a,b) \widehat{\sigma}_{a,b}(j)$  is also asymptotically Gaussian for all  $\Delta \geq 0$ . Next the proof of [Roueff and Taqqu \(2009a\)](#) is unchanged since the above inequalities on the variances of  $\widehat{\sigma}_{a,b}(j)$  are satisfied. It leads to the desired variance. The variance expression is detailed in next subsection.

## B.1 Expression of the covariance

Let us specify the variance of  $S_n$ ,

$$\begin{aligned}
& \text{Var}(S_n) \\
&= \frac{4 \log(2)^2}{n} 2\pi \sum_{a,b,a',b'} v_a v_{a'} G_{a,b}^{0-1} G_{a',b'}^{0-1} \sum_{u=0}^{j_1-j_0} \sum_{u'=0}^{j_1-j_0} n_{j_0+u}^{-1/2} n_{j_0+u'}^{-1/2} (j_0+u - \langle \mathcal{J} \rangle) (j_0+u' - \langle \mathcal{J} \rangle) \\
& \qquad \qquad \qquad 2^{-u(d_a^0+d_b^0)-u'(d_{a'}^0+d_{b'}^0)} V_{u,u'}((a,b), (a',b')) \\
&= 4 \log(2)^2 2\pi \sum_{a,b,a',b'} v_a v_{a'} G_{a,b}^{0(-1)} G_{a',b'}^{0(-1)} (G_{aa'}^0 G_{bb'}^0 \mathcal{I}(d_a^0 + d_{a'}^0, d_a^0 + d_{a'}^0) \\
& \qquad \qquad \qquad + G_{ab'}^0 G_{a'b}^0 \mathcal{I}^{(S)}(d_a^0 + d_{b'}^0, d_{a'}^0 + d_b^0)) ,
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{I}^{(S)}(\delta_1, \delta_2) &= \frac{1}{n} \sum_{u=0}^{j_1-j_0} \sum_{u'=0}^{j_1-j_0} n_{j_0+u}^{-1/2} n_{j_0+u'}^{-1/2} (j_0+u - \langle \mathcal{J} \rangle) (j_0+u' - \langle \mathcal{J} \rangle) \\
& \qquad \qquad \qquad 2^{(\delta_1+\delta_2)u \vee u' - |u-u'|/2} 2^{-u\delta_1 - u'\delta_2} \tilde{I}_{|u-u'|}(\delta_1, \delta_2) .
\end{aligned}$$

We can formulate this expression to recover a similar form than the one given in [Moulines et al \(2008\)](#) and [Roueff and Taqqu \(2009a\)](#). The arguments are the same than those used in the proof of Proposition 10 in [Moulines et al \(2008\)](#), but are recalled here to explicit the form of the variance. First, as  $n_j \sim N_X 2^{-j}$  and  $n \sim N_X 2^{-j_0} (2 - 2^{-(j_1-j_0)})$ , we can express  $\mathcal{I}^{(S)}(\delta_1, \delta_2)$  as

$$\begin{aligned}
\mathcal{I}^{(S)}(\delta_1, \delta_2) &= \frac{1}{2 - 2^{j_1-j_0}} \sum_{v'=0}^{\Delta} (j_0 + v' - \langle \mathcal{J} \rangle)^2 2^{-v'} \tilde{I}_0(\delta_1, \delta_2) \\
& \quad + \frac{1}{2 - 2^{j_1-j_0}} \sum_{v=1}^{j_1-j_0} \sum_{v'=0}^{\Delta-u} (v + v' - \eta_\Delta) (v' - \eta_\Delta) 2^{-v-v'} (2^{-v\delta_1} + 2^{-v\delta_2}) \tilde{I}_v(\delta_1, \delta_2) ,
\end{aligned}$$

where we set  $v = |u - u'|$  and  $v' = u \wedge u'$ . Lemma 13 of [Moulines et al \(2008\)](#) states that  $j_0 - \langle \mathcal{J} \rangle \rightarrow \eta_\Delta$  when  $N_X \rightarrow \infty$ . Using this result and the definition of  $\kappa_\Delta$ , we obtain

$$\begin{aligned}
\mathcal{I}^{(S)}(\delta_1, \delta_2) &= \kappa_\Delta \tilde{I}_0(\delta_1, \delta_2) \\
& \quad + \frac{2}{2 - 2^{j_1-j_0}} \sum_{v=1}^{j_1-j_0} \sum_{v'=0}^u (v + v' - \eta_\Delta) (v' - \eta_\Delta) 2^{-v-v'} (2^{-v\delta_1} + 2^{-v\delta_2}) \tilde{I}_v(\delta_1, \delta_2) .
\end{aligned}$$

Setting  $\mathcal{I}(\delta_1, \delta_2) = \frac{2}{\kappa_\Delta^2} \mathcal{I}^{(S)}(\delta_1, \delta_2)$ , we recognize the expression (8) given in Theorem 3.1.

When  $\Delta$  goes to infinity, the sequence  $\kappa_\Delta$  converges to 2. Additionally, for all  $u \geq 0$ ,

$$\frac{1}{\kappa_\Delta} \sum_{i=0}^{\Delta-u} \frac{2^{-i}}{2 - 2^{-\Delta}} (i - \eta_\Delta) (i + u - \eta_\Delta) \rightarrow 1 \text{ when } \Delta \rightarrow \infty .$$

We deduce the asymptotic form (9) when  $\Delta = j_1 - j_0 \rightarrow \infty$ , as in page 30 of [Moulines et al \(2008\)](#).

## C Proof of Theorem 3.2

Remark that  $\widehat{G}_{a,b}(\mathbf{d}) = 2^{\langle \mathcal{J} \rangle (d_a - d_a^0 + d_b - d_b^0)} \overline{G}_{a,b}(\mathbf{d})$  and  $\widehat{G}_{a,b}(\mathbf{d}^0) = \overline{G}_{a,b}(\mathbf{d}^0)$  with  $\overline{G}(\mathbf{d})$  defined in (32). As  $2^{\langle \mathcal{J} \rangle u} - 1 = j_0 u \log(2)(1 + o(1))$  when  $u \rightarrow 0$ ,

$$\widehat{G}_{a,b}(\widehat{\mathbf{d}}) - \overline{G}_{a,b}(\widehat{\mathbf{d}}) = j_0 (\widehat{d}_a - d_a^0 + \widehat{d}_b - d_b^0) \log(2)(1 + o(1)) \overline{G}_{a,b}(\widehat{\mathbf{d}}).$$

Since  $j_0 (\widehat{d}_a - d_a^0 + \widehat{d}_b - d_b^0) = o_{\mathbb{P}}(1)$ , it is sufficient to establish the asymptotic distribution of  $\overline{G}_{a,b}(\widehat{\mathbf{d}})$ . More precisely, we want to prove that  $\sqrt{n} \text{vect} \left( \overline{G}_{a,b}(\widehat{\mathbf{d}}) - G_{a,b}^0, a, b = 1, \dots, p \right)$  converges in distribution to a centered Gaussian distribution. We decompose  $\sqrt{n}(\overline{G}_{a,b}(\widehat{\mathbf{d}}) - G_{a,b}^0)$  as

$$\sqrt{n}(\overline{G}_{a,b}(\widehat{\mathbf{d}}) - G_{a,b}^0) = \sqrt{n} \left( \overline{G}_{a,b}(\mathbf{d}^0) - G_{a,b}^0 \right) + \sqrt{n} \left( \overline{G}_{a,b}(\widehat{\mathbf{d}}) - \overline{G}_{a,b}(\mathbf{d}^0) \right).$$

We will establish that the first term converges to the desired distribution, while the second one is negligible.

### C.1 Study of $\sqrt{n} \left( \overline{G}(\mathbf{d}^0) - \mathbf{G}^0 \right)$

Consider  $T_0 = \text{vect} \left( \sqrt{n} \left( \overline{G}_{a,b}(\mathbf{d}^0) - \mathbb{E}(\overline{G}_{a,b}(\mathbf{d}^0)) \right), a, b = 1, \dots, n \right)$ . Recall that

$$\overline{G}_{a,b}(\mathbf{d}^0) = \frac{1}{n} \sum_{j=j_0}^{j_1} n_j 2^{-j(d_a^0 + d_b^0)} \widehat{\sigma}_{a,b}(j).$$

Using inequality (3),

$$\mathbb{E}(\overline{G}_{a,b}(\mathbf{d}^0)) = G_{a,b}^0(j) + O\left(2^{j_0 \beta}\right),$$

and consequently  $\sqrt{n} \left( \mathbb{E}(\overline{G}_{a,b}(\mathbf{d}^0)) - G_{a,b}^0(j) \right) = o(1)$ .

We can write  $\sqrt{n} (T_{0,a,b} - \mathbb{E}(T_{0,a,b}))$  as

$$\sqrt{n} \left( \overline{G}_{a,b}(\mathbf{d}^0) - \mathbb{E}(\overline{G}_{a,b}(\mathbf{d}^0)) \right) = \sqrt{N_X 2^{-j_0}} 2^{-j_0(d_a^0 + d_b^0)} \sum_{u=0}^{j_1 - j_0} w_u (\widehat{\sigma}_{a,b}(j) - \sigma_{a,b}(j)),$$

with  $w_u = (N_X 2^{-j_0})^{-1/2} n_{j_0+u} / \sqrt{n}$ . Applying Theorem 3 of [Roueff and Taqqu \(2009a\)](#), we obtain that  $T_0 - \mathbb{E}(T_0)$  converges as  $n$  goes to infinity to a centered Gaussian distribution, with variance

$$\begin{aligned} \text{Cov}(T_{0,a,b}, T_{0,a',b'}) &= \sum_{u=0}^{\Delta} \sum_{u'=0}^{\Delta} 2^{-u(d_a^0 + d_b^0) - u'(d_{a'}^0 + d_{b'}^0)} \frac{\sqrt{n_{j_0+u} n_{j_0+u'}}}{n} V_{u,u'}((a,b), (a',b')) \\ &= 2\pi \left( G_{a,a'}^0 G_{b,b'}^0 \mathcal{I}_{\Delta}^G(d_a + d_{a'}, d_b + d_{b'}) + G_{a,b'}^0 G_{b,a'}^0 \mathcal{I}_{\Delta}^G(d_a + d_{b'}, d_b + d_{a'}) \right), \end{aligned}$$

where

$$\mathcal{I}_{\Delta}^G(\delta_1, \delta_2) = \sum_{u=0}^{\Delta} \sum_{u'=0}^{\Delta} 2^{(u \vee u' - u)\delta_1 + (u \vee u' - u')\delta_2} \frac{2^{-u/2 - u'/2 - |u - u'|/2}}{2 - 2^{-\Delta}} \widetilde{I}_{|u' - u|}(\delta_1, \delta_2).$$



Quantity  $\mathcal{I}^G(\delta_1, \delta_2)$  can be simplified as

$$\begin{aligned}\mathcal{I}_\Delta^G(\delta_1, \delta_2) &= \sum_{u=0}^{\Delta} \frac{2^{-u}}{2-2^{-\Delta}} \tilde{I}_0(\delta_1, \delta_2) + \sum_{u=1}^{\Delta} (2^{u\delta_1} + 2^{u\delta_2}) 2^{-u} \tilde{I}_u(\delta_1, \delta_2) \sum_{v=0}^{\Delta-u} \frac{2^{-v}}{2-2^{-\Delta}} \\ &= \tilde{I}_0(\delta_1, \delta_2) + \sum_{u=1}^{\Delta} (2^{u\delta_1} + 2^{u\delta_2}) 2^{-u} \frac{2-2^{-\Delta+u}}{2-2^{-\Delta}} \tilde{I}_u(\delta_1, \delta_2).\end{aligned}$$

When  $\Delta$  goes to infinity,

$$\mathcal{I}_\infty^G(\delta_1, \delta_2) = \tilde{I}_0(\delta_1, \delta_2) + \sum_{u=1}^{\infty} (2^{u\delta_1} + 2^{u\delta_2}) 2^{-u} \tilde{I}_u(\delta_1, \delta_2).$$

## C.2 Study of $\sqrt{n} \left( \overline{G}(\hat{\mathbf{d}}) - \overline{G}(\mathbf{d}^0) \right)$

A Taylor expansion at order one at  $\mathbf{d}^0$  gives

$$\sqrt{n} \left( \overline{G}(\hat{\mathbf{d}}) - \overline{G}(\mathbf{d}^0) \right) = \sqrt{n} \left( \frac{\partial \overline{G}}{\partial \mathbf{d}} \Big|_{\mathbf{d}^0} \right) (\hat{\mathbf{d}} - \mathbf{d}^0) + \sqrt{n} (\hat{\mathbf{d}} - \mathbf{d}^0)^T \left( \frac{\partial^2 \overline{G}}{\partial \mathbf{d} \partial \mathbf{d}^T} \Big|_{\overline{\mathbf{d}}} \right) (\hat{\mathbf{d}} - \mathbf{d}^0),$$

with  $\left\| \overline{\mathbf{d}} - \mathbf{d}^0 \right\| \leq \left\| \hat{\mathbf{d}} - \mathbf{d}^0 \right\|$ . Section E.2.4. of [Achard and Gannaz \(2016a\)](#) states that, when  $2^{-j_0\beta} + N_X^{-1/2} 2^{j_0/2} \rightarrow 0$ ,

$$\frac{\partial^2 \overline{G}}{\partial \mathbf{d} \partial \mathbf{d}^T} \Big|_{\mathbf{d}^0} = O_{\mathbb{P}}(1).$$

Thus

$$\sqrt{n} (\hat{\mathbf{d}} - \mathbf{d}^0)^T \left( \frac{\partial^2 \overline{G}}{\partial \mathbf{d} \partial \mathbf{d}^T} \Big|_{\overline{\mathbf{d}}} \right) (\hat{\mathbf{d}} - \mathbf{d}^0) = o_{\mathbb{P}}(1).$$

Next, using the derivative of  $\overline{G}(\mathbf{d})$  given in (35), we have:

$$\sqrt{n} \left( \frac{\partial \overline{G}}{\partial \mathbf{d}} \Big|_{\mathbf{d}^0} (\hat{\mathbf{d}} - \mathbf{d}^0) \right)_{a,b} = -\frac{\log(2)}{\sqrt{n}} \sum_{j=j_0}^{j_1} n_j (j - \langle \mathcal{J} \rangle) 2^{-j(d_a^0 + d_b^0)} \hat{\sigma}_{a,b}(j) (\hat{d}_a - d_a^0 + \hat{d}_b - d_b^0).$$

Similarly to what was done in Section B, we can establish that  $\frac{\log(2)}{\sqrt{n}} \sum_{j=j_0}^{j_1} n_j (j - \langle \mathcal{J} \rangle) 2^{-j(d_a^0 + d_b^0)} \hat{\sigma}_{a,b}(j)$  is asymptotically Gaussian. Since  $(\hat{d}_a - d_a^0 + \hat{d}_b - d_b^0) \rightarrow 0$  we conclude that this term goes to 0, which concludes the proof.

## D Proof of Proposition 3.4

From the proof of Theorem 3.1 and Theorem 3.2, we can extract the following results:

$$\begin{aligned}
& \left( (\mathbf{G}^{0-1} \circ \mathbf{G}^0 + \mathbf{I}_p) \sqrt{n}(\widehat{\mathbf{d}} - \mathbf{d}^0) \right)_{a'} \\
&= \sum_{b'=1}^p \sqrt{N_X} 2^{-j_0} 2^{-j_0(d_{a'}^0 + d_{b'}^0)} \sum_{u=0}^{j_1-j_0} w_{u,(a',b')}^{(d)} (\widehat{\sigma}_{a',b'}(j) - \sigma_{a',b'}(j))(1 + o_{\mathbb{P}}(1)) + o_{\mathbb{P}}(1), \\
& \sqrt{n} \left( \widehat{G}_{a,b}(\widehat{\mathbf{d}}) - G_{a,b}^0 \right) \\
&= \sqrt{N_X} 2^{-j_0} 2^{-j_0(d_a^0 + d_b^0)} \sum_{u=0}^{j_1-j_0} w_{u,(a,b)}^{(G)} (\widehat{\sigma}_{a,b}(j) - \sigma_{a,b}(j))(1 + o_{\mathbb{P}}(1)) + o_{\mathbb{P}}(1),
\end{aligned}$$

with

$$\begin{aligned}
w_{u,(a',b')}^{(d)} &= (\kappa_{j_1-j_0} 2 \log(2))^{-1} (2 - 2^{-\Delta})^{-1/2} 2^{-u} (j_0 + u - \langle \mathcal{J} \rangle) G_{a',b'}^{0(-1)} 2^{-u(d_{a'}^0 + d_{b'}^0)} \\
w_{u,(a,b)}^{(G)} &= (2 - 2^{-\Delta})^{-1/2} 2^{-u} 2^{-u(d_a^0 + d_b^0)}.
\end{aligned}$$

Hence, linear combinations of  $\sqrt{n}(\widehat{d}_{a'} - d_{a'})$  and  $\sqrt{n}(\widehat{G}_{a,b}(\widehat{\mathbf{d}}) - G_{a,b}^0)$  can be written as  $\sum_{\ell,m} \sqrt{N_X} 2^{-j_0} 2^{-j_0(d_\ell^0 + d_m^0)} \sum_{u=0}^{j_1-j_0} w_{u,(\ell,m)} (\widehat{\sigma}_{\ell,m}(j) - \sigma_{\ell,m}(j))(1 + o_{\mathbb{P}}(1)) + o_{\mathbb{P}}(1)$  with  $w_{u,(\ell,m)}$  linear combination of  $(w_{u,(a,b)}^{(d)})_{a,b}$  and  $(w_{u,(a,b)}^{(G)})_{a,b}$ . Theorem 3 of [Roueff and Taqqu \(2009a\)](#) gives the joint convergence to a Gaussian distribution.

It remains to explicit the asymptotic covariance between  $\widehat{\mathbf{d}}$  and  $\widehat{\mathbf{G}}(\widehat{\mathbf{d}})$ . Asymptotically, the covariance between  $\sqrt{n}(\widehat{d}_\ell - d_\ell^0)$  and  $\sqrt{n}(\widehat{G}_{a,b}(\widehat{\mathbf{d}}) - G_{a,b}^0)$  is

$$\begin{aligned}
W_{\ell,(a,b)}^{d,G(\Delta)} &= 2\pi (\kappa_\Delta 2 \log(2))^{-1} (2 - 2^{-\Delta})^{-1} \sum_{u=0}^{\Delta} \sum_{u'=0}^{\Delta} 2^{-u-u'-|u-u'|/2} (j_0 + u - \langle \mathcal{J} \rangle) \\
& \quad 2^{(u \wedge u' - u')(d_a^0 + d_b^0)} \sum_{m=1}^p \left( (\mathbf{G}^{0-1} \circ \mathbf{G}^0 + \mathbf{I}_p)^{-1} \right)_{\ell,m} 2^{(u \wedge u' - u)(d_\ell^0 + d_m^0)} \\
& \quad \left( G_{\ell,a}^0 G_{m,b}^0 \widetilde{I}_{|u-u'|}(d_\ell + d_a, d_m + d_b) + G_{\ell,b}^0 G_{a,m}^0 \widetilde{I}_{|u-u'|}(d_\ell + d_b, d_m + d_a) \right) \\
&= \pi (2 \log(2))^{-1} \sum_{m=1}^p \left( (\mathbf{G}^{0-1} \circ \mathbf{G}^0 + \mathbf{I}_p)^{-1} \right)_{\ell,m} \left( G_{\ell,a}^0 G_{m,b}^0 \mathcal{I}_\Delta^{d,G}(d_\ell + d_a, d_m + d_b) \right. \\
& \quad \left. + G_{\ell,b}^0 G_{a,m}^0 \mathcal{I}_\Delta^{d,G}(d_\ell + d_b, d_m + d_a) \right), \tag{36}
\end{aligned}$$

with

$$\begin{aligned}
\mathcal{I}_\Delta^{d,G}(\delta_1, \delta_2) &= \frac{2}{\kappa_\Delta} \sum_{u=0}^{\Delta} \frac{2^{-u}}{2 - 2^{-\Delta}} (u - \eta_\Delta) \widetilde{I}_{|u-u'|}(\delta_1, \delta_2) \\
& \quad + \frac{1}{\kappa_\Delta} \sum_{u=1}^{\Delta} 2^{-u} \widetilde{I}_u(\delta_1, \delta_2) \sum_{v=0}^{\Delta-u} \frac{2^{-v}}{2 - 2^{-\Delta}} \left( 2^{-u\delta_1} (u - \eta_\Delta) + 2^{-u\delta_2} (u + v - \eta_\Delta) \right) \\
&= \frac{1}{\kappa_\Delta} \sum_{u=1}^{\Delta} 2^{-u} \widetilde{I}_u(\delta_1, \delta_2) \left( (2^{-u\delta_1} + 2^{-u\delta_2}) \frac{2 - 2^{-\Delta-u}}{2 - 2^{-\Delta}} (u - \eta_\Delta) + 2^{-u\delta_2} \eta_{\Delta-u} \right). \tag{37}
\end{aligned}$$

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## References

- Achard S, Gannaz I (2016a) Multivariate wavelet whittle estimation in long-range dependence. *Journal of Time Series Analysis* 37(4):476–512
- Achard S, Gannaz I (2016b) Wavelet Whittle estimation in multivariate time series models: application to fMRI data. In: *Conference of the International Society for Non-Parametric Statistics*, Springer, pp 271–285
- Achard S, Gannaz I (2019) Wavelet-based and Fourier-based multivariate Whittle estimation: multiwave. *Journal of Statistical Software, Articles* 89(6):1–31
- Achard S, Bassett DS, Meyer-Lindenberg A, Bullmore E (2008) Fractal connectivity of long-memory networks. *Physical Review E* 77(3):036,104
- Achard S, Delon-Martin C, Vértes PE, Renard F, Schenck M, Schneider F, Heinrich C, Kremer S, Bullmore ET (2012) Hubs of brain functional networks are radically reorganized in comatose patients. *Proceedings of the National Academy of Sciences* 109(50):20,608–20,613
- Achard S, Borgnat P, Gannaz I, Roux M (2019) Wavelet-based graph inference using multiple testing. In: *Wavelets and Sparsity XVIII, International Society for Optics and Photonics*, vol 11138, p 1113811
- Baek C, Kechagias S, Pipiras V (2020) Asymptotics of bivariate local whittle estimators with applications to fractal connectivity. *Journal of Statistical Planning and Inference* 205:245–268
- Becq G, Barbier E, Achard S (2020a) Brain networks of rats under anesthesia using resting-state fMRI: comparison with dead rats, random noise and generative models of networks. *Journal of Neural Engineering* URL <http://iopscience.iop.org/10.1088/1741-2552/ab9fec>
- Becq G, Habet T, Collomb N, Faucher M, Delon-Martin C, Coizet V, Achard S, Barbier EL (2020b) Functional connectivity is preserved but reorganized across several anesthetic regimes. *NeuroImage* 219:116,945, DOI <https://doi.org/10.1016/j.neuroimage.2020.116945>
- Billingsley P (1999) *Convergence of probability measures*. John Wiley & Sons
- Didier G, Pipiras V, et al (2011) Integral representations and properties of operator fractional brownian motions. *Bernoulli* 17(1):1–33
- Düker MC, Pipiras V (2019) Asymptotic results for multivariate local whittle estimation with applications. In: *2019 IEEE 8th International Workshop on Computational Advances in Multi-Sensor Adaptive Processing (CAMSAP)*, IEEE, pp 584–588

- Gençay R, Selçuk F, Whitcher BJ (2001) An introduction to wavelets and other filtering methods in finance and economics. Academic Press
- Kechagias S, Pipiras V (2014) Definitions and representations of multivariate long-range dependent time series. *Journal of Time Series Analysis* DOI 10.1111/jtsa.12086
- Lobato IN (1997) Consistency of the averaged cross-periodogram in long memory series. *Journal of Time Series Analysis* 18(2):137–155
- Lobato IN (1999) A semiparametric two-step estimator in a multivariate long memory model. *Journal of Econometrics* 90(1):129–153
- Maxim V, Şendur L, Fadili MJ, Suckling J, Gould R, Howard R, Bullmore ET (2005) Fractional Gaussian noise, functional MRI and Alzheimer’s disease. *NeuroImage* 25:141–158
- McCoy E, Walden A (1996) Wavelet analysis and synthesis of stationary long-memory processes. *Journal of computational and Graphical statistics* 5(1):26–56
- Moulines E, Roueff F, Taqqu MS (2007) On the spectral density of the wavelet coefficients of long-memory time series with application to the log-regression estimation of the memory parameter. *Journal of Time Series Analysis* 28(2):155–187
- Moulines E, Roueff F, Taqqu MS (2008) A wavelet Whittle estimator of the memory parameter of a nonstationary gaussian time series. *The Annals of Statistics* pp 1925–1956
- Nielsen FS (2011) Local whittle estimation of multi-variate fractionally integrated processes. *Journal of Time Series Analysis* 32(3):317–335
- Pawela CP, Biswal BB, Cho YR, Kao DS, Li R, Jones SR, Schulte ML, Matloub HS, Hudetz AG, Hyde JS (2008) Resting-state functional connectivity of the rat brain. *Magnetic Resonance in Medicine* 59(5):1021–1029, DOI 10.1002/mrm.21524, URL <http://dx.doi.org/10.1002/mrm.21524>
- Pipiras V, Kechagias S, Baek C (2017) Semiparametric, parametric, and possibly sparse models for multivariate long-range dependence. *Proceedings Volume 10394, Wavelets and Sparsity XVII*
- Robinson PM (1995) Gaussian semiparametric estimation of long range dependence. *The Annals of Statistics* 23(5):1630–1661
- Robinson PM (2005) Robust covariance matrix estimation: Hac estimates with long memory/antipersistence correction. *Econometric Theory* 21(01):171–180
- Robinson PM (2008) Multiple local Whittle estimation in stationary systems. *The Annals of Statistics* 36(05):2508–2530
- Roueff F, Taqqu MS (2009a) Asymptotic normality of wavelet estimators of the memory parameter for linear processes. *Journal of Time Series Analysis* 30(5):534–558
- Roueff F, Taqqu MS (2009b) Central limit theorems for arrays of decimated linear processes. *Stochastic processes and their applications* 119(9):3006–3041
- Sela RJ, Hurvich CM (2008) Computationally efficient methods for two multivariate fractionally integrated models. *Journal of Time Series Analysis* 30:6, DOI 10.1111/jtsa.12086

- Shimotsu K (2007) Gaussian semiparametric estimation of multivariate fractionally integrated processes. *Journal of Econometrics* 137(2):277–310
- Whitcher B, Jensen MJ (2000) Wavelet estimation of a local long memory parameter. *Exploration Geophysics* 31(1/2):94–103
- Whitcher B, Guttorp P, Percival DB (2000) Wavelet analysis of covariance with application to atmospheric time series. *Journal of Geophysical Research: Atmospheres* 105(D11):14,941–14,962