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**ON A CLASS OF CLOSED COCYCLES FOR ALGEBRAS OF
NON-FORMAL, POSSIBLY UNBOUNDED,
PSEUDODIFFERENTIAL OPERATORS**

JEAN-PIERRE MAGNOT

ABSTRACT. In this article, we consider algebras \mathcal{A} of non-formal pseudodifferential operators over S^1 which contain $C^\infty(S^1)$, understood as multiplication operators. We apply a construction of Chern-Weil type forms in order to get $2k$ -closed cocycles. For $k = 1$, we obtain a cocycle on the algebra of (maybe non classical) pseudodifferential operators with the same cohomology class as the Schwinger cocycle on the algebra of Classical pseudodifferential operators, previously extended and studied by the author on algebras of the same type. We also prove non-triviality in Hochschild cohomology for $k = 2$.

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INTRODUCTION

We present here a construction of a family of cocycles on the Lie algebra of maybe unbounded, maybe non classical, non formal pseudodifferential operators $PDO(S^1, V)$. Cocycles on algebras of pseudodifferential operators have been studied from the viewpoint of algebras of formal symbols, see e.g. [11, 14] for cocycles related to our study, or algebras of non-formal but classical pseudo-differential operators, see e.g. [30]. In the last study, the notion of renormalized trace plays an important role, as well as in e.g. the works [4, 5, 25] where these cocycles are shown to be linked with anomalies in physics via differential geometric considerations.

In our works [17, 19, 20], we made more precise the link between various aspects:

- the Kravchenko-Khesin cocycle [14] on formal pseudodifferential operators over S^1
- the index cocycle on the restricted linear group GL_{res} defined in [29]
- the approach by Radul [30]
- the Schwinger cocycle [32], see e.g. [4, 25],

and we showed [20] that the Schwinger cocycle and the index cocycle could be extended to the algebra $PDO(S^1, V)$ of maybe unbounded, maybe on classical, non formal pseudodifferential operators over S^1 .

We come back to this program in the present work by adding a new idea to our investigations: integrate the classical formulas for Chern-Weil forms $\text{tr}\Omega^k$ in order to define (closed) $2k$ -cocycles on $PDO(S^1, V)$. For this task, we choose a left-invariant connection 1-form on the formal Lie group of invertible pseudodifferential operators $PDO^*(S^1, V)$ with a curvature 2-form with values on smoothing operators. This enables us to consider tr as the classical trace of trace class operators, even if technical steps of our investigations require zeta-renormalized traces along the lines

of [28, 33]. In this framework, classical computations of Chern-Weil forms apply and we show that $\text{tr}\Omega^k$ is a $2k$ -cocycle on $PDO(S^1, V)$ for $k \in \mathbb{N}^*$.

For $k = 1$, we recover via $\text{tr}\Omega$ a 2-cocycle related to the cohomology class of the Schwinger cocycle. We have to remark that we are here able to prove that these two cocycles have the same cohomology class (up to a factor) on the algebra of *classical* pseudodifferential operators, but we have no information of this kind for $PDO(S^1, V)$ (except that $\text{tr}\Omega$ is not a coboundary).

For $k = 2$, we show that $\text{tr}(\Omega^2)$ is a cocycle, with non trivial cohomology class, for a large family of Lie subalgebras of $PDO(S^1, V)$, including $PDO(S^1, V)$. To our best knowledge, this cocycle is new.

1. PRELIMINARIES

We shall start with a description of (non-formal!) pseudo-differential operator groups and algebras which we consider in this work. We shall specialize below to the trivial complex vector bundle $S^1 \times V$ in which V is a d -dimensional complex vector space. The following definition appears in [3, Section 2.1].

Definition 1.1. The graded algebra of differential operators acting on the space of smooth sections $C^\infty(S^1, V)$ is the algebra $DO(S^1, V)$ generated by:

- Elements of $C^\infty(S^1, M_d(\mathbb{C}))$
- Covariant derivation operators

$$\nabla_X : g \in C^\infty(S^1, E) \mapsto \nabla_X g$$

where ∇ is a smooth connection on E and X is a smooth vector field on S^1 .

We assign as usual the order 0 to smooth function multiplication operators. The derivation operators and vector fields have the order 1. A differential operator of order k has the form $P(u)(x) = \sum p_{i_1 \dots i_r} \nabla_{x_{i_1}} \dots \nabla_{x_{i_r}} u(x)$, $r \leq k$, In local coordinates (the coefficients $p_{i_1 \dots i_r}$ can be matrix-valued). We denote by $DO^k(S^1), k \geq 0$, the differential operators of order less or equal than k . The algebra $DO(E)$ is filtered by the order. It is a subalgebra of the algebra of classical pseudo-differential operators $Cl(S^1, V)$ that we describe shortly hereafter, focusing on its necessary aspects. This is an algebra that contains, for example, the square root of the Laplacian

$$(1.1) \quad |D| = \Delta^{1/2} = \int_{\Gamma} \lambda^{1/2} (\Delta - \lambda Id)^{-1} d\lambda,$$

where $\Delta = -\frac{d^2}{dx^2}$ is the positive Laplacian and Γ is a contour around the spectrum of the Laplacian, see e.g. [31, 28] for an exposition on contour integrals of pseudodifferential operators. $Cl(S^1, V)$ contains also the inverse of $Id + \Delta$, and all smoothing operators on $L^2(S^1, V)$. Pseudodifferential operators (maybe non-classical) are linear operators acting on $C^\infty(S^1, V)$ which reads locally as

$$A(f) = \int e^{ix \cdot \xi} \sigma(x, \xi) \hat{f}(\xi) d\xi$$

where $\sigma \in C^\infty(T^*S^1, M_n(\mathbb{C}))$ satisfying additional estimates on its partial derivatives and \hat{f} means the Fourier transform of f . Basic facts on pseudo-differential operators defined on a vector bundle $E \rightarrow S^1$ can be found e.g. in [8]. We note by $PDO(S^1, V)$ the space of maybe non classical, maybe unbounded, pseudodifferential operators acting on $C^\infty(S^1, V)$. A pseudo-differential operator of order o is called

classical if and only if its symbols σ have an asymptotic expansion $\sigma(x, \xi) \sim_{|\xi| \rightarrow +\infty} \sum_{j=-\infty}^o \sigma_j(x, \xi)$, where the maps $\sigma_j : S^1 \times \mathbb{R}^* \rightarrow \mathbb{C}$, called **partial symbols**, are j -positively homogeneous, i.e. $\forall t > 0, (x, \xi) \in S^1 \times \mathbb{R}^*, (\sigma)_j(x, t\xi) = t^j(\sigma)_j(x, \xi)$.

Pseudodifferential operators can be also described by their kernel

$$K(x, y) = \int_{\mathbb{R}} e^{i(x-y)\xi} \sigma(x, \xi) d\xi$$

which is off-diagonal smooth. Pseudodifferential operators with infinitely smooth kernel (or "smoothing" operators), i.e. that are maps: $L^2 \rightarrow C^\infty$ form a two-sided ideal that we note by $Cl^{-\infty}(S^1, V)$. Their symbols are those which are in the Schwartz space $\mathcal{S}(T^*S^1, M_n(\mathbb{C}))$. The quotient $\mathcal{FCl}(S^1, V) = Cl(S^1, V)/Cl^{-\infty}(S^1, V)$ of the algebra of pseudo-differential operators by $Cl^{-\infty}(S^1, V)$ forms the algebra of formal pseudo-differential operators.

Remark 1.2. Through identification of $\mathcal{FCl}(S^1, V)$ with the corresponding space of formal symbols, the space $\mathcal{FCl}(S^1, V)$ is equipped with the natural locally convex topology inherited from the space of formal symbols. A formal symbol σ_k is a smooth function in $C^\infty(T^*S^1 \setminus S^1, M_n(\mathbb{C}))$ which is k -homogeneous (for $k > 0$), and hence with an element of $C^\infty(S^1, M_n(\mathbb{C}))^2$ evaluating σ_k at $\xi = 1$ and $\xi = -1$. Identify $Cl^d(S^1, V)$ with

$$\prod_{k \leq d} C^\infty(S^1, M_n(\mathbb{C}))^2,$$

the vector space $Cl^d(S^1, V)$ is a Fréchet space, and hence

$$Cl(S^1, V) = \cup_{d \in \mathbb{Z}} Cl^d(S^1, V)$$

is a locally convex topological algebra.

We have to precise that the classical topology on non-formal classical pseudo-differential operators $Cl(S^1, V)$ is finer than the one obtained by pull-back from $\mathcal{FCl}(S^1, V)$. A "useful" topology on $Cl(S^1, V)$ needs to ensure that partial symbols **and** off-diagonal smooth kernels converge. The topology on spaces of classical pseudo differential operators has been described by Kontsevich and Vishik in [12]; see also [4, 28, 33] for descriptions. This is a Fréchet topology on each space $Cl^d(S^1, E)$. However, passing to the quotients $\mathcal{FCl}^d(S^1, E) = Cl^d(S^1, E)/Cl^{-\infty}(S^1, E)$, the push-forward topology coincides with the topology of $\mathcal{FCl}^d(S^1, V)$ described at the beginning of this remark.

Finally, we equip the space $PDO(S^1, V)$ with a topology derived:

- from the pull-back of the evaluation maps $PDO(S^1, V) \times C^\infty(S^1, V) \rightarrow C^\infty(S^1, V)$
- from the pull-back of the topology on symbols via the symbol maps [33]

1.0.1. *The splitting with induced by the connected components of $T^*S^1 \setminus S^1$.* In this section, we define two ideals of the algebra $\mathcal{FCl}(S^1, V)$, that we call $\mathcal{FCl}_+(S^1, V)$ and $\mathcal{FCl}_-(S^1, V)$, such that $\mathcal{FCl}(S^1, V) = \mathcal{FCl}_+(S^1, V) \oplus \mathcal{FCl}_-(S^1, V)$. This decomposition is explicit in [10, section 4.4., p. 216], and we give an explicit description here following [17, 19].

Definition 1.3. Let σ be a partial symbol of order o on E . Then, we define, for $(x, \xi) \in T^*S^1 \setminus S^1$,

$$\sigma_+(x, \xi) = \begin{cases} \sigma(x, \xi) & \text{if } \xi > 0 \\ 0 & \text{if } \xi < 0 \end{cases} \quad \text{and} \quad \sigma_-(x, \xi) = \begin{cases} 0 & \text{if } \xi > 0 \\ \sigma(x, \xi) & \text{if } \xi < 0 \end{cases}$$

We define $p_+(\sigma) = \sigma_+$ and $p_-(\sigma) = \sigma_-$.

The maps $p_+ : \mathcal{FCl}(S^1, V) \rightarrow \mathcal{FCl}(S^1, V)$ and $p_- : \mathcal{FCl}(S^1, V) \rightarrow \mathcal{FCl}(S^1, V)$ are clearly smooth algebra morphisms (yet non-unital morphisms) that leave the order invariant and are also projections (since multiplication on formal symbols is expressed in terms of pointwise multiplication of tensors).

Definition 1.4. We define $\mathcal{FCl}_+(S^1, V) = \text{Im}(p_+) = \text{Ker}(p_-)$ and $\mathcal{FCl}_-(S^1, V) = \text{Im}(p_-) = \text{Ker}(p_+)$.

Since p_+ is a projection, we have the splitting

$$\mathcal{FCl}(S^1, V) = \mathcal{FCl}_+(S^1, V) \oplus \mathcal{FCl}_-(S^1, V).$$

Let us give another characterization of p_+ and p_- . The operator $D = -i \frac{d}{dx}$ splits $C^\infty(S^1, \mathbb{C}^n)$ into three spaces :

- its kernel E_0 , built of constant maps
- E_+ , the vector space spanned by eigenvectors related to positive eigenvalues
- E_- , the vector space spanned by eigenvectors related to negative eigenvalues.

The L^2 -orthogonal projection on E_0 is a smoothing operator, which has null formal symbol. By the way, concentrating our attention on the formal symbol of operators, we can ignore this projection and hence we work on $E_+ \oplus E_-$. When dealing with non-formal operators, we shall set $p_+ = p_{E_+}$. The following elementary result will be useful for the sequel.

Lemma 1.5. [17, 19]

Let p_{E_+} (resp. p_{E_-}) be the projection on E_+ (resp. E_-), then $\sigma(p_{E_+}) = \frac{1}{2}(Id + \frac{\xi}{|\xi|})$ and $\sigma(p_{E_-}) = \frac{1}{2}(Id - \frac{\xi}{|\xi|})$.

From this, we have the following result.

Proposition 1.6. [17, 19] Let $A \in \mathcal{FCl}(S^1, V)$. $p_+(A) = \sigma(p_{E_+}) \circ A = A \circ \sigma(p_{E_+})$ and $p_-(A) = \sigma(p_{E_-}) \circ A = A \circ \sigma(p_{E_-})$.

Notation. For shorter notations, we note by $A_\pm = p_\pm(A)$ the formal operators defined from another viewpoint by

$$\sigma(A_+)(x, \xi) \quad (\text{resp. } \sigma(A_-)(x, \xi)) = \begin{cases} \sigma(A)(x, \xi) & \text{if } \xi > 0 \quad (\text{resp. } \xi < 0) \\ 0 & \text{if } \xi < 0 \quad (\text{resp. } \xi > 0) \end{cases}$$

1.1. Renormalized traces of classical pseudodifferential operator. $S^1 \times V$ is equipped with an Hermitian products $\langle \cdot, \cdot \rangle$, which induces the following L^2 -inner product on $C^\infty(S^1, V)$:

$$\forall u, v \in C^\infty(S^1, V), \quad (u, v)_{L^2} = \int_{S^1} \langle u(x), v(x) \rangle dx,$$

where dx is the Riemannian volume.

Definition 1.7. Q is a **weight** of order $s > 0$ on E if and only if Q is a classical, elliptic, self-adjoint, positive pseudo-differential operator acting on smooth sections of E .

Recall that, under these assumptions, the weight Q has a real discrete spectrum, and that all its eigenspaces are finite dimensional. For such a weight Q of order q , one can define the complex powers of Q [31], see e.g. [4] for a fast overview of

technicalities. The powers Q^{-s} of the weight Q are defined for $Re(s) > 0$ using with a contour integral,

$$Q^{-s} = \int_{\Gamma} \lambda^s (Q - \lambda Id)^{-1} d\lambda,$$

where Γ is a contour around the real positive axis. Let A be a log-polyhomogeneous pseudo-differential operator. The map $\zeta(A, Q, s) = s \in \mathbb{C} \mapsto \text{tr}(AQ^{-s}) \in \mathbb{C}$, defined for $Re(s)$ large, extends on \mathbb{C} to a meromorphic function [16]. When A is classical, $\zeta(A, Q, \cdot)$ has a simple pole at 0 with residue $\frac{1}{q} \text{res}_W A$, where res_W is the Wodzicki residue ([34], see also [10]). Notice that the Wodzicki residue extends the Adler trace [2] on formal symbols. Following textbooks [28, 33] for the renormalized trace of classical operators, we define

Definition 1.8. Let A be a log-polyhomogeneous pseudo-differential operator. The finite part of $\zeta(A, Q, s)$ at $s = 0$ is called the renormalized trace $\text{tr}^Q A$. If A is a classical pseudo-differential operator,

$$\text{tr}^Q A = \lim_{s \rightarrow 0} (\text{tr}(AQ^{-s}) - \frac{1}{qs} \text{res}_W(A)).$$

If A is trace class acting on $L^2(S^1, \mathbb{C}^k)$, $\text{tr}^Q(A) = \text{tr}(A)$. The functional tr^Q is of course not a trace. In this formula, it appears that the Wodzicki residue $\text{res}_W(A)$.

Proposition 1.9.

- (i) *The Wodzicki residue res_W is a trace on the algebra of classical pseudo-differential operators $Cl(S^1, E)$, i.e. $\forall A, B \in Cl(S^1, V)$, $\text{res}_W[A, B] = 0$.*
- (ii) *(local formula for the Wodzicki residue) Moreover, if $A \in Cl(S^1, V)$,*

$$\text{res}_W A = \frac{1}{2\pi} \int_{S^1} \int_{|\xi|=1} \text{tr} \sigma_{-1}(x, \xi) d\xi dx = \frac{1}{2\pi} \sum_{\xi=\pm 1} \int_{S^1} \text{tr} \sigma_{-1}(x, \xi) d\xi dx.$$

In particular, res_W does not depend on the choice of Q .

Since tr^Q is a linear extension of the classical trace tr of trace-class operators acting on $L^2(S^1 V)$, it has weaker properties. Let us summarize some of them which are of interest for our work following first [4], completed by [22] for the third point.

Proposition 1.10. • *Given two (classical) pseudo-differential operators A and B , given a weight Q ,*

$$(1.2) \quad \text{tr}^Q[A, B] = -\frac{1}{q} \text{res}(A[B, \log Q]).$$

- *Under the previous notations, if C is a classical elliptic injective operator or a diffeomorphism, $\text{tr}^{C^{-1}QC}(C^{-1}AC)$ is well-defined and equals $\text{tr}^Q A$.*

Since tr^Q is not tracial, let us give one more property on the renormalized trace of the bracket, from e.g. [24].

Proposition 1.11.

$$\forall (A, B) \in Cl^{-\infty}(S^1, V) \times Cl(S^1, V), \quad \text{tr}^Q[A, B] = 0.$$

1.2. Groups of invertible PDOs. Let us consider the infinite dimensional group $PDO^*(S^1, V) \subset PDO(S^1, V)$ of operators $A \in PDO(S^1, V)$ with an inverse in $PDO(S^1, V)$. Let us consider here the space $PDO(S^1, V) \times PDO^*(S^1, V)$ as an heuristic analog to the full tangent space of $PDO^*(S^1, V)$. Concerning $Cl(S^1, V)$, one can define the same way $Cl^*(S^1, V)$ and $Cl(S^1, V) \times Cl^*(S^1, V)$. Indeed, for a classical (maybe infinite dimensional) Lie group G with Lie algebra \mathfrak{g} , we have $TG \sim \mathfrak{g} \times G$, when G acts on the left on \mathfrak{g} by Adjoint action.

We mention here only an heuristic correspondence because of the following facts:

- The groups $PDO^*(S^1, V)$ and $Cl^*(S^1, V)$ carry no atlas, as is was first mentioned in [1], see e.g. [24]. Then, in order to deal rigorously with them, one has to consider Gelfand's formal geometry or, from another viewpoint, diffeological Lie groups.
- As mentioned in [24] there is no exponential map from paths on $Cl(S^1, V)$ to paths on $Cl^*(S^1, V)$. The same obstruction holds obviously with $PDO(S^1, V)$ and $PDO^*(S^1, V)$. This means that the "tangent space at identity" of $PDO^*(S^1, V)$ should not be defined as a domain for an exponential maps, but as a generalized version of a kinetic tangent space along the lines of [15] (generalized because we are not here in the c^∞ setting).

Hence, in the sequel, when dealing with left invariant objects $PDO(S^1, V) \times PDO^*(S^1, V)$ for the action of $PDO^*(S^1, V)$, we shall circumvent the technical difficulties by restricting our investigations to $PDO(S^1, V)$, ignoring which one of the two already mentionned frameworks have been chosen.

2. A FAMILY OF COCYCLES

Definition 2.1. We define on $PDO(S^1, V) \times PDO^*(S^1, V)$, understood as a trivial vector bundle, the left-invariant connection 1-form

$$\theta : a \in PDO(S^1, V) \mapsto \theta(a) = a \circ p_+.$$

We remark that, concerning formal symbols,

$$\sigma(\theta(a)) = \sigma(a)_+$$

which implies straightway the following property:

Proposition 2.2. *Let Ω the curvature of θ . Then Ω is a $PDO^{-\infty}(S^1, V)$ -valued 2-form.*

This is the main property to get the following theorem

Theorem 2.3. *The Chern-Weil like forms*

$$\text{tr}(\Omega^k)$$

define closed $2k$ -cocycles in Hochschild cohomology of $PDO(S^1, V)$.

Proof. The forms Ω^k , and hence $\text{tr}(\Omega^k)$, are left-invariant $2k$ -forms in de Rham cohomology. Adapting slightly the arguments of [18, Theorem 1], we remark that

$$d\text{tr}\Omega^k = \text{tr}\nabla\Omega^k = 0$$

since Ω takes values in the two-sided ideal $Cl^{-\infty}(S^1, V)$. Restricting $\text{tr}(\Omega^k)$ to left-invariant vector fields over $PDO^*(S^1, V)$, $\text{tr}\Omega^k$ is now a $2k$ -form on $PDO(S^1, V)$ and the de Rham differential coincides with the differential in Hochschild cohomology. Thus, $\text{tr}(\Omega^k)$ is closed in Hochschild cohomology. \square

Remark 2.4. One may wonder whether an Ambrose-Singer type theorem would apply here, in order to reduce the connection θ to a principal subbundle of frames over $PDO^*(S^1, V)$ with structure group a group of Fredholm type invertible operators with Lie algebra in $PDO(S^1, V)$. In this case, [18, Theorem 1] would apply and would show directly that $\text{tr}(\Omega^k)$ is closed. To our best knowledge, there only exists an infinite dimensional version of an Ambrose-Singer theorem when the structure group before reduction is regular (in the sense of the existence of an exponential map, see [15, 21, 26, 27] for definitions in various settings). In our case, this does not seem to be the case since [23] suggests that even $PDO^*(S^1, V)$ may not be regular. Some related works are actually in progress to clarify these aspects.

This result remains valid for any Lie subalgebra $\mathcal{A} \subset PDO(S^1, V)$. Let us now give a key elementary lemma about non-exactness, already applied in [20]:

Lemma 2.5. *Let \mathcal{A} be a Lie subalgebra of $PDO(S^1, V)$. Let c be a cocycle on \mathcal{A} . Let \mathcal{B} be a commutative Lie subalgebra of \mathcal{A} , i.e. $[\mathcal{B}, \mathcal{B}] = \{0\}$. If c is non vanishing on \mathcal{B} , then c is not exact.*

3. CASE STUDIES

3.1. $\text{tr}\Omega$ is cohomologous to the Schwinger cocycle on $Cl(S^1, V)$.

Theorem 3.1. *On $Cl(S^1, V)$, $\text{tr}\Omega$ has the same cohomology class as $\frac{1}{2}c_s$, where c_s is the Schwinger cocycle. By the way it has non-trivial Hochschild cohomology class on $PDO(S^1, V)$.*

Proof. First, let $(X, Y) \in Cl(S^1, V)^2$. We have that

$$\text{tr}\Omega(X, Y) = \text{tr}^\Delta [\theta_X, \theta_Y] - \text{tr}^\Delta \theta_{[X, Y]}.$$

The term $\text{tr}^\Delta \theta_{[X, Y]}$ is a coboundary. Let us calculate $\text{tr}^\Delta [\theta_X, \theta_Y]$. For this, we remark that $\sigma(\theta_X) = \sigma_+(X)$ thus

$$\text{tr}^\Delta [\theta_X, \theta_Y] = -\frac{i}{2\pi} \text{res} \sigma_+(X) [\sigma_+(Y), \log \Delta].$$

The last term thus can be identified with the pull-back of the Kravchenko-Khesin-Radul cocycle on $Cl(S^1, V)$ so that it has the same cohomology class as the Schwinger cocycle following [19].

If $\text{tr}\Omega$ was a coboundary on $PDO(S^1, V)$, it would be also a coboundary on $Cl(S^1, V)$. So that, $\text{tr}\Omega$ has non-trivial Hochschild cohomology class on $PDO(S^1, V)$. \square

3.2. $\text{tr}(\Omega^2)$ is a 4-cocycle with non trivial cohomology class.

Theorem 3.2. *Let \mathcal{A} be a Lie algebra such that*

$$C^\infty(S^1, \mathbb{C}) \otimes Id_V \subset \mathcal{A} \subset PDO(S^1, V).$$

Then $\text{tr}(\Omega^2)$ is a 4-cocycle with non-trivial cohomology class.

Proof. Let $(z_n)_{n \in \mathbb{Z}}$ be the Fourier L^2 -orthonormal base of $L^2(S^1, \mathbb{C})$. In this proof, traces will be calculated in this preferred base.

$$\forall (m, n) \in \mathbb{Z}^2, [z^m, z^n] = 0 \quad \Leftrightarrow \quad \Omega(z^m, z^n) = [\theta_{z^m}, \theta_{z^n}].$$

Let $(k, m, n) \in \mathbb{Z}^3$. We compute:

$$\begin{aligned} \Omega(z^m, z^n)z^k &= [\theta_{z^m}, \theta_{z^n}]z^k \\ &= (z^m p_+ z^n - z^n p_+ z^m) p_+ z^k \\ &= \begin{cases} 0 & \text{if } k \leq 0 \text{ or } ((m+k > 0) \text{ and } (n+k > 0)) \\ z^{k+m+n} & \text{if } k > 0 \text{ and } (m+k \leq 0) \text{ and } (n+k > 0) \\ -z^{k+m+n} & \text{if } k > 0 \text{ and } (m+k > 0) \text{ and } (n+k \leq 0) \end{cases} \end{aligned}$$

Let $(k, m, n, p, q) \in \mathbb{Z}^5$. The previous calculations are preliminary to compute now:

$$\begin{aligned} \Omega(z^m, z^n)\Omega(z^p, z^q)z^k &= \begin{cases} 0 & \text{if } k \leq 0 \text{ or } ((m+k > 0) \text{ and } (n+k > 0)) \\ \Omega(z^m, z^n)z^{k+p+q} & \text{if } k > 0 \text{ and } (p+k \leq 0) \text{ and } (q+k > 0) \\ -\Omega(z^m, z^n)z^{k+m+n} & \text{if } k > 0 \text{ and } (p+k > 0) \text{ and } (q+k \leq 0) \end{cases} \\ &= \begin{cases} 0 & \text{if } \mathcal{A}_0 \\ z^{k+m+n+p+q} & \text{if } \mathcal{A}_1 \\ -z^{k+m+n+p+q} & \text{if } \mathcal{A}_{-1} \end{cases} \end{aligned}$$

where the first condition in the list reads as

$$\mathcal{A}_0 = k \leq 0 \text{ or } ((p+k > 0) \text{ and } (q+k > 0))$$

$$\text{or } k+p+q < 0 \text{ or } ((m+k+p+q > 0) \text{ and } (n+k+p+q > 0)),$$

the second one reads as:

$$\mathcal{A}_1 = k > 0 \text{ and } k+p+q > 0 \text{ and}$$

$$((p+k \leq 0 \text{ and } q+k > 0 \text{ and } k+m+p+q \leq 0 \text{ and } k+n+p+q > 0) \text{ or}$$

$$(p+k > 0 \text{ and } q+k \leq 0 \text{ and } k+m+p+q > 0 \text{ and } k+n+p+q \leq 0))$$

and the third one reads as:

$$\mathcal{A}_{-1} = k > 0 \text{ and } k+p+q > 0 \text{ and}$$

$$((p+k \leq 0 \text{ and } q+k > 0 \text{ and } k+m+p+q > 0 \text{ and } k+n+p+q \leq 0) \text{ or}$$

$$(p+k > 0 \text{ and } q+k \leq 0 \text{ and } k+m+p+q \leq 0 \text{ and } k+n+p+q > 0))$$

We now turn to traces. We remark from explicit formulas that the operator $\Omega(z^m, z^n)\Omega(z^p, z^q)$, for fixed $(m, n, p, q) \in \mathbb{Z}^4$, is not only smoothing, it is moreover with finite rank. Recall that $d = \dim V$. We note by n_1 the number of integers k satisfying \mathcal{A}_1 , and by n_{-1} the number of integers k satisfying \mathcal{A}_{-1} .

$$\begin{aligned} \text{tr}(\Omega(z^m, z^n)\Omega(z^p, z^q)) &= d \sum_{k \in \mathbb{Z}} (\Omega(z^m, z^n)\Omega(z^p, z^q)z^k, z^k)_{L^2} \\ &= \begin{cases} 0 & \text{if } n+m+p+q \neq 0 \\ d(n_1 - n_{-1}) & \text{if } n+m+p+q = 0 \end{cases} \end{aligned}$$

Then, in order to calculate $\text{tr}(\Omega^2)(z^m, z^n, z^p, z^q)$, we need to skew-symmetrize with respect to the 4 variables the results obtained. Let us consider $(m, n, p, q) = (-2, 2, -3, 3)$ and let us calculate $\text{tr}(\Omega(z^{s(m)}, z^{s(n)})\Omega(z^{s(p)}, z^{s(q)}))$ for $s \in \mathfrak{G}_4$.

Then, computing the corresponding numbers $n_1(s)$ and $n_{-1}(s)$, we get the following properties:

- $\forall s \in \mathfrak{G}_4, (n_1(s), n_{-1}(s)) \subset \{0; 2\}^2$.
- if $\epsilon(s) = 1$, then $n_{-1}(s) = 0$
- if $\epsilon(s) = -1$, then $n_1(s) = 0$
- $\text{tr}(\Omega(z^{-2}, z^2)\Omega(z^{-3}, z^3)) = 2d > 0$.

By the way

$$\mathrm{tr}(\Omega^2)(z^{-2}, z^2, z^{-3}, z^3) = \frac{1}{24} \sum_{s \in \mathfrak{S}_4} \epsilon(s) \mathrm{tr} \left(\Omega(z^{s(m)}, z^{s(n)}) \Omega(z^{s(p)}, z^{s(q)}) \right)$$

is a sum of non-negative terms, with at least one positive one. Thus

$$\mathrm{tr}(\Omega^2)(z^{-2}, z^2, z^{-3}, z^3) \neq 0.$$

Since $\mathrm{tr}(\Omega^2)$ is ono-vanishing on $C^\infty(S^1, \mathbb{C}) \otimes Id_V$ applying Lemma 2.5 we get that for any Lie algebra \mathcal{A} such that

$$C^\infty(S^1, \mathbb{C}) \otimes Id_V \subset \mathcal{A} \subset PDO(S^1, V),$$

$\mathrm{tr}(\Omega^2)$ is a cocycle with non-vanishing cohomology class. \square

4. CONCLUSION

The family of cocycles on $PDO(S^1, V)$ that we produced show that there can exist some non-trivial $2k$ -cocycles on $PDO(S^1, V)$. Beside the 2 -cocycle $\mathrm{tr}\Omega$ which is cohomologous to the well-known Schwinger cocycle on $Cl(S^1, V)$ we produced a non-trivial 4 -cocycle new to our knowledge. Algebraically, the full description of the Hochschild cohomology of various Lie subalgebras of $PDO(S^1, V)$ (especially those with some unbounded operators) needs to be investigated. From a geometric viewpoint, the meaning of the higher Chern-Weil forms that we describe here, intrinsically linked with the sign of the Dirac operator, carry interpretations that can be only heuristic since the classical differential geometry (with atlases) fail to apply.

We have here pointed out some open questions that may be investigated in the future from various viewpoints. Our hope is to see these open problems solved in a very next future.

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