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# Solvability by radicals

Rodney Coleman, Laurent Zwald

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## Abstract

In this note we present one of the fundamental theorems of algebra, namely Galois's theorem concerning the solution of polynomial equations. We will begin with a study of cyclic extensions, before moving onto radical extensions.

## 1 Cyclic extensions

We say that a finite extension  $E$  of a field  $F$  is cyclic if the Galois group  $Gal(E/F)$  is cyclic. In this section we aim to present some elementary properties of such extensions. We begin with a preliminary result known as *Hilbert's theorem 90*.

**Theorem 1** *Let  $E/F$  be a finite cyclic Galois extension of degree  $n$ . We suppose that  $\sigma$  is a generator of the Galois group  $Gal(E/F)$ . If  $\alpha \in E$ , then  $N_{E/F}(\alpha) = 1$  if and only if there exists  $\beta \in E^*$  such that  $\alpha = \frac{\beta}{\sigma(\beta)}$ .*

PROOF If  $\alpha = \frac{\beta}{\sigma(\beta)}$ , then

$$N_{E/F}(\alpha) = \alpha\sigma(\alpha) \cdots \sigma^{n-1}(\alpha) = \frac{\beta\sigma(\beta)\sigma^2(\beta) \cdots \sigma^{n-1}(\beta)}{\sigma(\beta)\sigma^2(\beta) \cdots \sigma^{n-1}(\beta)\sigma^n(\beta)} = 1,$$

because  $\sigma^n(\beta) = \beta$ .

Now suppose that  $N_{E/F}(\alpha) = 1$ , i.e.,  $\alpha\sigma(\alpha) \cdots \sigma^{n-1}(\alpha) = 1$ . We define a finite sequence  $(\delta_i)_{i=0}^{n-1}$  as follows:

$$\delta_0 = \alpha, \delta_1 = \alpha\sigma(\alpha), \delta_2 = \alpha\sigma(\alpha)\sigma^2(\alpha), \dots, \delta_{n-1} = \alpha\sigma(\alpha) \cdots \sigma^{n-1}(\alpha) = N_{E/F}(\alpha) = 1.$$

From Result [REDUCdedcor1](#) the characters  $1, \sigma, \dots, \sigma^{n-1}$  form an independant set and so there exists  $\gamma \in E$  such that

$$\delta_0\gamma + \delta_1\sigma(\gamma) + \cdots + \delta_{n-2}\sigma^{n-2}(\gamma) + \sigma^{n-1}(\gamma) \neq 0.$$

We note this sum  $\beta$ . Then

$$\begin{aligned} \sigma(\beta) &= \sigma(\delta_0)\sigma(\gamma) + \sigma(\delta_1)\sigma^2(\gamma) + \cdots + \sigma(\delta_{n-2})\sigma^{n-1}(\gamma) + \sigma^n(\gamma) \\ &= \alpha^{-1}(\delta_1\sigma(\gamma) + \delta_2\sigma^2(\gamma) + \cdots + \delta_{n-1}\sigma^{n-1}(\gamma)) + \sigma^n(\gamma), \end{aligned}$$

because  $\alpha^{-1}\delta_i = \sigma(\delta_{i-1})$ . As  $\sigma^n = \text{id}_E$ , we have

$$\sigma^n(\gamma) = \gamma = \alpha^{-1}\delta_0\gamma,$$

hence  $\sigma(\beta) = \alpha^{-1}\beta$ . □

The theorem which we have just proved is often refered to as the 'multiplicative' version of Hilbert's theorem 90 to distinguish from the 'additive' version, which we will now present.

**Theorem 2** *If  $E/F$  is a finite cyclic Galois extension of degree  $n$ , then  $T_{E/F}(\alpha) = 0$  if and only if there exists  $\beta \in E$  such that  $\alpha = \beta - \sigma(\beta)$ .*

PROOF Let  $\sigma$  be a generator of the Galois group  $G = \text{Gal}(E/F)$ , so that  $\sigma^n = \text{id}_n$ .

If  $\alpha = \beta - \sigma(\beta)$ , then

$$T_{E/F}(\alpha) = \sum_{k=0}^{n-1} \sigma^k(\alpha) = \sum_{k=0}^{n-1} \sigma^k(\beta) - \sum_{k=1}^n \sigma^k(\beta) = 0,$$

because  $\sigma^n(\beta) = \beta$ .

Conversely, suppose that  $T_{E/F}(\alpha) = 0$ . As  $T_{E/F} \neq 0$ , we may find an element  $x$  such that  $T_{E/F}(x) \neq 0$ . Let

$$w = \alpha\sigma(x) + (\alpha + \sigma(\alpha))\sigma^2(x) + \cdots + (\alpha + \sigma(\alpha) + \cdots + \sigma^{n-2}(\alpha))\sigma^{n-1}(x).$$

Then

$$\sigma(w) = \sigma(\alpha)\sigma^2(x) + (\sigma(\alpha) + \sigma^2(\alpha))\sigma^3(x) + \cdots + (\sigma(\alpha) + \sigma^2(\alpha) + \cdots + \sigma^{n-1}(\alpha))\sigma^n(x).$$

Since  $T_{E/F}(\alpha) = 0$ , we have

$$-\alpha = \sigma(\alpha) + \cdots + \sigma^{n-1}(\alpha),$$

so the last summand in the expression for  $\sigma(w)$  is  $-\alpha x$ , hence

$$w - \sigma(w) = \alpha(x + \sigma(x) + \sigma^2(x) + \cdots + \sigma^{n-1}(x)) = \alpha T_{E/F}(x).$$

Setting  $\beta = \frac{w}{T_{E/F}(x)}$ , we obtain

$$\beta - \sigma(\beta) = \frac{w}{T_{E/F}(x)} - \frac{\sigma(w)}{\sigma(T_{E/F}(x))} = \frac{w - \sigma(w)}{T_{E/F}(x)},$$

because  $T_{E/F}(x) \in F$ . Therefore

$$\beta - \sigma(\beta) = \frac{\alpha T_{E/F}(x)}{T_{E/F}(x)} = \alpha,$$

as required.  $\square$

In the following we consider extensions  $E/F$  of a particular sort, namely where the field  $F$  contains a primitive  $n$ th root of unity, for some positive integer  $n$ .

**Theorem 3** *Let  $E/F$  be a cyclic Galois extension of degree  $n$ , where  $F$  contains a primitive  $n$ th root of unity  $\zeta$ . Then there exists an irreducible polynomial  $f(X) = -\alpha + X^n \in F[X]$  such that  $E$  is a splitting field of  $f$ .*

PROOF Let  $\sigma$  be a generator of the Galois group  $\text{Gal}(E/F)$ . Since  $\zeta \in F$ , we have  $N_{E/F}(\zeta) = \zeta^n = 1$  and so, by Hilbert's theorem 90 (multiplicative version), there is an element  $\beta \in E^*$  such that  $\zeta = \frac{\beta}{\sigma(\beta)}$ . Then

$$\begin{aligned} \sigma(\beta) &= \zeta^{-1}\beta \\ \sigma^2(\beta) &= \sigma(\zeta)^{-1}\sigma(\beta) = \zeta^{-1}(\zeta^{-1}\beta) = \zeta^{-2}\beta \\ \sigma^3(\beta) &= \sigma(\zeta)^{-2}\sigma(\beta) = \zeta^{-2}(\zeta^{-1}\beta) = \zeta^{-3}\beta \end{aligned}$$

and generally  $\sigma^m(\beta) = \zeta^{-m}\beta$ , for all  $m \in \mathbf{N}^*$ . As the elements  $\sigma^i(\beta)$ , for  $0 \leq m < n$  are distinct, the automomorphisms  $\text{id}_{F(\beta)}, \sigma|_{F(\beta)}, \dots, \sigma|_{F(\beta)}^{n-1}$  form a set of  $n$  distinct elements of the Galois group  $\text{Gal}(F(\beta)/F)$ . Then  $[F(\beta) : F] \geq n$  and

$$n = |\text{Gal}(E/F)| = [E : F] = [E : F(\beta)][F(\beta) : F] \implies [F(\beta) : F] \leq n.$$

Therefore  $[F(\beta) : F] = n$ , which implies that  $E = F(\beta)$ .

We claim that  $\beta^n \in F$ . Now

$$\sigma(\beta^n) = \sigma(\beta)^n = \zeta^{-n}\beta^n = \beta^n,$$

and it follows that  $\beta^n$  belongs to the fixed field of  $\text{Gal}(E/F)$ , i.e.,  $F$ . Thus there exists  $\alpha \in F$  such that  $\beta^n = \alpha$ . We have shown that  $E = F(\beta)$ , where  $\beta$  is a root of the polynomial  $f(X) = -\alpha + X^n$ . The roots of  $f$  have the form  $\zeta^i\beta$ , for  $0 \leq i < n$  and  $\zeta \in F$  by hypothesis, so  $E$  contains all the roots of  $f$  and hence is a splitting field of the polynomial  $f$ . Since  $[F(\beta) : F] = n$ , the degree of the minimal polynomial  $m(\beta, F)$  is  $n$ , hence  $f = m(\beta, F)$ ; it follows that  $f$  is irreducible.  $\square$

The theorem which we have just proved has a converse.

**Theorem 4** *Let  $F$  be a field containing an  $n$ th primitive root of unity  $\zeta$  and  $E$  a splitting field of the polynomial  $f(X) = -\alpha + X^n \in F[X]$ , with  $\alpha \neq 0$ . Then  $E = F(\beta)$ , where  $\beta$  is a root of  $f$ , and the Galois group  $G = \text{Gal}(E/F)$  is cyclic of order dividing  $n$ . The order of  $G$  is equal to  $n$  if and only if  $f$  is irreducible.*

**PROOF** If  $\beta$  is root of  $f$ , then the roots of  $f$  have the form  $\zeta^i\beta$ , for  $0 \leq i < n$ . As  $\zeta \in F$ , all the roots of  $f$  lie in  $F(\beta)$ , so  $F(\beta)$  is a splitting field of  $f$  and so  $E = F(\beta)$ .

Let  $\sigma \in G = \text{Gal}(E/F)$ . As  $\sigma$  permutes the roots of  $f$ , we may write  $\sigma(\beta) = \zeta^{k(\sigma)}\beta$ , where  $k(\sigma)$  is uniquely determined modulo  $n$ . We thus obtain a mapping

$$\phi : G \longrightarrow (\mathbf{Z}_n, +), \sigma \longmapsto [k(\sigma)],$$

which is clearly a group homomorphism. If  $\phi(\sigma) = [0]$ , then  $\sigma(\beta) = \beta$ , which implies that  $\sigma$  is the identity on  $F(\beta)$  and so  $\phi$  is a monomorphism. Since  $G$  is isomorphic to a subgroup of the cyclic group  $(\mathbf{Z}_n, +)$ ,  $G$  is cyclic of order dividing  $n$ .

If  $f$  is irreducible, then  $G$  acts transitively on the roots of  $f$  (Result 4). Thus there exists  $\sigma \in G$  such that  $\sigma(\beta) = \zeta^i\beta$ , for  $0 \leq i < n$ . It follows that  $\phi$  is surjective, so  $G$  is isomorphic to  $(\mathbf{Z}_n, +)$ .

Now suppose that  $f$  is reducible. As  $f$  has no multiple roots,  $f$  has two distinct irreducible factors, so  $G$  does not act transitively on the roots of  $f$  (Result 5). Thus there exist roots  $\zeta^i\beta$  and  $\zeta^j\beta$ , with  $j > i$ , for which there exists no element  $\sigma \in G$  such that  $\sigma(\zeta^i\beta) = \zeta^j\beta$ , i.e.,  $\sigma(\beta) = \zeta^{j-i}\beta$ . It follows that  $\phi$  is not surjective and so  $|G| < n$ .  $\square$

**Corollary 1** *Let  $p$  be prime,  $F$  a field containing a primitive  $p$ th root of unity and  $E$  a splitting field of the polynomial  $f(X) = -\alpha + X^p \in F[X]$ , with  $\alpha \neq 0$ . Then either  $f$  is irreducible and  $\text{Gal}(E/F) \simeq \mathbf{Z}_p$  or  $f$  splits in  $F$  and  $\text{Gal}E/F = \{\text{id}_F\}$ .*

**PROOF** The order of  $\text{Gal}(E/F) = p$  or 1. In the first case, from Theorem 4,  $f$  is irreducible and  $\text{Gal}(E/F) \simeq \mathbf{Z}_p$ , because  $\text{Gal}(E/F)$  is cyclic. Suppose now that the order of  $\text{Gal}(E/F)$  is 1. Then  $[E : F] = 1$ . There exists  $\beta \in E$  such that  $E = F(\beta)$ , so  $[F(\beta) : F] = 1$ , which implies that  $\beta \in F$ . It now follows that  $f$  splits in  $F$ .  $\square$

## 2 Radical extensions

Suppose that  $E/F$  is a field extension such that  $E = F(\alpha)$  for some  $\alpha \in E$ . If there exists  $n \in \mathbf{N}^*$  such that  $\alpha^n \in F$ , then we say that  $E$  is a *pure extension* of type  $n$ . In this case  $\alpha$  is a root of a polynomial of the form  $f(X) = -c + X^n \in F[X]$ . A chain of extensions

$$F = F_0 \subset F_1 \subset \cdots \subset F_r = E,$$

where each extension  $F_{i+1}/F_i$  is a pure extension is called a *radical chain* and we say that  $E/F$  is a *radical extension*. Using Result 2 and Result 3, we see that a radical extension is a finite algebraic extension.

**Proposition 1** • **a.** *If  $E/F$  is a radical extension and  $Z$  an intermediate field, then  $E/Z$  is a radical extension.*

• **b.** *If  $E/F$  is a field extension and  $Z_1, Z_2$  intermediate fields such that  $Z_1/F, Z_2/F$  are radical extensions, then  $Z_1Z_2/F$  is a radical extension.*

**PROOF a.** We have the radical chain

$$F \subset F(\alpha_1) \subset F(\alpha_1, \alpha_2) \subset \cdots \subset F(\alpha_1, \dots, \alpha_r) = E,$$

with  $\alpha_i^{n_i} \in F(\alpha_1, \dots, \alpha_{i-1})$ , for some  $n_i \in \mathbf{N}^*$ . Then

$$Z \subset Z(\alpha_1) \subset Z(\alpha_1, \alpha_2) \subset \cdots \subset Z(\alpha_1, \dots, \alpha_r) = E$$

and

$$\alpha_i^{n_i} \in F(\alpha_1, \dots, \alpha_{i-1}) \subset Z(\alpha_1, \dots, \alpha_{i-1}).$$

It follows that  $E/Z$  is a radical extension.

**b.** We have the radical extensions

$$F \subset F(\alpha_1) \subset F(\alpha_1, \alpha_2) \subset \cdots \subset F(\alpha_1, \dots, \alpha_r) = Z_1$$

and

$$F \subset F(\beta_1) \subset F(\beta_1, \beta_2) \subset \cdots \subset F(\beta_1, \dots, \beta_s) = Z_2.$$

Then

$$F \subset F(\alpha_1) \subset \cdots \subset F(\alpha_1, \dots, \alpha_r) \subset F(\alpha_1, \dots, \alpha_r, \beta_1) \subset F(\alpha_1, \dots, \alpha_r, \beta_1, \dots, \beta_s) = Z_1Z_2.$$

The only possible difficulty in seeing that we have a radical chain is in finding a power of  $\beta_1$  in  $F(\alpha_1, \dots, \alpha_r)$ . However, there exists  $m_1 \in \mathbf{N}^*$  such that  $\beta_1^{m_1} \in F \subset F(\alpha_1, \dots, \alpha_r)$ , so we indeed have a radical chain.  $\square$

**Remark** We may easily extend **b.** to any finite number of intermediate fields.

We now consider the normal closure of a radical extension.

**Proposition 2** *If  $E/F$  is a radical extension and  $N$  the normal closure of  $E$  over  $F$ , then  $N/F$  is a radical extension.*

PROOF From Result 12 the normal closure of  $E$  over  $F$  can be written as a compositum of fields  $Z_1, \dots, Z_r$  isomorphic to  $E$  contained in a field extension of  $F$ . As  $E$  is a radical extension of  $F$ , so are the  $Z_i$ . From the remark after Proposition 1 the normal closure of  $E$  over  $F$  is a radical extension of  $F$ .  $\square$

We may insert fields into a radical chain so that each extension  $F_{i+1}/F_i$  is pure of type  $p_i$ , for some prime number  $p_i$ . Let us see why this is the case. Suppose that  $L/K$  is a pure extension of type  $n$  and  $L = K(\alpha)$ , where  $\alpha \in L$  and  $\alpha^n \in K$ . If  $n = p_1 \cdots p_s$  is the prime factorization of  $n$ , then

$$K \subset K(\alpha^{p_1 \cdots p_{s-1}}) \subset \cdots \subset K(\alpha^{p_1 p_2}) \subset K(\alpha^{p_1}) \subset K(\alpha) = L.$$

Then we have  $\alpha^{p_1} \in K(\alpha^{p_1})$ ,  $(\alpha^{p_1})^{p_2} \in K(\alpha^{p_1 p_2})$ , and so on. This proves our claim. In this case we say that the radical extension is of prime type.

### 3 Solvability by radicals

Let  $F$  be a field and  $f$  a polynomial over  $F$ . We say that  $f$  is *solvable by radicals* if its splitting field  $L$  is contained in a radical extension  $E$  of  $F$ . In this section we aim first to show that under certain conditions solvability by radicals implies the solvability of the Galois group. (In an appendix we revise the notion of a solvable group.) Later we will prove the converse.

For certain results we need the presence of a primitive root of unity. We might be tempted to think that, for a given  $n \in \mathbf{N}^*$ , if a field does not contain a primitive  $n$ th root of unity, then we can simply add this root. However, this is not the case. Suppose that the field  $F$  has characteristic  $p \neq 0$  and  $p|n$ , say  $n = pm$ . Let  $\zeta$  be a primitive  $n$ th root of unity, then

$$0 = \zeta^n - 1 = (\zeta^m - 1)^p,$$

so  $\zeta^m - 1 = 0$ . This implies that the order of  $\zeta$  is strictly less than  $n$ , a contradiction. So there is no primitive  $n$ th root of unity.

On the other hand, if the characteristic of  $F$  does not divide  $n$ , this problem does not occur. If  $f(X) = -1 + X^n$ , then

$$f'(X) = nX^{n-1} \neq 0 \implies \gcd(f, f') = 1,$$

so  $f$  has no multiple roots ( $f$  is strongly separable). Therefore there are  $n$  distinct  $n$ th roots of unity. As the group of  $n$ th roots of unity is cyclic, it has a generator, namely a primitive  $n$ th root of unity. To be certain that we can find primitive  $n$ th roots of unity, we may suppose that the characteristic of a field is 0. This also ensures that all extensions are separable.

Our next step is to establish a preliminary result, which will play an important role in the following. To do this we need a lemma.

**Lemma 1** Let  $F \subset K \subset E$  be a chain of fields, with  $K$  the splitting field of a polynomial  $f \in F[X]$ . If  $\sigma \in \text{Gal}(E/F)$ , then  $\sigma|_K \in \text{Gal}(K/F)$ .

PROOF It is sufficient to show that  $\sigma(K) = K$ . If  $\alpha_1, \dots, \alpha_n$  are the distinct roots of  $f$ , then  $K = F(\alpha_1, \dots, \alpha_n)$ . For each  $\alpha_i$ , the element  $\sigma(\alpha_i)$  is a root of  $f$ , so

$$\sigma(K) = \sigma(F(\alpha_1, \dots, \alpha_n)) = F(\sigma(\alpha_1), \dots, \sigma(\alpha_n)) = K,$$

as required.  $\square$

EXTSOLVsolvth1

**Theorem 5** Let  $F \subset K \subset E$  be a chain of fields, with  $K$  the splitting field of a polynomial  $f \in F[X]$  and  $E$  the splitting field of a polynomial  $g \in F[X]$ . Then  $Gal(E/K)$  is a normal subgroup of  $Gal(E/F)$  and

$$Gal(E/F)/Gal(E/K) \simeq Gal(K/F).$$

PROOF We define a mapping  $\phi : Gal(E/F) \rightarrow Gal(K/F)$  by  $\phi(\sigma) = \sigma|_K$ . (From Lemma [EXTSOLVsolvlem1](#) we know that  $\sigma|_K \in Gal(K/F)$ .) Clearly  $\phi$  is a homomorphism. As  $\phi(\sigma) = id_K$  if and only if  $\sigma$  fixes all the elements of  $K$ , the kernel of  $\phi$  is the Galois group  $Gal(E/K)$ . Therefore  $Gal(E/K)$  is a normal subgroup of  $Gal(E/F)$ .

We claim that  $\phi$  is surjective. Let  $\tau \in Gal(K/F)$ . As  $F \subset K$ , the polynomial  $f$  belongs to  $K[X]$ . Using Result [EXTSOLVsolvth2](#), with  $K' = K$  and  $f^* = f$ , we obtain an automorphism  $\tilde{\tau}$  of  $E$  extending  $\tau$ . Thus  $\phi(\tilde{\tau}) = \tau$ . It follows that  $\phi$  is surjective and

$$Gal(E/F)/Gal(E/K) \simeq Gal(K/F),$$

by the first isomorphism theorem. □

We now consider polynomials which are solvable by radicals. Let  $F$  be a field and  $f \in F[X]$  solvable by radicals. We will suppose that the characteristic of  $F$  is 0. The reason for doing so is twofold: 1. Extensions of  $F$  are separable; 2. Roots of unity of any order may be found in some extension. By hypothesis there is a radical chain

$$F = F_0 \subset F_1 \subset \dots \subset F_r = E$$

such that the splitting field  $L$  of  $f$  is included in  $E$ . From Proposition [EXTSOLVradprop2](#) we may suppose that  $E$  is a Galois extension of  $F$ , hence the splitting field of a polynomial  $g \in F[X]$ . We have also seen at the end of the previous section that we may suppose that each extension  $F_{i+1}/F_i$  may be taken to be pure of prime type, for some prime  $p_i$ . In the next proposition we will suppose that the field  $F$  contains a primitive  $p_i$ th root of unity, for all the primes  $p_i$ . This is not in general the case. However, we will show further on that we may replace the radical chain by another for which this is the case.

EXTSOLVprop1

**Proposition 3** If  $F$  contains a primitive  $p_i$ th root of unity for all  $i$ , then the Galois group of  $f$  is solvable.

PROOF We set

$$G_i = Gal(F_r/F_i),$$

for  $i = 0, \dots, r$ . We consider  $[F_i : F_{i-1}]$ . As  $\alpha_i$  is a root of the polynomial  $f_i(X) = -\alpha_i^{p_i} + X^{p_i} \in F_{i-1}[X]$ , the degree of the minimal polynomial  $m(\alpha_i, F_{i-1})$  must be 1 or  $p_i$ . We can exclude the first case, because this would imply that  $F_i = F_{i-1}$ . Now

$$[F_{i-1}(\alpha_i) : F_{i-1}] = \deg m(\alpha_i, F_{i-1}) \implies [F_{i-1}(\alpha_i) : F_{i-1}] = p_i.$$

As  $F_i$  is a Galois extension of  $F_{i-1}$ , we have  $|Gal(F_i/F_{i-1})| = p_i$  and so the extension  $F_i/F_{i-1}$  is cyclic of degree  $p_i$ . We may apply Theorem [EXTSOLVcycyth3](#). As  $F \subset F_{i-1}$ ,  $F_{i-1}$  contains a  $p_i$ th primitive root of unity and it follows that  $F_i$  is the splitting field of a polynomial  $f_i$  over  $F_{i-1}$ . (Thus  $F_i$  is a Galois extension of  $F_{i-1}$ .)

We now consider the sequence

$$G_0 \supset G_1 \supset \dots \supset G_r = \{id\}.$$

We claim this sequence is normal, i.e.,  $G_{i-1} \triangleright G_i$ , for all  $i$ . We have  $F_{i-1} \subset F_i \subset F_r$ . Since  $F_r$  is a normal extension of  $F$ ,  $F_r$  is the splitting field of a polynomial  $g \in F[X]$ . Given that  $F \subset F_r$ , we have  $g \in F_{i-1}[X]$  and we may apply Theorem [EXTSOLVsolvth1](#), which ensures us that  $G_{i-1} \triangleright G_i$ . Hence the sequence is normal. From Theorem [EXTSOLVsolvth1](#) again, the quotient group  $G_{i-1}/G_i$  is isomorphic to  $Gal(F_i/F_{i-1})$ , which is cyclic. We have shown that the Galois group  $Gal(F_r/F_0)$  is a solvable.

To conclude we consider the chain of fields

$$F \subset L \subset F_r,$$

where  $L$  is the splitting field of  $f$ . Using Theorem [EXTSOLVsolvth1](#) again, we obtain the isomorphism

$$Gal(L/F) \simeq Gal(F_r/F)/Gal(F_r/L).$$

As  $Gal(L/F)$  is isomorphic to a quotient of a solvable group,  $Gal(L/F)$  is itself solvable.  $\square$

We now eliminate the hypothesis concerning the existence of  $p_i$ th primitive roots of unity in the field  $F$ . Our aim is to replace the initial radical chain by another satisfying the conditions of Proposition [EXTSOLVprop1](#).

**Theorem 6** *Let  $F$  be a field of characteristic 0 and  $f \in F[X]$  solvable by radicals. If  $L$  is a splitting field of  $f$ , then  $Gal(L/F)$  is a solvable group.*

PROOF By hypothesis there is a radical chain

$$F = F_0 \subset F_1 \subset \dots \subset F_r = E,$$

with  $L \subset F_r$ . We may assume that  $F_r$  is the splitting field of some polynomial  $g \in F[X]$  and that  $F_i/F_{i-1}$  is a pure extension of prime type  $p_i$ . In addition, each extension  $F_i/F_{i-1}$  is a Galois extension of degree  $p_i$ , because  $F \subset F_{i-1}$  implies that  $g \in F_{i-1}[X]$ .

Let  $m$  be the lcm of the  $p_i$  and  $\zeta$  an  $m$ th primitive root of unity. We construct a new chain by adding  $\zeta$  to each element of our first chain:

$$F = F_0 \subset F_0(\zeta) \subset F_1(\zeta) \subset \dots \subset F_r(\zeta).$$

We notice that  $F_0(\zeta)$  contains all the  $p_i$ th primitive roots of unity. For example, if  $m = p_1 \cdots p_s$ , then  $\zeta^{p_2 \cdots p_s}$  is a primitive  $p_1$ th root of unity. To simplify the notation, let us write  $F'_i$  for  $F_i(\zeta)$ , for  $i = 0, \dots, r$ .

We consider the extensions  $F'_i/F'_{i-1}$ . From Result [GALGRPcompex1](#), we have  $[F'_i F'_{i-1} : F'_{i-1}] | [F_i : F_{i-1}]$ . However,

$$F_i = F_{i-1}(\alpha_i) \implies F_i F_{i-1}(\zeta) = F_{i-1}(\alpha_i, \zeta) = F_i(\zeta) = F'_i,$$

so  $[F'_i : F'_{i-1}] | [F_i : F_{i-1}]$ , which implies that  $[F'_i : F'_{i-1}]$  has the value 1 or  $p_i$ . In the first case, we can eliminate  $F'_i$  (or  $F'_{i-1}$ ), so we can assume that  $[F'_i : F'_{i-1}] = p_i$ . Since  $F_i$  and  $F'_{i-1}$  are extensions of  $F_{i-1}$  and  $F_i$  is a finite Galois extension of  $F_{i-1}$ , by Result [GALGRPcomp1](#), the compositum  $F_i F'_{i-1}$  is a Galois extension of  $F'_{i-1}$ , i.e.,  $F'_i$  is a Galois extension of  $F'_{i-1}$ . We now may apply Theorem [EXTSOLVcycchs](#): There exists an irreducible polynomial  $f_{i-1}(X) = -c_{i-1} + X^{p_i} \in F'_{i-1}[X]$  such that  $F'_i$  is a splitting field of  $f_{i-1}$ . Let  $\alpha'_i$  be a root of  $f_{i-1}$ . Then  $f_{i-1}$  splits in  $F'_{i-1}(\alpha'_i)$ , because  $F'_{i-1}$  contains a primitive  $p_i$ th root of unity. This implies that  $F'_i \subset F'_{i-1}(\alpha'_i)$ . As the reverse inclusion is clear, we have  $F'_i = F'_{i-1}(\alpha'_i)$ . We have shown that there exists  $\alpha'_i \in F'_i$  such that  $F'_i = F'_{i-1}(\alpha'_i)$ , with  $\alpha_i^{p_i} \in F'_{i-1}$ . Hence the extension  $F'_r/F'_0$  is radical of prime type.

Our next step is to show that  $Gal(F'_r/F'_0)$  is solvable. Since  $F'_r$  is the splitting field of the polynomial  $h(X) = (-1 + X^m)g(X) \in F'_0[X]$ ,  $F'_r$  is a Galois extension of  $F'_0$  and  $Gal(F'_r/F'_0)$  is the Galois group of  $h$ . From Proposition [EXTSOLVprop1](#) we deduce that  $Gal(F'_r/F'_0)$  is solvable.



We now show that  $Gal(F'_r/F)$  is solvable. We consider the chain of fields

$$F_0 \subset F'_0 \subset F'_r.$$

Since  $F'_0$  and  $F'_r$  are each the splitting field of a polynomial in  $F_0[X]$ , from Theorem [EXTSOLVsolvth1](#),  $Gal(F'_r/F'_0)$  is a normal subgroup of  $Gal(F'_r/F_0)$  and

$$Gal(F'_r/F_0)/Gal(F'_r/F'_0) \simeq Gal(F'_0/F_0).$$

From Result [GALPOLYctomcoria](#),  $Gal(F'_0/F_0)$  is abelian, hence solvable. Therefore the quotient group  $Gal(F'_r/F_0)/Gal(F'_r/F'_0)$  is solvable. However, we have also seen that the subgroup  $Gal(F'_r/F'_0)$  is solvable. It follows that  $Gal(F'_r/F_0)$  is solvable, i.e.,  $Gal(F'_r/F)$  is solvable.

Let  $L$  be a splitting field in  $F'_r$ . (All the roots of  $f$  lie in  $F'_r$ , so a splitting field of  $f$  is contained in  $F'_r$ .) We are now in a position to show that  $Gal(L/F)$  is solvable. Once again we use Theorem [EXTSOLVsolvth1](#). We consider the chain of fields

$$F \subset L \subset F'_r.$$

$L$  and  $F'_r$  are both splitting fields of polynomials over  $F$ , hence  $Gal(F'_r/L)$  is a normal subgroup of  $Gal(F'_r/F)$  and

$$Gal(F'_r/F)/Gal(F'_r/L) \simeq Gal(L/F).$$

The left hand side of the expression is a quotient of a solvable group, hence solvable. Therefore the Galois group  $Gal(L/F)$  is solvable, as required.  $\square$

**Example** The Galois group of the polynomial  $f(X) = -1 - 4X + X^5 \in \mathbf{Q}[X]$  is the symmetric group  $S_5$ , which is not solvable, hence  $f$  is not solvable by radicals.

We now prove the converse of Theorem [EXTSOLVsolvth1](#). We need a preliminary result.

**Lemma 2** *Let  $F$  be a field,  $f \in F[X]$  and  $E$  a splitting field of  $f$ . If  $F^*$  is an extension of  $F$ , then  $f \in F^*[X]$ . If  $E^*$  is an extension of  $F^*$  and  $E^*$  is a splitting field of  $f$  containing  $E$  and  $\sigma \in Gal(E^*/F^*)$ , then  $\sigma|_E \in Gal(E/F)$  and the mapping*

$$\phi : Gal(E^*/F^*) \longrightarrow Gal(E/F), \sigma \longmapsto \sigma|_E$$

*is a monomorphism.*

PROOF By hypothesis we have

$$E = F(\alpha_1, \dots, \alpha_n) \quad \text{and} \quad E^* = F^*(\alpha_1, \dots, \alpha_n),$$

where the  $\alpha_i$  are the roots of  $f$ . If  $\sigma \in Gal(E^*/F^*)$ , then  $\sigma$  permutes the roots of  $f$  and fixes  $F^*$ , hence  $F$ ; therefore  $\sigma|_E \in Gal(E/F)$ . Clearly  $\phi$  is a homomorphism. If  $\sigma|_E = \text{id}_E$ , then  $\sigma|_E(\alpha_i) = \alpha_i$ , for all  $i$ . Thus  $\sigma(\alpha_i) = \alpha_i$ , for all  $i$ , which implies that  $\sigma = \text{id}_{E^*}$ . It follows that  $\phi$  is injective.  $\square$

Now for the theorem.

**Theorem 7** *Let  $F$  be a field of characteristic 0 and  $E$  a finite Galois extension of  $F$ . If the Galois group  $Gal(E/F)$  is solvable, then  $E$  can be embedded in a radical extension.*

PROOF From Corollary [3](#) <sup>APPSOLVcor1</sup> there is a normal subgroup  $H$  of prime index, say  $p$ , in  $G$ . Let  $\zeta$  be a primitive  $p$ th root of unity, which exists in some extension of  $F$ , because  $F$  has characteristic 0.

We will first suppose that  $\zeta \in F$  and prove the theorem by induction on  $n = [E : F]$ . If  $n = 1$ , then  $E = F$  and there is nothing to prove. (It is sufficient to take any radical extension of  $F$ , for example  $F(\alpha)$ , where  $\alpha$  is a root of the polynomial  $f(X) = -c + X^p \in F[X]$ .)

Now let us suppose that the result is true up to  $n - 1$ . From Result [8](#) <sup>NORMprop2</sup> we know that the extension  $E/E^H$  is normal, ( $E^H$  is the subfield of  $E$  fixed by the elements of  $H$ ), hence Galois, because the extension is also separable. Now  $Gal(E/E^H)$  is a solvable group, being a subgroup of the Galois group  $Gal(E/F)$ , which we note  $G$ , and  $[E : E^H] < n$ . By the induction hypothesis there is a radical chain

$$E^H \subset F_1 \subset \cdots \subset F_s, \tag{1}$$

EXTSOLVsolveq1

where  $E \subset F_s$ .

Next we observe that  $E^H/F$  is a Galois extension, because  $H$  is a normal subgroup of  $G$ , and  $[E^H : F] = [G : H] = p$  (Result [11](#)). From the proof of Theorem [3](#) <sup>EXTSOLVcycctn3</sup> there exists  $\beta \in E^H$  such that  $E^H = F(\beta)$  and  $\beta^p \in F$ , i.e.,  $E^H$  is a pure extension of  $F$  and we may lengthen the chain [\(1\)](#) by adding the prefix  $F \subset E^H$ . We thus obtain the radical extension  $F_s/F$  (and  $E \subset F_s$ ).

We now consider the general case. We set  $F^* = F(\zeta)$  and  $E^* = E(\zeta)$ . As  $E$  is a normal extension of  $F$ ,  $E$  is the splitting field of a polynomial  $f \in F[X]$ . It is not difficult to see that  $E^*$  is the splitting field of the polynomial  $g(X) = (-1 + X^p)f(X) \in F[X]$ . The polynomial  $g$  belongs to  $F^*[X]$  and  $E^*$  is its splitting field. Thus  $E^*$  is a normal extension of  $F^*$ , hence a Galois extension. We set  $G^* = Gal(E^*/F^*)$ . From Lemma [2](#) <sup>EXTSOLVsolvlem2</sup> there is a monomorphism from  $G^*$  into  $G$ , so we may consider that  $G^*$  is a subgroup of the solvable group  $G$ , hence is solvable. We may now apply the first part of the proof: there is a radical extension  $F_t^*$  of  $F^*$  containing  $E^*$ : we have the radical chain

$$F^* \subset F_1^* \subset \cdots \subset F_t^*,$$

with  $E^* \subset F_t^*$ . As  $E \subset E^*$ , we have  $E \subset F_t^*$ . Since  $F^* = F(\zeta)$ ,  $F^*$  is a pure extension of  $F$  and we may lengthen the radical chain by adding the prefix  $F \subset F^*$ , providing us with the radical extension  $F_t^*/F$  (and  $E \subset F_t^*$ ).  $\square$

**Corollary 2** *If  $F$  is a field of characteristic 0 and  $f \in F[X]$  whose Galois group is solvable, then  $f$  is solvable by radicals.*

**Example** The Galois group of  $f(X) = -1 + X + X^3 \in \mathbf{Q}[X]$  is the symmetric group  $S_3$ , hence  $f$  is solvable by radicals.

### Basic results from Galois theory

REDUCdedcor1

**Result 1** *A set of distinct automorphisms  $\{\sigma_1, \dots, \sigma_n\}$  on a field  $F$  is independant, i.e., if  $\lambda_1, \dots, \lambda_n \in F$  and*

$$\lambda_1 \sigma_1(x) + \cdots + \lambda_n \sigma_n(x) = 0,$$

*for all  $x \in F^*$ , then  $\lambda_1 = \cdots = \lambda_n = 0$ .*

corFEalgext1a

**Result 2** *If  $K$  is a finite extension of  $F$  and  $E$  a finite extension of  $K$ , then  $E$  is a finite extension of  $F$  and*

$$[E : F] = [E : K][K : F],$$

*where  $[A : B]$  is the degree of  $A$  over  $B$ .*

propFEalgext3

**Result 3** If  $K$  is an algebraic extension of  $F$ ,  $E$  an extension of  $K$  and  $\alpha \in E$  is algebraic over  $K$ , then  $\alpha$  is algebraic over  $F$ .

thGALPOLYirred2

**Result 4** Let  $f$  be a separable polynomial in  $F[X]$  of degree  $n$  with Galois group  $G = \text{Gal}(E/F)$ . If  $f$  is irreducible, then

- a.  $n$  divides the order of  $G$ ;
- b. the action of  $G$  on  $A$ , the set of roots of  $f$ , is transitive.

propGALPOLYirred1

**Result 5** Let  $f \in F[X]$ , with  $\deg f \geq 2$ , and  $G$  be its Galois group. If  $f$  has two distinct irreducible factors, then the action of  $G$  on  $A$ , the set of roots of  $f$ , is not transitive.

GALPOLYctomcor1a

**Result 6** If  $E$  is a cyclotomic extension of  $F$ , then the Galois group  $G(E/F)$  is abelian.

thSPLIT2

**Result 7** Let  $F$  and  $F'$  be fields,  $\sigma : F \rightarrow F'$  an isomorphism,  $f \in F[X]$  and  $f' \in F'[X]$  the polynomial corresponding to  $f$ . If  $E$  is a splitting field of  $f$  and  $E'$  a splitting field of  $f'$ , then there is an isomorphism  $\tilde{\sigma} : E \rightarrow E'$  extending  $\sigma$ .

NORMprop2

**Result 8** Suppose that  $K$  is an extension of  $F$  and  $E$  an extension of  $K$ , with  $E$  normal over  $F$ . Then  $E$  is normal over  $K$ .

thGALGRPcomp1

**Result 9** Let  $K$  and  $L$  be extensions of  $F$  in  $E$ , where  $K$  is a finite Galois extension of  $F$ . Then

- a.  $KL$  is a finite Galois extension of  $L$ ;
- b. If  $\sigma \in \text{Gal}(KL/L)$ , then the restriction of  $\sigma$  to  $K$  belongs to  $\text{Gal}(K/F)$  and the mapping

$$\phi : \text{Gal}(KL/L) \rightarrow \text{Gal}(K/F), \sigma \mapsto \sigma|_K$$

is a monomorphism;

GALGRPcompex1

**Result 10** If  $K$  and  $L$  are extensions of  $F$ , then  $[KL : L]$  divides  $[K : F]$ .

thGALGRPfund3b

**Result 11** Suppose that  $E$  is a finite Galois extension of  $F$  and  $G$  the associated Galois group. Then  $K$  is a normal extension of  $F$  if and only if  $H = \text{Gal}(E/K)$  is a normal subgroup of  $G$ . In this case the Galois group  $\text{Gal}(K/F)$  is isomorphic to the quotient group  $G/H$ .

In addition, for any subgroup  $H$  (not necessarily normal),

$$[K : F] = [G : H] \quad \text{and} \quad [E : K] = |H|.$$

ALGRPnormclosth1

**Result 12** Let  $E$  be a finite extension of  $F$  and  $N$  the normal closure of  $E$  over  $F$  in an algebraic closure  $C$  of  $F$ . Then

$$N = \prod_{\sigma \in \text{Gal}(N/F)} \sigma(E).$$

## 4 Appendix: Solvable groups

In this appendix we revise the basic properties of solvable groups, which may or may not be known to the reader.

A *normal series* or *normal sequence* of a group  $G$  is a sequence of subgroups of the form

$$G = G_0 \triangleright G_1 \triangleright G_2 \triangleright \cdots \triangleright G_n = \{e\},$$

where  $e$  is the identity of  $G$  and  $G_{i+1}$  is a proper normal subgroup of  $G_i$ . We do not require the subgroups to be normal subgroups of  $G$ . The quotient groups  $G_i/G_{i+1}$  are called the factors of the normal series. A group is said to be *solvable* if it has a normal series with abelian factors. In this case we say that the normal series is solvable. An abelian group  $G$  is clearly solvable: we only need to take the sequence  $G = G_0 \triangleright \{e\}$ .

Let  $G$  be a group. The *commutator* of two elements  $x, y \in G$ , which we note  $[x, y]$ , is the product  $xyx^{-1}y^{-1}$ . The *commutator subgroup* of  $G$ , written  $G'$ , is the subgroup generated by the commutators. From the observation that

$$g[x, y]g^{-1} = [gxg^{-1}, gyg^{-1}]$$

for all  $g \in G$ , we deduce that  $G'$  is a normal subgroup of  $G$ . There is no difficulty in seeing that the quotient group  $G/G'$  is abelian.

APPSOLVprop1

**Proposition 4** *If  $H$  is a normal subgroup of  $G$ , then  $G/H$  is abelian if and only if  $G' \subset H$ .*

PROOF If  $G/H$  is abelian, then

$$xyH = xHyH = yHxH = yxH \implies [x, y] \in H.$$

As all the commutators lie in  $H$ , we have  $G' \subset H$ .

On the other hand, if  $G' \subset H$ , then by the third isomorphism theorem

$$(G/G')/(H/G') \simeq G/H,$$

so  $G/H$  is isomorphic to the quotient group of an abelian group and is thus abelian.  $\square$

The *higher commutator subgroups* are defined by induction:

$$G^{(0)} = G \quad \text{and} \quad G^{(i+1)} = G^{(i)'}$$

i.e.,  $G^{(i+1)}$  is the commutator subgroup of  $G^{(i)}$  and so is a normal subgroup of  $G^{(i)}$ .

APPSOLVprop2

**Proposition 5** *A group  $G$  is solvable if and only if there is a nonnegative integer  $n$  such that  $G^{(n)} = \{e\}$ .*

PROOF If  $G^{(n)} = \{e\}$ , then the series

$$G = G^{(0)} \triangleright G^{(1)} \triangleright G^{(2)} \triangleright \dots \triangleright G^{(n)} = \{e\}$$

is solvable, so  $G$  is a solvable group.

If  $G$  is solvable, then there exists a solvable series

$$G = G_0 \triangleright G_1 \triangleright G_2 \triangleright \dots \triangleright G_n = \{e\}$$

We prove, by induction on  $i$ , that  $G_i \supset G^{(i)}$ , for  $i = 0, \dots, n$ . First, for  $i = 0$ , we have  $G_0 = G = G^{(0)}$ . Suppose now, that  $G_i \supset G^{(i)}$ . Then  $G'_i \supset G^{(i)'} = G^{(i+1)}$ . However,  $G_i/G_{i+1}$  is abelian, hence, from Proposition 4, we have  $G_{i+1} \supset G'_i$  and so  $G_{i+1} \supset G^{(i+1)}$ . This finishes the induction step. It follows that  $G^{(n)} = \{e\}$ .  $\square$

Now we come to two fundamental results.

**Theorem 8** *If  $G$  is a solvable group, then every subgroup and quotient group is solvable.*

PROOF Let  $H$  be a subgroup of the solvable group  $G$ . By induction it is easy to prove that  $H^{(i)} \subset G^{(i)}$ , for all  $i$ . There exists a nonnegative integer  $n$  such that  $G^{(n)} = \{e\}$ . It follows that  $H^{(n)} = \{e\}$  and so, by Proposition 5,  $H$  is solvable.

We now consider the case of a quotient group. We will prove a more general result: if  $G$  is a solvable group and  $\phi : G \rightarrow C$  is a surjective homomorphism, then  $C$  is solvable. To do so, we will show that  $\phi(G^{(i)}) = C^{(i)}$ , for all  $i$ . We will use an induction argument. For  $i = 0$ , the result is evident. As the image of a commutator is a commutator,  $\phi(G') \subset C'$ . On the other hand, if  $u, v \in C$ , then there exist  $x, y \in G$  such that  $\phi(x) = u$ ,  $\phi(y) = v$  and we have  $\phi[x, y] = [u, v]$ ; it follows that  $\phi(G') = C'$ . Thus the result is true for  $i = 1$ . Suppose now that  $\phi(G^{(i)}) = C^{(i)}$ . Then  $\phi$  restricted to  $G^{(i)}$  provides us with a surjective homomorphism of  $G^{(i)}$  onto  $C^{(i)}$ . From the previous argument, we have  $\phi(G^{(i)'}) = C^{(i)'}$ , i.e.,  $\phi(G^{(i+1)}) = C^{(i+1)}$ . This completes the induction step. To finish we notice that there is an  $n$  such that  $G^{(n)} = \{e_G\}$ . This implies that  $C^{(n)} = \{e_C\}$  and so  $C$  is solvable.  $\square$

This theorem has a converse.

**Theorem 9** *Let  $G$  be a group with a normal subgroup  $H$ . If  $H$  and  $G/H$  are solvable, then so is  $G$ .*

PROOF As  $G/H$  is solvable, there is a solvable series composed of subgroups of  $G/H$ :

$$G/H = G_0^* \triangleright G_1^* \triangleright \cdots \triangleright G_s^* = \{H/H\},$$

where  $G_i/H = G_i^*$ . By the correspondance theorem, there is a normal series

$$G = G_0 \triangleright G_1 \triangleright \cdots \triangleright G_s = H.$$

By the third isomorphism theorem, we have  $G_i/G_{i+1} \simeq (G_i/H)/(G_{i+1}/H)$ , so the factors  $G_i/G_{i+1}$  are abelian. As  $H$  is solvable, we have a solvable series composed of subgroups of  $H$ :

$$H = H_0 \triangleright H_1 \triangleright \cdots \triangleright H_t = \{e_H\} = \{e_G\}.$$

Putting the two series together gives us a solvable series of  $G$ .  $\square$

We may refine the solvable series so as to obtain factor groups of prime order. To prove this we need a preliminary result. We recall that a group  $G$  is *simple* if  $G \neq \{e\}$  and its only normal subgroups are  $\{e\}$  and  $G$  itself.

APPSOLV1em1

**Lemma 3** *If  $G$  is a finite abelian simple group, then  $G$  is cyclic of prime order.*

PROOF If  $x \in G$  and  $x \neq e$ , then  $\langle x \rangle$  is a normal subgroup, because  $G$  is abelian. As  $x \neq e$ , we have  $\langle x \rangle = G$ , so  $G$  is cyclic.

If  $G$  is not of prime order and  $|G| = n$ , then  $G$  has a (unique) subgroup of order  $\frac{n}{d}$ , for every positive divisor  $d$  of  $n$ . As  $G$  is abelian, these subgroups are normal, so we have a contradiction to the simplicity of  $G$ . Thus  $G$  is of prime order.  $\square$

APPSOLVth3

**Theorem 10** *If  $G$  is a solvable group, then  $G$  has a solvable series whose factor groups are cyclic of prime order.*

PROOF As  $G$  is solvable,  $G$  has a solvable series

$$G = G_0 \triangleright \cdots \triangleright G_i \triangleright G_{i+1} \triangleright \cdots \triangleright G_n = \{e\}.$$

If  $G_{i+1}$  is not a maximal normal subgroup of  $G_i$ , then we can insert a normal subgroup  $H$  between  $G_{i+1}$  and  $G_i$ . As the quotient group  $H/G_{i+1}$  is a subgroup of  $G_i/G_{i+1}$ , it is abelian. Continuing the process we obtain a solvable series such that  $G_{i+1}$  is always a maximal normal subgroup of  $G_i$ .

Now  $G_i/G_{i+1}$  is finite and abelian. If this factor group is not simple, then it has a proper nontrivial normal subgroup. By the correspondence theorem there exists a nontrivial subgroup, which is normal in  $G_i$  and lies strictly between  $G_{i+1}$  and  $G_i$ ; thus  $G_i$  is not maximal, a contradiction. Therefore  $G_i/G_{i+1}$  is finite, abelian and simple. By Lemma 3,  $G_i/G_{i+1}$  is cyclic of prime order.  $\square$

APPSOLVcor1

**Corollary 3** *A solvable group  $G$  has a normal subgroup of prime index.*

PROOF From Theorem 10 we may suppose that in the solvable series of  $G$  the index of  $G_1$  in  $G_0$  is a prime number  $p$ . As  $G_0 = G$ , we have the result.  $\square$

### Solvability of the symmetric groups

The symmetric groups  $S_n$  are solvable if and only if  $n \leq 4$ . If  $n = 1$  or  $n = 2$ , then  $S_n$  is abelian, hence solvable. For  $S_3$  it is sufficient to consider the series

$$\langle e \rangle \subset A_3 \subset S_3.$$

For  $S_4$  we may take the chain

$$\langle e \rangle \subset V \subset A_4 \subset S_4,$$

where

$$V = \{e, (12)(34), (13)(24), (14)(23)\}.$$

(The only possible difficulty here is in seeing that  $V \triangleleft A_4$ ; however, it is sufficient to notice that conjugation preserves the cycle structure.)

For the next result we will suppose that it is known that, for  $n \geq 5$ ,  $A_n$  is simple, i.e., has no normal subgroups other than  $\{e\}$  or  $A_n$  itself.

**Theorem 11** *For  $n \geq 5$ ,  $S_n$  is not solvable.*

PROOF For  $n \geq 5$ ,  $A_n$  is simple, therefore  $A'_n = A_n$  or  $A'_n = \{e\}$ . However, the second case is not possible, because  $A_n$  is not abelian for  $n \geq 4$ . Thus  $A'_n = A_n$ , which implies that  $A_n^{(k)} = A_n$ , for all  $k$ . It follows that  $A_n$  is not solvable. As a subgroup of a solvable group is solvable,  $S_n$  cannot be solvable.  $\square$