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Asymptotic normality of wavelet covariances and multivariate wavelet Whittle estimators

Irène Gannaz

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Abstract

Multivariate processes with long-range dependence properties can be encountered in many fields of application. Two fundamental characteristics in such frameworks are long-range dependence parameters and correlations between component time series. We consider multivariate long-range dependent linear processes, not necessarily Gaussian. We show that the covariances between the wavelet coefficients in this setting are asymptotically Gaussian. We also study the asymptotic distributions of the estimators of the long-range dependence parameter and the long-run covariance by a wavelet-based Whittle procedure. We prove the asymptotic normality of the estimators, and we provide an explicit expression for the asymptotic covariances. An empirical illustration of this result is proposed on a real dataset of rat brain connectivity.

Keywords: Multivariate processes, long-range dependence, covariance, wavelets, asymptotic normality, cerebral connectivity

1. Introduction

Univariate long-range dependent processes are processes with an autocovariance function with a power-law decay or equivalently a spectral density diverging at the zero frequency with a power-law rate. Univariate long-range dependence (LRD) has encountered much interest and is used widely in applications. See, for example, [1, 2, 3] and references therein.

Data is often recorded by multiple sensors where multivariate modeling brings better

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representation and can increase the consistency of inference. Multivariate processes with LRD properties are found in a wide range of applications, such as geoscience [4], finance [5], or neuroscience [6]. Extensions of univariate LRD models to multivariate frameworks were initiated by [7], and this topic has met great interest over the last decades. Several models have been proposed, such as multivariate autoregressive fractionally integrated moving average (ARFIMA) models [8, 9, 10]. In [10], Kechagias and Pipiras provide properties in the time and spectral domains of linear representations of multivariate long-range dependent processes. A nonlinear example of multivariate long-range dependent processes was also proposed by Didier and Pipiras in [11], where a multivariate Brownian motion was defined.

The specificity of the multivariate setting is that, in addition to LRD properties, it helps identify the correlation structure between the processes. The coupling between each component is characterized by the long-run covariance matrix [7]. A key point for real data application is the development of statistical tests on LRD parameters and on long-run covariance. For example, as illustrated here on a real data example, these characteristics are intrinsically related to the brain activity recordings in neuroscience. Some work has shown that their distributions can be modified by pathologies (see *e.g.* [12] and [13]). A statistical test may be useful to rigorously assess such observations.

We focus on semiparametric estimators, which are more robust to model misspecification [7]. A common estimation procedure in this framework is Whittle estimation, which is based on a Fourier decomposition of the processes [14, 15, 16]. The authors prove the consistency and the asymptotic distribution of their estimators. More recently, the asymptotic normality of estimators has been provided by Baek et al. [17], in prolongation of [18], in a multivariate framework where components can be co-integrated. An estimation with a Lasso penalty is also proposed in this setting by Pipiras et al. [19], and Düker and Pipiras [20] establish the asymptotic normality of this procedure.

As an alternative to Fourier, wavelet-based estimators can be used. Wavelet transforms are interesting especially because wavelet analysis performs an implicit differentiation, which offers the possibility to consider non-stationary processes. Wavelet-based Whittle estimation was introduced by Moulines et al. in [21] for univariate long-range dependent time series. It was generalized to the multivariate setting by Achard and Gannaz [6]. Estimators are consistent and have theoretical rates comparable to Fourier-based estimators. The numerical performances of wavelet-based and Fourier-based estimators are also similar, as illustrated in [22].

This paper considers linear processes, not necessarily Gaussian, with long-range dependence. The two main characteristics we are interested in are the long-range dependence parameters, which measure the LRD behavior of the processes, and long-run covariance, which captures the dependence structure between the components. Long-range dependence parameters and long-run covariance are estimated jointly with the procedure described in [6]. The aim of this paper is to establish the asymptotic normality of these estimators. Roueff and Taqqu [23] prove that this asymptotic normality is acquired in the univariate setting, but no result exists for wavelet-based estimation in the multivariate setting.

We first state that the sample covariances between the wavelet coefficients at a given scale are asymptotically Gaussian. We recover the univariate results of [23] but also provide the behavior of sample wavelet covariance between two processes, with possible different LRD parameters.

Asymptotic distributions of the estimators of long-range dependence parameters and long-run covariance of [6] are then obtained. The results highlight that multivariate estimation of LRD parameters decreases variance with respect to an estimation of LRD parameters component by component. Long-run covariance can be estimated by the sample wavelet correlation at a unique scale or by wavelet-based Whittle procedure, which aggregates the sample wavelet covariance at numerous scales. We highlight that, not surprisingly, Whittle estimation converges at a better rate. We also prove the asymptotic normality of the Whittle estimator. Moreover, test procedures can be built from the asymptotic normality theorems, for LRD parameters, and for long-run covariance.

The paper is organized as follows. Section 2 introduces the specific framework of our study. The LRD properties of the processes are described, and assumptions on a linear representation of the time series are given. The properties of the wavelet representation of the processes are also synthesized. The asymptotic behavior of the covariance between wavelet coefficients is provided in Section 3. Wavelet-based Whittle estimation is considered in Section 4. The asymptotic normality of the estimators is established. Section 5 illustrates the asymptotic normality of the estimators on a real data example, with the study of functional magnetic resonance images (fMRI) of a dead rat and a live rat.

2. The semiparametric multivariate long-range dependence framework

Let $\mathbf{X} = \{X_a(k), k \in \mathbb{Z}, a = 1, \dots, p\}$ be a multivariate stochastic process. We consider a process \mathbf{X} with long-range dependence parameters $\mathbf{d} = (d_1, d_2, \dots, d_p)$. The stationary framework corresponds to LRD parameters $d_i \in (-1.2, 1/2)$. In this case, following [14, 15], we suppose that the cross-spectral density satisfies: for all $\lambda \in [-\pi, \pi]$,

$$\text{for all } (a, b), f_{a,b}(\lambda) = \frac{1}{2\pi} \Omega_{a,b} (1 - e^{-i\lambda})^{-d_a} (1 - e^{i\lambda})^{-d_b} f_{a,b}^S(\lambda).$$

The functions $f_{a,b}^S(\cdot)$ correspond to the short-range dependence behavior of the process. This modelling is semiparametric since, if it imposes the LRD behavior, short-range dependence is left nonparametric through functions $f^S(\cdot)$. Some assumptions on $f^S(\cdot)$ are needed, which will be detailed below.

The LRD parameters, \mathbf{d} , model the long-run dynamics of the process. This model is a multivariate extension of a scalar fractionally integrated process (the so-called I(d) process), and for any $a \in \{1, \dots, p\}$, the time series X_a exhibits long-range dependence whenever $0 < d_a < 1/2$. The case $-1/2 < d_a < 0$ corresponds to antipersistence, where the spectral density $f_a(\cdot)$ tends toward 0 at the origin. The case $d_a = 0$ is the weak-dependence case, where the spectral density $f_a(\cdot)$ tends toward a positive constant at the origin. See [14, 15]. For simplicity, the term LRD is used throughout the paper, regardless of the values of \mathbf{d} .

Wavelet analysis performs an implicit differentiation, which offers the possibility to consider non stationary processes, that is, LRD parameters d_i possibly higher than $1/2$. Let \mathbb{L} denote the difference operator, $\mathbb{L}\mathbf{X}(t) = \mathbf{X}(t+1) - \mathbf{X}(t)$. The k th difference operator, \mathbb{L}^k , $k \in \mathbb{N}$, is defined by k recursive applications of \mathbb{L} . Introduce $\mathbf{D} \in \mathbb{N}^p$. We suppose that the multivariate process $\mathbf{Z} = \{\mathbb{L}^{D_a} X_a(k), k \in \mathbb{Z}, a = 1, \dots, p\}$ is covariance stationary with a spectral density matrix given by, for all $\lambda \in [-\pi, \pi]$:

$$\text{for all } (a, b) \in \{1, \dots, p\}^2, f_{a,b}^{(D_a, D_b)}(\lambda) = \frac{1}{2\pi} \Omega_{a,b} (1 - e^{-i\lambda})^{-d_a^*} (1 - e^{i\lambda})^{-d_b^*} f_{a,b}^S(\lambda),$$

where the long-range dependence parameters of \mathbf{Z} are given by $d_a^* \in (-1/2, 1/2)$ for all $a = 1, \dots, p$.

Let the overline be the conjugate operator and \circ be the Hadamard product. For any vector $\mathbf{v} \in \mathbb{R}^p$, $\text{diag}(\mathbf{v})$ stands for the $p \times p$ matrix with entries \mathbf{v} in the diagonal and 0 elsewhere.

The LRD assumption can be expressed as follows:

(M1) The generalized spectral density of the multivariate process \mathbf{X} is, for all $\lambda \in [-\pi, \pi]$,

$$\mathbf{f}(\lambda) = \mathbf{\Omega} \circ (\mathbf{\Lambda}^0(\mathbf{d})\mathbf{f}^S(\lambda)\overline{\mathbf{\Lambda}^0(\mathbf{d})}), \quad \text{with } \mathbf{\Lambda}^0(\mathbf{d}) = \text{diag}((1 - e^{-i\lambda})^{-\mathbf{d}}), \quad (1)$$

where $\mathbf{d} = \mathbf{D} + \mathbf{d}^*$, $\mathbf{D} \in \mathbb{N}^p$, $\mathbf{d}^* \in (-1/2, 1/2)^p$.

The matrix $\mathbf{\Omega}$ is called *fractal connectivity* by [24] or *long-run covariance* matrix by [25]. Similar to [26, 6] we introduce some regularity assumptions on the short-range dependence, modeled by function $\mathbf{f}^S(\cdot)$.

The space $\mathcal{H}_p(\beta, L)$ is defined as the class of non-negative symmetric functions $\mathbf{g}(\cdot) : [\pi, \pi] \rightarrow \mathbb{C}^{p \times p}$ such that $\mathbf{g}(0) = \mathbf{1}_{p \times p}$ and such that

$$\sup_{\lambda \in (-\pi, \pi)} |\mathbf{g}(\lambda) - \mathbf{1}_{p \times p}| \leq L|\lambda|^\beta,$$

with $\mathbf{1}_{p \times p}$ the $p \times p$ matrix with all entries equal to 1. We suppose that the following assumption is fulfilled:

(M2) $\mathbf{f}^S(\cdot) \in \mathcal{H}_p(\beta, L)$ with $0 < \beta \leq 2$ and $0 < L$.

Assumption (M2) imposes that $\mathbf{f}^S(0)$ has constant entries equal to 1. This assumption is necessary to make $\mathbf{\Omega}$ identifiable in (M1).

When λ tends toward 0, the spectral density matrix can be approximated at the first order by

$$\mathbf{f}(\lambda) \sim \tilde{\mathbf{\Lambda}}(\mathbf{d})\mathbf{\Omega}\overline{\tilde{\mathbf{\Lambda}}(\mathbf{d})}, \quad \text{with } \tilde{\mathbf{\Lambda}}(\mathbf{d}) = \text{diag}(|\lambda|^{-\mathbf{d}}e^{-i\pi\mathbf{d}/2}), \quad (2)$$

where \sim means that the ratio of the left- and right-hand sides converges to one.

Lobato [14] uses $\tilde{\mathbf{\Lambda}}(\mathbf{d}) = \text{diag}(\lambda^{-\mathbf{d}})$ as an approximation of $\mathbf{f}(\cdot)$ whereas Shimotsu [15] chooses to approximate $\mathbf{f}(\cdot)$ using $\tilde{\mathbf{\Lambda}}(\mathbf{d}) = \text{diag}(\lambda^{-\mathbf{d}}e^{-i(\pi-\lambda)\mathbf{d}/2})$, which corresponds to a second-order approximation due to the remaining term λ in the exponential. We refer to [6, Section 2.1] and references therein for examples of processes satisfying approximation (2).

2.1. Linear decomposition

We suppose hereafter that the multivariate process admits a linear representation.

(M3) There exists
a sequence $\{\mathbf{A}^{(\mathbf{D})}(u)\}_{u \in \mathbb{Z}}$ in $\mathbb{R}^{p \times p}$ such that $\sum_{u \in \mathbb{Z}} \max_{a,b=1,\dots,p} |A_{a,b}^{(\mathbf{D})}(u)|^2 < \infty$
and

$$\forall t \in \mathbb{Z}, \quad (\mathbb{L}^{D_a} X_a(t))_{a=1,\dots,p} = \sum_{u \in \mathbb{Z}} \mathbf{A}^{(\mathbf{D})}(t+u) \boldsymbol{\varepsilon}(u)$$

with $\boldsymbol{\varepsilon}(t)$ weak white noise process, in \mathbb{R}^p . Let \mathcal{F}_{t-1} denote the σ -field of events generated by $\{\boldsymbol{\varepsilon}(s), s \leq t-1\}$. Assume that $\boldsymbol{\varepsilon}$ satisfies $\mathbb{E}[\boldsymbol{\varepsilon}(t) | \mathcal{F}_{t-1}] = 0$, $\mathbb{E}[\varepsilon_a(t) \varepsilon_b(t) | \mathcal{F}_{t-1}] = \mathbb{1}_{a=b}$ and $\mathbb{E}[\varepsilon_a(t) \varepsilon_b(t) \varepsilon_c(t) \varepsilon_d(t) | \mathcal{F}_{t-1}] = \mu_{a,b,c,d}$ with $|\mu_{a,b,c,d}| \leq \mu_\infty < \infty$, for all $a, b, c, d = 1, \dots, p$.

Define for all $\lambda \in \mathbb{R}$, $\mathbf{A}^{(\mathbf{D})^*}(\lambda) = \sum_{t \in \mathbb{Z}} \mathbf{A}^{(\mathbf{D})}(t) e^{i\lambda t}$ the Fourier series associated to $\{\mathbf{A}^{(\mathbf{D})}(u)\}_{u \in \mathbb{Z}}$. That is, $\mathbf{A}^{(\mathbf{D})^*}(\lambda) = (A_{a,b}^{(\mathbf{D})^*}(\lambda))_{a,b=1,\dots,p}$ with

$$A_{a,b}^{(\mathbf{D})^*}(\lambda) = (2\pi)^{-1/2} \sum_{t \in \mathbb{Z}} A_{a,b}^{(\mathbf{D})}(t) e^{-i\lambda t}, \quad \lambda \in \mathbb{R}.$$

We add the following assumption:

(M4) For all $(a, b) \in \{1, \dots, p\}^2$, for all $\lambda \in \mathbb{R}$, the sequence $(2^{-j d_a} |A_{a,b}^{(\mathbf{D})^*}(2^{-j} \lambda)|)_{j \geq 0}$ is convergent as j goes to infinity.

This assumption is necessary for technical reasons. It does not seem restrictive.

An example of a process that satisfies these assumptions is the causal multivariate linear representations with trigonometric power law coefficients proposed in [10].

2.2. Wavelet representation

We introduce a discrete wavelet transform. Write $L^2(\mathbb{R})$ the set of square-integrable functions with respect to the Lebesgue measure. Let $\phi(\cdot)$ and $\psi(\cdot)$ be two functions of $L^2(\mathbb{R})$. Their Fourier transforms are given by $\widehat{\phi}(\lambda) = \int_{-\infty}^{\infty} \phi(t) e^{-i\lambda t} dt$ and $\widehat{\psi}(\lambda) = \int_{-\infty}^{\infty} \psi(t) e^{-i\lambda t} dt$, for all $\lambda \in \mathbb{R}$. We suppose that $\phi(\cdot)$ and $\psi(\cdot)$ satisfy the following assumptions:

- (W1) The functions $\phi(\cdot)$ and $\psi(\cdot)$ are integrable, have compact supports, $\int_{\mathbb{R}} \phi(t)dt = 1$ and $\int_{\mathbb{R}} \psi^2(t)dt = 1$.
- (W2) There exists $\alpha > 1$ such that $\sup_{\lambda \in \mathbb{R}} |\widehat{\psi}(\lambda)|(1 + |\lambda|)^\alpha < \infty$.
- (W3) The mother wavelet $\psi(\cdot)$ has $M > 1$ vanishing moments.
- (W4) The function $\sum_{k \in \mathbb{Z}} k^\ell \phi(\cdot - k)$ is polynomial with degree ℓ for all $\ell = 1, \dots, M - 1$.
- (W5) For all $i = 1, \dots, p$, $(1 + \beta)/2 - \alpha < d_i \leq M$.

Recall that β in (W5) is the regularity of the short-range dependence behavior introduced in (M2).

These assumptions are the same as the ones considered in [26, 21, 6]. Assumptions (W1)–(W4) are usual when considering that $\phi(\cdot)$ and $\psi(\cdot)$ are respectively the scaling-function and the wavelet-function associated with a multiresolution analysis [27]. They are satisfied, for example, by Daubechies wavelets. These wavelets are parametrized by the number of vanishing moments M . Assumption (W2) holds with α an increasing function of M going to infinity (see [28]). Assumptions (W1)–(W5) are fulfilled by Daubechies wavelet basis with sufficiently large M .

Assumption (W3) implies that the wavelet transform performs an implicit differentiation of order M and makes it possible to consider nonstationary processes. In Fourier analysis, tapering procedures are necessary to consider nonstationary frameworks, see *e.g.* [29, 30], and references therein.

At a given resolution $j \geq 0$, for $k \in \mathbb{Z}$, we define the dilated and translated functions $\phi_{j,k}(\cdot) = 2^{-j/2}\phi(2^{-j} \cdot - k)$ and $\psi_{j,k}(\cdot) = 2^{-j/2}\psi(2^{-j} \cdot - k)$. The wavelet coefficients of the process \mathbf{X} are defined by

$$\mathbf{W}(j, k) = \int_{\mathbb{R}} \widetilde{\mathbf{X}}(t)\psi_{j,k}(t)dt \quad j \geq 0, k \in \mathbb{Z},$$

where $\widetilde{\mathbf{X}}(t) = \sum_{k \in \mathbb{Z}} \mathbf{X}(k)\phi(t - k)$. For given $j \geq 0$ and $k \in \mathbb{Z}$, $\mathbf{W}(j, k)$ is a p -dimensional vector $\mathbf{W}(j, k) = (W_1(j, k) \ W_2(j, k) \ \dots \ W_p(j, k))^T$ where $W_a(j, k) = \int_{\mathbb{R}} \widetilde{X}_a(t)\psi_{j,k}(t)dt$, $a = 1, \dots, p$. Throughout the paper, we adopt the same convention as in [26] and [21]; that is, large values of the scale index j correspond to coarse scales (low frequencies). The index k is a location parameter, and $\mathbf{W}(j, k)$ captures information at scale j and location k on the behavior of the process \mathbf{X} .

In practice, let $\mathbf{X}(1), \dots, \mathbf{X}(N_X)$ denote the observations of the process \mathbf{X} . Since the wavelets have a compact support, only a finite number n_j of coefficients are non-null at each scale j . Suppose without loss of generality that the support of $\psi(\cdot)$ is included in $[0, T_\psi]$ with $T_\psi \geq 1$. For every $j \geq 0$, define

$$n_j := \max(0, 2^{-j}(N_X - T_\psi + 1) - T_\psi + 1).$$

At each scale j , the non-zero coefficients belong to $\{\mathbf{W}(j, k), k = 0, \dots, n_j\}$.

Let j_0 be the minimal scale and $j_1 = j_0 + \Delta$ the maximal scale which are considered in the estimation procedure. Following [21, 6], the asymptotic behavior is given for N_X and j_0 going to infinity. Results obtained in [21, 6, 22] state that optimal rates in estimation are obtained when j_0 is high enough to remove the scales affected by low-range dependence. In practice, the number of scales Δ is finite. Yet, considering the asymptotic behavior, two cases may be distinguished: either the number of scales Δ is finite and fixed when j_0 goes to infinity, or $\Delta = j_1 - j_0$ goes to infinity. The latter case seems natural, for example, when one takes all available scales above j_0 in estimation.

In the following, n will denote the number of wavelet coefficients used for estimation and $\langle \mathcal{J} \rangle$ the mean of scales, that is,

$$n = \sum_{j=j_0}^{j_1} n_j \quad \text{and} \quad \langle \mathcal{J} \rangle = \frac{1}{n} \sum_{j=j_0}^{j_1} n_j j.$$

Define also

$$\eta_\Delta := \sum_{u=0}^{\Delta} u \frac{2^{-u}}{2 - 2^{-\Delta}} \quad \text{and} \quad \kappa_\Delta := \sum_{u=0}^{\Delta} (u - \eta_\Delta)^2 \frac{2^{-u}}{2 - 2^{-\Delta}}.$$

These sequences converge respectively to 1 and to 2 when Δ goes to infinity [21, Lemma 13].

Moulines et al. state that under assumptions (W1)–(W5), the wavelet coefficient process $\{\mathbf{W}(j, k), k \in \mathbb{Z}\}$ is covariance stationary for any given $j \geq 0$ [26]. Let

$$\begin{aligned} D_{u,\tau}(\lambda; \delta) &= \sum_{t \in \mathbb{Z}} |\lambda + 2t\pi|^{-\delta} \overline{\widehat{\psi}(\lambda + 2t\pi)} 2^{u/2} \widehat{\psi}(2^u(\lambda + 2t\pi)) e^{-i2^u\tau(\lambda + 2t\pi)}, \\ \widetilde{D}_{u,\infty}(\lambda; \delta) &= \sum_{\tau=0}^{2^{-u}-1} D_{u,\tau}(\lambda; \delta). \end{aligned} \tag{3}$$

Moulines, Roueff and Taqqu [26] establish that $D_{u,\tau}(\lambda; \delta)$ is an approximation of the cross-spectral density between wavelet coefficients $\{\mathbf{W}(j, k), k \in \mathbb{Z}\}$ and $\{\mathbf{W}(j + u, 2^u k + \tau), \tau = 0, \dots, 2^u - 1, k \in \mathbb{Z}\}$. The parameter δ captures the long-range dependence of the processes. Indeed, the cross-spectral density of $(W_a(j, k), W_b(j + u, 2^j k' + \tau))$ is approximated by $D_{u,\tau}(\lambda; d_a + d_b)$. Function $\tilde{D}_{u,\infty}(\lambda; \delta)$ allows us to consider between-scales dependence.

For $u \geq 0$, $(\delta_1, \delta_2) \in (-\alpha, M)^2$, define

$$I_u(\delta_1, \delta_2) = \int_{-\pi}^{\pi} \overline{\tilde{D}_{u,\infty}(\lambda; \delta_1)} \tilde{D}_{u,\infty}(\lambda; \delta_2) d\lambda, \quad (4)$$

where $\tilde{D}_{u,\infty}(\lambda; \delta_2)$ is defined in (3). $I_u(\delta_1, \delta_2)$ will naturally appear when studying the covariance between sample wavelet covariances.

3. Asymptotic normality of sample wavelet covariances and correlations

Define $\hat{\sigma}_{a,b}(j)$ as the empirical covariance of the wavelet coefficients at a given scale $j \geq 0$, between components a and b , and let $\sigma_{a,b}(j)$ denote the theoretical covariance,

$$\begin{aligned} \hat{\sigma}_{a,b}(j) &= \frac{1}{n_j} \sum_{k=0}^{n_j-1} W_a(j, k) W_b(j, k), \\ \sigma_{a,b}(j) &= \mathbb{E}[W_a(j, k) W_b(j, k)]. \end{aligned}$$

Let $\hat{\Sigma}(j) = (\hat{\sigma}_{a,b}(j))_{a,b=1,\dots,p}$ and $\Sigma(j) = (\sigma_{a,b}(j))_{a,b=1,\dots,p}$ be the two associated matrices in $\mathbb{R}^{p \times p}$. In the following, for any matrix $\mathbf{M} \in \mathbb{C}^{p \times p}$, the maximal entry will be denoted by $\|\mathbf{M}\|_{\infty} = \max_{a,b=1,\dots,p} |M_{a,b}|$.

Proposition 2 in [6] proposes an approximation of the wavelet covariance at a given scale. It is recalled below.

Proposition 1 ([6]). *Suppose assumptions (M1)–(M2) and (W1)–(W5) hold. For all $j \geq 0$, for all $\lambda \in (-\pi, \pi)$,*

$$\|\Lambda(j)(\mathbf{d})^{-1} \Sigma(j) \Lambda(j)(\mathbf{d})^{-1} - \mathbf{G}\|_{\infty} \leq CL2^{-\beta j}. \quad (5)$$

with constant C depending on β , $\min_{\ell} d_{\ell}$, $\max_{\ell} d_{\ell}$, $\max_{\ell, m} |\Omega_{\ell, m}|$, ϕ and ψ and

$$\mathbf{\Lambda}_j(\mathbf{d}) = \text{diag}(2^{j\mathbf{d}}), \quad (6)$$

$$G_{a,b} = \Omega_{a,b} \cos(\pi(d_a - d_b)/2) K(d_a + d_b), \quad a, b = 1, \dots, p, \quad (7)$$

$$K(\delta) = \int_{-\infty}^{\infty} |\lambda|^{-\delta} |\widehat{\psi}(\lambda)|^2 d\lambda, \quad \delta \in (-\alpha, M).$$

For $u \geq 0$, $(\delta_1, \delta_2) \in (-\alpha, M)^2$, let us introduce $\widetilde{I}_u(\delta_1, \delta_2)$ as

$$\widetilde{I}_u(\delta_1, \delta_2) = 2\pi \frac{I_u(\delta_1, \delta_2)}{K(\delta_1)K(\delta_2)},$$

with $I_u(\delta_1, \delta_2)$ defined in (4). We also define $\mathbf{G} \cdot \widetilde{\mathbf{I}} \cdot \mathbf{G}(u) \in \mathbb{R}^{p^2 \times p^2}$ as:

$$\begin{aligned} \mathbf{G} \cdot \widetilde{\mathbf{I}} \cdot \mathbf{G}(u, u') &= \text{diag}\left(\text{vec}\left(\mathbf{\Lambda}_{u \wedge u'}(\mathbf{d})^{-1} \mathbf{G} \mathbf{\Lambda}_{u \wedge u'}(\mathbf{d})^{-1}\right)\right) \\ &\quad \left(\widetilde{\mathbf{I}}_{|u-u'|}(d_a + d_b, d_{a'} + d_{b'})_{(a,b),(a',b') \in \{1, \dots, p\}^2}\right) \text{diag}\left(\text{vec}\left(\mathbf{\Lambda}_{u \wedge u'}(\mathbf{d})^{-1} \mathbf{G} \mathbf{\Lambda}_{u \wedge u'}(\mathbf{d})^{-1}\right)\right). \end{aligned} \quad (8)$$

Remark 1. Observe that

$$\widetilde{I}_0(\delta_1, \delta_2) = \frac{2\pi \int_{-\pi}^{\pi} g_{\psi}(\lambda; \delta_1) g_{\psi}(\lambda; \delta_2) d\lambda}{\left(\int_{-\pi}^{\pi} g_{\psi}(\lambda; \delta_1) d\lambda\right) \left(\int_{-\pi}^{\pi} g_{\psi}(\lambda; \delta_2) d\lambda\right)}$$

where $g_{\psi}(\lambda; \delta) = \sum_{t \in \mathbb{Z}} |\lambda + 2t\pi|^{-\delta} |\widehat{\psi}(\lambda + 2t\pi)|^2$. It is straightforward that $\widetilde{I}_0(\delta_1, \delta_2) \leq 2\pi$. Cauchy-Schwarz's inequality on the denominator also provides $\widetilde{I}_0(\delta_1, \delta_1) \geq 1$.

Here and subsequently, $\xrightarrow{\mathcal{L}}$ denotes a convergence in distribution. The asymptotic distribution of the sample wavelet covariance process is given in the following theorem.

Theorem 2. For all $j_0 \geq 0$, $u \geq 0$, define

$$\begin{aligned} \widehat{\mathbf{T}}(j_0 + u) &= \text{vec}\left(2^{-(j_0+u)(d_a+d_b)} \widehat{\sigma}_{a,b}(j_0 + u), \quad a, b = 1, \dots, p\right) \\ \vec{\mathbf{G}} &= \text{vec}(G_{a,b}, \quad a, b = 1, \dots, p) \end{aligned}$$

where $\text{vec}(\mathbf{M})$ denotes the operation which transforms a matrix $\mathbf{M} \in \mathbb{R}^{p^1 \times p^2}$ in a vector of $\mathbb{R}^{p^1 p^2}$. Suppose assumptions (M1)–(M4) and (W1)–(W5) hold. Let $2^{-j_0 \beta} \rightarrow 0$ and $N_X^{-1} 2^{j_0} \rightarrow 0$. Then for all $\Delta \in \mathbb{N}$,

$$\left\{ \sqrt{n_{j_0+u}} \left(\widehat{\mathbf{T}}(j_0 + u) - \vec{\mathbf{G}} \right), \quad u = 0, \dots, \Delta \right\} \xrightarrow{j_0 \rightarrow \infty} \{ \mathbf{Q}(u), \quad u = 0, \dots, \Delta \},$$

where $\mathbf{Q}(\cdot)$ is a centered Gaussian process with covariance function $\text{Cov}(Q_{a,b}(u), Q_{a',b'}(u')) = V_{(a,b),(a',b')}(u, u')$ where

$$V_{(a,b),(a',b')}(u, u') = 2^{-|u-u'|/2} \left(\mathbf{G} \cdot \tilde{\mathbf{I}} \cdot \mathbf{G}_{(a,a'),(b,b')}(u, u') + \mathbf{G} \cdot \tilde{\mathbf{I}} \cdot \mathbf{G}_{(a,b'),(a',b)}(u, u') \right), \quad (9)$$

and $\mathbf{G} \cdot \tilde{\mathbf{I}} \cdot \mathbf{G}(u, u')$ is defined in (8).

The proof is given in Appendix C. It is similar to the one of the univariate setting given in [23, Theorem 2]. It relies on decimated processes and limit theorems developed in [31].

Remark 2. In the univariate setting, we obtain the same result as [23, Theorem 2]. The authors use a different normalization, by $\sqrt{N_X 2^{-j_0}}$ rather than by $\sqrt{n_{j_0+u}}$. The correspondence between the results follows first from the equivalence $\sqrt{n_{j_0+u}} \sim \sqrt{N_X 2^{-j_0-u}}$ and second from approximation (5) for the expectancy term.

The main difference in the multivariate case is that the LRD properties in two processes can be different. It introduces a bias term through the presence of the cosine term in \mathbf{G} (7) and slightly modifies the variance through terms $\tilde{\mathbf{I}}_u(\cdot, \cdot)$ (8).

Remark 3. In [32], Whitcher et al. establish the asymptotic normality for wavelet correlations of bivariate multivariate time series with long-range dependence. The advantage of Theorem 2 is to provide an explicit form of the asymptotic variance.

Remark 4. As already pointed out by Roueff and Taqqu [23], the covariance of the wavelet coefficients involves between-scales correlations which do not vanish when the sample size goes to infinity. This fact contrasts with the behavior of Fourier periodogram or Fourier-based Whittle estimation. In the variance formulation (9), these correlations appear through quantities $\{\tilde{\mathbf{I}}_u(\delta_1, \delta_2), u \geq 0, (a, b) \in \{1, \dots, p\}^2\}$.

We can deduce the asymptotic normality for sample wavelet correlations by means of delta method. We do not present here the multivariate result for the sake of brevity, except when the matrix \mathbf{G} is diagonal since formulas are more simple. We focus on the pointwise result to highlight the specificity of our setting.

Corollary 3. Let $(a, b) \in \{1, \dots, p\}^2$, $a \neq b$, and $j \geq j_0 \geq 0$. Define

$$\hat{\rho}_{a,b}(j) = \frac{\hat{\sigma}_{a,b}(j)}{\sqrt{\hat{\sigma}_{a,a}(j)\hat{\sigma}_{b,b}(j)}} \text{ and } r_{a,b} = \frac{G_{a,b}}{\sqrt{G_{a,a}G_{b,b}}}.$$

Then, under conditions of Theorem 2,

$$\sqrt{n_j} (\widehat{\rho}_{a,b}(j) - r_{a,b}) \xrightarrow{j \rightarrow \infty} \mathcal{N} \left(0, V_{a,b}^{(\rho)} \right)$$

with

$$\begin{aligned} V_{a,b}^{(\rho)} = & \left(\widetilde{I}_0(2d_a, 2d_b) + \widetilde{I}_0(d_a + d_b, d_a + d_b)(r_{a,b}^2 + r_{a,b}^4) \right. \\ & - (\widetilde{I}_0(2d_a, 2d_b) + \widetilde{I}_0(2d_b, d_a + d_b)) 2r_{a,b}^2 \\ & \left. - (\widetilde{I}_0(2d_a, 2d_a) + \widetilde{I}_0(2d_b, 2d_b)) r_{a,b}^2 / 2 \right). \quad (10) \end{aligned}$$

When all off-diagonal entries of \mathbf{G} are equal to 0,

$$\begin{aligned} & \sqrt{n_j} \text{vec}(\widehat{\rho}_{a,b}(j), 1 \leq a < b \leq p) \\ & \xrightarrow{j \rightarrow \infty} \mathcal{N}_{p(p-1)/2} \left(0, \text{diag} \left(\text{vec} \left(2^{-j(2d_a+2d_b)} \widetilde{I}_0(2d_a, 2d_b), 1 \leq a < b \leq p \right) \right) \right). \end{aligned}$$

Remark 5. Let $\widehat{\rho}$ denote the sample correlation of a bivariate-Gaussian-distributed n -sample with correlation ρ . Then $\sqrt{n}(\widehat{\rho} - \rho) \xrightarrow{n \rightarrow \infty} \mathcal{N} \left(0, (1 - \rho^2)^2 \right)$; see *e.g.* [33, Theorem 4.2.4]. When parameters d_a and d_b are equal, Corollary 3 entails that

$$\sqrt{n_j} (\widehat{\rho}_{a,b}(j) - r_{a,b}) \xrightarrow{j \rightarrow \infty} \mathcal{N} \left(0, \widetilde{I}_0(2d_a, 2d_a)(1 - r_{a,b}^2)^2 \right).$$

We recover a similar form of the asymptotic distribution, up to a normalization constant.

Remark 6. In [32, Section 4.2.] Whitcher et al. use the convergence of the Fisher transform of $\widehat{\rho}_{a,b}(j)$ to a standard Gaussian distribution at a rate $\sqrt{n_j}$, when the correlation $r_{a,b}(j)$ is equal to zero. The result is true if we suppose that between-scale wavelet coefficients are independent, which is asymptotically satisfied when the regularity of the wavelet goes to infinity [34]. Corollary 3 illustrates that an additional normalization by $\widetilde{I}_0(2d_a, 2d_b)^{-1/2}$ of $\widehat{\rho}_{a,b}(j)$ is necessary.

Some computed values of $\widetilde{I}_0(2d_a, 2d_b)$ are displayed in Table 1. It shows that $\widetilde{I}_0(2d_a, 2d_b)$ indeed decreases when the regularity increases. But between-scale wavelet coefficients dependence may not be negligible if the regularity is not high enough. For example, in the absence of long-range dependence, when $d_a = d_b = 0$, $\widetilde{I}_0(0, 0) = 1.62$ for Daubechies wavelets with $M = 4$ vanishing moments. Hence, in real data application, the approximation in [32] may lead to false positives.

		$\mathbf{d} = (d_1, d_2)$				
		(0, 0)	(0, 0.1)	(0, 0.2)	(0, 0.3)	(0, 0.4)
M= 1	$\alpha= 1.00$	5.43	5.56	5.67	5.77	5.86
M= 2	$\alpha= 1.34$	2.65	2.66	2.67	2.68	2.69
M= 3	$\alpha= 1.64$	1.85	1.86	1.86	1.86	1.87
M= 4	$\alpha= 1.91$	1.62	1.62	1.61	1.61	1.61
M= 5	$\alpha= 2.18$	2.05	2.04	2.04	2.03	2.02
M= 6	$\alpha= 2.43$	1.90	1.91	1.92	1.93	1.94
M= 7	$\alpha= 2.68$	1.22	1.23	1.24	1.25	1.26
M= 8	$\alpha= 2.93$	1.01	1.01	1.01	1.01	1.01

Table 1: Values of $\tilde{I}_0(2d_1, 2d_2)$ with respect to $\mathbf{d} = (d_1, d_2)$ for Daubechies's wavelets with different values of vanishing moments M in (W3). Parameter α characterizes the regularity of the wavelets in (W2).

Remark 7. The wavelet correlation at a given scale is also known as *wavelet coherence*. It is used in some applications, as in environmental studies by [32], or in neurosciences in [35]. In such real data applications, the crucial point is the use of test procedures. In particular, the test of the nullity of the correlations is essential. Corollary 3 shows that the asymptotic distribution depends on the long-range dependence parameters \mathbf{d} . Plugging in a consistent estimator of parameter \mathbf{d} in (10) allows for a test procedure to be built. For instance, one can use the wavelet Whittle estimator described in Section 4 below.

4. Asymptotic normality of the parameters estimates

For clarity, the true parameters are denoted with an exponent 0 in this part.

The wavelet-based local Whittle procedure proposes to estimate the parameters by maximizing a pseudo-likelihood given by a Gaussian approximation of the wavelet coefficients $\{\mathbf{W}(j, k), j \geq 0, k = 0, \dots, n_j\}$. Moulines et al. [21] and Achard and Gannaz [6] prove that the wavelet-based Whittle approximation provides consistent estimators even for non-Gaussian processes. The Whittle procedure can also be applied in multivariate cases, which is not possible for example with the regression of the wavelet log-scalogram [36, 24].

Let $\widehat{\mathbf{d}}$ and $\widehat{\boldsymbol{\Omega}}$ be the wavelet Whittle estimators as defined in [6, Section 3.3]. They maximize the objective function

$$\begin{aligned} \mathcal{L}(\mathbf{G}(\mathbf{d}), \mathbf{d}) = & \frac{1}{n} \sum_{j=j_0}^{j_1} \left[n_j \log \det (\boldsymbol{\Lambda}_j(\mathbf{d}) \mathbf{G}(\mathbf{d}) \boldsymbol{\Lambda}_j(\mathbf{d})) \right. \\ & \left. + \sum_{k=0}^{n_j-1} \mathbf{W}_{j,k}^T (\boldsymbol{\Lambda}_j(\mathbf{d}) \mathbf{G}(\mathbf{d}) \boldsymbol{\Lambda}_j(\mathbf{d}))^{-1} \mathbf{W}_{j,k} \right], \end{aligned}$$

where the superscript T denotes the transpose operator and $\boldsymbol{\Lambda}_j(\mathbf{d})$ and the matrix $\mathbf{G}(\mathbf{d})$ are defined respectively in (6) and (7).

The function $\mathcal{L}(\cdot, \cdot)$ corresponds to the negative log-likelihood of $\{\mathbf{W}(j, k), j \geq 0, k = 0, \dots, n_j\}$ under a Gaussian assumption, where Proposition 1 is used for a parametrization of the variance at each scale. The estimation of the vector of long-range dependence parameters \mathbf{d} satisfies $\widehat{\mathbf{d}} = \underset{\mathbf{d} \in \mathbb{R}^p}{\operatorname{argmin}} R(\mathbf{d})$, with

$$R(\mathbf{d}) = \log \det(\widehat{\mathbf{G}}(\mathbf{d})) + 2 \log(2) \left(\frac{1}{n} \sum_{j=j_0}^{j_1} j n_j \right) \left(\sum_{\ell=1}^p d_\ell \right). \quad (11)$$

The covariance matrix $\boldsymbol{\Omega}$ is estimated by

$$\begin{aligned} \widehat{\boldsymbol{\Omega}}_{a,b} = & \widehat{G}_{a,b}(\widehat{\mathbf{d}}) / (\cos(\pi(\widehat{d}_a - \widehat{d}_b)/2) K(\widehat{d}_a + \widehat{d}_b)), \quad a, b = 1, \dots, p, \\ \text{where } \widehat{\mathbf{G}}(\mathbf{d}) = & \frac{1}{n} \sum_{j=j_0}^{j_1} n_j \boldsymbol{\Lambda}_j(\mathbf{d})^{-1} \widehat{\boldsymbol{\Sigma}}(j) \boldsymbol{\Lambda}_j(\mathbf{d})^{-1}. \end{aligned} \quad (12)$$

We introduce

$$\begin{aligned} \mathcal{I}_\Delta^d(\delta_1, \delta_2) = & \frac{2}{\kappa_\Delta} \widetilde{I}_0(\delta_1, \delta_2) \\ & + \frac{2}{\kappa_\Delta^2} \sum_{u=1}^{\Delta} (2^{u\delta_1} + 2^{u\delta_2}) 2^{-u} \frac{2 - 2^{-\Delta+u}}{2 - 2^{-\Delta}} ((u + \eta_{\Delta-u} - \eta_\Delta)(\eta_{\Delta-u} - \eta_\Delta) + \kappa_{\Delta-u}) \widetilde{I}_u(\delta_1, \delta_2) \end{aligned} \quad (13)$$

if $\Delta < \infty$,

$$\mathcal{I}_\infty^d(\delta_1, \delta_2) = \widetilde{I}_0(\delta_1, \delta_2) + \sum_{u=1}^{\infty} (2^{u\delta_1} + 2^{u\delta_2}) 2^{-u} \widetilde{I}_u(\delta_1, \delta_2), \quad \text{if } \Delta = \infty. \quad (14)$$

Define also

$$\mathbf{G} \cdot \mathcal{I}^{\mathbf{d}} \cdot \mathbf{G}(\Delta) = \text{diag}\left(\text{vec}(\mathbf{G}^0)\right) \left(\mathcal{I}_{\Delta}^{\mathbf{d}}(d_a^0 + d_b^0, d_{a'}^0 + d_{b'}^0)_{(a,b),(a',b') \in \{1, \dots, p^2\}}\right) \text{diag}\left(\text{vec}(\mathbf{G}^0)\right). \quad (15)$$

The asymptotic normality of the estimator of the long-range dependence parameters is established by our next theorem.

Theorem 4. *Suppose assumptions (M1)–(M4) and (W1)–(W5) hold. Let $j_0 < j_1 \leq j_N$ with $j_N = \max\{j, n_j \geq 1\}$ such that*

$$j_1 - j_0 \rightarrow \Delta \in \{1, \dots, \infty\}, \log(N_X)^2(N_X 2^{-j_0(1+2\beta)} + N_X^{-1/2} 2^{j_0/2}) \rightarrow 0.$$

Then $\sqrt{n}(\widehat{\mathbf{d}} - \mathbf{d}^0)$ converges in distribution to a centered Gaussian distribution with a variance equal to

$$\mathbf{V}^{(\mathbf{d})}(\Delta) = \frac{1}{2 \log(2)^2} (\mathbf{G}^{0-1} \circ \mathbf{G}^0 + \mathbf{I}_p)^{-1} \Upsilon(\Delta) (\mathbf{G}^{0-1} \circ \mathbf{G}^0 + \mathbf{I}_p)^{-1}, \quad (16)$$

where \mathbf{I}_p is the identity matrix in $\mathbb{R}^{p \times p}$ and with entry (a, a') of $\Upsilon^{(\Delta)}$, for $(a, a') \in \{1, \dots, p\}^2$, given by

$$\Upsilon_{a,a'}(\Delta) = \sum_{b,b'=1, \dots, p} (G^{0-1})_{a,b} (G^{0-1})_{a',b'} (\mathbf{G} \cdot \mathcal{I}^{\mathbf{d}} \cdot \mathbf{G}_{(a,a'),(b,b')}(\Delta) + \mathbf{G} \cdot \mathcal{I}^{\mathbf{d}} \cdot \mathbf{G}_{(a,b'),(a',b)}(\Delta)) \quad (17)$$

where quantities $\mathbf{G} \cdot \mathcal{I}^{\mathbf{d}} \cdot \mathbf{G}(\Delta)$ are defined by (15).

The proof is given in Appendix F.

Remark 8. In the univariate setting, we recover [23, Theorem 5], using the equality

$$\sum_{v=0}^{\Delta-u} \frac{2^{-v}}{2 - 2^{-\Delta}} (v - \eta_{\Delta})(u + v - \eta_{\Delta}) = \frac{2 - 2^{-\Delta+u}}{2 - 2^{-\Delta}} ((u + \eta_{\Delta-u} - \eta_{\Delta})(\eta_{\Delta-u} - \eta_{\Delta}) + \kappa_{\Delta-u}),$$

in (13). Observe that the result is also normalized by \sqrt{n} rather than $\sqrt{N_X 2^{-j_0}}$.

Remark 9. The condition on j_0 and j_1 , that is, $\log(N_X)^2(N_X 2^{-j_0(1+2\beta)} + N_X^{-1/2} 2^{j_0/2}) \rightarrow 0$ is more restrictive than the condition required for the consistency of the estimators given in [6, Theorem 6]. Roueff and Taqqu [23, Theorem 5] obtain a similar result in the

univariate setting. As illustrated in [22], the condition $\log(N_X)^2 N_X^{-1/2} 2^{j_0/2} \rightarrow 0$ means that the highest frequencies, which are affected by the short-range dependence, should be removed from the estimation. The additional condition $\log(N_X)^2 N_X 2^{-j_0(1+2\beta)} \rightarrow 0$ prevents us from choosing the scale $j_0 = N_X^{1/(1+2\beta)}$ giving the minimax rate [6, Corollary 7]. Yet a near minimax rate is possible, with only a logarithmic loss, choosing, for example, $j_0 = \log(N_X)^3 N_X^{1/(1+2\beta)}$.

Remark 10. If the vector \mathbf{d}^0 has all entries equal to d^0 , the resulting covariance is

$$\frac{1}{4\log(2)^2} \mathcal{I}_\Delta(2d^0, 2d^0) (\mathbf{G}^{0-1} \circ \mathbf{G}^0 + \mathbf{I}_p)^{-1}.$$

We recognize a form of asymptotic variance similar to the ones given by [14], [15] and [20] with Fourier-based Whittle estimators. Note that they use a different approximation of spectral density at zero frequency. Lobato [14] and Shimotsu [15] consider respectively $G_{a,b}^0 = \Omega_{a,b} e^{i\pi(d_a - d_b)/2}$ and $\mathbf{G}^0 = \mathbf{\Omega}$. Düker and Pipiras's modelling in [20] is more general and does not suppose a linear representation of the time series. Their result is valid for a general form of matrix $\mathbf{\Omega}$. Additionally, Baek et al. [17] establish asymptotic normality of estimators in a bivariate model with possible co-integration.

Remark 11. Consider the bivariate setting with $\mathbf{\Omega} = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$ and $d_1 = d_2 = d$. Let \widehat{d}_1^U and \widehat{d}_2^U be the wavelet Whittle estimators obtained by separately considering the components $\{X_1(k), k = 1, \dots, N_X\}$ and $\{X_2(k), k = 1, \dots, N_X\}$ in. That is, for $i = 1, 2$,

$$\widehat{d}_i^U = \underset{d_i \in \mathbb{R}}{\operatorname{argmin}} R_i(d_i) \text{ with } R_i(d_i) = \log\left(\frac{1}{n} \sum_{j=j_0}^{j_1} n_j 2^{-2jd_i} \widehat{\sigma}_{ii}(j)\right) + 2 \log(2) \left(\frac{1}{n} \sum_{j=j_0}^{j_1} j n_j\right) d_i.$$

According to Theorem 4 \widehat{d}_i^U , $i = 1, 2$, are asymptotically normal, with the same asymptotic variance $\sigma^2(d, j_1 - j_0) = V^{(d)}(j_1 - j_0)$, given by (16).

Let now $\widehat{\mathbf{d}}$ be the bivariate wavelet Whittle estimator defined in (11). Theorem 4 provides the asymptotic normality of $\widehat{\mathbf{d}}$ with the asymptotic variance given by (16), which is equal to

$$\mathbf{V}^{(d)}(j_1 - j_0) = (\mathbf{G}^{0-1} \circ \mathbf{G}^0 + \mathbf{I}_p)^{-1} 2 \sigma^2(d, j_1 - j_0) = \begin{pmatrix} 1 - \rho^2/2 & \rho^2/2 \\ \rho^2/2 & 1 - \rho^2/2 \end{pmatrix} \sigma^2(d, j_1 - j_0).$$

This result proves that we reduce the entrywise variance when we perform multivariate estimation instead of univariate estimation. A similar conclusion was obtained for Fourier-based estimation by [14] and [16]. Achard and Gannaz [22] support this assertion

on simulated data. In real data application, [37] also establishes that the multivariate approach performs better than the univariate one, comparing their application on fMRI data where subjects were scanned twice.

Remark 12. Quantities $\mathcal{I}_\Delta(\delta_1, \delta_2)$ are computable for given $\delta_1, \delta_2, \Delta$. Hence, plugging in (16)–(17) consistent estimators of \mathbf{d} and \mathbf{G} , for example $\widehat{\mathbf{d}}$ and $\widehat{\mathbf{G}}(\widehat{\mathbf{d}})$ [6, Theorem 6], a test procedure on parameters \mathbf{d} can be built.

We now study the asymptotic behavior of the estimation of long-run covariance. We show the asymptotic normality of $\widehat{\mathbf{G}}(\widehat{\mathbf{d}})$, defined in (12).

Write

$$\begin{aligned}\mathcal{I}_\Delta^G(\delta_1, \delta_2) &= \widetilde{I}_0(\delta_1, \delta_2) + \sum_{u=1}^{\Delta} (2^{u\delta_1} + 2^{u\delta_2}) 2^{-u} \frac{2 - 2^{-\Delta+u}}{2 - 2^{-\Delta}} \widetilde{I}_u(\delta_1, \delta_2) && \text{if } \Delta < \infty, \\ \mathcal{I}_\infty^G(\delta_1, \delta_2) &= \widetilde{I}_0(\delta_1, \delta_2) + \sum_{u=1}^{\infty} (2^{u\delta_1} + 2^{u\delta_2}) 2^{-u} \widetilde{I}_u(\delta_1, \delta_2) && \text{if } \Delta = \infty.\end{aligned}$$

Let us also define

$$\mathbf{G} \cdot \mathcal{I}^{\mathbf{G}} \cdot \mathbf{G}(\Delta) = \text{diag}\left(\text{vec}(\mathbf{G}^0)\right) \left(\mathcal{I}_\Delta^G(d_a^0 + d_b^0, d_{a'}^0 + d_{b'}^0)_{(a,b),(a',b') \in \{1, \dots, p\}^2}\right) \text{diag}\left(\text{vec}(\mathbf{G}^0)\right). \quad (18)$$

We are now in a position to formulate the asymptotic distribution of $\widehat{G}(\widehat{\mathbf{d}})$.

Theorem 5. *Suppose Assumptions (M1)–(M4) and (W1)–(W5) hold. Let*

$$j_1 - j_0 \rightarrow \Delta \in \{1, \dots, \infty\}, \log(N_X)^2 (N_X 2^{-j_0(1+2\beta)} + N_X^{-1/2} 2^{j_0/2}) \rightarrow 0.$$

Then $\text{vec}\left(\sqrt{n}\left(\widehat{\mathbf{G}}(\widehat{\mathbf{d}}) - \mathbf{G}^0\right)\right)$ converges in distribution to a centered Gaussian distribution with a variance equal to $\mathbf{V}^{\mathbf{G}(\Delta)}$, with

$$V_{(a,b),(a',b')}^{(\mathbf{G})}(\Delta) = \mathbf{G} \cdot \mathcal{I}^{\mathbf{G}} \cdot \mathbf{G}_{(a,a'),(b,b')}(\Delta) + \mathbf{G} \cdot \mathcal{I}^{\mathbf{G}} \cdot \mathbf{G}_{(a,b'),(a',b)}(\Delta) \quad (19)$$

where quantities $\mathbf{G} \cdot \mathcal{I}^{\mathbf{G}} \cdot \mathbf{G}(\Delta)$ are defined by (18).

The proof is given in Appendix G.

We can deduce a convergence result for correlations.

Corollary 6. *Let $(a, b) \in \{1, \dots, p\}^2$, $a \neq b$. Define*

$$\widehat{r}_{a,b} = \frac{\widehat{G}_{a,b}(\widehat{\mathbf{d}})}{\sqrt{\widehat{G}_{a,a}(\widehat{\mathbf{d}})\widehat{G}_{b,b}(\widehat{\mathbf{d}})}} \text{ and } r_{a,b} = \frac{G_{a,b}^0}{\sqrt{G_{a,a}^0 G_{b,b}^0}}.$$

Then, under conditions of Theorem 5,

$$\sqrt{n} (\widehat{r}_{a,b} - r_{a,b}) \xrightarrow{j \rightarrow \infty} \mathcal{N} \left(0, V_{a,b}^{(\mathbf{r})}(\Delta) \right)$$

with

$$\begin{aligned} V_{a,b}^{(\mathbf{r})}(\Delta) &= \mathcal{I}_{\Delta}^G(2d_a, 2d_b) + \mathcal{I}_{\Delta}^G(d_a + d_b, d_a + d_b)(r_{a,b}^2 + r_{a,b}^4) \\ &\quad - (\mathcal{I}_{\Delta}^G(2d_a, 2d_b) + \mathcal{I}_{\Delta}^G(2d_b, d_a + d_b)) 2r_{a,b}^2 - (I_{\Delta}^G(2d_a, 2d_a) + \mathcal{I}_{\Delta}^G(2d_b, 2d_b)) r_{a,b}^2/2. \end{aligned}$$

When all off-diagonal entries of \mathbf{G} are equal to 0,

$$\sqrt{n} \text{vec}(\widehat{r}_{a,b}, 1 \leq a < b \leq p) \xrightarrow{j \rightarrow \infty} \mathcal{N}_{p(p-1)/2} \left(0, \text{diag}(\text{vec}(\widetilde{I}_{\Delta}^G(2d_a, 2d_b), 1 \leq a < b \leq p)) \right). \quad (20)$$

The proof is based on delta method, and it is similar to the proof of Corollary 3. It is thus omitted. The covariance structure of $\text{vec}(\widehat{r}_{a,b}, a, b = 1, \dots, p)$ can also be deduced from Theorem 5, but it is not displayed here.

Remark 13. The result is very similar to the one presented in Corollary 3. For all $(a, b) \in \{1, \dots, p\}^2$, the sequence $(\sqrt{n_j}(\rho_{a,b}(j) - r_{a,b}))_{j \geq 0}$ converges in distribution as j goes to infinity. The strength of Corollary 6 is that all the scales are used to estimate $r_{a,b}$, which reduces the variance. Indeed, $(\sqrt{n}(\widehat{r}_{a,b} - r_{a,b}))_{j \geq 0}$ converges in distribution as j goes to infinity, with $n = \sum_{j=j_0}^{j_1} n_j$.

Remark 14. When the LRD parameters are equal, *i.e.* $d_a = d_b$, Corollary 6 provides a more simple form, which is

$$\sqrt{n}(\widehat{r}_{a,b} - r_{a,b}) \xrightarrow{j \rightarrow \infty} \mathcal{N} \left(0, \mathcal{I}_{\Delta}^G(2d_a, 2d_a)(1 - r_{a,b}^2)^2 \right).$$

Remark 15. The asymptotic variances $\mathbf{V}^{(\mathbf{G})}(\Delta)$ and $\mathbf{V}^{(\mathbf{r})}(\Delta)$, given respectively in Theorem 5 and Corollary 6, depend on parameters \mathbf{d} . Test procedures can be built by plugging in $\mathbf{V}^{(\mathbf{G})}(\Delta)$ and $\mathbf{V}^{(\mathbf{r})}(\Delta)$ the estimator $\widehat{\mathbf{d}}$, which is consistent [6, Theorem 6].

Remark 16. In [20], Düker and Pipiras propose a global test for non-connectivity. That is, a test of $(H_0) \forall a \neq b, r_{a,b} = 0$ against $(H_1) \exists a \neq b, r_{a,b} \neq 0$. A similar test can be developed in our setting, based on (20). Another possibility is to perform the $p(p-1)/2$ tests of $(H_{0,a,b}) r_{a,b} = 0$ against $(H_{1,a,b}) r_{a,b} \neq 0$, for $1 \leq a < b \leq p$ and to apply a multiple testing correction on the p-values, for instance, Bonferroni's or Sidak's [38]. This approach may be less powerful than the previous one if we are interested in the global test, but it provides information on which correlations are significant.

We can go further than Theorem 4 and Theorem 5 by giving the joint distribution of estimators $\widehat{\mathbf{d}}$ and $\widehat{\mathbf{G}}(\widehat{\mathbf{d}})$.

Proposition 7. *Suppose assumptions of Theorem 5 hold.*

Let $\mathbf{T} = \left(\widehat{\mathbf{d}} - \mathbf{d}^0, \text{vec} \left(\widehat{\mathbf{G}}(\widehat{\mathbf{d}}) - \mathbf{G}^0 \right) \right)$.

Then $\sqrt{n} \mathbf{T}$ converges in distribution to a centered Gaussian distribution.

A proof is given in Appendix H. An explicit form of the asymptotic covariance term is given in (H.1)-(H.2). It is not displayed here to gain in clarity.

Remark 17. Baek et al. [17] and Düker and Pipiras [20] also find that the estimates of long-range dependence parameters and long-run covariance converge jointly to a Gaussian distribution in a Fourier-based Whittle estimation framework. As stated before, they consider a more general model, allowing for a complex-valued matrix $\mathbf{\Omega}$.

5. Illustration on real data

We illustrate here the asymptotically Gaussian behavior on real data rather than on simulations. We consider fMRI recordings on dead and live rats. The dataset is freely available at <https://zenodo.org/record/2452871> [39, 40]. The duration of scanning is 30 minutes with a time repetition of 0.5 second so that $N_X = 3,600$ time points are

available at the end of experience. After preprocessing as described in [41], we extracted $p = 51$ time series, each one being associated with a brain region of the rat. fMRI recordings of brain activity are based on the hemodynamic response to a magnetic field, which may create some temporal and spatial dependence. They suffer from different sources of noise, including system-related instabilities, subject motion, or physiological fluctuations [42]. Additionally, during the preprocessing step, we aggregate the time series of each voxel to obtain a unique time series for each brain region. This aggregation step may create LRD properties [43]. Our claim is that long-range dependence and long-run covariance are closely related to brain activity and not to recording artifacts or preprocessing. We would like to check this assertion on the dataset. This means that we expect $\mathbf{d}^0 = 0$ and a diagonal matrix \mathbf{G}^0 for a dead rat but not for a live one.

We estimate \mathbf{d} and \mathbf{G} by wavelet-based Whittle estimation, using *multiwave* package [22]. We follow the procedure described in [22, Section 5.2] to choose the scales. Estimation is performed taking $j_0 = 4$ and $j_1 = 9$, which is the maximal scale; that is, we remove the frequencies above 0.12 Hz.

Based on Theorem 4, for each rat, we can test if the LRD parameters are significant for each brain region. That is, for all $a = 1, \dots, p$, we test

$$(H_{0a}^{(\mathbf{d})}) d_a = 0 \text{ against } (H_{1a}^{(\mathbf{d})}) d_a \neq 0,$$

replacing \mathbf{d} and \mathbf{G} respectively by \hat{d} and $\hat{\mathbf{G}}(\hat{\mathbf{d}})$ in $\mathbf{V}^{(\mathbf{d})}$. We consider a level $\alpha' = 5\%$ and apply Bonferroni's multiple testing correction, *i.e.* each test is applied with a level α'/p to ensure that the probability to have a false positive on the p tests is equal to α' .

Next Corollary 6 allows to test the significance of the long-run correlation between each pair of brain regions. For all $1 \leq a < b \leq p$, we test

$$(H_{0a,b}^{(\mathbf{r})}) r_{a,b} = 0 \text{ against } (H_{1a,b}^{(\mathbf{r})}) r_{a,b} \neq 0.$$

Similarly, we apply Bonferroni's multiple testing correction and we consider a level $\alpha'/(p(p-1)/2)$ for each test.

The tests have been applied on one dead rat and one live rat. The results are displayed in Figure 1 as graphs. Figure 1 shows that, indeed, we can conclude that $\mathbf{d}^0 = 0$ and that off-diagonal entries of \mathbf{G}^0 are equal to zero for the dead rat. For the live rat, six brain regions (over 51) have a significant LRD parameter, and 483 correlations (over 1275 of $\{r_{a,b}, 1 \leq a < b \leq p\}$) are significant. These observations tend to confirm that long-range dependence and long-run covariance result from brain activity.

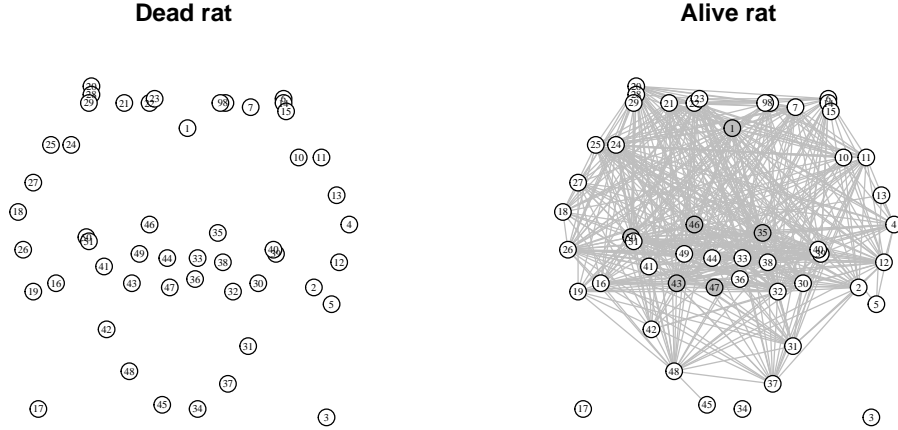


Figure 1: Inferred graphs of cerebral activity for a dead rat (left) and a live rat (right). Each vertex of the graph corresponds to a brain region. Colored vertices are regions where the LRD parameter d_a is significant, *i.e.* where the null hypothesis ($H_{0_a}^{(r)}$) is rejected. Two vertices a, b are connected by an edge if the long-run correlation is significant, *i.e.* if the null hypothesis ($H_{0_{a,b}}^{(r)}$) is rejected.

Conclusion

In this paper, we consider a multivariate process with long-range dependence properties, with a linear representation. We first establish that the covariance between wavelet coefficients is asymptotically Gaussian. The variance is explicitly given, and the convergence is established under mild assumptions on the wavelet transform and on the process. The asymptotic normality for the wavelet-based Whittle estimators defined in [6] is also established.

These results allow to perform statistical tests on the LRD parameters and on the long-run covariance. We propose an application on fMRI data, where we have recordings on a dead rat and alive one. The tests of significance on the LRD parameters and on the long-run correlations highlight that these characteristics are intrinsically linked to brain activity.

Appendix A. Expression of wavelet coefficients

Let $(a, b) \in \{1, \dots, p\}^2$. The objective is to study the asymptotic normality of the sample wavelet covariance $\{\widehat{\sigma}_{a,b}(j_0 + u), u = 1, \dots, \Delta\}$ when j_0 goes to infinity. To this end, we introduce a new indexing of wavelet coefficients similar to that in [23, pages 543 and 544]. This new indexing enables to approximate the sequence of wavelet coefficients with m -dependent variables and to use the results on linear decimated processes of [31]. The section is structured as follows. In Appendix A.1, we give a linear representation of wavelet coefficients, the new indexing of the coefficients is defined in Appendix A.2. Appendix A.3 finally introduces the approximation by a m -dependent process.

Appendix A.1. Linear representation of wavelet coefficients

Consider a scale $j \geq 0$ and $k \in \mathbb{Z}$. Define, for all $l \in \mathbb{Z}$, $h_{j,l} = \int_{\mathbb{R}} \phi(t+l)2^{-j/2}\psi(2^{-j}t)dt$, the discrete wavelet filter associated to $(\phi(\cdot), \psi(\cdot))$. Then under (W1), the vector of wavelet coefficients $\mathbf{W}(j, k)$ defined in Section 2.2 can be written as

$$\mathbf{W}(j, k) = \sum_{l \in \mathbb{Z}} h_{j,2^j k - l} \mathbf{X}(l),$$

with $\mathbf{W}(j, k) \in \mathbb{R}^p$. For all $\lambda \in \mathbb{R}$ let us denote

$$\mathbb{H}_j(\lambda) = \sum_{l \in \mathbb{Z}} h_{j,l} e^{-i\lambda l} = \int_{\mathbb{R}} \sum_{l \in \mathbb{Z}} \phi(t+l) 2^{-j/2} \psi(2^{-j}t) dt,$$

the discrete Fourier transform of $\{h_{j,l}, l \in \mathbb{Z}\}$.

Suppose that the multivariate process $\mathbf{X} = \{X_a(k), k \in \mathbb{Z}, a = 1, \dots, p\}$ satisfies Assumption (M3). To express wavelet coefficients, we introduce, for all $\lambda \in (-\pi, \pi)$,

$$\mathbf{A}^*(\lambda) = \text{diag}((1 - e^{i\lambda})^{-\mathbf{D}}) \mathbf{A}^{(\mathbf{D})^*}(\lambda), \quad (\text{A.1})$$

and $\{\mathbf{A}(t), t \in \mathbb{Z}\} \in \ell^2(\mathbb{Z})$ such that

$$\mathbf{A}^*(\lambda) = (2\pi)^{-1/2} \sum_{t \in \mathbb{Z}} \mathbf{A}(t) e^{i\lambda t}.$$

The function $\mathbf{A}^*(\lambda)$ satisfies

$$\mathbf{A}^*(\lambda)\overline{\mathbf{A}^*(\lambda)}^T = \mathbf{f}(\lambda), \quad (\text{A.2})$$

where $\mathbf{f}(\cdot)$ is defined in (M1).

For all $\lambda \in (-\pi, \pi)$, let

$$\mathbf{A}^*(j; \lambda) = \mathbb{H}_j(\lambda)\mathbf{A}^*(\lambda),$$

with $\mathbf{A}^*(\lambda)$ defined in (A.1). Let us also define $\{\mathbf{A}(j; t), t \in \mathbb{Z}\} \in \ell^2(\mathbb{Z})$ such that

$$\mathbf{A}^*(j; \lambda) = (2\pi)^{-1/2} \sum_{t \in \mathbb{Z}} \mathbf{A}(j; t) e^{i\lambda t}.$$

Then, the wavelet coefficients can be written as

$$\mathbf{W}(j, k) = \sum_{l \in \mathbb{Z}} \mathbf{A}(j; 2^j k - l) \boldsymbol{\varepsilon}(l).$$

Implicit differentiation by wavelet representation. As the wavelet ψ admits M vanishing moments under (W3), \mathbb{H}_j can be factorized as $\mathbb{H}_j(\lambda) = (1 - e^{i\lambda})^M \tilde{\mathbb{H}}_j(\lambda)$, with $\tilde{\mathbb{H}}_j$ trigonometric polynomial, $\tilde{\mathbb{H}}_j(\lambda) = \sum_{t \in \mathbb{Z}} \tilde{h}_{j,t} e^{it\lambda}$. It results that

$$W_a(j, k) = \sum_{l \in \mathbb{Z}} \tilde{h}_{j, 2^j k - l} (\mathbb{L}^M X_a)(l).$$

Appendix A.2. New indexing of wavelet coefficients

Let $j \geq 0$ and $k \in \{0, \dots, n_j - 1\}$. We introduce the new indexing proposed by [23]. Let $u = j - j_0$, $u \in \{0, \dots, \Delta\}$, and define (i, s) such that $k = 2^{\Delta-u}(s-1) + i$, with $i \in \{2^{\Delta-u}, \dots, 2^{\Delta-u+1} - 1\}$ and $s \in \mathbb{Z}$. We have $2^j k = 2^{j_1}(s-1) + 2^{u-\Delta}i$. Index i varies from 1 to $N = 2^{\Delta+1} - 1$ and each couple (j, k) corresponds to a unique couple (i, s) . We can rewrite wavelet coefficients as

$$\boldsymbol{\Lambda}_{j_0}(\mathbf{d})^{-1} \mathbf{W}(j, k) = \sum_{t \in \mathbb{Z}} \boldsymbol{\nu}^{(i, j_0)}(2^{j_1} s - t) \boldsymbol{\varepsilon}(t)$$

with $\boldsymbol{\Lambda}_{j_0}(\mathbf{d})$ is defined in (1) and

$$\boldsymbol{\nu}^{(i, j_0)}(t) = \boldsymbol{\Lambda}_{j_0}(\mathbf{d})^{-1} \mathbf{A}(j; 2^j(i - 2^{j_1-j}) + t), \quad j = j_1 - \lfloor \log_2(i) \rfloor,$$

where $\lfloor \log_2(i) \rfloor = \Delta - u$ is the integer part of $\log_2(i)$. Write

$$\mathbf{Z}^{(i,s,j_0)} = \sum_{t \in \mathbb{Z}} \mathbf{V}^{(i,j_0)}(2^{j_1} s - t) \boldsymbol{\varepsilon}(t) = \boldsymbol{\Lambda}_{j_0}(\mathbf{d})^{-1} \mathbf{W}(j, 2^{-\lfloor \log_2(i) \rfloor} (s-1) + i).$$

$\mathbf{W}(j, k)$, $\mathbf{V}^{(i,j_0)}$, $\mathbf{Z}^{(i,s,j_0)}$ belong respectively to \mathbb{R}^p , $\mathbb{R}^{p \times p}$ and \mathbb{R}^p .

For all $u = 0, \dots, \Delta$, denoting $j = j_0 + u$, the empirical variance satisfies

$$\begin{aligned} \boldsymbol{\Lambda}_{j_0}(\mathbf{d})^{-1} \widehat{\boldsymbol{\Sigma}}(j) \boldsymbol{\Lambda}_{j_0}(\mathbf{d})^{-1} &= \frac{1}{n_j} \sum_{k=0}^{n_j-1} \boldsymbol{\Lambda}_{j_0}(\mathbf{d})^{-1} \mathbf{W}(j, k) \mathbf{W}(j, k)^T \boldsymbol{\Lambda}_{j_0}(\mathbf{d})^{-1} \\ &= \frac{\sqrt{n_{j_1}}}{n_j} \sum_{i=2^{\Delta-u}}^{2^{\Delta-u+1}-1} \underbrace{n_{j_1}^{-1/2} \sum_{s=0}^{n_{j_1}-1} \mathbf{V}^{(i,s,j_0)}}_{\mathcal{S}^{(i,j_0)}} + n_j^{-1} \underbrace{\sum_{i=2^{\Delta-u}}^{T_\psi(2^{\Delta-u}-1)} \mathbf{V}^{(i,n_{j_1}(j))}}_{\mathcal{R}(j)} \end{aligned} \quad (\text{A.3})$$

with

$$\mathbf{V}^{(i,s,j_0)} = \mathbf{Z}^{(i,s,j_0)} \mathbf{Z}^{(i,s,j_0)T} = (\mathbf{Z}_a^{(i,s,j_0)} \mathbf{Z}_b^{(i,s,j_0)})_{a,b=1,\dots,p}.$$

Indeed, when $s \in \{0, \dots, n_{j_1} - 1\}$ and $i \in \{2^{\Delta-u}, \dots, 2^{\Delta-u+1} - 1\}$, index $k = 2^{\Delta-u}(s-1) + i$ varies in $\{0, \dots, 2^{\Delta-u}n_{j_1} - 1\}$, and when $s = n_{j_1}$ and $i \in \{2^{\Delta-u}, \dots, T_\psi(2^{\Delta-u}-1)\}$, index k varies from $2^{\Delta-u}n_{j_1}$ to $2^{\Delta-u}(n_{j_1} - 1) + T_\psi(2^{\Delta-u}-1) = 2^{-j}(N_X - T_\psi + 1) - T_\psi = n_j - 1$. That is,

$$\begin{aligned} \{k = 0, \dots, n_j - 1\} &= \{k = 2^{\Delta-u}(s-1) + i, s = 0, \dots, n_{j_1} - 1, i = 2^{\Delta-u}, \dots, 2^{\Delta-u+1} - 1\} \\ &\cup \{k = 2^{\Delta-u}(s-1) + i, s = n_{j_1}, i = 2^{\Delta-u}, \dots, T_\psi(2^{\Delta-u}-1)\}. \end{aligned}$$

The proof of Theorem 2 consists in establishing first the asymptotic normality of $\{\mathcal{S}^{(i,j_0)}, i = 1, \dots, N\}_{j_0 \geq 0}$ when j_0 goes to infinity, and second that $(\mathcal{R}(j))_{j \geq 0}$ is negligible. To prove the asymptotic normality of $\{\mathcal{S}^{(i,j_0)}, i = 1, \dots, N\}_{j_0 \geq 0}$, we will need to approximate the variables $\{\mathbf{V}^{(i,j_0)}, i = 1, \dots, N\}_{j_0 \geq 0}$ by m -dependent variables.

Appendix A.3. Approximation by a m -dependent process

Following [31], we introduce a non-negative infinitely differentiable function $H(\cdot)$ defined on \mathbb{R} such that $H(0) = 1$ and $H(t) = 0$ if $|t| > 1/2$. Write $\widehat{H}(\cdot)$ its Fourier transform,

$\widehat{H}(\lambda) = \int_{-\infty}^{\infty} H(t)e^{-i\lambda t} dt$. Since H is supposed infinitely derivable, when $|\lambda|$ tends to infinity, $\widehat{H}(\lambda)$ decreases to 0 faster than any polynomial. Hence, there exists $c_H > 0$ such that $|\widehat{H}(\lambda)| \leq c_H |\lambda|^{-\delta_v - 1}$ for all $|\lambda| \geq 1$, with δ_v defined in Lemma 10. Additionally, $\frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{H}(\lambda) d\lambda = H(0) = 1$.

Let us define, for all $t \in \mathbb{R}$, for all $\lambda \in \mathbb{R}$,

$$\begin{aligned} \mathbf{v}^{(i,j_0)(m)}(t) &= H(2^{-j_1}t/m) \mathbf{v}^{(i,j_0)}(t), \\ \mathbf{v}^{(i,j_0)(m)*}(\lambda) &= (2\pi)^{-1/2} \sum_{t \in \mathbb{Z}} \mathbf{v}^{(i,j_0)(m)}(t) e^{-i\lambda t} = \frac{m}{2\pi} \int_{-\infty}^{\infty} \widehat{H}(m\xi) \mathbf{v}^{(i,j)*}(\lambda - 2^{-j_1}\xi) d\xi, \\ \mathbf{z}^{(i,s,j_0)(m)} &= \sum_{t \in \mathbb{Z}} \mathbf{v}^{(i,j)(m)}(2^{j_1}s - t) \boldsymbol{\varepsilon}(t). \end{aligned}$$

Then for all $a, b = 1, \dots, p$, vectors $\mathbb{Z}_{a,b}^{(s,j_0)(m)} = \left(\mathbf{z}_a^{(1,s,j_0)(m)}, \dots, \mathbf{z}_a^{(N,s,j_0)(m)}, \mathbf{z}_b^{(1,s,j_0)(m)}, \dots, \mathbf{z}_b^{(N,s,j_0)(m)} \right)^T \in \mathbb{R}^N$ are m -dependent relatively to index s . That is, for all $q \geq 1$, for all (s_1, \dots, s_q) such that $s_{r+1} \geq s_r + m$ for $r = 1, \dots, q$, vectors $\mathbb{Z}_{a,b}^{(s_1,j_0)(m)}, \dots, \mathbb{Z}_{a,b}^{(s_q,j_0)(m)}$ are independent.

We will next study sequences $\{\mathcal{S}^{(i,j_0)(m)}, i = 1, \dots, N\}_{j_0 \geq 0}$ which are defined as follows:

$$\mathbf{y}^{(i,s,j_0)(m)} = \mathbf{z}^{(i,s,j_0)(m)} \mathbf{z}^{(i,s,j_0)(m)T}, \quad (\text{A.4})$$

$$\mathcal{S}^{(i,j_0)(m)} = n_{j_1}^{-1/2} \sum_{s=0}^{n_{j_1}-1} \mathbf{y}^{(i,s,j_0)(m)}. \quad (\text{A.5})$$

The outline of the proof of Theorem 2 is first to prove the asymptotic normality of $\{\mathcal{S}^{(i,j_0)(m)}, i = 1, \dots, N\}_{j_0 \geq 0}$ with the use of the results on decimated m -dependent processes of [23]. Next a similar result for $\{\mathcal{S}^{(i,j_0)}, i = 1, \dots, N\}_{j_0 \geq 0}$ is deduced by letting m go to infinity.

Appendix B. Notations and technical lemmas

This section provides some technical results on the quantities introduced in the wavelet representation and in the approximation by a m -dependent process, respectively in Appendix A.2 and Appendix A.3. These results will be used for the proof of Theorem 2.

Appendix B.1. Useful inequalities concerning the linear wavelet representation

We first give two lemmas, respectively on the behavior of the spectral density $f(\cdot)$ and of the function $A^*(\cdot)$.

Lemma 8. *Suppose (M1)–(M2) hold. Then there exists $C_f > 0$ depending on L , β and Ω such that for all $a, b = 1, \dots, p$, for all $\lambda \in (-\pi, \pi)$,*

$$|f_{a,b}(\lambda)| \leq C_f |\lambda|^{-d_a - d_b}.$$

Proof. Let $(a, b) \in \{1, \dots, p\}^2$ and $\lambda \in (-\pi, \pi)$. By (M1),

$$|f_{a,b}(\lambda)| \leq \max_{\ell, m} |\Omega_{\ell, m}| |1 - e^{i\lambda}|^{-d_a - d_b} |f_{a,b}^S(\lambda)|.$$

From Assumption (M2), $|f_{a,b}^S(\lambda)| \leq L(1 + \pi^\beta)$. Additionally, $|1 - e^{i\lambda}| = |2 \sin(\lambda/2)| \leq |\lambda|$. Lemma 8 follows with $C_f = L(1 + \pi^\beta) \max_{a,b=1,\dots,p} |\Omega_{a,b}|$. \square

Lemma 9. *Suppose (M1)–(M3) hold. Then there exists $C_A > 0$ depending on L , β and Ω such that for all $(a, b) \in \{1, \dots, p\}^2$,*

$$\left| \left(\mathbf{A}^*(\lambda) \overline{\mathbf{A}^*(\lambda)}^T \right)_{a,b} \right| \leq C_A |\lambda|^{-d_a - d_b}.$$

Proof. The lemma is straightforward combining (A.2) and Lemma 8. \square

Appendix B.2. Preliminary results on $(\mathbf{v}^{(i,j_0)})$

Define

$$\mathbf{v}^{(i,j_0)*}(\lambda) = (2\pi)^{-1/2} \sum_{t \in \mathbb{Z}} \mathbf{v}^{(i,j_0)}(t) e^{-i\lambda t}, \quad \lambda \in \mathbb{R}.$$

Observe that $[\mathbf{v}^{(i,j_0)}(2^{j_1}s - t)]^*(\lambda) = \overline{\mathbf{v}^{(i,j_0)*}(\lambda)} e^{-i2^{j_1}s\lambda}$.

For all $i, i' = 1, \dots, N$, for all $\lambda \in \mathbb{R}$, let us define also $\mathbf{W}^{(i,i')*}(\lambda) = (\mathcal{W}_{a,b}^{(i,i')*}(\lambda))_{a,b=1,\dots,p}$ with

$$\begin{aligned} \mathbf{W}^{(i,i')*}(\lambda) &= 2^{(u-\Delta)/2 + (u'-\Delta)/2} \overline{\widehat{\psi}(2^{u-\Delta}\lambda)} \widehat{\psi}(2^{u'-\Delta}\lambda) e^{i(2^{u-\Delta}(i-1) - 2^{u'-\Delta}(i'-1))\lambda} \\ &\quad \text{diag}(|\lambda|^{-\mathbf{d}} e^{-i \text{sgn}(\lambda) \pi \mathbf{d}/2}) \mathbf{\Lambda}_\Delta(\mathbf{d}) \mathbf{\Omega} \mathbf{\Lambda}_\Delta(\mathbf{d}) \text{diag}(|\lambda|^{-\mathbf{d}} e^{i \text{sgn}(\lambda) \pi \mathbf{d}/2}), \end{aligned} \quad (\text{B.1})$$

where $u = \Delta - \lfloor \log_2(i) \rfloor$, $u' = \Delta - \lfloor \log_2(i') \rfloor$.

We begin by providing some results on the behavior of $(\mathbf{v}^{(i,j_0)*}(\cdot))$ in a sequence of lemmas.

Lemma 10. *Suppose assumptions of Theorem 20 hold. Suppose $\Delta < \infty$. Then there exists $\delta_v > 1/2$ such that for all $j \geq j_0$, $j - j_0 \leq \Delta$, we have*

$$\sup_{|\lambda| < \pi} \left\| \mathbf{v}^{(i,j_0)*}(\lambda) \overline{\mathbf{v}^{(i,j_0)*}(\lambda)}^T \right\|_{\infty} \leq C_v 2^j (1 + 2^j |\lambda|)^{-2\delta_v},$$

with $j = j_0 + \Delta - \lfloor \log_2(i) \rfloor$ and $C_v = C_A C_{H1}^2 2^{\Delta(d_a+d_b)} < \infty$, depending on $L, \beta, \mathbf{\Omega}, \mathbf{d}, \Delta, \phi(\cdot)$ and $\psi(\cdot)$.

Proof. Recall that the Fourier transform $\mathbf{A}^*(\cdot)$ was defined in (A.1). Observe that

$$\mathbf{v}^{(i,j_0)*}(\lambda) \overline{\mathbf{v}^{(i,j_0)*}(\lambda)}^T = (2\pi)^{-1} |\mathbb{H}_j(\lambda)|^2 \mathbf{\Lambda}_{j_0}(\mathbf{d})^{-1} \mathbf{A}^*(\lambda) \overline{\mathbf{A}^*(\lambda)}^T \mathbf{\Lambda}_{j_0}(\mathbf{d})^{-1}.$$

Lemma 9 yields

$$\left| \left(\mathbf{v}^{(i,j_0)*}(\lambda) \overline{\mathbf{v}^{(i,j_0)*}(\lambda)}^T \right)_{a,b} \right| \leq C_A (2\pi)^{-1} |\mathbb{H}_j(\lambda)|^2 2^{(j-j_0)(d_a+d_b)} |2^j \lambda|^{-d_a-d_b}.$$

From (I.1) we get, for all $a, b = 1, \dots, p$,

$$\begin{aligned} & \left| \left(\mathbf{v}^{(i,j_0)*}(\lambda) \overline{\mathbf{v}^{(i,j_0)*}(\lambda)}^T \right)_{a,b} \right| \\ & \leq C_A C_{H1}^2 2^{(j-j_0)(d_a+d_b)} 2^j \left(\frac{2^j |\lambda|}{1 + 2^j |\lambda|} \right)^{2M-d_a-d_b} (1 + 2^j |\lambda|)^{-2\alpha-d_a-d_b}. \end{aligned}$$

Therefore, since $d_a + d_b < 2M$ and $0 \leq \frac{2^j |\lambda|}{1 + 2^j |\lambda|} \leq 1$,

$$\left| \left(\mathbf{v}^{(i,j_0)*}(\lambda) \overline{\mathbf{v}^{(i,j_0)*}(\lambda)}^T \right)_{a,b} \right| \leq C_A C_{H1}^2 2^{(j-j_0)(d_a+d_b)} 2^j (1 + 2^j |\lambda|)^{-2\alpha-d_a-d_b}.$$

Lemma 10, hence, holds with $C_v = C_A C_{H1}^2 2^{(j-j_0)M}$ and $\delta_v = \alpha + \max_{a=1, \dots, p} d_a$. Assumption (W5) ensures that $\delta_v > 1/2$. \square

The following lemma provides some convergence results on $(\mathbf{v}^{(i,j_0)*}(\cdot))$.

Lemma 11. For all $i = 1, \dots, N$, for all $\lambda \in \mathbb{R}$, there exist $\Phi^{(i,j_0)}(\lambda) \in (-\pi, \pi)^{p \times p}$ and $\mathcal{V}^{(i,\infty)*}(\lambda) \in \mathbb{C}^{p \times p}$, such that

$$2^{-j_1/2} \mathcal{V}^{(i,j_0)*}(2^{-j_1} \lambda) e^{-i\Phi^{(i,j_0)}(2^{-j_1} \lambda)} \xrightarrow{j_0 \rightarrow \infty} \mathcal{V}^{(i,\infty)*}(\lambda), \quad (\text{B.2})$$

$$2^{-j_1} \mathcal{V}^{(i,j_0)*}(2^{-j_1} \lambda) \overline{\mathcal{V}^{(i',j_0)*}(2^{-j_1} \lambda)}^T \xrightarrow{j_0 \rightarrow \infty} \mathcal{W}^{(i,i')*}(\lambda), \quad (\text{B.3})$$

where $\mathcal{W}^{(i,i')*}(\lambda)$ is defined in (B.1).

Proof.

Proof of (B.2). Let $\Phi^{(i,j_0)}(\lambda)$ be the arguments of $\mathcal{V}^{(i,j_0)*}(\lambda)$. Let $(a, b) \in \{1, \dots, p\}^2$. From (I.2), we have

$$\begin{aligned} & \left| 2^{-j_1/2} \mathcal{V}_{a,b}^{(i,j_0)*}(2^{-j_1} \lambda) e^{-i\Phi_{a,b}^{(i,j_0)}(2^{-j_1} \lambda)} - 2^{-\Delta/2 + \Delta d_a} \left| \widehat{\phi}(2^{-j_1} \lambda) \widehat{\psi}(2^{u-\Delta} \lambda) 2^{-j_1 d_a} A_{a,b}^*(2^{-j_1} \lambda) \right| \right| \\ & \leq C_{H2} 2^{j(1/2 - \alpha - M) + (u - \Delta)M} |\lambda|^M 2^{-\Delta/2 + \Delta d_a} \left| 2^{-j_1 d_a} A_{a,b}^*(2^{-j_1} \lambda) \right|. \end{aligned}$$

Lemma 9 gives the inequality

$$\left| 2^{-j_1 d_a} A_{a,b}^*(2^{-j_1} \lambda) \right|^2 \leq 2^{-2j_1 d_a} \sum_{a'=1}^p \left| A_{a,a'}^*(2^{-j_1} \lambda) \right|^2 \leq C_A |\lambda|^{-2d_a},$$

for all $a, b = 1, \dots, p$. Hence,

$$\begin{aligned} & \left| 2^{-j_1/2} \mathcal{V}_{a,b}^{(i,j_0)*}(2^{-j_1} \lambda) e^{-i\Phi_{a,b}^{(i,j_0)}(2^{-j_1} \lambda)} - 2^{-\Delta/2 + \Delta d_a} \left| \widehat{\phi}(2^{-j_1} \lambda) \widehat{\psi}(2^{u-\Delta} \lambda) 2^{-j_1 d_a} A_{a,b}^*(2^{-j_1} \lambda) \right| \right| \\ & \leq C_{H2} C_A |\lambda|^{M - 2d_a} 2^{-\Delta/2 + \Delta d_a + (u - \Delta)M} 2^{j(1/2 - \alpha - M)}. \end{aligned}$$

Since $1/2 - \alpha - M < 0$, we obtain that the right-hand side goes to 0 when j_0 goes to infinity. By continuity, $\left| \widehat{\phi}(2^{-j_1} \lambda) \right|$ tends to $\left| \widehat{\phi}(0) \right| = 1$ when j_0 goes to infinity. We conclude (B.2) by Assumption (M4), which supposes that $2^{-j_1 d_a} A_{a,b}^*(2^{-j_1} \lambda)$ converges when j_1 goes to infinity.

Proof of (B.3). By equality (A.2), we get

$$\begin{aligned} & \left\| 2^{-j_1} \mathbf{v}^{(i,j_0)*}(2^{-j_1}\lambda) \overline{\mathbf{v}^{(i',j_0)*}(2^{-j_1}\lambda)}^T - 2^{-j_1+j/2+j'/2} |\widehat{\phi}(2^{-j_1}\lambda)|^2 \overline{\widehat{\psi}(2^{j-j_1}\lambda)} \widehat{\psi}(2^{j'-j_1}\lambda) \right. \\ & \quad \left. \mathbf{\Lambda}_{j_0}(\mathbf{d})^{-1} \mathbf{f}(2^{-j_1}\lambda) \mathbf{\Lambda}_{j_0}(\mathbf{d})^{-1} e^{i(2^{u-\Delta}i - 2^{u'-\Delta}i')\lambda} \right\|_{\infty} \\ & \leq 2^{-j_1} \left\| \mathbb{H}_j(2^{-j_1}\lambda) \overline{\mathbb{H}_{j'}(2^{-j_1}\lambda)} \right. \\ & \quad \left. - 2^{j/2+j'/2} |\widehat{\phi}(2^{-j_1}\lambda)|^2 \overline{\widehat{\psi}(2^{j-j_1}\lambda)} \widehat{\psi}(2^{j'-j_1}\lambda) \mathbf{\Lambda}_{j_0}(\mathbf{d})^{-1} \mathbf{f}(2^{-j_1}\lambda) \mathbf{\Lambda}_{j_0}(\mathbf{d})^{-1} \right\|_{\infty} \end{aligned}$$

Inequality (I.3) gives that the right-hand side can be bounded by

$$C_{H3} 2^{-j_1} 2^{(j+j')(1/2-\alpha-M)} 2^{(u+u'-2\Delta)M} |\lambda|^{2M} \left\| \mathbf{\Lambda}_{j_0}(\mathbf{d})^{-1} \mathbf{f}(2^{-j_1}\lambda) \mathbf{\Lambda}_{j_0}(\mathbf{d})^{-1} \right\|_{\infty}.$$

With Lemma 8, the bound becomes

$$\max_{a,b=1,\dots,p} C_{H3} C_f 2^{-(2j_0+u+u')(\alpha+M)} 2^{(u+u'-2\Delta)(M+1/2)} 2^{\Delta(d_a+d_b)} |\lambda|^{2M-d_a-d_b}.$$

This term goes to 0 when j_0 goes to infinity uniformly for $\lambda \in (-\pi, \pi)$.

As $\left| \widehat{\phi}(2^{-j_1}\lambda) \right| \xrightarrow{j_0 \rightarrow \infty} \left| \widehat{\phi}(0) \right| = 1$ and as $\mathbf{f}(2^{-j_1}\lambda)$ satisfies approximation (2), we obtain convergence (B.3). \square

We introduce some useful notations. For $(i, i') \in \{1, \dots, N\}^2$, $t \in \mathbb{Z}$, $\lambda \in \mathbb{R}$, let

$$\begin{aligned} \widetilde{\mathbf{v}}^{(i,\infty)}(t) &= \frac{1}{\sqrt{2\pi}} \int \mathbf{v}^{(i,\infty)*}(\lambda) e^{i\lambda t} d\lambda, \\ \widetilde{\mathbf{w}}^{(i,i')*}(\lambda) &= \sum_{t \in \mathbb{Z}} \mathbf{w}^{(i,i')*}(\lambda + 2t\pi), \end{aligned} \quad (\text{B.4})$$

where $\mathbf{v}^{(i,\infty)*}(\cdot)$ and $\mathbf{w}^{(i,i')*}(\cdot)$ have been defined respectively in (B.2) and in (B.1). We also define, for $(s, s') \in \{1, \dots, N\}^2$, $(a, b, a', b') \in \{1, \dots, p\}^4$,

$$\Theta^{((i,s),(i',s'))} = \int_{-\infty}^{\infty} \mathbf{w}^{(i,i')*}(\lambda) e^{-i(s-s')\lambda} d\lambda, \quad (\text{B.5})$$

$$\Gamma_{(a,b),(a',b')}^{(i,i')} = 2\pi \int_{-\pi}^{\pi} \overline{\widetilde{\mathcal{W}}_{a,a'}^{(i,i')*}(\lambda)} \widetilde{\mathcal{W}}_{b,b'}^{(i,i')*}(\lambda) d\lambda + 2\pi \int_{-\pi}^{\pi} \overline{\widetilde{\mathcal{W}}_{a,b'}^{(i,i')*}(\lambda)} \widetilde{\mathcal{W}}_{b,a'}^{(i,i')*}(\lambda) d\lambda. \quad (\text{B.6})$$

We can first state a result on Θ . This result is useful to show that $\Gamma^{(i,i')}$ is the asymptotic covariance matrix of $(\mathcal{S}^{(i,j_0)}, \mathcal{S}^{(i',j_0)})$ as j_0 goes to infinity. This will be proved in Lemma 16.

Lemma 12. For all $(i, i') \in \{1, \dots, N\}^2$,

$$\forall j_0 \geq 0, \forall a = 1, \dots, p, \sup_{s \in \mathbb{Z}} \sum_{t \in \mathbb{Z}} \sum_{a'=1}^p \mathcal{V}_{a,a'}^{(i,j_0)} (2^{j_1} s - t)^2 < \infty, \quad (\text{B.7})$$

$$\forall (s, s') \in \mathbb{Z}^2, \sum_{t \in \mathbb{Z}} \mathcal{V}^{(i,j_0)}(2^{j_1} s - t) \mathcal{V}^{(i',j_0)}(2^{j_1} s' - t)^T \xrightarrow{j_0 \rightarrow \infty} \Theta^{((i,s),(i',s'))}, \quad (\text{B.8})$$

with $\Theta^{((i,s),(i',s'))}$ defined in (B.5).

Moreover, for all $(a, b) \in \{1, \dots, p\}^2$, for all $i \in \{1, \dots, N\}$,

$$\sup_{t \in \mathbb{Z}} \sum_{s \in \mathbb{Z}} \mathcal{V}_{a,b}^{(i,j_0)} (2^{j_1} s - t)^2 \xrightarrow{j_0 \rightarrow \infty} 0. \quad (\text{B.9})$$

Proof.

Proof of (B.7). By Parseval's identity and a change of variable, for all $(a, a') \in \{1, \dots, p\}^2$,

$$\sum_{t \in \mathbb{Z}} \mathcal{V}_{a,a'}^{(i,j_0)} (2^{j_1} s - t)^2 = \int_{-\pi}^{\pi} |\mathcal{V}_{a,a'}^{(i,j_0)*}(\lambda)|^2 d\lambda \leq C_v \int_{-\infty}^{\infty} (1 + |\lambda|)^{-2\delta_v} d\lambda,$$

which implies (B.7).

Proof of (B.8). Applying Parseval's theorem and the change of variable $\lambda \rightarrow 2^{j_1} \lambda$, we get the equality

$$\begin{aligned} \sum_{t \in \mathbb{Z}} \mathcal{V}^{(i,j_0)}(2^{j_1} s - t) \mathcal{V}^{(i',j_0)}(2^{j_1} s' - t)^T \\ = \int_{-2^{j_1} \pi}^{2^{j_1} \pi} 2^{-j_1} \mathcal{V}^{(i,j_0)*}(2^{-j_1} \lambda) \overline{\mathcal{V}^{(i',j_0)*}(2^{-j_1} \lambda)}^T e^{i(s-s')\lambda} d\lambda. \end{aligned}$$

The function under the integral converges to $\mathcal{W}^{(i,i')*}(\lambda) e^{i(s-s')\lambda}$ by (B.3). Convergence under the integral can be applied thanks to dominated convergence, by Lemma 10. This gives (B.8).

Proof of (B.9). Observe that

$$\mathcal{V}_{a,b}^{(i,j_0)}(2^{j_1}s - t) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \overline{\mathcal{V}_{a,b}^{(i,j_0)*}(\lambda)} e^{-i(2^{j_1}s+t)\lambda} d\lambda.$$

Since the function $\lambda \rightarrow \mathcal{V}_{a,b}^{(i,j_0)*}(\lambda)$ is 2π -periodic, [31, Lemma 4], leads to

$$\begin{aligned} \mathcal{V}_{a,b}^{(i,j_0)}(2^{j_1}s - t) &= \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} 2^{-j_1} \sum_{l=0}^{2^{j_1}-1} \mathcal{V}_{a,b}^{(i,j_0)*}(2^{-j_1}(\lambda + 2\pi l)) e^{-i(2^{j_1}s-t)(2^{-j_1}(\lambda+2\pi l))} d\lambda \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \left[2^{-j_1} \sum_{l=0}^{2^{j_1}-1} \mathcal{V}_{a,b}^{(i,j_0)*}(2^{-j_1}(\lambda + 2\pi l)) e^{-it 2^{-j_1}(\lambda+2\pi l)} \right] e^{is\lambda} d\lambda. \end{aligned}$$

Parseval's identity entails that

$$\sum_{s \in \mathbb{Z}} \left| \mathcal{V}_{a,b}^{(i,j_0)}(2^{j_1}s - t) \right|^2 = \int_{-\pi}^{\pi} \left| 2^{-j_1} \sum_{l=0}^{2^{j_1}-1} \mathcal{V}_{a,b}^{(i,j_0)*}(2^{-j_1}(\lambda + 2\pi l)) e^{-it 2^{-j_1}(\lambda+2\pi l)} \right|^2 d\lambda.$$

Hence,

$$\sum_{s \in \mathbb{Z}} \left| \mathcal{V}_{a,b}^{(i,j_0)}(2^{j_1}s - t) \right|^2 \leq 2^{-j_1} \int_{-\pi}^{\pi} \left(\sum_{l=0}^{2^{j_1}-1} 2^{-j_1/2} \left| \mathcal{V}_{a,b}^{(i,j_0)*}(2^{-j_1}(\lambda + 2\pi l)) \right| \right)^2 d\lambda.$$

Lemma 10 implies that

$$\sum_{s \in \mathbb{Z}} \left| \mathcal{V}_{a,b}^{(i,j_0)}(2^{j_1}s - t) \right|^2 \leq C_v 2^{-j_1} \int_{-\pi}^{\pi} \left(\sum_{l=0}^{2^{j_1}-1} (1 + |\lambda + 2\pi l|)^{-\delta_v} \right)^2 d\lambda.$$

We can deduce the following inequalities

$$\begin{aligned} \sum_{s \in \mathbb{Z}} \left| \mathcal{V}_{a,b}^{(i,j_0)}(2^{j_1}s - t) \right|^2 &\leq C_v 2^{-j_1} \int_{-\pi}^{\pi} \left(1 + \sum_{l=0}^{2^{j_1}-2} (1 + |2\pi l|)^{-\delta_v} \right)^2 d\lambda \\ &\leq C_v 2^{-j_1} 2\pi \left(1 + \int_0^{2^{j_1}-2} (1 + |2\pi \xi|)^{-\delta_v} d\xi \right)^2 \\ &\leq C_v 2^{-j_1} 2\pi \left(1 + (1 + |2\pi 2^{j_1}|)^{1-\delta_v} \right)^2 \\ &\leq C_v 4\pi 2^{-j_1} \left(1 + (2\pi)^2 2^{j_1(2-2\delta_v)} \right). \end{aligned}$$

The right-hand side goes to 0 when j_1 goes to infinity since $\delta_v > 1/2$. □

Appendix B.3. Preliminary results on the m -dependent processes

We define similar quantities than in Appendix B.2 in the m -dependent setting. That is,

$$\begin{aligned}
\mathbf{v}^{(i,\infty)(m)}(t) &= H(2^{-j_1}t/m) \mathbf{v}^{(i,\infty)}(t), \\
\mathbf{v}^{(i,\infty)(m)*}(\lambda) &= (2\pi)^{-1/2} \sum_{t \in \mathbb{Z}} \mathbf{v}^{(i,\infty)(m)}(t) e^{-i\lambda t}, \\
\mathbf{w}^{(i,i')(m)*}(\lambda) &= \mathbf{v}^{(i,\infty)(m)*}(\lambda) \overline{\mathbf{v}^{(i',\infty)(m)*}(\lambda)}^T, \\
\widetilde{\mathbf{w}}^{(i,i')(m)*}(\lambda) &= \sum_{t \in \mathbb{Z}} \mathbf{w}^{(i,i')*}(\lambda + 2t\pi).
\end{aligned} \tag{B.10}$$

We also denote

$$\Theta^{((i,s),(i',s'))(m)} = \int_{-\infty}^{\infty} \mathbf{w}^{(i,i')(m)*}(\lambda) e^{-i(s-s')\lambda} d\lambda, \tag{B.11}$$

$$\Gamma_{(a,b),(a',b')}^{(i,i')(m)} = 2\pi \int_{-\pi}^{\pi} \overline{\widetilde{\mathcal{W}}_{a,a'}^{(i,i')(m)*}(\lambda)} \widetilde{\mathcal{W}}_{b,b'}^{(i,i')(m)*}(\lambda) d\lambda + 2\pi \int_{-\pi}^{\pi} \overline{\widetilde{\mathcal{W}}_{a,b'}^{(i,i')(m)*}(\lambda)} \widetilde{\mathcal{W}}_{b,a'}^{(i,i')(m)*}(\lambda) d\lambda. \tag{B.12}$$

We will prove in Appendix C.3 that $\mathbf{\Gamma}^{(i,i')(m)}$ is the asymptotic covariance matrix of $(\mathcal{S}^{(i,j_0)(m)}, \mathcal{S}^{(i',j_0)(m)})$ as j_0 goes to infinity.

We now provide some general results on the behavior of $(\mathbf{v}^{(i,j_0)(m)})$ and $(\mathbf{v}^{(i,j_0)(m)*})$, in much the same way as in Lemma 10, Lemma 11 and Lemma 12 for $(\mathbf{v}^{(i,j_0)})$ and $(\mathbf{v}^{(i,j_0)*})$.

Lemma 13. *Suppose assumptions of Theorem 20 hold. Suppose $\Delta < \infty$. Then there exists $\delta_v > 1/2$ such that for all $j \geq j_0$, $j - j_0 \leq \Delta$, we have*

$$\left\| \mathbf{v}^{(i,j_0)(m)*}(\lambda) \right\|_{\infty} \leq C_{vm} 2^{j/2} (1 + 2^j |\lambda|)^{-\delta_v}.$$

with $j = j_0 + \Delta - \lfloor \log_2(i) \rfloor$ and $C_{vm} < \infty$, depending on $m, L, \beta, \mathbf{\Omega}, \mathbf{d}, \Delta, \phi(\cdot)$ and $\psi(\cdot)$.

Proof. The lemma follows from Lemma 10 and [31, Lemma 5]. \square

Lemma 14. *Suppose assumptions of Theorem 20 hold. For all $i = 1, \dots, N$, for all $m \geq 1$, sequences $\{\mathcal{V}_{a,b}^{(i,j_0)(m)}, a, b = 1, \dots, p, j_0 \geq 0\}$ verify the following properties:*

$$\forall j_0 \geq 0, \forall a = 1, \dots, p, \sum_{t \in \mathbb{Z}} \sum_{a'=1}^p \mathcal{V}_{a,a'}^{(i,j_0)(m)} (2^{j_1} s - t)^2 < \infty,$$

for all $(a, b) \in \{1, \dots, p\}^2$, for all $i \in \{1, \dots, N\}$,

$$\sup_{t \in \mathbb{Z}} \sum_{s \in \mathbb{Z}} \mathcal{V}_{a,b}^{(i,j_0)(m)} (2^{j_1} s - t)^2 \xrightarrow{j_0 \rightarrow \infty} 0.$$

Moreover for all $m \geq 1$, for all $(a, b) \in \{1, \dots, p\}^2$, for all $(i, i') \in \{1, \dots, N\}^2$ and $(s, s') \in \{0, \dots, n_{j_1} - 1\}^2$,

$$\sum_{t \in \mathbb{Z}} \mathcal{V}^{(i,j_0)(m)} (2^{j_1} s - t) \mathcal{V}^{(i',j_0)(m)} (2^{j_1} s' - t)^T \xrightarrow{j_0 \rightarrow \infty} \Theta^{((i,s),(i',s'))(m)}, \quad (\text{B.13})$$

with $\Theta^{((i,s),(i',s'))(m)}$ defined in (B.11).

The proof is similar to that of Lemma 12 and it is thus omitted.

Appendix B.4. Asymptotic variance of $(\mathcal{S}^{(i,j_0)}, i = 1, \dots, N)_{j_0 \geq 0}$

Lemma 15 below studies the behavior of $\{\Theta^{((i,s),(i',s'))}, i, i' = 1, \dots, N, s, s' \geq 0\}$ and $\{\Theta^{((i,s),(i',s'))(m)}, i, i' = 1, \dots, N, s, s' \geq 0\}$ when summing over the parameters (s, s') . It is used next to prove Lemma 16 which establishes that the asymptotic covariances of $(\mathcal{S}^{(i,j_0)}, i = 1, \dots, N)_{j_0 \geq 0}$ are equal to $(\Gamma^{(i,i')}, i, i' = 1, \dots, N)$ when j_0 goes to infinity.

Lemma 15. *Suppose conditions of Theorem 2 hold. For all $(a, b, a', b') \in \{1, \dots, p\}^4$, for all $(i, i') \in \{1, \dots, N\}$,*

$$\lim_{\ell \rightarrow \infty} \ell^{-1} \sum_{s, s' = 0, \dots, \ell-1} \Theta_{a,b}^{((i,s),(i',s'))} \Theta_{a',b'}^{((i,s),(i',s'))} = 2\pi \int_{-\pi}^{\pi} \overline{\widetilde{\mathcal{W}}_{a,b}^{(i,i')*}(\lambda)} \widetilde{\mathcal{W}}_{a',b'}^{(i,i')*}(\lambda) d\lambda, \quad (\text{B.14})$$

$$\lim_{\ell \rightarrow \infty} \ell^{-1} \sum_{s, s' = 0, \dots, \ell-1} \Theta_{a,b}^{((i,s),(i',s'))(m)} \Theta_{a',b'}^{((i,s),(i',s'))(m)} = 2\pi \int_{-\pi}^{\pi} \overline{\widetilde{\mathcal{W}}_{a,b}^{(i,i')(m)*}(\lambda)} \widetilde{\mathcal{W}}_{a',b'}^{(i,i')(m)*}(\lambda) d\lambda, \quad (\text{B.15})$$

where $\widetilde{\mathcal{W}}_{a',b'}^{(i,i')*}(\cdot)$ and $\widetilde{\mathcal{W}}_{a',b'}^{(i,i')(m)*}(\cdot)$ are defined respectively in (B.4) and in (B.10).

Proof. We only prove (B.14), since the proof of (B.15) is similar. Quantity $\Theta^{((i,s),(i',s'))}$ can be written as

$$\Theta^{((i,s),(i',s'))} = \int_{-\pi}^{\pi} \widetilde{\mathcal{W}}^{(i,i')*}(\lambda) e^{-i(s-s')\lambda} d\lambda,$$

Hence, setting $v = s - s'$,

$$\begin{aligned} & \ell^{-1} \sum_{s,s'=0,\dots,\ell-1} \Theta_{a,b}^{((i,s),(i',s'))} \Theta_{a',b'}^{((i,s),(i',s'))} \\ &= \sum_{v \in \mathbb{Z}} \ell^{-1} (\ell - |v|)_+ \left(\int_{-\pi}^{\pi} \widetilde{\mathcal{W}}_{a,b}^{(i,i')*}(\lambda) e^{-iv\lambda} d\lambda \right) \left(\int_{-\pi}^{\pi} \widetilde{\mathcal{W}}_{a',b'}^{(i,i')*}(\lambda) e^{-iv\lambda} d\lambda \right), \end{aligned}$$

with $(\ell - |v|)_+ = \ell - |v|$ if $\ell - |v| \geq 0$ and 0 otherwise. Lemma 33 entails that, when ℓ goes to infinity, the above term converges to

$$2\pi \int_{-\pi}^{\pi} \overline{\widetilde{\mathcal{W}}_{a,b}^{(i,i')*}(\lambda)} \widetilde{\mathcal{W}}_{a',b'}^{(i,i')*}(\lambda) d\lambda.$$

This is precisely the assertion of the lemma. \square

We can deduce from Lemma 15 that for all $i, i' = 1, \dots, N$, $\Gamma^{(i,i')}$ is the asymptotic covariance between $\mathcal{S}^{(i,j_0)}$ and $\mathcal{S}^{(i',j_0)}$.

Lemma 16. *For all $m \geq 1$, for all $(a, b, a', b') \in \{1, \dots, p\}^4$, for all $(i, i') \in \{1, \dots, N\}^2$,*

$$\lim_{j_0 \rightarrow \infty} \text{Cov}(\mathcal{S}_{a,b}^{(i,j_0)}, \mathcal{S}_{a',b'}^{(i',j_0)}) = \Gamma_{(a,b),(a',b')}^{(i,i')},$$

with $\Gamma^{(i,i')}$ defined in (B.6).

Proof. We first decompose $\text{Cov}(\mathcal{S}_{a,b}^{(i,j_0)}, \mathcal{S}_{a',b'}^{(i',j_0)})$ in two terms and next study separately the two terms.

Step 1. Decomposition of $\text{Cov}(\mathcal{S}_{a,b}^{(i,j_0)}, \mathcal{S}_{a',b'}^{(i',j_0)})$. Easy calculation shows that

$$\text{Cov}(\mathcal{Y}^{(i,s,j_0)}, \mathcal{Y}^{(i',s',j_0)}) = T_{(a,a'),(b,b')}^{((i,s),(i',s'),j_0)} + T_{(a,b'),(b,a')}^{((i,s),(i',s'),j_0)} - R_{(a,b),(a',b')}^{((i,s),(i',s'),j_0)},$$

with

$$\begin{aligned} T_{(a,a'),(b,b')}^{((i,s),(i',s'),j_0)} &= \left(\sum_{t_1 \in \mathbb{Z}} (\mathcal{V}(2^{j_1} s - t_1) \mathcal{V}^{(i',j_0)}(2^{j_1} s' - t_1)^T)_{a,a'} \right) \\ &\quad \left(\sum_{t_2 \in \mathbb{Z}} (\mathcal{V}^{(i,j_0)}(2^{j_1} s - t_2) \mathcal{V}^{(i',j_0)}(2^{j_1} s' - t_2)^T)_{b,b'} \right), \end{aligned}$$

a similar expression for $T_{(a,b'),(b,a')}^{((i,i'),s,s',j_0)}$, and

$$\begin{aligned}
& R_{(a,b),(a',b')}^{((i,s),(i',s'),j_0)} \\
&= \sum_{a_1,a_2,a_3,a_4=1,\dots,p} \mu_{a_1,a_2,a_3,a_4} \sum_{t \in \mathbb{Z}} \mathcal{V}_{a,a_1}^{(i,j_0)}(2^{j_1}s-t) \mathcal{V}_{b,a_2}^{(i,j_0)}(2^{j_1}s-t) \mathcal{V}_{a',a_3}^{(i',j_0)}(2^{j_1}s'-t) \\
&\quad \mathcal{V}_{b',a_4}^{(i',j_0)}(2^{j_1}s'-t) \\
&- \sum_{a_1,a_2=1,\dots,p} \sum_{t \in \mathbb{Z}} \mathcal{V}_{a,a_1}^{(i,j_0)}(2^{j_1}s-t) \mathcal{V}_{b,a_1}^{(i,j_0)}(2^{j_1}s-t) \mathcal{V}_{a',a_2}^{(i',j_0)}(2^{j_1}s'-t) \mathcal{V}_{b',a_2}^{(i',j_0)}(2^{j_1}s'-t) \\
&- \sum_{a_1,a_2=1,\dots,p} \sum_{t \in \mathbb{Z}} \mathcal{V}_{a,a_1}^{(i,j_0)}(2^{j_1}s-t) \mathcal{V}_{b,a_2}^{(i,j_0)}(2^{j_1}s-t) \mathcal{V}_{a',a_1}^{(i',j_0)}(2^{j_1}s'-t) \mathcal{V}_{b',a_2}^{(i',j_0)}(2^{j_1}s'-t) \\
&- \sum_{a_1,a_2=1,\dots,p} \sum_{t \in \mathbb{Z}} \mathcal{V}_{a,a_1}^{(i,j_0)}(2^{j_1}s-t) \mathcal{V}_{b,a_2}^{(i,j_0)}(2^{j_1}s-t) \mathcal{V}_{a',a_2}^{(i',j_0)}(2^{j_1}s'-t) \mathcal{V}_{b',a_1}^{(i',j_0)}(2^{j_1}s'-t).
\end{aligned}$$

Hence,

$$\begin{aligned}
\text{Cov}(\mathcal{S}^{(i,j_0)}, \mathcal{S}^{(i',j_0)}) &= \frac{1}{n_{j_1}} \sum_{s=0}^{n_{j_1}-1} \sum_{s'=0}^{n_{j_1}-1} T_{(a,a'),(b,b')}^{((i,s),(i',s'),j_0)} + \frac{1}{n_{j_1}} \sum_{s=0}^{n_{j_1}-1} \sum_{s'=0}^{n_{j_1}-1} T_{(a,b'),(b,a')}^{((i,s),(i',s'),j_0)} \\
&\quad - \frac{1}{n_{j_1}} \sum_{s=0}^{n_{j_1}-1} \sum_{s'=0}^{n_{j_1}-1} R_{(a,b),(a',b')}^{((i,s),(i',s'),j_0)}. \quad (\text{B.16})
\end{aligned}$$

We shall now study separately the terms in the right-hand side.

Step 2. Study of $R_{(a,b)}^{((i,s),(i',s'),j_0)}$. Let us study first $R_{(a,b),(a',b')}^{((i,s),(i',s'),j_0)}$. Cauchy-Schwarz's inequality yields to:

$$\begin{aligned}
& \frac{1}{\ell} \sum_{s=0}^{\ell-1} \sum_{s'=0}^{\ell-1} |R_{(a,b),(a',b')}^{((i,s),(i',s'),j_0)}| \\
&\leq (\mu_\infty + 3) \sum_{a_1,a_2,a_3,a_4=1,\dots,p} \sup_{t \in \mathbb{Z}} \left(\sum_{s \in \mathbb{Z}} \mathcal{V}_{a,a_1}^{(i,j_0)}(2^{j_1}s-t)^2 \right)^{1/2} \sup_{t \in \mathbb{Z}} \left(\sum_{s \in \mathbb{Z}} \mathcal{V}_{b,a_2}^{(i,j_0)}(2^{j_1}s-t)^2 \right)^{1/2} \\
&\quad \sup_{s' \in \mathbb{Z}} \left(\sum_{t \in \mathbb{Z}} \mathcal{V}_{a',a_3}^{(i',j_0)}(2^{j_1}s'-t)^2 \right)^{1/2} \sup_{s' \in \mathbb{Z}} \left(\sum_{t \in \mathbb{Z}} \mathcal{V}_{b',a_4}^{(i',j_0)}(2^{j_1}s'-t)^2 \right)^{1/2}.
\end{aligned}$$

The right-hand side does not depend on ℓ . Results (B.7) and (B.9) in Lemma 12 imply that it converges to 0 when j_0 goes to infinity. Hence,

$$\frac{1}{n_{j_1}} \sum_{s=0}^{n_{j_1}-1} \sum_{s'=0}^{n_{j_1}-1} \left| R_{(a,b),(a',b')}^{((i,s),(i',s'),j_0)} \right| \xrightarrow{j_0 \rightarrow \infty} 0. \quad (\text{B.17})$$

Step 3. Study of $T_{(a,b)}^{((i,s),(i',s'),j_0)}$. First observe that

$$\begin{aligned} & \sum_{t_1 \in \mathbb{Z}} \mathbf{v}(2^{j_1} s - t_1) \mathbf{v}^{(i,j_0)}(2^{j_1} s' - t_1)^T \\ &= \int_{-\pi}^{\pi} 2^{-j_1} \mathbf{v}^{(i,j_0)*}(\lambda) \overline{\mathbf{v}^{(i',j_0)*}(\lambda)}^T e^{i2^{j_1}(s-s')\lambda} d\lambda \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} (2\pi)^{1/2} 2^{-j_1} \sum_{q=0}^{2^{j_1}-1} \mathbf{v}^{(i,j_0)*}(2^{-j_1}(\lambda + 2\pi q)) \overline{\mathbf{v}^{(i',j_0)*}(2^{-j_1}(\lambda + 2\pi q))}^T e^{i2^{j_1}(s-s')\lambda} d\lambda. \end{aligned}$$

Last equality was obtained by [31, Lemma 4], since the function $\lambda \rightarrow \mathbf{v}^{(i,j_0)*}(2^{-j_1}(\lambda + 2\pi q)) \overline{\mathbf{v}^{(i',j_0)*}(2^{-j_1}(\lambda + 2\pi q))}^T e^{i2^{j_1}(s-s')\lambda}$ is 2π -periodic.

For all functions g_1, g_2 in $L^2(-\pi, \pi)$, $Q \in \mathbb{N}$, set

$$L_Q(g_1, g_2) = \sum_{q \in \mathbb{Z}} \left(1 - \frac{|q|}{Q}\right)_+ \left(\int_{-\pi}^{\pi} g_1(\lambda) d\lambda \right) \left(\int_{-\pi}^{\pi} g_2(\lambda) d\lambda \right).$$

For all $\ell \in \mathbb{N}$,

$$\frac{1}{\ell} \sum_{s=0}^{\ell-1} \sum_{s'=0}^{\ell-1} T_{(a,b),(a',b')}^{((i,s),(i',s'),j_0)} = L_{\ell}(g_{a,a'}^{(j_0)}, g_{b,b'}^{(j_0)}),$$

where

$$g_{a,a'}^{(j_0)}(\lambda) = (2\pi)^{1/2} 2^{-j_1} \sum_{q=0}^{2^{j_1}-1} \left(\mathbf{v}^{(i,j_0)*}(2^{-j_1}(\lambda + 2\pi q)) \overline{\mathbf{v}^{(i',j_0)*}(2^{-j_1}(\lambda + 2\pi q))}^T \right)_{a,a'},$$

and a similar definition of $g_{b,b'}^{(j_0)}(\lambda)$. We omit the dependence on i, i' temporarily to simplify notations.

Introduce

$$g_{a,a'}^{(\infty)}(\lambda) = (2\pi)^{1/2} \widetilde{\mathcal{W}}_{a,a'}(\lambda) = (2\pi)^{1/2} \sum_{q \in \mathbb{Z}} \mathcal{W}_{a,a'}(\lambda + 2\pi q).$$

We have

$$\begin{aligned} & \left| L_\ell(g_{a,a'}^{(j_0)}, g_{b,b'}^{(j_0)}) - L_\ell(g_{a,a'}^{(\infty)}, g_{b,b'}^{(\infty)}) \right| \\ &= \left| L_\ell(g_{a,a'}^{(j_0)} - g_{a,a'}^{(\infty)}, g_{b,b'}^{(j_0)} - g_{b,b'}^{(\infty)}) + L_\ell(g_{a,a'}^{(\infty)}, g_{b,b'}^{(j_0)} - g_{b,b'}^{(\infty)}) + L_\ell(g_{a,a'}^{(j_0)} - g_{a,a'}^{(\infty)}, g_{b,b'}^{(\infty)}) \right| \\ &\leq M_\ell(g_{a,a'}^{(j_0)} - g_{a,a'}^{(\infty)}) M_\ell(g_{b,b'}^{(j_0)} - g_{b,b'}^{(\infty)}) + M_\ell(g_{a,a'}^{(\infty)}) M_\ell(g_{b,b'}^{(j_0)} - g_{b,b'}^{(\infty)}) \\ &\quad + M_\ell(g_{a,a'}^{(j_0)} - g_{a,a'}^{(\infty)}) M_\ell(g_{b,b'}^{(\infty)}), \end{aligned}$$

with $M_Q(g_1)$ defined in Lemma 32. Last inequality results from Cauchy-Schwarz's inequality, which entails that $L_Q(g_1, g_2) \leq M_Q(g_1) M_Q(g_2)$.

Applying Lemma 32,

$$M_\ell(g_{a,a'}^{(\infty)}) \leq \left(\int_{-\pi}^{\pi} \left| g_{a,a'}^{(\infty)}(\lambda) \right|^2 d\lambda \right)^{1/2}, \quad (\text{B.18})$$

$$M_\ell(g_{a,a'}^{(j_0)} - g_{a,a'}^{(\infty)}) \leq \left(\int_{-\pi}^{\pi} \left| g_{a,a'}^{(j_0)}(\lambda) - g_{a,a'}^{(\infty)}(\lambda) \right|^2 d\lambda \right)^{1/2}. \quad (\text{B.19})$$

The two bounds in (B.18) and (B.19) do not depend on ℓ . The right-hand side of (B.18) is finite since by (B.3) and Lemma 10, $|g_{a,a'}(\lambda)| = (2\pi)^{1/2} \left| \widetilde{\mathcal{W}}_{a,a'}(\lambda) \right| \leq (2\pi)^{1/2} C_v (1 + |\lambda|)^{-2\delta_v}$, with $\delta_v > 1/2$. Next, notice that

$$\begin{aligned} & \int_{-\pi}^{\pi} \left| g_{a,a'}^{(j_0)}(\lambda) - g_{a,a'}^{(\infty)}(\lambda) \right|^2 d\lambda \\ &= 2\pi \int_{-\infty}^{\infty} \left| 2^{-j_1} \left(\mathbf{V}^{(i,j_0)*} (2^{-j_1} \lambda) \overline{\mathbf{V}^{(i,j_0)*} (2^{-j_1} \lambda)^T} \right)_{a,a'} - \mathcal{W}_{a,a'}^{(i,i')*}(\lambda) \right|^2 d\lambda. \end{aligned}$$

Convergence (B.3) and inequality (B.19) ensure that the integral goes to 0 when j_0 goes to infinity by dominated convergence. Therefore, the right-hand side of (B.19) goes to 0 when j_0 goes to infinity. It results that $\left| L_\ell(g_{a,a'}^{(j_0)}, g_{b,b'}^{(j_0)}) - L_\ell(g_{a,a'}^{(\infty)}, g_{b,b'}^{(\infty)}) \right|$ can be bounded by a quantity which is independent of ℓ and which goes to 0 when j_0 goes to infinity. Consequently, $\left| L_{n_{j_1}}(g_{a,a'}^{(j_0)}, g_{b,b'}^{(j_0)}) - L_{n_{j_1}}(g_{a,a'}^{(\infty)}, g_{b,b'}^{(\infty)}) \right| \xrightarrow{j_0 \rightarrow \infty} 0$.

Observe that

$$\begin{aligned}
L_{n_{j_1}}(g_{a,a'}^{(j_0)}, g_{b,b'}^{(j_0)}) &= n_{j_1}^{-1} \sum_{s=0}^{n_{j_1}-1} \sum_{s'=0}^{n_{j_1}-1} T_{(a,a'),(b,b')}^{((i,s),(i',s'),j_0)} \\
L_{n_{j_1}}(g_{a,a'}^{(\infty)}, g_{b,b'}^{(\infty)}) &= n_{j_1}^{-1} \sum_{s=0}^{n_{j_1}-1} \sum_{s'=0}^{n_{j_1}-1} \Theta_{a,a'}^{((i,s),(i',s'))} \Theta_{b,b'}^{((i,s),(i',s'))} \\
&\xrightarrow{j_0 \rightarrow \infty} 2\pi \int_{-\pi}^{\pi} \overline{\widetilde{\mathcal{W}}_{a,a'}^{(i,i')*}(\lambda)} \widetilde{\mathcal{W}}_{b,b'}^{(i,i')*}(\lambda) d\lambda,
\end{aligned}$$

where the last convergence is given by Lemma 15. Hence,

$$n_{j_1}^{-1} \sum_{s=0}^{n_{j_1}-1} \sum_{s'=0}^{n_{j_1}-1} T_{(a,a'),(b,b')}^{((i,s),(i',s'),j_0)} \xrightarrow{j_0 \rightarrow \infty} 2\pi \int_{-\pi}^{\pi} \overline{\widetilde{\mathcal{W}}_{a,a'}^{(i,i')*}(\lambda)} \widetilde{\mathcal{W}}_{b,b'}^{(i,i')*}(\lambda) d\lambda. \quad (\text{B.20})$$

Similarly,

$$n_{j_1}^{-1} \sum_{s=0}^{n_{j_1}-1} \sum_{s'=0}^{n_{j_1}-1} T_{(a,b'),(b,a')}^{((i,s),(i',s'),j_0)} \xrightarrow{j_0 \rightarrow \infty} 2\pi \int_{-\pi}^{\pi} \overline{\widetilde{\mathcal{W}}_{a,b'}^{(i,i')*}(\lambda)} \widetilde{\mathcal{W}}_{b,a'}^{(i,i')*}(\lambda) d\lambda. \quad (\text{B.21})$$

Step 4. End of the proof. Lemma 16 follows from (B.16), (B.17), (B.20) and (B.21). \square

Appendix B.5. Convergence of $\Gamma^{(m)}$ to Γ

We proceed to show that $\Gamma^{(i,i')(m)}$ goes to $\Gamma^{(i,i')}$ when m goes to infinity, for all $i, i' = 1, \dots, N$. That is, the asymptotic variance of $\mathcal{S}^{(\cdot, j_0)(m)}$ when j_0 goes to infinity converges to the asymptotic variance of $\mathcal{S}^{(\cdot, j_0)}$.

Lemma 17. *For all $i, i' = 1, \dots, N$,*

$$\lim_{m \rightarrow \infty} \Gamma^{(i,i')(m)} = \Gamma^{(i,i')}.$$

Proof. To study the limit of $\Gamma_{a,b}^{(i,i')(m)}$ when m goes to infinity, we will first prove that $\Gamma^{(i,i')}$ satisfies $\Gamma^{(i,i')} = \lim_{j_0 \rightarrow \infty} \lim_{m \rightarrow \infty} \widehat{\Gamma}^{(i,i',j_0)(m)}$, where $(\widehat{\Gamma}_{(a,b),(a',b')}^{(i,i',j_0)(m)})_{a,b,a',b'}$ are defined

by

$$\begin{aligned}\widehat{\Gamma}_{(a,b),(a',b')}^{(i,i',j_0)(m)} &= 2\pi \int_{-\pi}^{\pi} \overline{\widetilde{\mathcal{W}}_{a,a'}^{(i,i',j_0)(m)*}}(\lambda) \widetilde{\mathcal{W}}_{b,b'}^{(i,i',j_0)(m)*}(\lambda) d\lambda \\ &\quad + 2\pi \int_{-\pi}^{\pi} \overline{\widetilde{\mathcal{W}}_{a,b'}^{(i,i',j_0)(m)*}}(\lambda) \widetilde{\mathcal{W}}_{b,a'}^{(i,i',j_0)(m)*}(\lambda) d\lambda,\end{aligned}$$

with

$$\begin{aligned}\widetilde{\mathcal{W}}^{(i,i',j_0)(m)*}(\lambda) &= \sum_{t \in \mathbb{Z}} \widehat{\mathcal{W}}^{(i,i',j_0)(m)*}(\lambda + 2t\pi), \\ \widehat{\mathcal{W}}^{(i,i',j_0)(m)*}(\lambda) &= 2^{-j_1} \mathcal{V}^{(i,j_0)(m)*}(2^{-j_1}\lambda) \overline{\mathcal{V}^{(i',j_0)(m)*}(2^{-j_1}\lambda)}^T.\end{aligned}$$

Notice that

$$\int_{-\pi}^{\pi} \overline{\widetilde{\mathcal{W}}_{a,a'}^{(i,i',j_0)(m)*}}(\lambda) \widetilde{\mathcal{W}}_{b,b'}^{(i,i',j_0)(m)*}(\lambda) d\lambda = \int_{-\infty}^{\infty} \overline{\widehat{\mathcal{W}}_{a,a'}^{(i,i',j_0)(m)*}}(\lambda) \mathcal{W}_{b,b'}^{(i,i',j_0)(m)*}(\lambda) d\lambda.$$

When j_0 goes to infinity, $\widehat{\mathcal{W}}^{(i,i',j_0)(m)*}(\lambda)$ converges to $\mathcal{W}^{(i,i')(m)*}(\lambda)$. The convergence under the integral is obtained by dominated convergence thanks to Lemma 10. It results that $\lim_{j_0 \rightarrow \infty} \widehat{\Gamma}^{(i,i',j_0)(m)} = \Gamma^{(i,i')(m)}$.

Let us now study the convergence of $(\widehat{\Gamma}^{(i,i',j_0)(m)})$ with respect to m . We introduce

$$\begin{aligned}\widehat{\mathcal{W}}^{(i,i',j_0)*}(\lambda) &= 2^{-j_1} \mathcal{V}^{(i,j_0)*}(2^{-j_1}\lambda) \overline{\mathcal{V}^{(i',j_0)*}(2^{-j_1}\lambda)}^T, \\ \widetilde{\mathcal{W}}^{(i,i',j_0)*}(\lambda) &= \sum_{t \in \mathbb{Z}} \widehat{\mathcal{W}}^{(i,i',j_0)*}(\lambda + 2t\pi),\end{aligned}$$

and

$$\widehat{\Gamma}_{(a,b),(a',b')}^{(i,i',j_0)} = 2\pi \int_{-\pi}^{\pi} \overline{\widetilde{\mathcal{W}}_{a,a'}^{(i,i',j_0)*}}(\lambda) \widetilde{\mathcal{W}}_{b,b'}^{(i,i',j_0)*}(\lambda) d\lambda + 2\pi \int_{-\pi}^{\pi} \overline{\widetilde{\mathcal{W}}_{a,b'}^{(i,i',j_0)*}}(\lambda) \widetilde{\mathcal{W}}_{b,a'}^{(i,i',j_0)*}(\lambda) d\lambda. \quad (\text{B.22})$$

Since $\mathcal{V}^{(i,s,j_0)*}(\cdot)$ is continuous, we can apply a convergence under the integral. Hence,

for all $\lambda \in \mathbb{R}$,

$$\begin{aligned} \mathbf{v}^{(i,j_0)(m)*}(\lambda) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \widehat{H}(u) \mathbf{v}^{(i,j_0)*}(\lambda - u/m) du \\ &\xrightarrow{m \rightarrow \infty} \mathbf{v}^{(i,j_0)*}(\lambda) \frac{1}{2\pi} \int_{-\pi}^{\pi} \widehat{H}(u) du = \mathbf{v}^{(i,j_0)*}(\lambda). \end{aligned}$$

Additionally Lemma 13 entails that

$$\forall j_0 \geq 0, \sup_{m \geq 1} \sup_{|\lambda| < \pi} 2^{-j/2} \left\| \mathbf{v}^{(i,s,j_0)(m)*}(\lambda) \right\|_{\infty} (1 + 2^j |\lambda|)^{\delta_v} < \infty.$$

Consequently, $\widehat{\Gamma}^{(i,i',j_0)(m)}$ converges uniformly in m to $\widehat{\Gamma}^{(i,i',j_0)}$. Moreover, $\widehat{\Gamma}^{(i,i',j_0)}$ converges to $\Gamma^{(i,i')}$ when j_0 goes to infinity, the convergence under the integral being obtained by continuity.

It results that

$$\lim_{m \rightarrow \infty} \lim_{j_0 \rightarrow \infty} \widehat{\Gamma}^{(i,i',j_0)(m)} = \lim_{j_0 \rightarrow \infty} \lim_{m \rightarrow \infty} \widehat{\Gamma}^{(i,i',j_0)(m)} = \Gamma^{(i,i')}.$$

Convergence of Lemma 17 follows. □

Appendix B.6. Sums of $\Gamma^{(i,i')}$

Due to decomposition (A.3), we will have to manipulate sums of covariances of $\{\mathcal{S}^{(i,j_0)}, i = 1, \dots, N\}$. By Lemma 16, the covariances are equal to $\{\Gamma^{(i,i')}, i, i' = 1, \dots, N\}$. The objective of this section is to give some results on these sums. Rather than using expressions (B.6), we use convergence (B.14). We first need the following Lemma on quantities $\{\Theta_{a,b}^{((i,s),(i',s'))}, i, i' = 1, \dots, N, s, s' \geq 0\}$.

Lemma 18. *Suppose conditions of Theorem 2 hold. Let $(a, b) \in \{1, \dots, p\}^2$, $(i, i') \in \{1, \dots, N\}^2$, $(s, s') \in \mathbb{N}^2$. Introduce*

$$\Xi_{a,b}^{((u,s),(u',s'))} = 2^{(\Delta-u)(1-d_a-d_b)} \int_{-\infty}^{\infty} g_{u'-u}(\lambda; d_a + d_b) e^{-i(k-2^{u'-u}k')\lambda} d\lambda,$$

with $g_{u'-u}(\xi; \delta) = \overline{\widehat{\psi}(\lambda)} \widehat{\psi}(2^{u'-u}\lambda) |\lambda|^{-\delta}$, and $k = i + 2^{\Delta-u}(s-1)$, $k' = i' + 2^{\Delta-u}(s'-1)$. Then, under assumptions of Theorem 20,

$$\Theta_{a,b}^{((i,s),(i',s'))} = \Omega_{a,b} \cos(\pi(d_a - d_b)/2) 2^{(u-\Delta)/2+(u'-\Delta)/2+\Delta} (d_a+d_b) \Xi_{a,b}^{((u,s),(u',s'))}.$$

Proof. Recall that $\{\Theta_{a,b}^{((i,s),(i',s'))}, a, b = 1, \dots, p, i = 1, \dots, N, s = 0, \dots, n_{j_1} - 1\}$ are defined in (B.5). For all $(a, b) \in \{1, \dots, p\}^2$,

$$\Theta_{a,b}^{((i,s),(i',s'))} = \overline{\Theta_{b,a}^{((i',s'),(i,s))}} = \frac{1}{2}(\Theta_{a,b}^{((i,s),(i',s'))} + \overline{\Theta_{b,a}^{((i,s),(i',s'))}}).$$

Hence,

$$\Theta^{((i,s),(i',s'))} = \int_{-\infty}^{\infty} \frac{1}{2}(\mathcal{W}^{(i,i')*}(\lambda) + \overline{\mathcal{W}^{(i',i)*}(\lambda)^T})e^{-i(s-s')\lambda}d\lambda.$$

Replacing $\mathcal{W}^{(i,i')*}(\lambda)$ and $\mathcal{W}^{(i',i)*}(\lambda)$ by their expression (B.1), we get

$$\Theta_{a,b}^{((i,s),(i',s'))} = \Omega_{a,b} \cos(\pi(d_a - d_b)/2) 2^{(u-\Delta)/2+(u'-\Delta)/2+\Delta} (d_a+d_b) \Xi_{a,b}^{((u,s),(u',s'))}$$

with

$$\begin{aligned} & \Xi_{a,b}^{((u,s),(u',s'))} \\ &= \int_{-\infty}^{\infty} \overline{\widehat{\psi}(2^{u-\Delta}\lambda)} \widehat{\psi}(2^{u'-\Delta}\lambda) |\lambda|^{-d_a-d_b} e^{-i(2^{u-\Delta}(i-1) - 2^{u'-\Delta}(i'-1))\lambda} e^{-i(s-s')\lambda} d\lambda, \\ &= 2^{(\Delta-u)(1-d_a-d_b)} \int_{-\infty}^{\infty} \overline{\widehat{\psi}(\lambda)} \widehat{\psi}(2^{u'-u}\lambda) |\lambda|^{-d_a-d_b} e^{-i(i+2^{\Delta-u}s)\lambda + i(2^{\Delta-u'}s'+i')2^{u'-u}\lambda} d\lambda. \end{aligned}$$

□

We are now in a position to give a useful result on $\mathbf{\Gamma}$. Namely, we consider the sum of $\{\mathbf{\Gamma}^{(i,i')}, i = 2^{\Delta-u}, \dots, 2^{\Delta-u+1} - 1, i' = 2^{\Delta-u'}, \dots, 2^{\Delta-u'+1} - 1\}$, which corresponds to the contribution of the scales $(j, j') = (j_0 + u, j_0 + u')$ to the asymptotic variance \mathbf{V} of the sample wavelet covariance.

Lemma 19. *Suppose conditions of Theorem 2 hold. For all $(a, b, a', b') \in \{1, \dots, p\}^4$, for all $\Delta \in \mathbb{N}$, $(u, u') \in \{0, \dots, \Delta\}^2$,*

$$\sum_{i=2^{\Delta-u}}^{2^{\Delta-u+1}-1} \sum_{i'=2^{\Delta-u'}}^{2^{\Delta-u'+1}-1} \Gamma_{(a,b),(a',b')}^{(i,i')} = 2^{\Delta-u} (\mathbf{G} \cdot \widetilde{\mathbf{I}} \cdot \mathbf{G}_{(a,a'),(b,b')}(u) + \mathbf{G} \cdot \widetilde{\mathbf{I}} \cdot \mathbf{G}_{(a,b'),(a',b)}(u)),$$

where $\mathbf{G} \cdot \widetilde{\mathbf{I}} \cdot \mathbf{G}(u)$ is defined in (8).

Proof. Quantities $\Gamma_{(a,b),(a',b')}^{(i,i')}$ can be expressed as:

$$\Gamma_{(a,b),(a',b')}^{(i,i')} = \lim_{\ell \rightarrow \infty} \ell^{-1} \sum_{s=0}^{\ell} \sum_{s'=0}^{\ell} \left(\Theta_{a,a'}^{((i,s),(i',s'))} \Theta_{b,b'}^{((i,s),(i',s'))} + \Theta_{a,b'}^{((i,s),(i',s'))} \Theta_{b,a'}^{((i,s),(i',s'))} \right), \quad (\text{B.23})$$

where $\{\Theta_{a,b}^{((i,s),(i',s'))}, a, b = 1, \dots, p, i = 1, \dots, N, s = 0, \dots, n_{j_1} - 1\}$ are noted in (B.5). Lemma 18 yields

$$\Theta_{a,b}^{((i,s),(i',s'))} = \Omega_{a,b} \cos(\pi(d_a - d_b)/2) 2^{(u-\Delta)/2+(u'-\Delta)/2+\Delta(d_a+d_b)} \Xi_{a,b}^{((u,s),(u',s'))}, \quad (\text{B.24})$$

with

$$\Xi_{a,b}^{((u,s),(u',s'))} = 2^{(\Delta-u)(1-d_a-d_b)} \int_{-\infty}^{\infty} g_{u'-u}(\lambda; d_a + d_b) e^{-i(k-2^{u'-u}k')\lambda} d\lambda,$$

and $g_{u'-u}(\xi; \delta) = \overline{\widehat{\psi}(\lambda)} \widehat{\psi}(2^{u'-u}\lambda) |\lambda|^{-\delta}$, $k = i + 2^{\Delta-u}(s-1)$, $k' = i' + 2^{\Delta-u'}(s'-1)$.

To get all values in \mathbb{Z} from $k - 2^{u'-u}k'$, we introduce $\tau \in \{0, \dots, 2^{-(u'-u)} - 1\}$. Then, when i, i', s and s' vary respectively in $\{2^{\Delta-u}, \dots, 2^{\Delta-u+1} - 1\}$, $\{2^{\Delta-u'}, \dots, 2^{\Delta-u'+1} - 1\}$, $\{0, \dots, \ell - 1\}$ and $\{0, \dots, \ell - 1\}$, quantity $q = k - 2^{u'-u}k' + 2^{u'-u}\tau$ takes all relative integers values in $\{-Q, \dots, Q\}$, with $Q = 2^{\Delta-u}(\ell - 1)$.

We have

$$\begin{aligned} & \sum_{i=2^{\Delta-u}}^{2^{\Delta-u+1}-1} \sum_{i'=2^{\Delta-u'}}^{2^{\Delta-u'+1}-1} \ell^{-1} \sum_{s=0}^{\ell-1} \sum_{s'=0}^{\ell-1} \Xi_{a,b}^{((u,s),(u',s'))} \Xi_{a',b'}^{((u,s),(u',s'))} \\ &= \frac{Q}{\ell} \sum_{-Q \leq q \leq Q} \left(1 - \frac{q}{Q}\right)_+ 2^{(\Delta-u)(2-d_a-d_b-d_{a'}-d_{b'})} \left(\int_{-\infty}^{\infty} \sum_{\tau=0}^{2^{-(u'-u)}-1} g(\lambda; d_a + d_b) e^{-i(q-2^{u'-u}\tau)\lambda} d\lambda \right) \\ & \quad \left(\int_{-\infty}^{\infty} \sum_{\tau=0}^{2^{-(u'-u)}-1} g(\lambda; d_{a'} + d_{b'}) e^{-i(q-2^{u'-u}\tau)\lambda} d\lambda \right) \\ &= 2^{\Delta-u} \sum_{-Q \leq q \leq Q} \left(1 - \frac{q}{Q}\right)_+ 2^{(\Delta-u)(2-d_a-d_b-d_{a'}-d_{b'})} \left(\int_{-\pi}^{\pi} 2^{-(u'-u)/2} \widetilde{D}_{u'-u,\infty}(\lambda; d_a + d_b) e^{iq\lambda} d\lambda \right) \\ & \quad \left(\int_{-\pi}^{\pi} 2^{-(u'-u)/2} \widetilde{D}_{u'-u,\infty}(\lambda; d_{a'} + d_{b'}) e^{iq\lambda} d\lambda \right) \end{aligned}$$

since $\tilde{D}_{u'-u;\infty}(\lambda; d_a + d_b) = \sum_{v=0}^{2^{u'-u}-1} 2^{(u'-u)/2} \sum_{t \in \mathbb{Z}} g_{u'-u}(\lambda + 2t\pi) e^{i2^{u'-u}\tau(\lambda + 2t\pi)}$. Applying Lemma 33, we obtain

$$\begin{aligned}
& \lim_{\ell \rightarrow \infty} \sum_{i=2^{\Delta-u}}^{2^{\Delta-u+1}-1} \sum_{i'=2^{\Delta-u'}}^{2^{\Delta-u'+1}-1} \ell^{-1} \sum_{s=0}^{\ell} \sum_{s'=0}^{\ell} \Xi_{a,b}^{((u,s),(u',s'))} \Xi_{a',b'}^{((u,s),(u',s'))} \\
&= 2^{\Delta-u} 2^{(\Delta-u)(2-d_a-d_b-d_{a'}-d_{b'})+u-u'} (2\pi) \int_{-\pi}^{\pi} \overline{\tilde{D}_{u'-u;\infty}(\lambda; d_a + d_b)} \tilde{D}_{u'-u;\infty}(\lambda; d_{a'} + d_{b'}) d\lambda \\
&= 2^{\Delta-u'+(\Delta-u)(2-d_a-d_b-d_{a'}-d_{b'})} \tilde{\mathcal{I}}_{u'-u}(d_a + d_b, d_{a'} + d_{b'}) K(d_a + d_b) K(d_{a'} + d_{b'}). \quad (\text{B.25})
\end{aligned}$$

Lemma 19 results from (B.23), (B.24) and (B.25). \square

Appendix C. Proof of Theorem 2

The asymptotic normality is given by Theorem 20 below. Proposition 1 enables to approximate $\{2^{-j(d_a+d_b)}\sigma_{a,b}(j), a, b = 1, \dots, p, j \geq 0\}$ by \mathbf{G} and hence entails Theorem 2.

Theorem 20. *Suppose Assumptions (M1)–(M4) and (W1) to (W5) hold. Let $2^{-j_0\beta} \rightarrow 0$ and $N_X^{-1}2^{j_0} \rightarrow 0$. Then*

$$\begin{aligned}
& \left\{ \sqrt{n_{j_0+u}} \text{vec} \left(\mathbf{\Lambda}_{j_0}(\mathbf{d})^{-1} \left(\widehat{\Sigma}(j_0 + u) - \Sigma(j_0 + u) \right) \mathbf{\Lambda}_{j_0}(\mathbf{d})^{-1} \right), u = 0, \dots, \Delta \right\} \\
& \xrightarrow{j_0 \rightarrow \infty} \mathcal{L} \left\{ \mathbf{Q}(u), u = 0, \dots, \Delta \right\}
\end{aligned}$$

where $\mathbf{Q}(\cdot)$ is the centered Gaussian process defined in Theorem 2.

The construction of the proof is adapted from that of [23, Theorem 2]. The proof has been structured as follows. Appendix A proposes a writing of wavelet coefficients as decimated linear processes, and provides an approximation by a m -dependent decimated linear processes, $m \geq 0$. Notations and technical results on the decimated decompositions are stated in Appendix B. They are useful for applying the propositions of [31], which lead to the asymptotic normality described in Theorem 20. The current section deals with this step.

Let us use the notations introduced in Appendix A.2. The sample wavelet covariances satisfy (A.3). It results that the vector of empirical covariances at different scales can be written as

$$2^{-j_0(d_a+d_b)} \begin{pmatrix} \widehat{\sigma}_{a,b}(j_0) \\ \vdots \\ \widehat{\sigma}_{a,b}(j_1) \end{pmatrix} = \sqrt{n_{j_1}} \mathbf{B}_{j_0} \begin{pmatrix} \mathcal{S}_{a,b}^{(1,j_0)} \\ \vdots \\ \mathcal{S}_{a,b}^{(N,j_0)} \end{pmatrix} + \begin{pmatrix} \mathcal{R}_{a,b}(j_0) \\ \vdots \\ \mathcal{R}_{a,b}(j_1) \end{pmatrix} \quad (\text{C.1})$$

with

$$\mathbf{B}_{j_0} = \begin{pmatrix} 0 \dots & \dots & \dots & 0 & \overbrace{n_{j_0}^{-1} \dots n_{j_0}^{-1}}^{2^\Delta \text{ times}} \\ 0 \dots & \dots & 0 & \overbrace{n_{j_0+1}^{-1} \dots n_{j_0+1}^{-1}}^{2^{\Delta-1} \text{ times}} & 0 \dots 0 \\ \vdots & & & \vdots & \vdots \\ n_{j_1}^{-1} & 0 \dots & \dots & \dots & \dots 0 \end{pmatrix}.$$

The objective is to show that $\text{vec}(\widehat{\sigma}_{a,b}(j), a, b = 1, \dots, p)$ is asymptotically Gaussian when j goes to infinity. The proof is divided into the following steps:

- Appendix C.1 establishes that the vector $\text{vec}(\mathcal{S}_{a,b}(i, j_0), a, b = 1, \dots, p)_{i=1, \dots, N}$ is asymptotically Gaussian when j_0 goes to infinity; the proof is based on the approximation by m -dependent processes introduced in Appendix A.3.
- Appendix C.2 proves that the terms $\text{vec}(\mathcal{R}_{a,b}(j), a, b = 1, \dots, p)_{j_0 \leq j \leq j_1}$ are negligible.

Appendix C.3 compiles all elements above to prove Theorem 20.

Appendix C.1. Asymptotic normality of $\mathcal{S}^{(i,j)}$

The asymptotic normality of $\text{vec}(\mathcal{S}^{(i,j_0)})$ is given by the following proposition.

Proposition 21. *Under conditions of Theorem 20,*

$$\left\{ \text{vec}(\mathcal{S}^{(i,j_0)}), i = 1, \dots, N \right\} \xrightarrow{j_0 \rightarrow \infty} \{ \mathbf{Q}^S(i), i = 1, \dots, N \},$$

where $\mathbf{Q}^S(\cdot)$ is a centered Gaussian process with covariance function $\text{Cov}(Q_{a,b}^S(i), Q_{a',b'}^S(i')) = \Gamma_{(a,b),(a',b')}^{(i,i')}$ defined in (B.6).

Recall that $\mathcal{S}^{(i,j_0)} = \frac{1}{n_{j_1}} \sum_{s=0}^{n_{j_1}-1} \mathcal{Y}^{(i,s,j_0)}$. The steps of the proof of the asymptotic normality of $\text{vec}\left(\mathcal{S}^{(i,j_0)}\right)$ are the following:

- We approximate $\{\mathcal{Y}_{a,b}^{(i,\cdot,j)}, i = 1, \dots, N\}$ by the m -dependent process $\{\mathcal{Y}_{a,b}^{(i,\cdot,j)(m)}, i = 1, \dots, N\}$ defined in Appendix A.3. We establish that $\{\mathcal{Y}_{a,b}^{(i,\cdot,j)(m)}, i = 1, \dots, N\}$ is asymptotically normal when j goes to infinity, using [31, Proposition 2].
- We obtain the asymptotic normality of $\sum_{a,b=1,\dots,p} u_{a,b} \mathcal{S}_{a,b}^{(i,j_0)(m)} = n_j^{-1} \sum_{s=0}^{n_j-1} \mathcal{Y}_{a,b}^{(i,s,j)(m)}$, thanks to [31, Proposition 3].
- The asymptotic normality for $\sum_{a,b=1,\dots,p} \nu_{a,b} \mathcal{S}_{a,b}^{(i,j_0)}$ is obtained by letting m goes to infinity, using [44, Theorem 3.2].

Appendix C.1.1. First step, approximation by a m -dependent process

We study variables $(\mathcal{Y}^{(i,s,j_0)(m)})_{i=1,\dots,N}$ defined in (A.4), in Appendix A.3. The objective of this step is to prove that variables $(\mathcal{Y}^{(i,s,j_0)(m)})_{i=1,\dots,N}$ are asymptotically Gaussian. They are defined from variables $(\mathcal{Z}_a^{(i,s,j_0)(m)}, a = 1, \dots, p)_{i=1,\dots,N}$ by $\mathcal{Y}^{(i,s,j_0)(m)} = \mathcal{Z}^{(i,s,j_0)(m)} \mathcal{Z}^{(i,s,j_0)(m)T}$. We will study first the behavior of $(\mathcal{Z}^{(i,s,j_0)(m)})_{i=1,\dots,N}$ and next deduce that variables $(\mathcal{Y}^{(i,s,j_0)(m)})_{i=1,\dots,N}$ are asymptotically Gaussian.

For all $a = 1, \dots, p$ and $s \in \mathbb{N}$, let

$$\mathbb{Z}_a^{(s,j_0)(m)} = \left(\mathcal{Z}_a^{(1,s,j_0)(m)}, \mathcal{Z}_a^{(2,s,j_0)(m)}, \dots, \mathcal{Z}_a^{(N,s,j_0)(m)} \right)^T.$$

$(\mathcal{Z}_a^{(i,s,j_0)(m)}, a = 1, \dots, p)_{i=1,\dots,N}$ are m -dependent decimated linear processes in \mathbb{R}^N . By [31, Proposition 2], we get that $\text{vec}\left(\mathbb{Z}_a^{(s,j_0)(m)}, a = 1, \dots, p\right)$ converges in distribution to $\text{vec}\left(\mathbb{Z}_a^{(s,\infty)(m)}, a = 1, \dots, p\right)$, which follows a centered Gaussian distribution with $\mathbb{Z}_a^{(s,\infty)(m)} = (\mathcal{Z}_a^{(i,s,\infty)(m)})_{i=1,\dots,N}$ and

$$\text{Cov}\left(\mathcal{Z}_a^{(i,s,\infty)(m)}, \mathcal{Z}_b^{(i',s',\infty)(m)}\right) = \Theta_{a,b}^{((i,s),(i',s'))(m)}.$$

For $s \in \mathbb{N}$, $\nu \in \mathbb{R}^{p \times p}$, write

$$\begin{aligned} \mathbb{Y}_{a,b}^{(s,j_0)(m)} &= \left(\mathcal{Y}_{a,b}^{(1,s,j_0)(m)}, \dots, \mathcal{Y}_{a,b}^{(N,s,j_0)(m)} \right)^T \\ \mathbf{Y}^{(s,j_0)(m)}(\nu) &= \sum_{a,b=1,\dots,p} \nu_{a,b} \mathbb{Y}_{a,b}^{(s,j_0)(m)} = \left(Y^{(1,s,j_0)(m)}(\nu), \dots, Y^{(N,s,j_0)(m)}(\nu) \right)^T \end{aligned}$$

The continuous mapping theorem implies that, when j_0 goes to infinity, $(\mathbf{Y}^{(s,j_0)(m)}(\boldsymbol{\nu}))_{s=0,\dots,n_{j_1}-1}$ converges in distribution to $(\mathbf{Y}^{(s,\infty)(m)}(\boldsymbol{\nu}))_{s=0,\dots,n_{j_1}-1}$ given by

$$\mathbf{Y}^{(s,\infty)(m)}(\boldsymbol{\nu}) = \sum_{a,b=1,\dots,p} \nu_{a,b} \mathbb{Y}_{a,b}^{(s,\infty)(m)} = (Y^{(1,s,\infty)(m)}(\boldsymbol{\nu}), \dots, Y^{(N,s,\infty)(m)}(\boldsymbol{\nu}))^T,$$

with $\mathbb{Y}_{a,b}^{(s,\infty)(m)} = (\mathcal{Y}_{a,b}^{(i,s,\infty)(m)})_{i=1,\dots,N}$ and $\mathcal{Y}_{a,b}^{(i,s,\infty)(m)} = \mathcal{Z}_a^{(i,s,\infty)(m)} \mathcal{Z}_b^{(i,s,\infty)(m)}$.

Appendix C.1.2. Second step: asymptotic normality of $\mathcal{S}^{(i,s,j_0)(m)}$

We first prove that conditions of [31, Proposition 3] are satisfied by $\{\mathbf{Y}^{(s,j_0)(m)}, s \in \mathbb{N}, j_0 \geq 0\}$.

Lemma 22. *For all $m \geq 1$, for all $\boldsymbol{\nu} \in \mathbb{R}^{p \times p}$,*

$$\sup_{i=1,\dots,N} \sup_{s \geq 0} \sup_{j_0 \geq 0} \mathbb{E}[Y^{(i,s,j_0)(m)}(\boldsymbol{\nu})] < \infty, \quad (\text{C.2})$$

$$\forall s, s' \geq 0, \lim_{j_0 \rightarrow \infty} \text{Cov}(\mathbf{Y}^{(s,j_0)(m)}(\boldsymbol{\nu}), \mathbf{Y}^{(s',j_0)(m)}(\boldsymbol{\nu})) = \text{Cov}(\mathbf{Y}^{(s,\infty)(m)}(\boldsymbol{\nu}), \mathbf{Y}^{(s',\infty)(m)}(\boldsymbol{\nu})), \quad (\text{C.3})$$

$$\lim_{\ell \rightarrow \infty} \lim_{j_0 \rightarrow \infty} \text{Cov}(\ell^{-1/2} \sum_{s=0}^{\ell-1} \mathbf{Y}^{(s,j_0)(m)}(\boldsymbol{\nu})) = \boldsymbol{\Gamma}^{(m)}(\boldsymbol{\nu}), \quad (\text{C.4})$$

with $\boldsymbol{\Gamma}^{(m)}(\boldsymbol{\nu}) = \sum_{a,b=1,\dots,p} \sum_{a',b'=1,\dots,p} \nu_{a,b} \nu_{a',b'} (\boldsymbol{\Gamma}_{(a,b),(a',b')}^{(i,i')(m)})_{i,i'=1,\dots,N}$ and $\boldsymbol{\Gamma}^{(m)}$ defined in (B.12).

Proof.

Proof of (C.2). Assertion (C.2) follows from the fact that $\mathbb{E}[\mathbb{Y}^{(s,\infty)(m)}(\boldsymbol{\nu})] = \sum_{a,b=1,\dots,p} \nu_{a,b} (\boldsymbol{\Theta}_{a,b}^{((i,s),(i,s))(m)})_{i=1,\dots,N}$.

Proof of (C.3). Vector $\left(\mathbb{Z}_a^{(s,\infty)(m)}, \mathbb{Z}_b^{(s,\infty)(m)}, \tilde{\mathbb{Z}}_{a'}^{(s',\infty)(m)}, \mathbb{Z}_{b'}^{(s',\infty)(m)}\right)^T$ follows a centered Gaussian distribution. We can therefore use Isserlis's theorem. We get

$$\begin{aligned} & \mathbb{E}(Y^{(i,s,\infty)(m)}(\boldsymbol{\nu})Y^{(i',s',\infty)(m)}(\boldsymbol{\nu})) \\ &= \sum_{a,b,a',b'=1,\dots,p} \nu_{a,b}\nu_{a',b'} \mathbb{E}(\mathcal{Z}_a^{(i,s,\infty)(m)} \mathcal{Z}_b^{(i,s,\infty)(m)} \mathcal{Z}_{a'}^{(i',s',\infty)(m)} \mathcal{Z}_{b'}^{(i',s',\infty)(m)}) \\ &= \sum_{a,b,a',b'=1,\dots,p} \nu_{a,b}\nu_{a',b'} \left[\Theta_{a,b}^{((i,s),(i,s))(m)} \Theta_{a',b'}^{((i',s'),(i',s'))(m)} \right. \\ & \quad \left. + \Theta_{a,a'}^{((i,s),(i',s'))(m)} \Theta_{b,b'}^{((i,s),(i',s'))(m)} + \Theta_{a,b'}^{((i,s),(i',s'))(m)} \Theta_{b,a'}^{((i,s),(i',s'))(m)} \right]. \end{aligned}$$

It results that:

$$\begin{aligned} & \text{Cov}(Y^{(i,s,\infty)(m)}(\boldsymbol{\nu}), Y^{(i',s',\infty)(m)}(\boldsymbol{\nu})) \\ &= \sum_{a,b,a',b'=1,\dots,p} \nu_{a,b}\nu_{a',b'} \left[\Theta_{a,a'}^{((i,s),(i',s'))(m)} \Theta_{b,b'}^{((i,s),(i',s'))(m)} + \Theta_{a,b'}^{((i,s),(i',s'))(m)} \Theta_{b,a'}^{((i,s),(i',s'))(m)} \right]. \end{aligned}$$

We deduce that it is sufficient to prove that, when j_0 goes to infinity, $\text{Cov}(Y^{(i,s,j_0)(m)}(\boldsymbol{\nu}), Y^{(i',s',j_0)(m)}(\boldsymbol{\nu}))$ converges to

$$\sum_{a,b,a',b'=1,\dots,p} \nu_{a,b}\nu_{a',b'} \left[\Theta_{a,a'}^{((i,s),(i',s'))(m)} \Theta_{b,b'}^{((i,s),(i',s'))(m)} + \Theta_{a,b'}^{((i,s),(i',s'))(m)} \Theta_{b,a'}^{((i,s),(i',s'))(m)} \right]$$

to obtain equality (C.3).

Following the proof of Lemma 16, we can write $\text{Cov}(Y^{(i,s,j_0)(m)}(\boldsymbol{\nu}), Y^{(i',s',j_0)(m)}(\boldsymbol{\nu}))$ as

$$\begin{aligned} & \text{Cov}(Y^{(i,s,j_0)(m)}(\boldsymbol{\nu}), Y^{(i',s',j_0)(m)}(\boldsymbol{\nu})) \\ &= \sum_{a,b,a',b'=1,\dots,p} \nu_{a,b}\nu_{a',b'} \left[T_{(a,a'),(b,b')}^{(i,i',s,s',j_0)(m)} + T_{(a,b'),(b,a')}^{(i,i',s,s',j_0)(m)} - \sum_{s'=0}^{n_{j_1}-1} \nu_{a,b}\nu_{a',b'} R_{(a,b),(a',b')}^{(i,i',s,s',j_0)(m)} \right], \end{aligned} \tag{C.5}$$

with

$$\begin{aligned} T_{(a,a'),(b,b')}^{(i,i',s,s',j_0)(m)}(\boldsymbol{\nu}) &= \left(\sum_{t_1 \in \mathbb{Z}} (\boldsymbol{\nu}^{(i,j_0)(m)}(2^{j_1}s - t_1) \boldsymbol{\nu}^{(i',j_0)(m)}(2^{j_1}s' - t_1)^T)_{a,a'} \right) \\ & \quad \left(\sum_{t_2 \in \mathbb{Z}} (\boldsymbol{\nu}^{(i,j_0)(m)}(2^{j_1}s - t_2) \boldsymbol{\nu}^{(i',j_0)(m)}(2^{j_1}s' - t_2)^T)_{b,b'} \right), \end{aligned}$$

and $\lim_{j_0 \rightarrow \infty} R_{(a,b),(a',b')}^{(i,i',s,s',j_0)(m)} = 0$. Conclusion follows from (B.13).

Proof of (C.4). The proof is based on decomposition (C.5) of $\text{Cov}(Y^{(i,s,j_0)(m)}, Y^{(i',s',j_0)(m)})$. Following the step 2 and the step 3 of the proof of Lemma 16, we can establish that

$$\begin{aligned} n_{j_1}^{-1} \sum_{s=0}^{n_{j_1}-1} \sum_{s'=0}^{n_{j_1}-1} R_{(a,a'),(b,b')}^{(i,i',s,s',j_0)(m)} &\xrightarrow{j_0 \rightarrow \infty} 0, \\ n_{j_1}^{-1} \sum_{s=0}^{n_{j_1}-1} \sum_{s'=0}^{n_{j_1}-1} T_{(a,b),(a',b')}^{(i,i',s,s',j_0)(m)} &\xrightarrow{j_0 \rightarrow \infty} 2\pi \int_{-\pi}^{\pi} \overline{\widetilde{\mathcal{W}}_{a,a'}^{(i,i')(m)*}(\lambda)} \widetilde{\mathcal{W}}_{b,b'}^{(i,i')(m)*}(\lambda) d\lambda, \\ n_{j_1}^{-1} \sum_{s=0}^{n_{j_1}-1} \sum_{s'=0}^{n_{j_1}-1} T_{(a,b'),(b,a')}^{(i,i',s,s',j_0)(m)} &\xrightarrow{j_0 \rightarrow \infty} 2\pi \int_{-\pi}^{\pi} \overline{\widetilde{\mathcal{W}}_{a,b'}^{(i,i')(m)*}(\lambda)} \widetilde{\mathcal{W}}_{b,a'}^{(i,i')(m)*}(\lambda) d\lambda. \end{aligned}$$

The proof is very similar and it is not detailed here for the sake of concision. It relies on Lemma 32, Lemma 13, Lemma 14 and Lemma 15. \square

We are now in a position to give the asymptotic normality of variables $\mathcal{S}^{(i,s,j_0)(m)}$, defined in (A.5).

Proposition 23. *Under conditions of Theorem 20,*

$$\left\{ \text{vec} \left(\mathcal{S}^{(i,j_0)(m)} \right), i = 1, \dots, N \right\} \xrightarrow{j_0 \rightarrow \infty} \left\{ \mathbf{Q}^{(m)}(i), i = 1, \dots, N \right\},$$

where $\mathbf{Q}^{(m)}(\cdot)$ is a centered Gaussian process with covariance function $\text{Cov}(Q_{a,b}^{(m)}(i), Q_{a',b'}^{(m)}(i')) = \Gamma_{(a,b),(a',b')}^{(i,i')(m)}$ defined in (B.12).

Proof. With results of Lemma 22, we can apply [31, Proposition 3], which gives the proposition. \square

Appendix C.1.3. Third step: proof of Proposition 21

For $i = 1, \dots, N$, when j_0 goes to infinity, $\text{vec} \left(\mathcal{S}^{(i,j_0)(m)} \right) = \text{vec} \left(n_j^{-1/2} \sum_{s=0}^{n_j-1} \mathcal{Y}^{(i,s,j_0)(m)} \right)$ converges to a $\mathcal{N}_{p^2}(0, \mathbf{\Gamma}^{(m)})$ distribution by Proposition 23. We want to deduce a similar

result for variables $(\mathcal{S}^{(i,j_0)})$. Lemma 17 establishes that $\lim_{m \rightarrow \infty} \Gamma^{(m)} = \Gamma$. Hence Lemma 24 below entails that we can apply [44, Theorem 3.2]. Proposition 21 follows.

It remains to prove the following lemma.

Lemma 24. *For all $i, i' = 1, \dots, N$,*

$$\lim_{m \rightarrow \infty} \lim_{j_0 \rightarrow \infty} \text{Var} \left(\mathcal{S}^{(i,j_0)(m)} - \mathcal{S}^{(i,j_0)} \right) = 0.$$

Proof. Lemma 16 and Proposition 23 state respectively that for all $(i, i') \in \{1, \dots, N\}^2$, for all $(a, b, a', b') \in \{1, \dots, p\}^4$, we have $\lim_{j_0 \rightarrow \infty} \text{Cov}(\mathcal{S}_{a,b}^{(i,j_0)}, \mathcal{S}_{a',b'}^{(i',j_0)}) = \Gamma_{(a,b),(a',b')}^{(i,i')}$ and $\lim_{j_0 \rightarrow \infty} \text{Cov}(\mathcal{S}_{a,b}^{(i,j_0)(m)}, \mathcal{S}_{a',b'}^{(i',j_0)(m)}) = \Gamma_{(a,b),(a',b')}^{(i,i')(m)}$. Additionally, by Lemma 17, $\lim_{m \rightarrow \infty} \Gamma^{(i,i')(m)} = \Gamma^{(i,i')}$. Consequently,

$$\lim_{m \rightarrow \infty} \lim_{j_0 \rightarrow \infty} \text{Cov}(\mathcal{S}_{a,b}^{(i,j_0)(m)}, \mathcal{S}_{a',b'}^{(i',j_0)(m)}) = \lim_{j_0 \rightarrow \infty} \text{Cov}(\mathcal{S}_{a,b}^{(i,j_0)}, \mathcal{S}_{a',b'}^{(i',j_0)}) = \Gamma_{(a,b),(a',b')}^{(i,i')}.$$

Hence it is sufficient to prove that $\lim_{m \rightarrow \infty} \lim_{j_0 \rightarrow \infty} \text{Cov}(\mathcal{S}_{a,b}^{(i,j_0)(m)}, \mathcal{S}_{a',b'}^{(i',j_0)}) = \Gamma_{(a,b),(a',b')}^{(i,i')}$. To this aim, we will prove that limits can be inverted, that is, $\lim_{m \rightarrow \infty} \lim_{j_0 \rightarrow \infty} \text{Cov}(\mathcal{S}_{a,b}^{(i,j_0)(m)}, \mathcal{S}_{a',b'}^{(i',j_0)}) = \lim_{j_0 \rightarrow \infty} \lim_{m \rightarrow \infty} \text{Cov}(\mathcal{S}_{a,b}^{(i,j_0)(m)}, \mathcal{S}_{a',b'}^{(i',j_0)})$.

We can establish that $\text{Cov}(\mathcal{S}_{a,b}^{(i,j_0)(m)}, \mathcal{S}_{a',b'}^{(i',j_0)})$ converges as j_0 goes to infinity to

$$2\pi \int_{-\pi}^{\pi} \overline{\widetilde{\mathcal{W}}_{a,a'}^{(i,i',j_0)(m,\infty)*}(\lambda)} \widetilde{\mathcal{W}}_{b,b'}^{(i,i',j_0)(m,\infty)*}(\lambda) d\lambda + 2\pi \int_{-\pi}^{\pi} \overline{\widetilde{\mathcal{W}}_{a,b'}^{(i,i',j_0)(m,\infty)*}(\lambda)} \widetilde{\mathcal{W}}_{b,a'}^{(i,i',j_0)(m,\infty)*}(\lambda) d\lambda, \quad (\text{C.6})$$

with

$$\begin{aligned} \widetilde{\mathcal{W}}_{a,b'}^{(i,i',j_0)(m,\infty)*}(\lambda) &= \sum_{t \in \mathbb{Z}} \mathcal{W}_{a,b'}^{(i,i',j_0)(m,\infty)*}(\lambda + 2t\pi) \\ \text{and } \mathcal{W}_{a,b'}^{(i,i',j_0)(m,\infty)*}(\lambda) &= 2^{-j_1} \sum_{a'=1}^p \mathcal{V}_{a,a'}^{(i,j_0)(m)*}(2^{-j_1} \lambda) \mathcal{V}_{b,a'}^{(i',j_0)*}(2^{-j_1} \lambda). \end{aligned}$$

The proof is very similar to the one carried out for proving (C.4) and it is thus omitted. Additionally (C.6) converges to $\Gamma_{(a,b),(a',b')}^{(i,i')}$ when j_0 goes to infinity.

Next we can study the limit of $\text{Cov}(\mathcal{S}_{a,b}^{(i,j_0)(m)}, \mathcal{S}_{a',b'}^{(i',j_0)})$ as m goes to infinity and state that it converges uniformly to $\widehat{\Gamma}_{(a,b),(a',b')}^{(i,i',j_0)}$ defined in (B.22). In the proof of Lemma 17, we have proved that $\lim_{j_0 \rightarrow \infty} \widehat{\Gamma}_{(a,b),(a',b')}^{(i,i',j_0)} = \widehat{\Gamma}_{(a,b),(a',b')}^{(i,i',j_0)}$. Hence

$$\lim_{m \rightarrow \infty} \lim_{j_0 \rightarrow \infty} \text{Cov}(\mathcal{S}_{a,b}^{(i,j_0)(m)}, \mathcal{S}_{a',b'}^{(i',j_0)}) = \lim_{j_0 \rightarrow \infty} \lim_{m \rightarrow \infty} \text{Cov}(\mathcal{S}_{a,b}^{(i,j_0)(m)}, \mathcal{S}_{a',b'}^{(i',j_0)}) = \Gamma_{(a,b),(a',b')}^{(i,i')}.$$

This concludes the proof. \square

Appendix C.2. Study of $\mathcal{R}(j)$

The following lemma gives the convergence of $\{\mathcal{R}_{a,b}(j), a, b = 1, \dots, p, j \geq 0\}$ to zero when j goes to infinity.

Lemma 25. $n_j^{1/2} \mathcal{R}_{a,b}(j) = n_j^{-1/2} \sum_{i=2^{\Delta-u}}^{T_\psi(2^{\Delta-u}-1)} \mathcal{Y}_{a,b}^{(i,n_{j_1},j_0)}$ goes to zero in probability as j_0 goes to infinity.

Proof. Hölder's inequality gives

$$\mathbb{E}[|\mathcal{Y}_{a,b}^{(i,n_{j_1},j_0)}|] \leq \sum_{a'=1}^p \left(\sum_{t \in \mathbb{Z}} \mathcal{V}_{a,a'}^{(i,j_0)} (2^{j_1} n_{j_1} - t)^2 \right)^{1/2} \left(\sum_{t \in \mathbb{Z}} \mathcal{V}_{b,a'}^{(i,j_0)} (2^{j_1} n_{j_1} - t)^2 \right)^{1/2}.$$

Using Parseval's equality and Lemma 10, we get

$$\begin{aligned} \mathbb{E}[|\mathcal{Y}_{a,b}^{(i,n_{j_1},j_0)}|] &\leq \sum_{a'=1}^p \left(\int_{-\pi}^{\pi} |\mathcal{V}_{a,a'}^{(i,j_0)*}(\lambda)|^2 d\lambda \right)^{1/2} \left(\int_{-\pi}^{\pi} |\mathcal{V}_{b,a'}^{(i,j_0)*}(\lambda)|^2 d\lambda \right)^{1/2} \\ &\leq C_v \int_{\mathbb{R}} (1 + |\xi|)^{-2\delta_v} d\xi. \end{aligned}$$

Since $\delta_v > 1/2$, $\mathbb{E}[|\mathcal{Y}_{a,b}^{(i,n_{j_1},j_0)}|]$ is finite. Thus $\mathcal{R}_{a,b}(j) = O_{\mathbb{P}}(n_j)$ using Markov's inequality. \square

Appendix C.3. Proof of Theorem 20

Equality (C.1) states that for all $a, b = 1, \dots, p$,

$$\begin{aligned} & 2^{-j_0(d_a+d_b)} \begin{pmatrix} \sqrt{n_{j_0}} \widehat{\sigma}_{a,b}(j_0) \\ \vdots \\ \sqrt{n_{j_0+\Delta}} \widehat{\sigma}_{a,b}(j_0 + \Delta) \end{pmatrix} \\ &= \sqrt{n_{j_1}} \begin{pmatrix} \sqrt{n_{j_0}} \\ \vdots \\ \sqrt{n_{j_0+\Delta}} \end{pmatrix} \mathbf{B}_{j_0} \begin{pmatrix} \mathcal{S}_{a,b}^{(1,j_0)} \\ \vdots \\ \mathcal{S}_{a,b}^{(N,j_0)} \end{pmatrix} + \begin{pmatrix} \sqrt{n_{j_0}} \mathcal{R}_{a,b}(j_0) \\ \vdots \\ \sqrt{n_{j_0+\Delta}} \mathcal{R}_{a,b}(j_0 + \Delta) \end{pmatrix}. \end{aligned}$$

By Proposition 25, $n_j^{1/2} \mathcal{R}_{a,b}(j)$ goes to 0 in probability when j goes to infinity.

Proposition 21 entails that the first term is asymptotically Gaussian and centered. We now explicit the asymptotic variance.

For all $0 \leq u' \leq u \leq \Delta$, $V_{(a,b),(a',b')}(u, u') = \lim_{j_0 \rightarrow \infty} \text{Cov}(n_{j_0+u}^{1/2} 2^{-j_0(d_a+d_b)} \widehat{\sigma}_{a,b}(j_0 + u), n_{j_0+u'}^{1/2} 2^{-j_0(d_{a'}+d_{b'})} \widehat{\sigma}_{a',b'}(j_0 + u'))$ satisfies

$$V_{(a,b),(a',b')}(u, u') = \lim_{j_0 \rightarrow \infty} n_{j_1} n_{j_0+u}^{1/2} n_{j_0+u'}^{1/2} (\mathbf{B}_{j_0} \text{Cov}(\mathcal{S}_{a,b}^{(i,j_0)}, \mathcal{S}_{a',b'}^{(i',j_0)})_{i,i'=1,\dots,N} \mathbf{B}_{j_0}^T)_{u,u'}.$$

Replacing \mathbf{B}_{j_0} by its expression, the equation above can be reformulated as

$$V_{(a,b),(a',b')}(u, u') = \lim_{j_0 \rightarrow \infty} n_{j_1} n_{j_0+u}^{-1/2} n_{j_0+u'}^{-1/2} \sum_{i=2^{\Delta-u}}^{2^{\Delta-u+1}-1} \sum_{i'=2^{\Delta-u'}}^{2^{\Delta-u'+1}-1} \text{Cov}(\mathcal{S}_{a,b}^{(i,j_0)}, \mathcal{S}_{a',b'}^{(i',j_0)})_{i,i'=1,\dots,N}.$$

Using the fact that $n_{j_1} n_{j_0+u}^{-1/2} n_{j_0+u'}^{-1/2} \sim 2^{-\Delta+u/2+u'/2}$, and using Proposition 21, it results that

$$V_{(a,b),(a',b')}(u, u') = 2^{-\Delta+u/2+u'/2} \sum_{i=2^{\Delta-u}}^{2^{\Delta-u+1}-1} \sum_{i'=2^{\Delta-u'}}^{2^{\Delta-u'+1}-1} \Gamma_{i,i'}((a,b), (a',b')),$$

with $\widetilde{\Gamma}((a,b), (a',b')) = (\Gamma_{(a,b),(a',b')}(i, i'))_{i,i'=1,\dots,N}$ defined in (B.6). We deduce from Lemma 19 that

$$V_{(a,b),(a',b')}(u, u') = 2^{-(u-u')/2} (\mathbf{G} \cdot \widetilde{\mathbf{I}} \cdot \mathbf{G}_{(a,a'),(b,b')}(u) + \mathbf{G} \cdot \widetilde{\mathbf{I}} \cdot \mathbf{G}_{(a,b),(a',b')}(u)).$$

Appendix D. Proof of Corollary 3

Let $(a, b) \in \{1, \dots, p\}^2$ and $j \geq 0$. We can write the correlation $\widehat{\rho}_{a,b}(j)$ as $\widehat{\rho}_{a,b}(j) = g(2^{-j(d_a+d_b)}\widehat{\sigma}_{a,b}(j), 2^{-j2d_a}\widehat{\sigma}_{a,a}(j), 2^{-j2d_b}\widehat{\sigma}_{b,b}(j))$ with $g(x_1, x_2, x_3) = x_1/\sqrt{x_2x_3}$. The vector $(\widehat{\sigma}_{a,a}(j), \widehat{\sigma}_{b,b}(j), \widehat{\sigma}_{a,b}(j))$ is asymptotically Gaussian by Theorem 20. By Delta method [33, Theorem 4.2.3.], we deduce that $\sqrt{n_j}(\widehat{\rho}_{a,b}(j) - r_{a,b})$ converges to a centered Gaussian distribution when j goes to infinity and that its asymptotic covariance satisfies:

$$\begin{aligned} & \lim_{j_0 \rightarrow \infty} \text{Cov}(\widehat{\rho}_{a,b}(j_0 + u), \widehat{\rho}_{a',b'}(j_0 + u')) \\ &= \frac{1}{\sqrt{G_{a,a}G_{b,b}}} \left(\frac{1}{\sqrt{G_{a',a'}G_{b',b'}}} V_{(a,b),(a',b')}(u, u') - \frac{r_{a',b'}}{2G_{a',a'}} V_{(a,b),(a',a')}(u, u') \right. \\ & \quad \left. - \frac{r_{a',b'}}{2G_{b',b'}} V_{(a,b),(b',b')}(u, u') \right) \\ & - \frac{r_{a,b}}{2G_{a,a}} \left(\frac{1}{\sqrt{G_{a',a'}G_{b',b'}}} V_{(a',a'),(a',b')}(u, u') - \frac{r_{a',b'}}{2G_{a',a'}} V_{(a,a),(a',a')}(u, u') \right. \\ & \quad \left. - \frac{r_{a',b'}}{2G_{b',b'}} V_{(a,a),(b',b')}(u, u') \right) \\ & - \frac{r_{a,b}}{2G_{b,b}} \left(\frac{1}{\sqrt{G_{a',a'}G_{b',b'}}} V_{(b,b),(a',b')}(u, u') - \frac{r_{a',b'}}{2G_{a',a'}} V_{(b,b),(a',a')}(u, u') \right. \\ & \quad \left. - \frac{r_{a',b'}}{2G_{b',b'}} V_{(b,b),(b',b')}(u, u') \right). \end{aligned}$$

We deduce first that

$$\begin{aligned} & \lim_{j \rightarrow \infty} \text{Var}(\widehat{\rho}_{a,b}(j)) \\ &= \frac{1}{G_{a,a}G_{b,b}} \left(V_{(a,b),(a,b)}(0, 0) - \frac{G_{a,b}}{2G_{a,a}} V_{(a,b),(a,a)}(0, 0) - \frac{G_{a,b}}{2G_{b,b}} V_{(a,b),(b,b)}(0, 0) \right) \\ & - \frac{G_{ab}}{2G_{a,a}^2G_{b,b}} \left(V_{(a,b),(a,a)}(0, 0) - \frac{G_{a,b}}{2G_{a,a}} V_{(a,a),(a,a)}(0, 0) - \frac{G_{a,b}}{2G_{b,b}} V_{(a,a),(b,b)}(0, 0) \right) \\ & - \frac{G_{ab}}{2G_{a,a}G_{b,b}^2} \left(V_{(a,b),(b,b)}(0, 0) - \frac{G_{a,b}}{2G_{a,a}} V_{(a,a),(b,b)}(0, 0) - \frac{G_{a,b}}{2G_{b,b}} V_{(b,b),(b,b)}(0, 0) \right), \end{aligned}$$

where we have used that $\mathbf{V}(u, u) = \mathbf{V}(0, 0)$ for all $u \in \mathbb{Z}$. Replacing also $V_{(a_1, a_2), (a_3, a_4)}(0, 0)$ for all $a_1, a_2, a_3, a_4 = 1, \dots, p$ by its expression given in (9), we obtain the asymptotic distribution of Corollary 3.

Second, suppose that the off-diagonal entries of \mathbf{G} are equal to zero. Then,

$$\lim_{j_0 \rightarrow \infty} \text{Cov}(\widehat{\rho}_{a,b}(j_0 + u), \widehat{\rho}_{a',b'}(j_0 + u')) = \frac{1}{\sqrt{G_{a,a}G_{b,b}G_{a',a'}G_{b',b'}}} V_{(a,b),(a',b')}(u, u').$$

Replacing $V_{(a,b),(a',b')}(u, u')$ by its expression, it results that the right-hand side is equal to 0 for all $(a, b) \notin \{(a', b'), (b', a')\}$, and is equal to $2^{-|u-u'|/2} 2^{-j(2d_a+2d_b)} \widetilde{I}_{|u-u'|}(2d_a, 2d_b)$ else.

Appendix E. Additional results on the sample wavelet covariance

The objective of this section is to prove that some linear combinations of sample wavelet covariances may be asymptotically Gaussian. Some conditions are given in the following proposition. It corresponds to [23, Theorem 3]. The arguments and the scheme of proof are the same. They are recalled here since the setting and the notations are slightly different.

Proposition 26. *Suppose assumptions of Theorem 2 hold. Let $\Delta \in \mathbb{N} \cup \{\infty\}$. Let $\{\boldsymbol{\omega}(u, j_0), u \in \mathbb{N}, j_0 \in \mathbb{N}\}$ be a sequence of $\mathbb{R}^{p \times p}$ such that for all $u \in \mathbb{N}$, $\boldsymbol{\omega}(u, j_0) \xrightarrow{j_0 \rightarrow \infty} \widetilde{\boldsymbol{\omega}}(u) \in \mathbb{R}$ and*

$$\sum_{u=0}^{\Delta} \sup_{j_0 \geq 0} \|\boldsymbol{\omega}(u, j_0)\|_{\infty} < \infty. \quad (\text{E.1})$$

Define

$$\mathbf{S}(\Delta, j_0) = \sum_{u=0}^{\Delta} \sqrt{n_{j_0+u}} \boldsymbol{\omega}(u, j_0) \boldsymbol{\Lambda}_{j_0+u}(\mathbf{d})^{-1} (\widehat{\boldsymbol{\Sigma}}(j_0 + u) - \boldsymbol{\Sigma}(j_0 + u)) \boldsymbol{\Lambda}_{j_0+u}(\mathbf{d})^{-1}.$$

Then $\text{vec}(\mathbf{S}(\Delta, j_0))$ converges in distribution to $\mathcal{N}_p(0, \mathbf{V}^{(S)}(\widetilde{\boldsymbol{\omega}}, \Delta))$ when j_0 goes to infinity, with

$$\mathbf{V}_{(a,b),(a',b')}^{(S)}(\widetilde{\boldsymbol{\omega}}, \Delta) = \sum_{u, u'=0, \dots, \Delta} 2^{-u(d_a^0+d_b^0+d_{a'}^0+d_{b'}^0)} \widetilde{\omega}_{a,b}(u) V_{(a,b),(a',b')}(u, u') \widetilde{\omega}_{a',b'}(u').$$

Proof. For all $\ell \geq 0$, introduce

$$\tilde{S}_{a,b}(\ell, j_0) = \sum_{u=0}^{\ell} \tilde{\omega}_{a,b}(u) \sqrt{n_{j_0+u}} 2^{-(j_0+u)(d_a+d_b)} (\hat{\sigma}_{a,b}(j_0+u) - \sigma_{a,b}(j_0+u)).$$

By Theorem 2, for all $0 \leq \ell < \infty$, $\tilde{S}_{a,b}(\ell, j_0)$ is asymptotically Gaussian, with distribution $\mathcal{N}_p(0, \mathbf{V}^{(S)}(\tilde{\omega}, \ell))$.

We will first establish the result when $\Delta = j_1 - j_0$ is finite and next when it is infinite.

- Δ finite.

Since $\tilde{S}_{a,b}(\Delta, j_0)$ is asymptotically Gaussian, it is sufficient to prove that $E \left| S_{a,b}(\Delta, j_0) - \tilde{S}_{a,b}(\Delta, j_0) \right|$ goes to 0 when j_0 goes to infinity. Using Lemma 27, we have

$$E \left| S_{a,b}(\Delta, j_0) - \tilde{S}_{a,b}(\Delta, j_0) \right| \leq C_{\sigma} \sum_{u=0}^{\Delta} |\omega_{a,b}(u, j_0) - \tilde{\omega}_{a,b}(u)|. \quad (\text{E.2})$$

We conclude with (E.1).

- Δ infinite.

The convergence of $S_{a,b}(\ell, j_0)$ has been established when ℓ is finite and (E.1)-(E.2) imply that

$$\lim_{\ell \rightarrow \infty} \lim_{j_0 \rightarrow \infty} \mathbb{E} |S_{a,b}(\ell, j_0) - S_{a,b}(\ell, j_0)| = 0.$$

Hence, it is sufficient to prove that $\lim_{\ell \rightarrow \infty} \lim_{j_0 \rightarrow \infty} \mathbb{E} |S_{a,b}(\infty, j_0) - S_{a,b}(\ell, j_0)| = 0$. Lemma 27 gives

$$E |S_{a,b}(\infty, j_0) - S_{a,b}(\ell, j_0)| \leq \sum_{u=\ell+1}^{\infty} |\omega_{a,b}(u, j_0)|.$$

The convergence is obtained using (E.1).

□

The proof of Proposition 26 above is based on the following lemma.

Lemma 27. *Suppose conditions of Theorem 20 hold. There exists C_σ depending on Ω , \mathbf{d} , $\phi(\cdot)$, $\psi(\cdot)$, L and β such that for all $(a, b) \in \{1, \dots, p\}$,*

$$\mathbb{E} |\widehat{\sigma}_{a,b}(j) - \sigma_{a,b}(j)| \leq C_\sigma (2^{j(d_a+d_b)} n_j^{-1/2}).$$

Proof. Since we only consider one scale, suppose momentarily that $j_0 = j = j_1$. Based on notations of Appendix A.2, equation (A.3), and Proposition 25,

$$2^{-j(d_a+d_b)} (\widehat{\sigma}_{a,b}(j) - \sigma_{a,b}(j)) = \frac{1}{\sqrt{n_j}} (\mathcal{S}_{a,b}(1, j) - \mathbb{E}(\mathcal{S}_{a,b}(1, j))) + O_{\mathbb{P}}(n_j).$$

Lemma 22 and Lemma 24 imply that $\text{Var}(\mathcal{S}_{a,b}(1, j))$ is finite. Lemma 27 is then straightforward. \square

Appendix F. Proof of Theorem 4

In this section the true parameters are denoted with an exponent 0.

Observe that conditions of [6, Proposition 6] are satisfied since we suppose assumption (M3). Consequently, under assumptions of Theorem 4, conditions of Theorem 6 of [6] hold. It entails that

$$\begin{aligned} \widehat{\mathbf{d}} - \mathbf{d}^0 &= O_{\mathbb{P}}(2^{-j_0\beta} + N_X 2^{j_0/2}), \\ \forall (a, b) \in \{1, \dots, p\}^2, \widehat{G}_{a,b}(\widehat{\mathbf{d}}) - G_{a,b}(\mathbf{d}^0) &= O_{\mathbb{P}}(\log(N)(2^{-j_0\beta} + N^{-1/2} 2^{j_0/2})). \end{aligned}$$

The proof of Theorem 4 is based on a Taylor expansion of the objective function. We first recall some useful results obtained [6] in Appendix F.1, next we give a normality result on the first derivative of the objective function in Appendix F.2. Appendix F.3 finally gives the proof of Theorem 4.

Appendix F.1. Some results about the objective function and its derivative

The objective function $R(\cdot)$ is equal to $R(\mathbf{d}) = \log \det \left(\mathbf{\Lambda}_{< \mathcal{J} >}(\mathbf{d}) \widehat{\mathbf{G}}(\mathbf{d}) \mathbf{\Lambda}_{< \mathcal{J} >}(\mathbf{d}) \right) - 1$. It is straightforward that $\widehat{\mathbf{d}} = \underset{\mathbf{d}}{\operatorname{argmin}} R(\mathbf{d})$ satisfies

$$\begin{aligned} \widehat{\mathbf{d}} = \underset{\mathbf{d}}{\operatorname{argmin}} \widetilde{R}(\mathbf{d}) \quad \text{with} \quad \widetilde{R}(\mathbf{d}) = \log \det \widetilde{\mathbf{G}}(\mathbf{d}) \\ \text{and} \quad \widetilde{\mathbf{G}}(\mathbf{d}) = \mathbf{\Lambda}_{< \mathcal{J} >}(\mathbf{d} - \mathbf{d}^0) \widehat{\mathbf{G}}(\mathbf{d}) \mathbf{\Lambda}_{< \mathcal{J} >}(\mathbf{d} - \mathbf{d}^0). \end{aligned}$$

The derivatives of the criterion $\widetilde{R}(\mathbf{d})$ are equal to

$$\begin{aligned} \frac{\partial \widetilde{R}(\mathbf{d})}{\partial d_a} &= \operatorname{trace} \left(\widetilde{\mathbf{G}}(\mathbf{d})^{-1} \frac{\partial \widetilde{\mathbf{G}}(\mathbf{d})}{\partial d_a} \right), \quad (\text{F.3}) \\ \frac{\partial^2 \widetilde{R}(\mathbf{d})}{\partial d_a \partial d_b} &= - \operatorname{trace} \left(\widetilde{\mathbf{G}}(\mathbf{d})^{-1} \frac{\partial \widetilde{\mathbf{G}}(\mathbf{d})}{\partial d_b} \widetilde{\mathbf{G}}(\mathbf{d})^{-1} \frac{\partial \widetilde{\mathbf{G}}(\mathbf{d})}{\partial d_a} \right) + \operatorname{trace} \left(\widetilde{\mathbf{G}}(\mathbf{d})^{-1} \frac{\partial^2 \widetilde{\mathbf{G}}(\mathbf{d})}{\partial d_a \partial d_b} \right). \end{aligned}$$

when $\widetilde{\mathbf{G}}(\mathbf{d})^{-1}$ exists.

For any $a = 1, \dots, p$, let \mathbf{i}_a be a $p \times p$ matrix whose a -th diagonal element is one and all other elements are zero. Let a and b be two indexes in $1, \dots, p$. The first derivative of $\widetilde{\mathbf{G}}(\mathbf{d})$ with respect to d_a , $\frac{\partial \widetilde{\mathbf{G}}(\mathbf{d})}{\partial d_a}$, is equal to

$$\begin{aligned} - \log(2) \frac{1}{n} \sum_{j=j_0}^{j_1} n_j (j - < \mathcal{J} >) \mathbf{\Lambda}_{< \mathcal{J} >}(\mathbf{d} - \mathbf{d}^0) \mathbf{\Lambda}_j(\mathbf{d})^{-1} \\ (\mathbf{i}_a \mathbf{\Sigma}(j) + \mathbf{\Sigma}(j) \mathbf{i}_a) \mathbf{\Lambda}_j(\mathbf{d})^{-1} \mathbf{\Lambda}_{< \mathcal{J} >}(\mathbf{d} - \mathbf{d}^0), \end{aligned}$$

with $\widehat{\mathbf{\Sigma}}(j) = (\widehat{\sigma}_{a', b'}(j))_{a', b'=1, \dots, p}$. Thus

$$\left. \frac{\partial \widetilde{\mathbf{G}}}{\partial d_a} \right|_{\mathbf{d}^0} = - \frac{\log(2)}{n} \sum_{j=j_0}^{j_1} n_j (j - < \mathcal{J} >) \mathbf{\Lambda}_j(\mathbf{d})^{-1} (\mathbf{i}_a \mathbf{\Sigma}(j) + \mathbf{\Sigma}(j) \mathbf{i}_a) \mathbf{\Lambda}_j(\mathbf{d})^{-1}. \quad (\text{F.4})$$

Appendix F.2. Asymptotic normality of the first derivative

The objective of this section is to prove that $\sqrt{n} \frac{\partial \tilde{R}}{\partial \mathbf{d}} \Big|_{\mathbf{d}^0}$ is asymptotically Gaussian.

Proposition 28. *Under assumptions of Theorem 4,*

$$\sqrt{n} \frac{\partial \tilde{R}}{\partial \mathbf{d}} \Big|_{\mathbf{d}^0} \xrightarrow{j_0 \rightarrow \infty} \mathcal{N}_p \left(0, 4 \log(2)^4 \kappa_\Delta^2 \mathbf{V}^{\mathbf{d}(\Delta)} \right).$$

where $\mathbf{V}^{\mathbf{d}(\Delta)}$ is defined in equation (17).

Proof. For any vector $\mathbf{v} = (v_a)_{a=1, \dots, p} \in \mathbb{R}^p$, we want to prove that $\mathbf{v}^T \frac{\partial \tilde{R}}{\partial \mathbf{d}} \Big|_{\mathbf{d}^0}$ converges to a centered Gaussian distribution with variance $\mathbf{v}^T (4 \log(2)^2 \kappa_\Delta^2 \mathbf{V}) \mathbf{v}$.

By (F.3),

$$\mathbf{v}^T \frac{\partial \tilde{R}(\mathbf{d})}{\partial d_a} = \sum_{a=1}^p v_a \text{trace} \left(\tilde{\mathbf{G}}(\mathbf{d})^{-1} \frac{\partial \tilde{\mathbf{G}}(\mathbf{d})}{\partial d_a} \right).$$

As expressed in [6, page 36], $\tilde{\mathbf{G}}(\mathbf{d})$ can be written as

$$\tilde{G}_{a,b}(\mathbf{d}^0) = G_{a,b}^0 + \sum_{j=j_0}^{j_1} \sum_{k=0}^{n_j} \frac{1}{n} \left(\frac{W_a(j,k) W_b(j,k)}{2^{j(d_a^0 + d_b^0)}} - G_{a,b}^0 \right).$$

Applying [6, Proposition 8], we get

$$\tilde{G}_{a,b}(\mathbf{d}^0) = G_{a,b}^0 + O_{\mathbb{P}}(2^{-j_0\beta} + N_X^{-1/2} 2^{-j_1/2}).$$

We introduce

$$\tilde{S}_{j_0} = \sqrt{n} \sum_{a=1}^p v_a \text{trace} \left(\mathbf{G}^{0-1} \frac{\partial \tilde{\mathbf{G}}(\mathbf{d})}{\partial d_a} \Big|_{\mathbf{d}^0} \right).$$

It is easily seen that

$$\sqrt{n} \mathbf{v}^T \frac{\partial \tilde{R}(\mathbf{d})}{\partial \mathbf{d}} \Big|_{\mathbf{d}^0} - \tilde{S}_{j_0} \xrightarrow{\mathbb{P}} 0. \quad (\text{F.5})$$

Reformulating (F.4), we get

$$\left(\frac{\partial \tilde{\mathbf{G}}(\mathbf{d})}{\partial d_a} \Big|_{\mathbf{d}} \right)_{a,b} = -2 \log(2) \frac{1}{n} \sum_{j=j_0}^{j_1} (j - \langle \mathcal{J} \rangle) 2^{-j(d_a^0 + d_b^0)} n_j \hat{\sigma}_{a,b}(j).$$

Thus we can write \tilde{S}_{j_0} as

$$\tilde{S}_{j_0} = \sum_{a,b=1,\dots,p} S_{a,b}^{(\mathbf{d})}(\Delta, j_0), \quad (\text{F.6})$$

$$\text{with } S_{a,b}^{(\mathbf{d})}(\Delta, j_0) = \sum_{u=0}^{\Delta} \omega_{a,b}^{(S)}(u, j_0) \sqrt{n_{j_0+u}} 2^{-(j_0+u)(d_a^0 + d_b^0)} \hat{\sigma}_{a,b}(j_0 + u)$$

$$\text{and } \omega_{a,b}^{(S)}(u, j_0) = -2 \log(2) \sqrt{\frac{n_{j_0+u}}{n}} (j_0 + u - \langle \mathcal{J} \rangle) v_a(G^{0-1})_{a,b}.$$

Lemma 13 of [21] states that $\langle \mathcal{J} \rangle - j_0 \rightarrow \eta_\Delta$ when $N_X \rightarrow \infty$. When $j_0 \rightarrow \infty$, $\omega_{a,b}^{(S)}(u, j_0)$ hence converges to $\tilde{\omega}_{a,b}^{(S)}(u) = -2 \log(2) \sqrt{\frac{2^{-u}}{2-2^{-\Delta}}} (u - \eta_\Delta) v_a(G^{0-1})_{a,b}$. Moreover (E.1) holds for $\Delta \in \mathbb{N} \cup \{\infty\}$.

Applying Proposition 26, $\text{vec}(\mathbf{S}^{(\mathbf{d})}(\Delta, j_0) - \mathbb{E}(\mathbf{S}^{(\mathbf{d})}(\Delta, j_0)))$ is asymptotically Gaussian, with distribution $\mathcal{N}_p(0, \mathbf{V}^{(S)}(\tilde{\omega}^{(S)}, \Delta))$. Consequently, $\tilde{S}_{j_0} - \mathbb{E}(\tilde{S}_{j_0})$ follows asymptotically the Gaussian distribution $\mathcal{N}_p(0, \sum_{a,b,a',b'=1,\dots,p} \mathbf{V}_{(a,b),(a',b')}^{(S)}(\tilde{\omega}^{(S)}, \Delta))$.

The end of the proof is divided into two steps. First we prove that $\mathbb{E}(\tilde{S}_{j_0})$ goes to 0 when j_0 goes to infinity, and next we establish that the asymptotic variance above, that is, $\sum_{a,b,a',b'=1,\dots,p} \mathbf{V}_{(a,b),(a',b')}^{(S)}(\tilde{\omega}^{(S)}, \Delta)$, is equal to $4 \log(2)^2 \kappa_\Delta^2 \mathbf{v}^T \mathbf{V}^{\mathbf{d}(\Delta)} \mathbf{v}$.

Convergence of $\mathbb{E}(\tilde{S}_{j_0})$ toward 0.

Taking the expectancy of \tilde{S}_{j_0} ,

$$\mathbb{E}(\tilde{S}_{j_0}) = -\frac{2 \log(2)}{\sqrt{n}} \sum_{j=j_0}^{j_1} n_j (j - \langle \mathcal{J} \rangle) \sum_{a,b=1,\dots,p} v_a(G^{0-1})_{a,b} 2^{-j(d_a^0 + d_b^0)} \sigma_{a,b}(j).$$

Since $\sum_{j=j_0}^{j_1} n_j (j - \langle \mathcal{J} \rangle) = 0$.

$$\mathbb{E}(\tilde{S}_{j_0}) = -\frac{2 \log(2)}{\sqrt{n}} \sum_{j=j_0}^{j_1} n_j (j - \langle \mathcal{J} \rangle) \sum_{a,b=1,\dots,p} v_a(G^{0-1})_{a,b} (2^{-j(d_a^0 + d_b^0)} \sigma_{a,b}(j) - G_{a,b}^0).$$

Using Proposition 1,

$$\mathbb{E}|\tilde{S}_{j_0}| = O(\sqrt{n}j_0 2^{-j_0\beta}).$$

Thus $\mathbb{E}(\tilde{S}_{j_0})$ converges to zero when $j_0^2 N_X 2^{-j_0(1+2\beta)} \rightarrow 0$.

Expression of $\sum_{a,b,a',b'=1,\dots,p} \mathbf{V}_{(a,b),(a',b')}^{(S)}(\tilde{\omega}^{(S)}, \Delta)$.

It remains to prove that $\sum_{a,b,a',b'=1,\dots,p} \mathbf{V}_{(a,b),(a',b')}^{(S)}(\tilde{\omega}^{(S)}, \Delta) = 4 \log(2)^2 \kappa_\Delta^2 \mathbf{u}^T \mathbf{V}^{\mathbf{d}(\Delta)} \mathbf{v}$. By expanding the expression,

$$\begin{aligned} & \sum_{a,b,a',b'=1,\dots,p} \mathbf{V}_{(a,b),(a',b')}^{(S)}(\tilde{\omega}^{(S)}, \Delta) \\ &= \frac{4 \log(2)^2}{2 - 2^{-\Delta}} \sum_{a,b,a',b'=1,\dots,p} v_a v_{a'} (G^{0-1})_{a,b} (G^{0-1})_{a',b'} \sum_{u=0}^{\Delta} \sum_{u'=0}^{\Delta} 2^{-u/2-u'/2} (u - \eta_\Delta)(u' - \eta_\Delta) \\ & \quad 2^{-u(d_a^0+d_b^0)-u'(d_{a'}^0+d_{b'}^0)} V_{(a,b),(a',b')}(u, u') \\ &= 4 \log(2)^2 \sum_{a,b,a',b'=1,\dots,p} v_a v_{a'} (G^{0-1})_{a,b} (G^{0-1})_{a',b'} (G_{aa'}^0 G_{bb'}^0 \mathcal{I}^{(S)}(d_a^0 + d_{a'}^0, d_b^0 + d_{b'}^0) \\ & \quad + G_{ab'}^0 G_{a'b}^0 \mathcal{I}^{(S)}(d_a^0 + d_{b'}^0, d_{a'}^0 + d_b^0)), \end{aligned}$$

where

$$\begin{aligned} \mathcal{I}^{(S)}(\delta_1, \delta_2) &= \frac{1}{2 - 2^{-\Delta}} \sum_{u=0}^{\Delta} \sum_{u'=0}^{\Delta} 2^{-u/2-u'/2} (u - \eta_\Delta)(u' - \eta_\Delta) \\ & \quad 2^{(\delta_1+\delta_2)u \vee u' - |u-u'|/2} 2^{-u\delta_1 - u'\delta_2} \tilde{I}_{|u-u'|}(\delta_1, \delta_2). \end{aligned}$$

We can formulate this expression to recover a similar form to that of [21, Theorem 5] and [23, Theorem 5]. The arguments are the same than those used in the proof of [21, Proposition 10], but are recalled here to explicit the form of the variance. We can express $\mathcal{I}^{(S)}(\delta_1, \delta_2)$ as:

$$\begin{aligned} \mathcal{I}^{(S)}(\delta_1, \delta_2) &= \frac{1}{2 - 2^\Delta} \sum_{\gamma'=0}^{\Delta} (\gamma' - \eta_\Delta)^2 2^{-\gamma'} \tilde{I}_0(\delta_1, \delta_2) \\ & \quad + \frac{1}{2 - 2^\Delta} \sum_{\gamma=1}^{\Delta} \sum_{\gamma'=0}^{\Delta-\gamma} (\gamma + \gamma' - \eta_\Delta)(\gamma' - \eta_\Delta) 2^{-\gamma-\gamma'} (2^{\gamma\delta_1} + 2^{\gamma\delta_2}) \tilde{I}_\gamma(\delta_1, \delta_2), \end{aligned}$$

where we set $\gamma = |u - u'|$ and $\gamma' = u \wedge u'$. We can use the equalities $\sum_{\gamma'=0}^{\Delta-\gamma} (\gamma' - \eta_\Delta)^2 2^{-\gamma'} = (2 - 2^{-\Delta+\gamma})(\kappa_{\Delta-\gamma} + (\eta_{\Delta-\gamma} - \eta_\Delta)^2)$ and $\sum_{\gamma'=0}^{\Delta-\gamma} (\gamma' - \eta_\Delta) 2^{-\gamma'} = (2 - 2^{-\Delta+\gamma})(\eta_{\Delta-\gamma} - \eta_\Delta)$. We obtain

$$\begin{aligned} \mathcal{I}^{(S)}(\delta_1, \delta_2) &= \kappa_\Delta \tilde{I}_0(\delta_1, \delta_2) \\ &+ \frac{1}{2 - 2^\Delta} \sum_{\gamma=1}^{\Delta} (2 - 2^{-\Delta+\gamma}) ((\gamma + \eta_{\Delta-\gamma} - \eta_\Delta)(\eta_{\Delta-\gamma} - \eta_\Delta) + \kappa_{\Delta-\gamma}) 2^{-\gamma} (2^{\gamma\delta_1} + 2^{\gamma\delta_2}) \tilde{I}_\gamma(\delta_1, \delta_2). \end{aligned}$$

We see that when $\Delta < \infty$, $\mathcal{I}^{(S)}(\delta_1, \delta_2) = \frac{\kappa_\Delta^2}{2} \mathcal{I}_\Delta(\delta_1, \delta_2)$, with $\mathcal{I}_\Delta(\delta_1, \delta_2)$ defined in (13). Hence, $\sum_{a,b,a',b'=1,\dots,p} \mathbf{V}_{(a,b),(a',b')}^{(S)}(\tilde{\omega}^{(S)}, \Delta) = 4 \log(2)^2 \kappa_\Delta^2 \mathbf{v}^T \mathbf{V}^{\mathbf{d}(\Delta)} \mathbf{v}$.

When Δ goes to infinity, the sequence κ_Δ converges to 2 and the sequence η_Δ converges to 1. We deduce the asymptotic form (14) when $\Delta \rightarrow \infty$ by dominated convergence. \square

Appendix F.3. Proof of Theorem 4

The Taylor expansion of $\frac{\partial \tilde{R}(\mathbf{d})}{\partial \mathbf{d}}$ at $\hat{\mathbf{d}}$ at the neighborhood of \mathbf{d}^0 gives

$$\sqrt{n}(\hat{\mathbf{d}} - \mathbf{d}^0) = \left(\frac{\partial^2 \tilde{R}}{\partial \mathbf{d} \partial \mathbf{d}^T} \Big|_{\bar{\mathbf{d}}} \right)^{-1} \sqrt{n} \frac{\partial \tilde{R}}{\partial \mathbf{d}} \Big|_{\mathbf{d}^0}, \quad (\text{F.7})$$

where $\bar{\mathbf{d}}$ is such that $\|\bar{\mathbf{d}} - \mathbf{d}^0\| \leq \|\hat{\mathbf{d}} - \mathbf{d}^0\|$.

It has already been established in [6, Equation (E10)] that, under assumptions of Theorem 4,

$$\frac{\partial^2 \tilde{R}(\mathbf{d})}{\partial \mathbf{d} \partial \mathbf{d}^T} \Big|_{\bar{\mathbf{d}}} \xrightarrow{\mathbb{P}} \kappa_{j_1-j_0} \log(2)^2 2(\mathbf{G}^{0-1} \circ \mathbf{G}^0 + \mathbf{I}_p). \quad (\text{F.8})$$

This justifies also that the matrix $\left(\frac{\partial^2 \tilde{R}}{\partial \mathbf{d} \partial \mathbf{d}^T} \Big|_{\bar{\mathbf{d}}} \right)$ in (F.7) is indeed invertible for sufficiently high N_X when $2^{-j_0\beta} + N_X^{-1/2} 2^{j_0/2} \rightarrow 0$.

Next Proposition 28 establishes that $\frac{\partial \tilde{R}}{\partial \mathbf{d}} \Big|_{\mathbf{d}^0}$ converges to a centered Gaussian distribution with variance $(2 \log(2))^2 \kappa_{j_1-j_0}^2 \mathbf{V}^{\mathbf{d}(\Delta)}$. Theorem 4 then follows with (F.7) and (F.8).

Appendix G. Proof of Theorem 5

In this section the true parameters are denoted with an exponent 0.

We have $\widehat{G}_{a,b}(\mathbf{d}) = 2^{\langle \mathcal{J} \rangle (d_a - d_a^0 + d_b - d_b^0)} \widetilde{G}_{a,b}(\mathbf{d})$ and $\widehat{G}_{a,b}(\mathbf{d}^0) = \overline{G}_{a,b}(\mathbf{d}^0)$ with $\widetilde{\mathbf{G}}(\mathbf{d})$ defined in (F.2). As $2^{\langle \mathcal{J} \rangle u} - 1 = j_0 u \log(2)(1 + o(1))$ when $u \rightarrow 0$, we deduce that

$$\widehat{G}_{a,b}(\widehat{\mathbf{d}}) - \widetilde{G}_{a,b}(\widehat{\mathbf{d}}) = j_0 (\widehat{d}_a - d_a^0 + \widehat{d}_b - d_b^0) \log(2)(1 + o(1)) \overline{G}_{a,b}(\widehat{\mathbf{d}}). \quad (\text{G.1})$$

Since $j_0 (\widehat{d}_a - d_a^0 + \widehat{d}_b - d_b^0) = o_{\mathbb{P}}(1)$, it is sufficient to establish the asymptotic distribution of $\widetilde{G}_{a,b}(\widehat{\mathbf{d}})$. More precisely, we want to prove that $\sqrt{n} \text{vec}(\overline{\mathbf{G}}(\widehat{\mathbf{d}}) - \mathbf{G}^0)$ converges in distribution to a centered Gaussian distribution. We decompose $\sqrt{n} \text{vec}(\widetilde{\mathbf{G}}(\widehat{\mathbf{d}}) - \mathbf{G}^0)$ as

$$\sqrt{n}(\widetilde{\mathbf{G}}(\widehat{\mathbf{d}}) - \mathbf{G}^0) = \sqrt{n}(\overline{\mathbf{G}}(\mathbf{d}^0) - \mathbf{G}^0) + \sqrt{n}(\widetilde{\mathbf{G}}(\widehat{\mathbf{d}}) - \widetilde{\mathbf{G}}(\mathbf{d}^0)). \quad (\text{G.2})$$

The first term converges to the desired distribution as established in Lemma 29, while the second one is negligible by Lemma 30.

Lemma 29 and Lemma 30 are given hereafter.

Appendix G.1. Study of $\sqrt{n}(\widetilde{\mathbf{G}}(\mathbf{d}^0) - \mathbf{G}^0)$

Lemma 29. *Under assumptions of Theorem 5, $\sqrt{n} \text{vec}(\overline{\mathbf{G}}(\mathbf{d}^0) - \mathbf{G}^0)$ converges as j_0 goes to infinity to a centered Gaussian distribution, with variance $\Theta^{\mathbf{G}(\Delta)}$ defined in (19).*

Proof. Consider $\mathbf{T}_0(j_0) = \text{vec}(\sqrt{n}(\overline{\mathbf{G}}(\mathbf{d}^0) - \mathbb{E}(\overline{\mathbf{G}}(\mathbf{d}^0))))$. Recall that

$$\widetilde{G}_{a,b}(\mathbf{d}^0) = \frac{1}{n} \sum_{j=j_0}^{j_1} n_j 2^{-j(d_a^0 + d_b^0)} \widehat{\sigma}_{a,b}(j).$$

Using inequality (5),

$$\mathbb{E}(\widetilde{G}_{a,b}(\mathbf{d}^0)) = G_{a,b}^0(j) + O(2^{j_0\beta}),$$

and consequently $\sqrt{n}(\mathbb{E}(\widetilde{G}_{a,b}(\mathbf{d}^0)) - G_{a,b}^0(j)) = o(1)$.

We can write $\sqrt{n} \text{vec}(\overline{\mathbf{G}}(\mathbf{d}^0) - \mathbf{G}^0)$ as

$$\begin{aligned} & \sqrt{n} \left(\tilde{G}_{a,b}(\mathbf{d}^0) - \mathbb{E}(\tilde{G}_{a,b}(\mathbf{d}^0)) \right) \\ &= \sum_{u=0}^{j_1-j_0} \omega^{(\mathbf{G})}(u, j_0) \sqrt{n_{j_0+u}} 2^{-(j_0+u)(d_a^0+d_b^0)} (\hat{\sigma}_{a,b}(j_0+u) - \sigma_{a,b}(j_0+u)), \quad (\text{G.3}) \end{aligned}$$

with $\omega^{(\mathbf{G})}(u, j_0) = \sqrt{n_{j_0+u}/n}$. The sequence $\omega^{(\mathbf{G})}(u)$ converges to $\tilde{\omega}^{(\mathbf{G})}(u) = 2^{-u/2}/\sqrt{2-2^{-\Delta}}$ when j_0 goes to infinity. Applying Proposition 26, we obtain that $\mathbf{T}_0(j_0) - \mathbb{E}(\mathbf{T}_0(j_0))$ converges as n goes to infinity to a centered Gaussian distribution, with variance

$$\begin{aligned} & \lim_{j_0 \rightarrow \infty} \text{Cov}(T_{0,a,b}(j_0), T_{0,a',b'}(j_0)) \\ &= \sum_{u=0}^{\Delta} \sum_{u'=0}^{\Delta} 2^{-u(d_a^0+d_b^0)-u'(d_{a'}^0+d_{b'}^0)} \frac{2^{-u/2-u'/2}}{2-2^{-\Delta}} V_{(a,b),(a',b')}(u, u') \\ &= 2\pi \left(G_{a,a'}^0 G_{b,b'}^0 \mathcal{I}_{\Delta}^G(d_a + d_{a'}, d_b + d_{b'}) + G_{a,b'}^0 G_{b,a'}^0 \mathcal{I}_{\Delta}^G(d_a + d_{b'}, d_b + d_{a'}) \right), \end{aligned}$$

where

$$\mathcal{I}_{\Delta}^G(\delta_1, \delta_2) = \sum_{u=0}^{\Delta} \sum_{u'=0}^{\Delta} 2^{(u \vee u' - u)\delta_1 + (u \vee u' - u')\delta_2} \frac{2^{-u/2-u'/2-|u-u'|/2}}{2-2^{-\Delta}} \tilde{I}_{|u'-u|}(\delta_1, \delta_2).$$

Quantity $\mathcal{I}^G(\delta_1, \delta_2)$ can be simplified as:

$$\begin{aligned} \mathcal{I}_{\Delta}^G(\delta_1, \delta_2) &= \sum_{u=0}^{\Delta} \frac{2^{-u}}{2-2^{-\Delta}} \tilde{I}_0(\delta_1, \delta_2) + \sum_{u=1}^{\Delta} (2^{u\delta_1} + 2^{u\delta_2}) 2^{-u} \tilde{I}_u(\delta_1, \delta_2) \sum_{v=0}^{\Delta-u} \frac{2^{-v}}{2-2^{-\Delta}} \\ &= \tilde{I}_0(\delta_1, \delta_2) + \sum_{u=1}^{\Delta} (2^{u\delta_1} + 2^{u\delta_2}) 2^{-u} \frac{2-2^{-\Delta+u}}{2-2^{-\Delta}} \tilde{I}_u(\delta_1, \delta_2). \end{aligned}$$

When Δ goes to infinity,

$$\mathcal{I}_{\infty}^G(\delta_1, \delta_2) = \tilde{I}_0(\delta_1, \delta_2) + \sum_{u=1}^{\infty} (2^{u\delta_1} + 2^{u\delta_2}) 2^{-u} \tilde{I}_u(\delta_1, \delta_2).$$

□

Appendix G.2. Study of $\sqrt{n} \left(\tilde{\mathbf{G}}(\hat{\mathbf{d}}) - \tilde{\mathbf{G}}(\mathbf{d}^0) \right)$

Lemma 30. *Under assumptions of Theorem 5, $\sqrt{n} \left(\tilde{\mathbf{G}}(\hat{\mathbf{d}}) - \tilde{\mathbf{G}}(\mathbf{d}^0) \right)$ tends to 0 in probability when j_0 goes to infinity.*

Proof. A Taylor expansion at order one at \mathbf{d}^0 gives

$$\sqrt{n} \left(\tilde{\mathbf{G}}(\hat{\mathbf{d}}) - \tilde{\mathbf{G}}(\mathbf{d}^0) \right) = \sqrt{n} \left(\frac{\partial \tilde{\mathbf{G}}}{\partial \mathbf{d}} \Big|_{\mathbf{d}^0} \right) (\hat{\mathbf{d}} - \mathbf{d}^0) + \sqrt{n} (\hat{\mathbf{d}} - \mathbf{d}^0)^T \left(\frac{\partial^2 \tilde{\mathbf{G}}}{\partial \mathbf{d} \partial \mathbf{d}^T} \Big|_{\bar{\mathbf{d}}} \right) (\hat{\mathbf{d}} - \mathbf{d}^0),$$

with $\|\bar{\mathbf{d}} - \mathbf{d}^0\| \leq \|\hat{\mathbf{d}} - \mathbf{d}^0\|$. Achard and Gannaz in [6, Section E.2.4.] state that, when $2^{-j_0\beta} + N_X^{-1/2} 2^{j_0/2} \rightarrow 0$,

$$\frac{\partial^2 \tilde{\mathbf{G}}}{\partial \mathbf{d} \partial \mathbf{d}^T} \Big|_{\mathbf{d}^0} = O_{\mathbb{P}}(1).$$

Thus

$$\sqrt{n} (\hat{\mathbf{d}} - \mathbf{d}^0)^T \left(\frac{\partial^2 \tilde{\mathbf{G}}}{\partial \mathbf{d} \partial \mathbf{d}^T} \Big|_{\bar{\mathbf{d}}} \right) (\hat{\mathbf{d}} - \mathbf{d}^0) = o_{\mathbb{P}}(1).$$

Next, using the derivative of $\bar{\mathbf{G}}(\mathbf{d})$ given in (F.4), we have:

$$\begin{aligned} & \sqrt{n} \left(\frac{\partial \tilde{\mathbf{G}}}{\partial \mathbf{d}} \Big|_{\mathbf{d}^0} (\hat{\mathbf{d}} - \mathbf{d}^0) \right)_{a,b} \\ &= -\frac{\log(2)}{\sqrt{n}} \sum_{j=j_0}^{j_1} n_j (j - \langle \mathcal{J} \rangle) 2^{-j(d_a^0 + d_b^0)} \hat{\sigma}_{a,b}(j) (\hat{d}_a - d_a^0 + \hat{d}_b - d_b^0). \end{aligned}$$

Similarly to what was done in Appendix F, we can establish that $\frac{\log(2)}{\sqrt{n}} \sum_{j=j_0}^{j_1} n_j (j - \langle \mathcal{J} \rangle) 2^{-j(d_a^0 + d_b^0)} \hat{\sigma}_{a,b}(j)$ is asymptotically Gaussian. Since $(\hat{d}_a - d_a^0 + \hat{d}_b - d_b^0) \rightarrow 0$ we conclude that this term goes to 0, which concludes the proof. \square

Appendix H. Proof of Proposition 7

In this section the true parameters are denoted with an exponent 0.

From the proof of Theorem 4 (using equations (F.7), (F.8), (F.5) and (F.6)) and Theorem 5 (using equations (G.1), (G.2), (G.3) and Lemma 30), we can extract the following results

$$\begin{aligned}
& \left((\mathbf{G}^{0-1} \circ \mathbf{G}^0 + \mathbf{I}_p) \sqrt{n} (\widehat{\mathbf{d}} - \mathbf{d}^0) \right)_{a'} \\
&= \sum_{b'=1}^p \sum_{u=0}^{j_1-j_0} \omega_{a',b'}^{(\mathbf{d})}(u, j_0) \sqrt{n_{j_0+u}} 2^{-(j_0+u)(d_{a'}^0+d_{b'}^0)} (\widehat{\sigma}_{a',b'}(j) - \sigma_{a',b'}(j)) (1 + o_{\mathbb{P}}(1)) + o_{\mathbb{P}}(1), \\
& \sqrt{n} \left(\widehat{G}_{a,b}(\widehat{\mathbf{d}}) - G_{a,b}^0 \right) \\
&= \sum_{u=0}^{j_1-j_0} \omega_{a,b}^{(\mathbf{G})}(u, j_0) \sqrt{n_{j_0+u}} 2^{-(j_0+u)(d_a^0+d_b^0)} (\widehat{\sigma}_{a,b}(j) - \sigma_{a,b}(j)) (1 + o_{\mathbb{P}}(1)) + o_{\mathbb{P}}(1),
\end{aligned}$$

with

$$\begin{aligned}
\omega_{(a',b')}^{(\mathbf{d})}(u, j_0) &= -(\kappa_{j_1-j_0} \log(2))^{-1} \sqrt{n_{j_0+u}/n} (j_0 + u - \langle \mathcal{J} \rangle) (G^{0-1})_{a',b'}, \\
\omega_{(a,b)}^{(\mathbf{G})}(u, j_0) &= \sqrt{n_{j_0+u}/n}.
\end{aligned}$$

Hence, linear combinations of $\sqrt{n}(\widehat{d}_{a'} - d_{a'})$ and $\sqrt{n} \left(\widehat{G}_{a,b}(\widehat{\mathbf{d}}) - G_{a,b}^0 \right)$ can be written as $\sum_{a'',b''=1,\dots,p} \sum_{u=0}^{j_1-j_0} \omega_{a'',b''}(u, j_0) \sqrt{n_{j_0+u}} 2^{-(j_0+u)(d_a^0+d_b^0)} (\widehat{\sigma}_{\ell,m}(j) - \sigma_{\ell,m}(j)) (1 + o_{\mathbb{P}}(1)) + o_{\mathbb{P}}(1)$ with $\omega_{a'',b''}(u, j_0)$ linear combination of $(\omega_{a''',b'''}^{(\mathbf{d})}(u, j_0))_{a''',b'''=1,\dots,p}$ and $(\mathcal{W}_{a''',b'''}^{(\mathbf{G})}(u, j_0))_{a''',b'''=1,\dots,p}$. Proposition 26 gives the joint convergence to a Gaussian distribution.

It remains to explicit the asymptotic covariance between $\widehat{\mathbf{d}}$ and $\widehat{\mathbf{G}}(\widehat{\mathbf{d}})$. Asymptotically,

the asymptotic covariance between $\sqrt{n}(\widehat{d}_{a'} - d_{a'}^0)$ and $\sqrt{n}(\widehat{G}_{a,b}(\widehat{\mathbf{d}}) - G_{a,b}^0)$ is

$$\begin{aligned}
V_{a',(a,b)}^{d,G(\Delta)} &= (\kappa_\Delta 2 \log(2))^{-1} (2 - 2^{-\Delta})^{-1} \sum_{u=0}^{\Delta} \sum_{u'=0}^{\Delta} 2^{-u-u'-|u'-u|/2} (j_0 + u - \langle \mathcal{J} \rangle) \\
&\quad 2^{(u \wedge u' - u')(d_a^0 + d_b^0)} \sum_{b'=1}^p ((\mathbf{G}^{0-1} \circ \mathbf{G}^0 + \mathbf{I}_p)^{-1})_{a',b'} 2^{(u \wedge u' - u)(d_{a'}^0 + d_{b'}^0)} \\
&\quad \left(G_{a',a}^0 G_{b',b}^0 \widetilde{I}_{|u-u'|}(d_{a'} + d_a, d_{b'} + d_b) + G_{a',b}^0 G_{a,b}^0 \widetilde{I}_{|u-u'|}(d_{a'} + d_b, d_{b'} + d_a) \right) \\
&= (4 \log(2))^{-1} \sum_{b'=1}^p ((\mathbf{G}^{0-1} \circ \mathbf{G}^0 + \mathbf{I}_p)^{-1})_{a',b'} \\
&\quad \left(G_{a',a}^0 G_{b',b}^0 \mathcal{I}_\Delta^{d,G}(d_{a'} + d_a, d_{b'} + d_b) + G_{a',b}^0 G_{a,b}^0 \mathcal{I}_\Delta^{d,G}(d_{a'} + d_b, d_{b'} + d_a) \right), \tag{H.1}
\end{aligned}$$

with

$$\begin{aligned}
\mathcal{I}_\Delta^{d,G}(\delta_1, \delta_2) &= \frac{2}{\kappa_\Delta} \sum_{u=0}^{\Delta} \frac{2^{-u}}{2 - 2^{-\Delta}} (u - \eta_\Delta) \widetilde{I}_{|u-u'|}(\delta_1, \delta_2) \\
&\quad + \frac{1}{\kappa_\Delta} \sum_{u=1}^{\Delta} 2^{-u} \widetilde{I}_u(\delta_1, \delta_2) \sum_{v=0}^{\Delta-u} \frac{2^{-v}}{2 - 2^{-\Delta}} (2^{u\delta_1}(u - \eta_\Delta) + 2^{u\delta_2}(u + v - \eta_\Delta)) \\
&= \frac{1}{\kappa_\Delta} \sum_{u=1}^{\Delta} 2^{-u} \widetilde{I}_u(\delta_1, \delta_2) \left((2^{u\delta_1} + 2^{u\delta_2}) \frac{2 - 2^{-\Delta-u}}{2 - 2^{-\Delta}} (u - \eta_\Delta) + 2^{u\delta_2} \eta_{\Delta-u} \right). \tag{H.2}
\end{aligned}$$

Appendix I. Technical lemmas

We first recall some inequalities on wavelet filters given in [26, Proposition 3].

Proposition 31. *Under (W1)–(W3), there exist positive constants C_{H1} , C_{H2} and C_{H3} only depending on ϕ and ψ , such that, for all $j, j' \geq 0$ and $\lambda \in (-\pi, \pi)$,*

$$|\mathbb{H}_j(\lambda)| \leq C_{H1} 2^{j/2} |2^{j/2} \lambda|^M (1 + 2^j |\lambda|)^{-\alpha - M}, \tag{I.1}$$

$$\left| \mathbb{H}_j(\lambda) - 2^{j/2} \widehat{\phi}(\lambda) \overline{\widehat{\psi}(2^j \lambda)} \right| \leq C_{H2} 2^{j/2 - j\alpha} |\lambda|^M, \tag{I.2}$$

$$\left| \mathbb{H}_j(\lambda) \overline{\mathbb{H}_{j'}(\lambda)} - 2^{j/2 + j'/2} |\widehat{\phi}(\lambda)|^2 \overline{\widehat{\psi}(2^j \lambda)} \widehat{\psi}(2^{j'} \lambda) \right| \leq C_{H3} 2^{(j+j')/2} 2^{-(j+j')\alpha} |\lambda|^{2M}. \tag{I.3}$$

Proof. Inequalities (I.1) and (I.2) are proved in [26, Proposition 3]. Next,

$$\begin{aligned} & \left| \mathbb{H}_j(\lambda) \overline{\mathbb{H}_{j'}(\lambda)} - 2^{j/2+j'/2} |\widehat{\phi}(\lambda)|^2 \overline{\widehat{\psi}(2^j \lambda)} \widehat{\psi}(2^{j'} \lambda) \right| \\ & \leq |\mathbb{H}_j(\lambda)| \left| \mathbb{H}_{j'}(\lambda) - 2^{j'/2} \widehat{\phi}(\lambda) \overline{\widehat{\psi}(2^{j'} \lambda)} \right| + |\mathbb{H}_{j'}(\lambda)| \left| \mathbb{H}_j(\lambda) - 2^{j/2} \widehat{\phi}(\lambda) \overline{\widehat{\psi}(2^j \lambda)} \right| \\ & \quad + \left| \mathbb{H}_{j'}(\lambda) - 2^{j'/2} \widehat{\phi}(\lambda) \overline{\widehat{\psi}(2^{j'} \lambda)} \right| \left| \mathbb{H}_j(\lambda) - 2^{j/2} \widehat{\phi}(\lambda) \overline{\widehat{\psi}(2^j \lambda)} \right|. \end{aligned}$$

Applying inequalities (I.1) and (I.2) to the right-hand side gives (I.3). \square

The following lemma is [26, Lemma 1]. It is used in the proofs of Lemma 16 and of Lemma 22.

Lemma 32 ([26]). *Let $Q \in \mathbb{N}$. For all function $g \in L^2(-\pi, \pi)$, write*

$$M_Q(g) = \left(\sum_{q \in \mathbb{Z}} (1 - |q|/Q)_+ \left(\int_{-\pi}^{\pi} g(\lambda) e^{-iq\lambda} d\lambda \right)^2 \right)^{1/2}.$$

Suppose g_1 and g_2 are \mathbb{C} -valued functions of $L^2((-\pi, \pi))$. Then,

$$|M_Q(g_1) - M_Q(g_2)|^2 \leq 2\pi \int_{-\pi}^{\pi} |g_1(\lambda) - g_2(\lambda)|^2 d\lambda.$$

Next, the following lemma states the convergence of a series of bivariate Fourier coefficients. It is used in the proofs of Lemma 15 and of Lemma 19.

Lemma 33. *Suppose $\{w_1^*(\lambda), \lambda \in (-\pi, \pi)\}$ and $\{w_2^*(\lambda), \lambda \in (-\pi, \pi)\}$ are \mathbb{C} -valued functions of $L^2((-\pi, \pi))$. Then*

$$\sum_{q \in \mathbb{Z}} (1 - |q|/Q)_+ \left(\int_{-\pi}^{\pi} w_1^*(\lambda) e^{-iq\lambda} d\lambda \right) \left(\int_{-\pi}^{\pi} w_2^*(\lambda) e^{-iq\lambda} d\lambda \right) \xrightarrow{Q \rightarrow \infty} 2\pi \int_{-\pi}^{\pi} \overline{w_1^*(\lambda)} w_2^*(\lambda) d\lambda.$$

Proof. Note that

$$\sum_{q \in \mathbb{Z}} (1 - |q|/Q)_+ \left(\int_{-\pi}^{\pi} w_1^*(\lambda) e^{-iq\lambda} d\lambda \right) \left(\int_{-\pi}^{\pi} w_2^*(\lambda) e^{-iq\lambda} d\lambda \right) = \sum_{q \in \mathbb{Z}} (1 - |q|/Q)_+ c_{1q} c_{2q},$$

with c_{1q} and c_{2q} the q^{th} Fourier coefficient respectively of functions w_1^* and w_2^* .

The sequence $(1 - |q|/Q)_+ c_{1q} c_{2q}$ converges to $c_{1q} c_{2q}$ when Q goes to infinity. By Cauchy-Schwarz's inequality and Parseval's equality,

$$\sum_{|q| \leq Q} |c_{1q} c_{2q}| \leq 2\pi \left(\int_{-\pi}^{\pi} |w_1^*(\lambda)|^2 d\lambda \right)^{1/2} \left(\int_{-\pi}^{\pi} |w_2^*(\lambda)|^2 d\lambda \right)^{1/2} < \infty.$$

Thus dominated convergence entails that the series $\sum_{q \in \mathbb{Z}} (1 - |q|/Q)_+ c_{1q} c_{2q}$ converges to $\sum_{q \in \mathbb{Z}} c_{1q} c_{2q}$. By Parseval's theorem,

$$\sum_{q \in \mathbb{Z}} c_{1q} c_{2q} = 2\pi \int_{-\pi}^{\pi} \overline{w_1^*(\lambda)} w_2^*(\lambda) d\lambda,$$

which concludes the proof. □

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