Solvability by radicals

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Abstract

In this note we present one of the fundamental theorems of algebra, namely Galois’s theorem concerning the solution of polynomial equations. We will begin with a study of cyclic extensions, before moving onto radical extensions.

1 Cyclic extensions

We say that a finite extension $E$ of a field $F$ is cyclic if the Galois group $\text{Gal}(E/F)$ is cyclic. In this section we aim to present some elementary properties of such extensions. We begin with a preliminary result known as Hilbert’s theorem 90.

**Theorem 1** Let $E/F$ be a finite cyclic Galois extension of degree $n$. We suppose that $\sigma$ is a generator of the Galois group $\text{Gal}(E/F)$. If $\alpha \in E$, then $N_{E/F}(\alpha) = 1$ if and only if there exists $\beta \in E^*$ such that $\alpha = \beta \sigma(\beta)$.

**Proof**

If $\alpha = \beta \sigma(\beta)$, then
$$N_{E/F}(\alpha) = \alpha \sigma(\alpha) \cdots \sigma^{n-1}(\alpha) = \frac{\beta \sigma(\beta) \sigma^2(\beta) \cdots \sigma^{n-1}(\beta)}{\sigma(\beta) \sigma^2(\beta) \cdots \sigma^{n-1}(\beta) \sigma^n(\beta)} = 1,$$

because $\sigma^n(\beta) = \beta$.

Now suppose that $N_{E/F}(\alpha) = 1$, i.e., $\alpha \sigma(\alpha) \cdots \sigma^{n-1}(\alpha) = 1$. We define a finite sequence $(\delta_i)_{i=0}^{n-1}$ as follows:

$$\delta_0 = \alpha, \delta_1 = \alpha \sigma(\alpha), \delta_2 = \alpha \sigma(\alpha) \sigma^2(\alpha), \ldots, \delta_{n-1} = \alpha \sigma(\alpha) \cdots \sigma^{n-1}(\alpha) = N_{E/F}(\alpha) = 1.$$

From Result 1, the characters $1, \sigma, \ldots, \sigma^{n-1}$ form an independent set and so there exists $\gamma \in E$ such that

$$\delta_0 \gamma + \delta_1 \sigma(\gamma) + \cdots + \delta_{n-2} \sigma^{n-2}(\gamma) + \sigma^{n-1}(\gamma) \neq 0.$$

We note this sum $\beta$. Then

$$\sigma(\beta) = \sigma(\delta_0) \sigma(\gamma) + \sigma(\delta_1) \sigma^2(\gamma) + \cdots + \sigma(\delta_{n-2}) \sigma^{n-2}(\gamma) + \sigma^{n}(\gamma)$$

$$= \alpha^{-1} (\delta_0 \sigma(\gamma) + \delta_1 \sigma^2(\gamma) + \cdots + \delta_{n-2} \sigma^{n-2}(\gamma) + \sigma^{n-1}(\gamma)) + \sigma^{n}(\gamma),$$

because $\alpha^{-1} \delta_i = \sigma(\delta_{i-1})$. As $\sigma^n = \text{id}_E$, we have

$$\sigma^n(\gamma) = \gamma = \alpha^{-1} \delta_0 \gamma,$$

hence $\sigma(\beta) = \alpha^{-1} \beta$.

The theorem which we have just proved is often referred to as the ‘multiplicative’ version of Hilbert’s theorem 90 to distinguish from the ‘additive’ version, which we will now present.
**Theorem 2** If $E/F$ is a finite cyclic Galois extension of degree $n$, then $T_{E/F}(\alpha) = 0$ if and only if there exists $\beta \in E$ such that $\alpha = \beta - \sigma(\beta)$.

**Proof** Let $\sigma$ be a generator of the Galois group $G = \text{Gal}(E/F)$, so that $\sigma^n = \text{id}_n$.

If $\alpha = \beta - \sigma(\beta)$, then

$$T_{E/F}(\alpha) = \sum_{k=0}^{n-1} \sigma^k(\alpha) = \sum_{k=0}^{n-1} \sigma^k(\beta) - \sum_{k=1}^{n} \sigma^k(\beta) = 0,$$

because $\sigma^n(\beta) = \beta$.

Conversely, suppose that $T_{E/F}(\alpha) = 0$. As $T_{E/F} \neq 0$, we may find an element $x$ such that $T_{E/F}(x) \neq 0$. Let

$$w = \alpha \sigma(x) + (\alpha + \sigma(\alpha)) \sigma^2(x) + \cdots + (\alpha + \sigma(\alpha) + \cdots + \sigma^{n-2}(\alpha)) \sigma^{n-1}(x).$$

Then

$$\sigma(w) = \sigma(\alpha) \sigma^2(x) + (\sigma(\alpha) + \sigma^2(\alpha)) \sigma^3(x) + \cdots + (\sigma(\alpha) + \sigma^2(\alpha) + \cdots + \sigma^{n-1}(\alpha)) \sigma^n(x).$$

Since $T_{E/F}(\alpha) = 0$, we have

$$-\alpha = \sigma(\alpha) + \cdots + \sigma^{n-1}(\alpha),$$

so the last summand in the expression for $\sigma(w)$ is $-\alpha x$, hence

$$w - \sigma(w) = \alpha \left( x + \sigma(x) + \sigma^2(x) + \cdots + \sigma^{n-1}(x) \right) = \alpha T_{E/F}(x).$$

Setting $\beta = \frac{w}{T_{E/F}(x)}$, we obtain

$$\beta - \sigma(\beta) = \frac{w}{T_{E/F}(x)} - \frac{\sigma(w)}{\sigma(T_{E/F}(x))} = \frac{w - \sigma(w)}{T_{E/F}(x)},$$

because $T_{E/F}(x) \in F$. Therefore

$$\beta - \sigma(\beta) = \frac{\alpha T_{E/F}(x)}{T_{E/F}(x)} = \alpha,$$

as required. \qed

In the following we consider extensions $E/F$ of a particular sort, namely where the field $F$ contains a primitive $n$th root of unity, for some positive integer $n$.

**Theorem 3** Let $E/F$ be a cyclic Galois extension of degree $n$, where $F$ contains a primitive $n$th root of unity $\zeta$. Then there exists an irreducible polynomial $f(X) = -\alpha + X^n \in F[X]$ such that $E$ is a splitting field of $f$.

**Proof** Let $\sigma$ be a generator of the Galois group $\text{Gal}(E/F)$. Since $\zeta \in F$, we have $N_{E/F}(\zeta) = \zeta^n = 1$ and so, by Hilbert’s theorem 90 (multiplicative version), there is an element $\beta \in E^*$ such that $\zeta = \frac{\beta}{\sigma(\beta)}$. Then

$$\sigma(\beta) = \zeta^{-1}\beta,$$

$$\sigma^2(\beta) = \sigma(\zeta^{-1})\sigma(\beta) = \zeta^{-1}(\zeta^{-1}\beta) = \zeta^{-2}\beta,$$

$$\sigma^3(\beta) = \sigma(\zeta^{-2})\sigma(\beta) = \zeta^{-2}(\zeta^{-1}\beta) = \zeta^{-3}\beta.$$
and generally $\sigma^m(\beta) = \zeta^{-m}\beta$, for all $m \in \mathbb{N}^*$. As the elements $\sigma^i(\beta)$, for $0 \leq m < n$ are distinct, the automorphisms $\text{id}_F, \sigma_1 F(\beta), \ldots, \sigma_{n-1} F(\beta)$ form a set of $n$ distinct elements of the Galois group $\text{Gal}(F(\beta)/F)$. Then $[F(\beta) : F] \geq n$ and

$$n = [\text{Gal}(E/F)] = [E : F] = [E : F(\beta)][F(\beta) : F] \implies [F(\beta) : F] \leq n.$$  

Therefore $[F(\beta) : F] = n$, which implies that $E = F(\beta)$.

We claim that $\beta^n \in F$. Now

$$\sigma(\beta^n) = \sigma(\beta)^n = \zeta^{-n}\beta^n = \beta^n,$$

and it follows that $\beta^n$ belongs to the fixed field of $\text{Gal}(E/F)$, i.e., $F$. Thus there exists $\alpha \in F$ such that $\beta^n = \alpha$. We have shown that $E = F(\beta)$, where $\beta$ is a root of the polynomial $f(X) = -\alpha + X^n$. The roots of $f$ have the form $\zeta^i\beta$, for $0 \leq i < n$ and $\zeta \in F$ by hypothesis, so $E$ contains all the roots of $f$ and hence is a splitting field of the polynomial $f$. Since $[F(\beta) : F] = n$, the degree of the minimal polynomial $m(\beta, F)$ is $n$, hence $f = m(\beta, F)$; it follows that $f$ is irreducible.

The theorem which we have just proved has a converse.

**Theorem 4** Let $F$ be a field containing an $n$th primitive root of unity $\zeta$ and $E$ a splitting field of the polynomial $f(X) = -\alpha + X^n \in F[X]$, with $\alpha \neq 0$. Then $E = F(\beta)$, where $\beta$ is a root of $f$, and the Galois group $G = \text{Gal}(E/F)$ is cyclic of order dividing $n$. The order of $G$ is equal to $n$ if and only if $f$ is irreducible.

**Proof** If $\beta$ is root of $f$, then the roots of $f$ have the form $\zeta^i\beta$, for $0 \leq i < n$. As $\zeta \in F$, all the roots of $f$ lie in $F(\beta)$, so $F(\beta)$ is a splitting field of $f$ and so $E = F(\beta)$.

Let $\sigma \in G = \text{Gal}(E/F)$. As $\sigma$ permutes the roots of $f$, we may write $\sigma(\beta) = \zeta^{k(\sigma)}\beta$, where $k(\sigma)$ is uniquely determined modulo $n$. We thus obtain a mapping

$$\phi : G \rightarrow (\mathbb{Z}_n, +), \sigma \mapsto [k(\sigma)],$$

which is clearly a group homomorphism. If $\phi(\sigma) = [0]$, then $\sigma(\beta) = \beta$, which implies that $\sigma$ is the identity on $F(\beta)$ and so $\phi$ is a monomorphism. Since $G$ is isomorphic to a subgroup of the cyclic group $(\mathbb{Z}_n, +)$, $G$ is cyclic of order dividing $n$.

If $f$ is irreducible, then $G$ acts transitively on the roots of $f$ (Result 4). Thus there exists $\sigma \in G$ such that $\sigma(\beta) = \zeta^i\beta$, for $0 \leq i < n$. It follows that $\phi$ is surjective, so $G$ is isomorphic to $(\mathbb{Z}_n, +)$.

Now suppose that $f$ is reducible. As $f$ has no multiple roots, $f$ has two distinct irreducible factors, so $G$ does not act transitively on the roots of $f$ (Result 5). Thus there exist roots $\zeta^i\beta$ and $\zeta^j\beta$, with $j > i$, for which there exists no element $\sigma \in G$ such that $\sigma(\zeta^i\beta) = \zeta^j\beta$, i.e., $\sigma(\beta) = \zeta^{j-i}\beta$. It follows that $\phi$ is not surjective and so $|G| < n$.

**Corollary 1** Let $p$ be prime, $F$ a field containing a primitive $p$th root of unity and $E$ a splitting field of the polynomial $f(X) = -\alpha + X^p \in F[X]$, with $\alpha \neq 0$. Then either $f$ is irreducible and $\text{Gal}(E/F) \cong \mathbb{Z}_p$, or $f$ splits in $F$ and $\text{Gal}(E/F) = \{\text{id}_F\}$.

**Proof** The order of $\text{Gal}(E/F) = p$ or 1. In the first case, from Theorem 4, $f$ is irreducible and $\text{Gal}(E/F) \cong \mathbb{Z}_p$, because $\text{Gal}(E/F)$ is cyclic. Suppose now that the order of $\text{Gal}(E/F)$ is 1. Then $[E : F] = 1$. There exists $\beta \in E$ such that $E = F(\beta)$, so $[F(\beta) : F] = 1$, which implies that $\beta \in F$. It now follows that $f$ splits in $F$. 

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2 Radical extensions

Suppose that $E/F$ is a field extension such that $E = F(\alpha)$ for some $\alpha \in E$. If there exists $n \in \mathbb{N}^*$ such that $\alpha^n \in F$, then we say that $E$ is a pure extension of type $n$. In this case $\alpha$ is a root of a polynomial of the form $f(X) = -c + X^n \in F[X]$. A chain of extensions

$$F = F_0 \subset F_1 \subset \cdots \subset F_r = E,$$

where each extension $F_{i+1}/F_i$ is a pure extension is called a radical chain and we say that $E/F$ is a radical extension. Using Result corFEalgext1 and Result propFEalgext3, we see that a radical extension is a finite algebraic extension.

**Proposition 1**

- **a.** If $E/F$ is a radical extension and $Z$ an intermediate field, then $E/Z$ is a radical extension.
- **b.** If $E/F$ is a field extension and $Z_1, Z_2$ intermediate fields such that $Z_1/F, Z_2/F$ are radical extensions, then $Z_1Z_2/F$ is a radical extension.

**Proof**

**a.** We have the radical chain

$$F \subset F(\alpha_1) \subset F(\alpha_1, \alpha_2) \subset \cdots \subset F(\alpha_1, \cdots, \alpha_r) = E,$$

with $\alpha_i^{n_i} \in F(\alpha_1, \cdots, \alpha_{i-1})$, for some $n_i \in \mathbb{N}^*$. Then

$$Z \subset Z(\alpha_1) \subset Z(\alpha_1, \alpha_2) \subset \cdots \subset Z(\alpha_1, \cdots, \alpha_r) = E$$

and

$$\alpha_i^{n_i} \in F(\alpha_1, \cdots, \alpha_{i-1}) \subset Z(\alpha_1, \cdots, \alpha_{i-1}).$$

It follows that $E/Z$ is a radical extension.

**b.** We have the radical extensions

$$F \subset F(\alpha_1) \subset F(\alpha_1, \alpha_2) \subset \cdots \subset F(\alpha_1, \cdots, \alpha_r) = Z_1$$

and

$$F \subset F(\beta_1) \subset F(\beta_1, \beta_2) \subset \cdots \subset F(\beta_1, \cdots, \beta_s) = Z_2.$$

Then

$$F \subset F(\alpha_1) \subset \cdots \subset F(\alpha_1, \cdots, \alpha_r) \subset F(\alpha_1, \cdots, \alpha_r, \beta_1) \subset F(\alpha_1, \cdots, \alpha_r, \beta_1, \cdots, \beta_s) = Z_1Z_2.$$ 

The only possible difficulty in seeing that we have a radical chain is in finding a power of $\beta_1$ in $F(\alpha_1, \cdots, \alpha_r)$. However, there exists $m_1 \in \mathbb{N}^*$ such that $\beta_1^{m_1} \in F \subset F(\alpha_1, \cdots, \alpha_r)$, so we indeed have a radical chain.

**Remark** We may easily extend **b.** to any finite number of intermediate fields.

We now consider the normal closure of a radical extension.

**Proposition 2** If $E/F$ is a radical extension and $N$ the normal closure of $E$ over $F$, then $N/F$ is a radical extension.
Lemma 1

PROOF From Result 1.2 the normal closure of $E$ over $F$ can be written as a compositum of fields $Z_1, \ldots, Z_r$, isomorphic to $E$ contained in a field extension of $F$. As $E$ is a radical extension of $F$, so are the $Z_i$. From the remark after Proposition 1.1 the normal closure of $E$ over $F$ is a radical extension of $F$. \hfill \Box

We may insert fields into a radical chain so that each extension $F_{i+1}/F_i$ is pure of type $p_i$, for some prime number $p_i$. Let us see why this is the case. Suppose that $L/K$ is a pure extension of type $n$ and $L = K(\alpha)$, where $\alpha \in L$ and $\alpha^n \in K$. If $n = p_1 \cdots p_s$ is the prime factorization of $n$, then

$$K \subset K(\alpha^{p_1}) \subset \cdots \subset K(\alpha^{p_1p_2}) \subset K(\alpha^{p_1p_2p_3}) \subset \cdots \subset K(\alpha) = L.$$ 

Then we have $\alpha^{p_1} \in K(\alpha^{p_1})$, $(\alpha^{p_1})^{p_2} \in K(\alpha^{p_1p_2})$, and so on. This proves our claim. In this case we say that the radical extension is of prime type.

3 Solvability by radicals

Let $F$ be a field and $f$ a polynomial over $F$. We say that $f$ is solvable by radicals if its splitting field $L$ is contained in a radical extension $E$ of $F$. In this section we aim first to show that under certain conditions solvability by radicals implies the solvability of the Galois group. (In an appendix we revise the notion of a solvable group.) Later we will prove the converse.

For certain results we need the presence of a primitive root of unity. We might be tempted to think that, for a given $n \in \mathbb{N}^*$, if a field does not contain a primitive $n$th root of unity, then we can simply add this root. However, this is not the case. Suppose that the field $F$ has characteristic $p \neq 0$ and $p|n$, say $n = pn$. Let $\zeta$ be a primitive $n$th root of unity, then

$$0 = \zeta^n - 1 = (\zeta^n - 1)^p,$$

so $\zeta^n - 1 = 0$. This implies that the order of $\zeta$ is strictly less than $n$, a contradiction. So there is no primitive $n$th root of unity.

On the other hand, if the characteristic of $F$ does not divide $n$, this problem does not occur. If $f(X) = -1 + X^n$, then

$$f'(X) = nX^{n-1} \neq 0 \implies \gcd(f, f') = 1,$$

so $f$ has no multiple roots ($f$ is strongly separable). Therefore there are $n$ distinct $n$th roots of unity. As the group of $n$th roots of unity is cyclic, it has a generator, namely a primitive $n$th root of unity. To be certain that we can find primitive $n$th roots of unity, we may suppose that the characteristic of a field is $0$. This also ensures that all extensions are separable.

Our next step is to establish a preliminary result, which will play an important role in the following. To do this we need a lemma.

Lemma 1 Let $F \subset K \subset E$ be a chain of fields, with $K$ the splitting field of a polynomial $f \in F[X]$. If $\sigma \in \text{Gal}(E/F)$, then $\sigma|_K \in \text{Gal}(K/F)$.

PROOF It is sufficient to show that $\sigma(K) = K$. If $\alpha_1, \ldots, \alpha_n$ are the distinct roots of $f$, then $K = F(\alpha_1, \ldots, \alpha_n)$. For each $\alpha_i$, the element $\sigma(\alpha_i)$ is a root of $f$, so

$$\sigma(K) = \sigma(F(\alpha_1, \ldots, \alpha_n)) = F(\sigma(\alpha_1), \ldots, \sigma(\alpha_n)) = K,$$

as required. \hfill \Box

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Theorem 5 Let $F \subset K \subset E$ be a chain of fields, with $K$ the splitting field of a polynomial $f \in F[X]$ and $E$ the splitting field of a polynomial $g \in F[X]$. Then $\text{Gal}(E/K)$ is a normal subgroup of $\text{Gal}(E/F)$ and

$$\text{Gal}(E/F)/\text{Gal}(E/K) \simeq \text{Gal}(K/F).$$

Proof We define a mapping $\phi : \text{Gal}(E/F) \rightarrow \text{Gal}(K/F)$ by $\phi(\sigma) = \sigma_{|K}$. (From Lemma 7 we know that $\sigma_{|K} \in \text{Gal}(K/F)$.) Clearly, $\phi$ is a homomorphism. As $\phi(\sigma) = \text{id}_K$ if and only if $\sigma$ fixes all the elements of $K$, the kernel of $\phi$ is the Galois group $\text{Gal}(E/K)$. Therefore $\text{Gal}(E/K)$ is a normal subgroup of $\text{Gal}(E/F)$.

We claim that $\phi$ is surjective. Let $\tau \in \text{Gal}(K/F)$. As $F \subset K$, the polynomial $f$ belongs to $K[X]$. Using Result 2, with $K' = K$ and $f^* = f$, we obtain an automorphism $\tilde{\tau}$ of $E$ extending $\tau$. Thus $\phi(\tilde{\tau}) = \tau$. It follows that $\phi$ is surjective and

$$\text{Gal}(E/F)/\text{Gal}(E/K) \simeq \text{Gal}(K/F),$$

by the first isomorphism theorem. \hfill \Box

We now consider polynomials which are solvable by radicals. Let $F$ be a field and $f \in F[X]$ solvable by radicals. We will suppose that the characteristic of $F$ is 0. The reason for doing so is twofold: 1. Extensions of $F$ are separable; 2. Roots of unity of any order may be found in some extension. By hypothesis there is a radical chain

$$F = F_0 \subset F_1 \subset \cdots \subset F_r = E$$

such that the splitting field $L$ of $f$ is included in $E$. From Proposition 5 we may suppose that $E$ is a Galois extension of $F$, hence the splitting field of a polynomial $g \in F[X]$. We have also seen at the end of the previous section that we may suppose that each extension $F_{i+1}/F_i$ may be taken to be pure of prime type, for some prime $p_i$. In the next proposition we will suppose that the field $F$ contains a primitive $p_i$th root of unity, for all the primes $p_i$. This is not in general the case. However, we will show further on that we may replace the radical chain by another for which this is the case.

Proposition 3 If $F$ contains a primitive $p_i$th root of unity for all $i$, then the Galois group of $f$ is solvable.

Proof We set

$$G_i = \text{Gal}(F_{i+1}/F_i),$$

for $i = 0, \ldots, r$. We consider $[F_i : F_{i-1}]$. As $a_i$ is a root of the polynomial $f_i(X) = -a_i^{p_i} + X^{p_i} \in F_{i-1}[X]$, the degree of the minimal polynomial $m(a_i, F_{i-1})$ must be 1 or $p_i$. We can exclude the first case, because this would imply that $F_i = F_{i-1}$. Now

$$[F_{i-1}(a_i) : F_{i-1}] = \deg m(a_i, F_{i-1}) \Rightarrow [F_{i-1}(a_i) : F_{i-1}] = p_i.$$

As $F_i$ is a Galois extension of $F_{i-1}$, we have $|\text{Gal}(F_i/F_{i-1})| = p_i$ and so the extension $F_i/F_{i-1}$ is cyclic of degree $p_i$. We may apply Theorem 5. As $F_i \subset F_{i-1}$, $F_i$ contains a $p_i$th primitive root of unity and it follows that $F_i$ is the splitting field of a polynomial $f_i$ over $F_{i-1}$. (Thus $F_i$ is a Galois extension of $F_{i-1}$.)

We now consider the sequence

$$G_0 \supset G_1 \supset \cdots \supset G_r = \{\text{id}\}.$$
We claim this sequence is normal, i.e., \( G_{i-1} \triangleright G_i \), for all \( i \). We have \( F_{i-1} \subset F_i \subset F_r \). Since \( F_r \) is a normal extension of \( F \), \( F_r \) is the splitting field of a polynomial \( g \in F[X] \). Given that \( F \subset F_r \), we have \( g \in F_{i-1}[X] \) and we may apply Theorem 6, which ensures us that \( G_{i-1} \triangleright G_i \). Hence the sequence is normal. From Theorem 6 again, the quotient group \( G_{i-1} / G_i \) is isomorphic to \( \text{Gal}(F_i / F_{i-1}) \), which is cyclic. We have shown that the Galois group \( \text{Gal}(F_r / F_0) \) is a solvable.

To conclude we consider the chain of fields

\[
F \subset L \subset F_r,
\]

where \( L \) is the splitting field of \( f \). Using Theorem 6 again, we obtain the isomorphism

\[
\text{Gal}(L/F) \simeq \text{Gal}(F_r/F)/\text{Gal}(F_r/L).
\]

As \( \text{Gal}(L/F) \) is isomorphic to a quotient of a solvable group, \( \text{Gal}(L/F) \) is itself solvable. \( \square \)

We now eliminate the hypothesis concerning the existence of \( p_i \)-th primitive roots of unity in the field \( F \). Our aim is to replace the initial radical chain by another satisfying the conditions of Proposition 6.

**Theorem 6** Let \( F \) be a field of characteristic 0 and \( f \in F[X] \) solvable by radicals. If \( L \) is a splitting field of \( f \), then \( \text{Gal}(L/F) \) is a solvable group.

**Proof** By hypothesis there is a radical chain

\[
F = F_0 \subset F_1 \subset \cdots \subset F_r = E,
\]

with \( L \subset F_r \). We may assume that \( F_r \) is the splitting field of some polynomial \( g \in F[X] \) and \( g \in F_{i-1}[X] \). In addition, each extension \( F_i/F_{i-1} \) is a Galois extension of degree \( p_i \), because \( F \subset F_r \) implies that \( g \in F_{i-1}[X] \).

Let \( m \) be the len of the \( p_i \) and \( \zeta \) an \( m \)-th primitive root of unity. We construct a new chain by adding \( \zeta \) to each element of our first chain:

\[
F = F_0 \subset F_0(\zeta) \subset F_1(\zeta) \subset \cdots \subset F_r(\zeta).
\]

We notice that \( F_0(\zeta) \) contains all the \( p_i \)-th primitive roots of unity. For example, if \( m = p_1 \cdots p_r \), then \( \zeta^{p_1 \cdots p_r} \) is a primitive \( p_i \)-th root of unity. To simplify the notation, let us write \( F_i' \) for \( F_i(\zeta) \), for \( i = 0, \ldots, r \).

We consider the extensions \( F_i'/F_{i-1}' \). From Result 10, we have \( [F_i:F_{i-1}'][F_i':F_{i-1}'] \). However,

\[
F_i = F_{i-1}(\alpha_i) \implies F_i F_{i-1}(\zeta) = F_{i-1}(\alpha_i, \zeta) = F_i(\zeta) = F_i',
\]

so \( [F_i':F_{i-1}'][F_i:F_{i-1}] \), which implies that \( [F_i':F_{i-1}'] \) has the value 1 or \( p_i \). In the first case, we can eliminate \( F_i' \) (or \( F_{i-1}' \)), so we can assume that \( [F_i':F_{i-1}'] = p_i \). Since \( F_i \) and \( F_{i-1} \) are extensions of \( F_{i-1} \) and \( F_i \) is a finite Galois extension of \( F_{i-1} \), by Result 6, the compositum \( F_i F_{i-1} \) is a Galois extension of \( F_{i-1}' \), i.e., \( F_i' \) is a Galois extension of \( F_{i-1}' \). We now may apply Theorem 7. There exists an irreducible polynomial \( f_{i-1}(X) = -c_{i-1} + X^{p_i} \in F_{i-1}[X] \) such that \( F_i' \) is a splitting field of \( f_{i-1} \). Let \( \alpha_i' \) be a root of \( f_{i-1} \). Then \( f_{i-1} \) splits in \( F_{i-1}'(\alpha_i') \), because \( F_{i-1}' \) contains a primitive \( p_i \)-th root of unity. This implies that \( F_i' \subset F_{i-1}'(\alpha_i') \). As the reverse inclusion is clear, we have \( F_i' = F_{i-1}'(\alpha_i') \). We have shown that there exists \( \alpha_i' \in F_i' \) such that \( F_i' = F_{i-1}'(\alpha_i') \), with \( \alpha_i^{p_i} \in F_{i-1}' \). Hence the extension \( F_i'/F_{i-1}' \) is radical of prime type.

Our next step is to show that \( \text{Gal}(F_i'/F_{i-1}') \) is solvable. Since \( F_i' \) is the splitting field of the polynomial \( h(X) = (-1 + X^{p_i})g(X) \in F_i'[X] \), \( F_i' \) is a Galois extension of \( F_{i-1}' \) and \( \text{Gal}(F_i'/F_{i-1}') \) is the Galois group of \( h \). From Proposition 6 again, we deduce that \( \text{Gal}(F_r'/F_0') \) is solvable.
We now show that $\text{Gal}(F_\sigma'/F)$ is solvable. We consider the chain of fields

$$F_0 \subset F'_0 \subset F'_\sigma.$$ 

Since $F'_0$ and $F'_\sigma$ are each the splitting field of a polynomial in $F_0[X]$, from Theorem 5, $\text{Gal}(F'_\sigma/F_0)$ is a normal subgroup of $\text{Gal}(F'_\sigma/F_0)$ and

$$\text{Gal}(F'_\sigma/F_0)/\text{Gal}(F'_\sigma/F'_0) \simeq \text{Gal}(F'_0/F_0).$$

From Result 6, $\text{Gal}(F_0/F_0)$ is abelian, hence solvable. Therefore the quotient group $\text{Gal}(F'_\sigma/F_0)/\text{Gal}(F'_\sigma/F'_0)$ is solvable. However, we have also seen that the subroup $\text{Gal}(F'_\sigma/F'_0)$ is solvable. It follows that $\text{Gal}(F'_\sigma/F_0)$ is solvable, i.e., $\text{Gal}(F'_\sigma/F)$ is solvable.

Let $L$ be a splitting field in $F'_\sigma$. (All the roots of $f$ lie in $F'_\sigma$, so a splitting field of $f$ is contained in $F'_\sigma$.) We are now in a position to show that $\text{Gal}(L/F)$ is solvable. Once again we use Theorem 5. We consider the chain of fields

$$F \subset L \subset F'_\sigma.$$ 

$L$ and $F'_\sigma$ are both splitting fields of polynomials over $F$, hence $\text{Gal}(F'_\sigma/L)$ is a normal subgroup of $\text{Gal}(F'_\sigma/F)$ and

$$\text{Gal}(F'_\sigma/F)/\text{Gal}(F'_\sigma/L) \simeq \text{Gal}(L/F).$$

The left hand side of the expression is a quotient of a solvable group, hence solvable. Therefore the Galois group $\text{Gal}(L/F)$ is solvable, as required. \qed

**Example** The Galois group of the polynomial $f(X) = -1 - 4X + X^5 \in \mathbb{Q}[X]$ is the symmetric group $S_5$, which is not solvable, hence $f$ is not solvable by radicals.

We now prove the converse of Theorem 5. We need a preliminary result.

**Lemma 2** Let $F$ be a field, $f \in F[X]$ and $E$ a splitting field of $f$. If $F^*$ is an extension of $F$, then $f \in F^*[X]$. If $E^*$ is an extension of $F^*$ and $E^*$ is a splitting field of $f$ containing $E$ and $\sigma \in \text{Gal}(E^*/F^*)$, then $\sigma|_E \in \text{Gal}(E/F)$ and the mapping

$$\phi : \text{Gal}(E^*/F^*) \longrightarrow \text{Gal}(E/F), \sigma \mapsto \sigma|_E$$

is a monomorphism.

**Proof** By hypothesis we have

$$E = F(\alpha_1, \ldots, \alpha_n) \quad \text{and} \quad E^* = F^*(\alpha_1, \ldots, \alpha_n),$$

where the $\alpha_i$ are the roots of $f$. If $\sigma \in \text{Gal}(E^*/F^*)$, then $\sigma$ permutes the roots of $f$ and fixes $F^*$, hence $F$; therefore $\sigma|_E \in \text{Gal}(E/F)$. Clearly $\phi$ is a homomorphism. If $\sigma|_E = \text{id}_E$, then $\sigma|_E(\alpha_i) = \alpha_i$, for all $i$. Thus $\sigma(\alpha_i) = \alpha_i$, for all $i$, which implies that $\sigma = \text{id}_{E^*}$. It follows that $\phi$ is injective. \qed

Now for the theorem.

**Theorem 7** Let $F$ be a field of characteristic 0 and $E$ a finite Galois extension of $F$. If the Galois group $\text{Gal}(E/F)$ is solvable, then $E$ can be embedded in a radical extension.
PROOF From Corollary 1 there is a normal subgroup \( H \) of prime index, say \( p \), in \( G \). Let \( \zeta \) be a primitive \( p \)th root of unity, which exists in some extension of \( F \), because \( F \) has characteristic \( 0 \).

We will first suppose that \( \zeta \in F \) and prove the theorem by induction on \( n = [E : F] \). If \( n = 1 \), then \( E = F \) and there is nothing to prove. (It is sufficient to take any radical extension of \( F \), for example \( F(\alpha) \), where \( \alpha \) is a root of the polynomial \( f(X) = -c + X^p \in F[X] \).)

Next we observe that \( E^H = F(\beta) \) and \( \beta \in F \), i.e., \( E^H \) is a pure extension of \( E \) and we may lengthen the chain (1) by adding the prefix \( F \subset E^H \). Thus we obtain the radical extension \( F_s/F \) (and \( E \subset F_s \)).

We now consider the general case. We set \( F^* = F(\zeta) \) and \( E^* = E(\zeta) \). As \( E \) is a normal extension of \( F \), \( E \) is the splitting field of a polynomial \( f \in F[X] \). It is not difficult to see that \( E^* \) is the splitting field of the polynomial \( g(X) = (-1 + X^p)f(X) \in F[X] \). The polynomial \( g \) belongs to \( F^*[X] \) and \( E^* \) is its splitting field. Thus \( E^* \) is a normal extension of \( F^* \), hence a Galois extension. We set \( G^* = Gal(E^*/F^*) \). From Lemma 2 there is a monomorphism from \( G^* \) into \( G \), so we may consider that \( G^* \) is a subgroup of the solvable group \( G \), hence is solvable. We may now apply the first part of the proof: there is a radical extension \( F_1^* \) of \( F^* \) containing \( E^* \): we have the radical chain

\[
F^* \subset F_1^* \subset \cdots \subset F_s^* ,
\]

with \( E^* \subset F_1^* \). As \( E \subset E^* \), we have \( F \subset F_1^* \). Since \( F^* = F(\zeta) \), \( F^* \) is a pure extension of \( F \) and we may lengthen the radical chain by adding the prefix \( F \subset F^* \), providing us with the radical extension \( F_1^*/F \) (and \( E \subset F_1^* \)). \( \square \)

**Corollary 2** If \( F \) is a field of characteristic \( 0 \) and \( f \in F[X] \) whose Galois group is solvable, then \( f \) is solvable by radicals.

**Example** The Galois group of \( f(X) = -1 + X + X^3 \in \mathbb{Q}[X] \) is the symmetric group \( S_3 \), hence \( f \) is solvable by radicals.

### Basic results from Galois theory

**Result 1** A set of distinct automorphisms \( \{ \sigma_1, \ldots, \sigma_n \} \) on a field \( F \) is indenpendant, i.e., if \( \lambda_1, \ldots, \lambda_n \in F \) and \( \lambda_1 \sigma_1(x) + \cdots + \lambda_n \sigma_n(x) = 0 \)

for all \( x \in F^* \), then \( \lambda_1 = \cdots = \lambda_n = 0 \).

**Result 2** If \( K \) is a finite extension of \( F \) and \( E \) a finite extension of \( K \), then \( E \) is a finite extension of \( F \) and

\[
[E : F] = [E : K][K : F],
\]

where \( [A : B] \) is the degree of \( A \) over \( B \).
Result 3 If $K$ is an algebraic extension of $F$, $E$ an extension of $K$ and $\alpha \in E$ is algebraic over $K$, then $\alpha$ is algebraic over $F$.

Result 4 Let $f$ be a separable polynomial in $F[X]$ of degree $n$ with Galois group $G = \text{Gal}(E/F)$. If $f$ is irreducible, then

- a. $n$ divides the order of $G$;
- b. the action of $G$ on $A$, the set of roots of $f$, is transitive.

Result 5 Let $f \in F[X]$, with $\deg f \geq 2$, and $G$ be its Galois group. If $f$ has two distinct irreducible factors, then the action of $G$ on $A$, the set of roots of $f$, is not transitive.

Result 6 If $E$ is a cyclotomic extension of $F$, then the Galois group $G(E/F)$ is abelian.

Result 7 Let $F$ and $F'$ be fields, $\sigma : F \to F'$ an isomorphism, $f \in F[X]$ and $f' \in F'[X]$ the polynomial corresponding to $f$. If $E$ is a splitting field of $F$ and $E'$ a splitting field of $f'$, then there is an isomorphism $\tilde{\sigma} : E \to E'$ extending $\sigma$.

Result 8 Suppose that $K$ is an extension of $F$ and $E$ an extension of $K$, with $E$ normal over $F$. Then $E$ is normal over $K$.

Result 9 Let $K$ and $L$ be extensions of $F$ in $E$, where $K$ is a finite Galois extension of $F$. Then

- a. $KL$ is a finite Galois extension of $L$;
- b. If $\sigma \in \text{Gal}(KL/L)$, then the restriction of $\sigma$ to $K$ belongs to $\text{Gal}(K/F)$ and the mapping

  $\phi : \text{Gal}(KL/L) \to \text{Gal}(K/F), \sigma \mapsto \sigma|_K$

  is a monomorphism;

Result 10 If $K$ and $L$ are extensions of $F$, then $[KL:L]$ divides $[K:F]$.

Result 11 Suppose that $E$ is a finite Galois extension of $F$ and $G$ the associated Galois group. Then $K$ is a normal extension of $F$ if and only if $H = \text{Gal}(E/K)$ is a normal subgroup of $G$. In this case the Galois group $\text{Gal}(K/F)$ is isomorphic to the quotient group $G/H$.

In addition, for any subgroup $H$ (not necessarily normal),

$$[K:F] = [G:H] \quad \text{and} \quad [E:K] = [H].$$

Result 12 Let $E$ be a finite extension of $F$ and $N$ the normal closure of $E$ over $F$ in an algebraic closure $C$ of $F$. Then

$$N = \prod_{\sigma \in \text{Gal}(N/F)} \sigma(E).$$

4 Appendix: Solvable groups

In this appendix we revise the basic properties of solvable groups, which may or may not be known to the reader.

A normal series or normal sequence of a group $G$ is a sequence of subgroups of the form

$$G = G_0 \triangleright G_1 \triangleright G_2 \triangleright \cdots \triangleright G_n = \{e\},$$

where

$$[G_i : G_{i+1}] = 1 \quad \text{for all} \quad i = 0, 1, \ldots, n-1.$$
where $e$ is the identity of $G$ and $G_{i+1}$ is a proper normal subgroup of $G_i$. We do not require the subgroups to be normal subgroups of $G$. The quotient groups $G_i/G_{i+1}$ are called the factors of the normal series. A group is said to be solvable if it has a normal series with abelian factors. In this case we say that the normal series is solvable. An abelian group $G$ is clearly solvable: we only need to take the sequence $G = G_0 \triangleright \{e\}$.

Let $G$ be a group. The commutator of two elements $x, y \in G$, which we note $[x, y]$, is the product $xyx^{-1}y^{-1}$. The commutator subgroup of $G$, written $G'$, is the subgroup generated by the commutators. From the observation that $g[x, y]g^{-1} = [gxg^{-1}, gyg^{-1}]$ for all $g \in G$, we deduce that $G'$ is a normal subgroup of $G$. There is no difficulty in seeing that the quotient group $G/G'$ is abelian.

**Proposition 4** If $H$ is a normal subgroup of $G$, then $G/H$ is abelian if and only if $G' \subset H$.

**Proof** If $G/H$ is abelian, then

$$xyH = xHyH = yHxH = yxH \implies [x, y] \in H.$$ 

As all the commutators lie in $H$, we have $G' \subset H$.

On the other hand, if $G' \subset H$, then by the third isomorphism theorem

$$(G/G')/(H/G') \cong G/H,$$

so $G/H$ is isomorphic to the quotient group of an abelian group and is thus abelian.

The higher commutator subgroups are defined by induction:

$$G^{(0)} = G \quad \text{and} \quad G^{(i+1)} = G'^{(i)},$$

i.e., $G^{(i+1)}$ is the commutator subgroup of $G^{(i)}$ and so is a normal subgroup of $G^{(i)}$.

**Proposition 5** A group $G$ is solvable if and only if there is a nonnegative integer $n$ such that $G^{(n)} = \{e\}$.

**Proof** If $G^{(n)} = \{e\}$, then the series

$$G = G^{(0)} \triangleright G^{(1)} \triangleright G^{(2)} \triangleright \cdots \triangleright G^{(n)} = \{e\}$$

is solvable, so $G$ is a solvable group.

If $G$ is solvable, then there exists a solvable series

$$G = G_0 \triangleright G_1 \triangleright G_2 \triangleright \cdots \triangleright G_n = \{e\}$$

We prove, by induction on $i$, that $G_i \supset G^{(i)}$, for $i = 0, \ldots, n$. First, for $i = 0$, we have $G_0 = G = G^{(0)}$. Suppose now, that $G_i \supset G^{(i)}$. Then $G'_i \supset G^{(i)} = G^{(i+1)}$. However, $G_i/G_{i+1}$ is abelian, hence, from Proposition 4, we have $G_{i+1} \supset G'_i$ and so $G_{i+1} \supset G^{(i+1)}$. This finishes the induction step. It follows that $G^{(n)} = \{e\}$.

Now we come to two fundamental results.

**Theorem 8** If $G$ is a solvable group, then every subgroup and quotient group is solvable.
Lemma 3

Theorem 10

PROOF Let \( H \) be a subgroup of the solvable group \( G \). By induction it is easy to prove that \( H^{(i)} \subset G^{(i)} \), for all \( i \). There exists a nonnegative integer \( n \) such that \( G^{(n)} = \{e\} \). It follows that \( H^{(n)} = \{e\} \) and so, by Proposition 4, \( H \) is solvable.

We now consider the case of a quotient group. We will prove a more general result: if \( G \) is a solvable group and \( \phi : G \rightarrow C \) is a surjective homomorphism, then \( C \) is solvable. To do so, we will show that \( \phi(G^{(i)}) = C^{(i)} \), for all \( i \). We will use an induction argument. For \( i = 0 \), the result is evident. As the image of a commutator is a commutator, \( \phi(G^i) \subseteq C^i \). On the other hand, if \( u, v \in C \), then there exist \( x, y \in G \) such that \( \phi(x) = u \), \( \phi(y) = v \) and we have \( \phi([x, y]) = [u, v] \); it follows that \( \phi(G^i) = C^i \). Thus the result is true for \( i = 1 \). Suppose now that \( \phi(G^{(i)}) = C^{(i)} \).

Then \( \phi \) restricted to \( G^{(i)} \) provides us with a surjective homomorphism of \( G^{(i)} \) onto \( C^{(i)} \). From the previous argument, we have \( \phi(G^{(i)}) = C^{(i)} \), i.e., \( \phi(G^{(i+1)}) = C^{(i+1)} \). This completes the induction step. To finish we notice that there is an \( n \) such that \( G^{(n)} = \{e_G\} \). This implies that \( C^{(n)} = \{e_C\} \) and so \( C \) is solvable.

This theorem has a converse.

Theorem 9

Let \( G \) be a group with a normal subgroup \( H \). If \( H \) and \( G/H \) are solvable, then so is \( G \).

PROOF As \( G/H \) is solvable, there is a solvable series composed of subgroups of \( G/H \):

\[
G/H = G_0^* \triangleright G_1^* \cdot \cdot \cdot \triangleright G_s^* = \{H/H\},
\]

where \( G_i/H = G_i^* \). By the correspondence theorem, there is a normal series

\[
G = G_0^* \triangleright G_1^* \cdot \cdot \cdot \triangleright G_s^* = H.
\]

By the third isomorphism theorem, we have \( G_i/G_{i+1} \cong (G_i/H)/(G_{i+1}/H) \), so the factors \( G_i/G_{i+1} \) are abelian. As \( H \) is solvable, we have a solvable series composed of subgroups of \( H \):

\[
H = H_0^* \triangleright H_1^* \cdot \cdot \cdot \triangleright H_t^* = \{e_H\} = \{e_G\}.
\]

Putting the two series together gives us a solvable series of \( G \).

We may refine the solvable series so as to obtain factor groups of prime order. To prove this we need a preliminary result. We recall that a group \( G \) is simple if \( G \neq \{e\} \) and its only normal subgroups are \( \{e\} \) and \( G \) itself.

Lemma 3

If \( G \) is a finite abelian simple group, then \( G \) is cyclic of prime order.

PROOF If \( x \in G \) and \( x \neq e \), then \( \langle x \rangle \) is a normal subgroup, because \( G \) is abelian. As \( x \neq e \), we have \( \langle x \rangle = G \), so \( G \) is cyclic.

If \( G \) is not of prime order and \( |G| = n \), then \( G \) has a (unique) subgroup of order \( \frac{n}{d} \), for every positive divisor \( d \) of \( n \). As \( G \) is abelian, these subgroups are normal, so we have a contradiction to the simplicity of \( G \). Thus \( G \) is of prime order.

Theorem 10

If \( G \) is a solvable group, then \( G \) has a solvable series whose factor groups are cyclic of prime order.

PROOF As \( G \) is solvable, \( G \) has a solvable series

\[
G = G_0 \triangleright \cdot \cdot \cdot \triangleright G_i \triangleright G_{i+1} \cdot \cdot \cdot \triangleright G_n = \{e\}.
\]
If $G_{i+1}$ is not a maximal normal subgroup of $G_i$, then we can insert a normal subgroup $H$ between $G_{i+1}$ and $G_i$. As the quotient group $H/G_{i+1}$ is a subgroup of $G_i/G_{i+1}$, it is abelian. Continuing the process we obtain a solvable series such that $G_{i+1}$ is always a maximal normal subgroup of $G_i$.

Now $G_i/G_{i+1}$ is finite and abelian. If this factor group is not simple, then it has a proper nontrivial normal subgroup. By the correspondence theorem there exists a nontrivial subgroup, which is normal in $G_i$ and lies strictly between $G_{i+1}$ and $G_i$; thus $G_i$ is not maximal, a contradiction. Therefore $G_i/G_{i+1}$ is finite, abelian and simple. By Lemma 4, $G_i/G_{i+1}$ is cyclic of prime order.

**Corollary 3** A solvable group $G$ has a normal subgroup of prime index.

**Proof** From Theorem 10 we may suppose that in the solvable series of $G$ the index of $G_1$ in $G_0$ is a prime number $p$. As $G_0 = G$, we have the result. \hfill $\square$

**Solvability of the symmetric groups**

The symmetric groups $S_n$ are solvable if and only if $n \leq 4$. If $n = 1$ or $n = 2$, then $S_n$ is abelian, hence solvable. For $S_3$ it is sufficient to consider the series

$$\langle e \rangle \subset A_3 \subset S_3.$$ 

For $S_4$ we may take the chain

$$\langle e \rangle \subset V \subset A_4 \subset S_4,$$

where

$$V = \{e, (12)(34), (13)(24), (14)(23)\}.$$

(The only possible difficulty here is in seeing that $V \triangleleft A_4$; however, it is sufficient to notice that conjugation preserves the cycle structure.)

For the next result we will suppose that it is known that, for $n \geq 5$, $A_n$ is simple, i.e., has no normal subgroups other than $\{e\}$ or $A_n$ itself.

**Theorem 11** For $n \geq 5$, $S_n$ is not solvable.

**Proof** For $n \geq 5$, $A_n$ is simple, therefore $A'_n = A_n$ or $A'_n = \{e\}$. However, the second case is not possible, because $A_n$ is not abelian for $n \geq 4$. Thus $A'_n = A_n$, which implies that $A_n^{(k)} = A_n$, for all $k$. It follows that $A_n$ is not solvable. As a subgroup of a solvable group is solvable, $S_n$ cannot be solvable. \hfill $\square$