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| Paul-Emile Paradan. Moment polytopes in real symplectic geometry I. 2020. hal-03065807

HAL Id: hal-03065807

<https://hal.science/hal-03065807>

Preprint submitted on 15 Dec 2020

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Moment polytopes in real symplectic geometry I

Paul-Emile Paradan*

December 14, 2020

Abstract

Let Z be the real part of a Kähler Hamiltonian manifold M . The O’Shea-Sjamaar’s Theorem tells us that the moment polytope $\Delta_{\mathfrak{p}}(Z)$ corresponds to the anti-invariant part of the Kirwan polytope $\Delta_{\mathfrak{u}}(M)$. The purpose of the present paper is to explain how to parameterize the equations of the facets of $\Delta_{\mathfrak{p}}(Z)$ in terms of *real Ressayre’s pairs* of Z .

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1 Introduction

Let $U_{\mathbb{C}}$ be a connected complex reductive group equipped with a complex-conjugate involution σ . Let U be the maximal compact subgroup of $U_{\mathbb{C}}$ that is associated with a Cartan involution commuting to σ : so U is invariant under the involution and we denote by K the connected component of the fixed point subgroup U^{σ} . At the level of Lie algebras we have a decomposition $\mathfrak{u} = \mathfrak{u}^{\sigma} \oplus \mathfrak{u}^{-\sigma}$ where $\mathfrak{u}^{\pm\sigma} = \{X \in \mathfrak{u}, \sigma(X) = \pm X\}$.

Let G be the connected component of the closed subgroup of $U_{\mathbb{C}}$ fixed by σ : it is a real reductive subgroup that is stable under the Cartan involution. Hence we have the isomorphism $K \times \mathfrak{p} \rightarrow G$, $(k, X) \mapsto ke^X$ where $\mathfrak{p} = i\mathfrak{u}^{-\sigma}$.

We consider now a proper Kähler Hamiltonian U -manifold (M, Ω, \mathbb{J}) . By that we mean :

- $U_{\mathbb{C}}$ acts holomorphically on the complex manifold (M, \mathbb{J}) ,
- the Kähler form Ω is U -invariant,
- there is a U -equivariant *proper* moment mapping $\Phi_{\mathfrak{u}} : M \rightarrow \mathfrak{u}^*$ satisfying the relation

$$(1) \quad d\langle \Phi_{\mathfrak{u}}, X \rangle = -\Omega(X_M, -), \quad \forall X \in \mathfrak{u}.$$

Here $X_M(m) = -X \cdot m = \frac{d}{dt}|_{t=0} e^{-tX} m$ is the vector field generated by $X \in \mathfrak{u}$.

In this paper we will give a particular attention to the K -equivariant map

$$\Phi_{\mathfrak{p}} : M \rightarrow \mathfrak{p}^*$$

defined by the relations $\langle \Phi_{\mathfrak{p}}, \beta \rangle = \langle \Phi_{\mathfrak{u}}, i\beta \rangle$, $\forall \beta \in \mathfrak{p}$. If we denote by $j : \mathfrak{p} \rightarrow \mathfrak{u}^{-\sigma}$ the isomorphism $X \mapsto iX$, we have $\Phi_{\mathfrak{p}} = j^* \circ \pi \circ \Phi_{\mathfrak{u}}$, where $j^* : (\mathfrak{u}^{-\sigma})^* \rightarrow \mathfrak{p}^*$ is the dual map and $\pi : \mathfrak{u}^* \rightarrow (\mathfrak{u}^{-\sigma})^*$ is the canonical projection. Thanks to (1), we see that β_M is the gradient vector field of the function $\langle \Phi_{\mathfrak{p}}, \beta \rangle$ for any $\beta \in \mathfrak{p}$.

We suppose now that the Kähler Hamiltonian U -manifold (M, Ω, \mathbb{J}) possesses a *real structure*, that is an anti-holomorphic involution $\tau : M \rightarrow M$ satisfying the following conditions : $\tau^*(\Omega) = -\Omega$ and

- $\tau(g \cdot m) = \sigma(g) \cdot \tau(m)$,
- $\Phi_{\mathfrak{u}}(\tau(m)) = -\sigma(\Phi_{\mathfrak{u}}(m))$,

for any $(g, m) \in U^{\mathbb{C}} \times M$.

Consider the submanifold $Z = \{m \in M, \tau(m) = m\}$: if Z is *non-empty*, it is a Lagrangian submanifold of (M, Ω) , not necessarily connected, stable under the action of the real reductive group G . In this context, we have the following important result [17].

Theorem 1.1 (O’Shea-Sjamaar)

$$\Phi_{\mathfrak{u}}(M) \cap (\mathfrak{u}^{-\sigma})^* = \Phi_{\mathfrak{u}}(Z) \underset{j^*}{\simeq} \Phi_{\mathfrak{p}}(Z).$$

Let us rephrase this theorem in terms of moment polytopes. Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subspace, and choose a σ -invariant maximal torus T of U with Lie algebra \mathfrak{t} containing $i\mathfrak{a}$. We can define Weyl chambers¹ $\mathfrak{t}_+^* \subset \mathfrak{t}^*$ and $\mathfrak{a}_+^* \subset \mathfrak{a}^*$ such that $\mathfrak{t}_+^* \cap (\mathfrak{u}^{-\sigma})^* \simeq_{j^*} \mathfrak{a}_+^*$ (see the Appendix in [17]).

We consider the moment polytopes

$$\Delta_{\mathfrak{u}}(M) = \Phi_{\mathfrak{u}}(M) \cap \mathfrak{t}_+^* \quad \text{and} \quad \Delta_{\mathfrak{p}}(Z) = \Phi_{\mathfrak{p}}(Z) \cap \mathfrak{a}_+^*.$$

The O’Shea-Sjamaar’s theorem can be rephrased by saying that the intersection $\Delta_{\mathfrak{u}}(M) \cap (\mathfrak{u}^{-\sigma})^*$ coincides with $\Delta_{\mathfrak{p}}(Z)$ through the linear isomorphism j^* .

The Convexity Theorem [1, 5, 10, 21] tells us that $\Delta_{\mathfrak{u}}(M)$ is a closed convex locally polyhedral subset. Thanks to Theorem 1.1, we see that $\Delta_{\mathfrak{p}}(Z)$ is also a closed convex locally polyhedral subset. In a previous article [18], we explained how to parameterize the facets of $\Delta_{\mathfrak{u}}(M)$ in terms of *Ressayre’s pairs*, a notion that N. Ressayre introduces in the algebraic setting [22].

The goal of this article is to show that we can adapt this notion in the real setting in order to parameterize the facets of $\Delta_{\mathfrak{p}}(Z)$. A future article [19] will be devoted to the study of examples.

Since the founding article of R. Richardson and P. J. Slodowy [20] where they showed that the Kempf-Ness theorem extends to representations of real reductive groups, many authors have studied extensions of geometric invariant theory to the real framework [6, 8, 15, 4, 3].

Let us cite the work of J. Lauret [12] and Heinzner-Schwarz-Stötzel [7] where they obtain a stratification theorem à la Kirwan-Ness for real reductive group actions on real manifolds. In the present work, the existence of real stratifications will be the main tool to parameterize the facets of the moment polytope $\Delta_{\mathfrak{p}}(Z)$. As we are working in a less general framework than that of [7], we propose another proof of their result based on the Kirwan-Ness stratification of the ambient complex manifold (see §3.4). This method will allow us to show that these real stratifications admit an open and dense stratum. This key point provides a simple proof of O’Shea-Sjamaar’s theorem in the Kähler setting (see §3.5).

¹ $\mathfrak{u}^*/U \simeq \mathfrak{t}_+^*$ and $\mathfrak{p}^*/K \simeq \mathfrak{a}_+^*$

2 Statement of the main result

For the rest of this section, we suppose that $Z \neq \emptyset$ and we fix a connected component \mathcal{Z} of Z : it is a Riemannian manifold equipped with an action of the groups K and G .

2.1 A refinement of O'Shea-Sjamaar's Theorem

The following result is proved in §3.5.

Theorem 2.1 *The following relations hold :*

- $\Delta_{\mathfrak{u}}(M) \cap (\mathfrak{u}^{-\sigma})^* \underset{j^*}{\simeq} \Delta_{\mathfrak{p}}(Z),$
- $\Delta_{\mathfrak{p}}(\mathcal{Z}) = \Delta_{\mathfrak{p}}(Z).$

The equality $\Delta_{\mathfrak{p}}(\mathcal{Z}) = \Delta_{\mathfrak{p}}(Z)$ was already obtained by L. O'Shea and R. Sjamaar when M is a polarized algebraic variety (see Corollary 5.11 in [17]).

2.2 Admissible elements

We start by introducing the notion of *admissible elements*. The group $\text{Hom}(U(1), T)$ admits a natural identification with the lattice $\Lambda := \frac{1}{2\pi} \ker(\exp : \mathfrak{t} \rightarrow T)$ of the vector space \mathfrak{t} . A vector $\gamma \in \mathfrak{a}$ is called *rational* if $i\gamma$ belongs to the \mathbb{Q} -vector space $\mathfrak{t}_{\mathbb{Q}}$ generated by Λ .

The stabilizer subgroups of $m \in M$ relatively to the K and G actions are denoted respectively by K_m and G_m : their Lie algebras are denoted by \mathfrak{k}_m and \mathfrak{g}_m . We will take a particular attention to the subspace $\mathfrak{p}_m = \{X \in \mathfrak{p}, X \cdot m = 0\} \subset \mathfrak{g}_m$. If $\gamma \in \mathfrak{a}$, we denote by \mathcal{Z}^γ the submanifold where the vector field $z \mapsto \gamma \cdot z$ vanishes.

Definition 2.2 *Let us define*

$$\dim_{\mathfrak{p}}(\mathcal{X}) := \min_{z \in \mathcal{X}} \dim(\mathfrak{p}_z)$$

for any subset $\mathcal{X} \subset \mathcal{Z}$. A non-zero element $\gamma \in \mathfrak{a}$ is called *admissible* if γ is rational, and if $\dim_{\mathfrak{p}}(\mathcal{Z}^\gamma) - \dim_{\mathfrak{p}}(\mathcal{Z}) \in \{0, 1\}$.

2.3 Real Ressayre's pair

The aim of this section is to introduce the notion of real Ressayre's pair on the G -manifold \mathcal{Z} .

For any $z \in \mathcal{Z}$, the infinitesimal action of \mathfrak{g} on \mathcal{Z} defines a real linear map

$$(2) \quad \begin{aligned} \rho_z : \mathfrak{g} &\longrightarrow \mathbf{T}_z \mathcal{Z} \\ X &\longmapsto X \cdot m \end{aligned}$$

that is equivariant under the action of the stabilizer subgroup G_z .

Definition 2.3 Consider a symmetric endomorphism $\mathcal{L}(\gamma)$ of an eucliden vector space E . We associate the eigenspace $E^{\gamma=a} = \{v \in E, \mathcal{L}(\gamma)v = av\}$ to any $a \in \mathbb{R}$. We have the decomposition $E = E^{\gamma>0} \oplus E^{\gamma=0} \oplus E^{\gamma<0}$ where $E^{\gamma>0} = \sum_{a>0} E^{\gamma=a}$, and $E^{\gamma<0} = \sum_{a<0} E^{\gamma=a}$.

Let Σ be the set of (non-zero) roots relative to the action of \mathfrak{a} on \mathfrak{g} . The choice of the Weyl chamber \mathfrak{a}_+^* induces a set Σ^+ of positive roots, and thus a decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{k}' \oplus \mathfrak{a} \oplus \mathfrak{n}$ where $\mathfrak{n} = \sum_{\alpha>0} \mathfrak{g}_\alpha$, $\mathfrak{n}_- = \sum_{\alpha<0} \mathfrak{g}_\alpha$ and \mathfrak{k}' is the centralizer subalgebra of \mathfrak{a} in \mathfrak{k} .

Definition 2.4 We denote by $Q \subset G$ the (minimal) parabolic subgroup with Lie algebra $\mathfrak{q} := \mathfrak{k}' \oplus \mathfrak{a} \oplus \mathfrak{n}$.

Consider $(x, \gamma) \in \mathcal{Z} \times \mathfrak{a}$ such that $x \in \mathcal{Z}^\gamma$. The infinitesimal action of γ defines symmetric endomorphisms $\mathcal{L}(\gamma) : \mathbf{T}_x \mathcal{Z} \rightarrow \mathbf{T}_x \mathcal{Z}$ and $\mathcal{L}(\gamma) : \mathfrak{g} \rightarrow \mathfrak{g}$ satisfying $\mathcal{L}(\gamma) \circ \rho_x = \rho_x \circ \mathcal{L}(\gamma)$. Thus the morphism (2) induces a real linear map

$$(3) \quad \rho_x^\gamma : \mathfrak{n}^{\gamma>0} \longrightarrow (\mathbf{T}_x \mathcal{Z})^{\gamma>0}.$$

Definition 2.5 Let $\gamma \in \mathfrak{a}$ be a non-zero element, and let $\mathcal{C} \subset \mathcal{Z}^\gamma$ be a connected component. The data (γ, \mathcal{C}) is called an **infinitesimal real Ressayre's pair** of \mathcal{Z} if $\exists x \in \mathcal{C}$, such as ρ_x^γ is an isomorphism. If furthermore we have $\dim_{\mathbb{P}}(\mathcal{C}) - \dim_{\mathbb{P}}(\mathcal{Z}) \in \{0, 1\}$, and γ is rational, we call (γ, \mathcal{C}) a regular infinitesimal real Ressayre's pair of \mathcal{Z} .

Remark 2.6 When $U = T$ is abelian, a couple (γ, \mathcal{C}) is an infinitesimal real Ressayre's pair when the vector bundle $(\mathbf{T}\mathcal{Z}|_{\mathcal{C}})^{\gamma>0}$ is equal to the "zero" bundle.

Let us now introduce a more restrictive notion, that of real Ressayre's pair.

Let $\gamma \in \mathfrak{a}$ be a non-zero element. The fonction $\langle \Phi_{\mathbb{P}}, \gamma \rangle : \mathcal{Z}^\gamma \rightarrow \mathbb{R}$ is locally constant. Let $\mathcal{C} = \mathcal{C}_1 \cup \dots \cup \mathcal{C}_p \subset \mathcal{Z}^\gamma$ be a union of connected components such that $\langle \Phi_{\mathbb{P}}, \gamma \rangle$ is constant on \mathcal{C} : we denote by $\langle \Phi_{\mathbb{P}}(\mathcal{C}), \gamma \rangle$ its value. We consider the real Białynicki-Birula's submanifold²

$$(4) \quad \mathcal{C}^- := \{z \in \mathcal{Z}, \lim_{t \rightarrow \infty} \exp(t\gamma)z \in \mathcal{C}\}.$$

We see that for any $x \in \mathcal{C}$, $(\mathbf{T}_x \mathcal{Z})^{\gamma \leq 0} = \mathbf{T}_x \mathcal{C}^-$. Consider now the real parabolic subgroup $P_\gamma \subset G$ defined by

$$(5) \quad P_\gamma = \{g \in G, \lim_{t \rightarrow \infty} \exp(t\gamma)g \exp(-t\gamma) \text{ exists}\}.$$

²See §3.1.

The submanifold \mathcal{C}^- is invariant under the action of P_γ , hence we can consider the manifold $Q \times_{Q \cap P_\gamma} \mathcal{C}^-$ and the smooth map

$$q_\gamma : Q \times_{Q \cap P_\gamma} \mathcal{C}^- \rightarrow \mathcal{Z}$$

that sends $[q, x]$ to qx . We immediately see that for any $x \in \mathcal{C}$, the tangent map $\mathbf{T}q_\gamma|_x$ is an isomorphism if and only if ρ_x^γ is an isomorphism.

Definition 2.7 *Let $\gamma \in \mathfrak{a}$ be a non-zero element, and let $\mathcal{C} \subset \mathcal{Z}^\gamma$ be a union of connected components such that $\langle \Phi_{\mathfrak{p}}, \gamma \rangle$ is constant on \mathcal{C} . The data (γ, \mathcal{C}) is called a **real Ressayre's pair** of \mathcal{Z} if the following conditions hold*

- *The image of q_γ contains a dense open subset of \mathcal{Z} .*
- *There exists a $Q \cap P_\gamma$ -invariant, open and dense subset $U \subset \mathcal{C}^-$, intersecting \mathcal{C} , so that q_γ defines a diffeomorphism $Q \times_{Q \cap P_\gamma} U \simeq QU$.*

If furthermore we have $\dim_{\mathfrak{p}}(\mathcal{C}) - \dim_{\mathfrak{p}}(\mathcal{Z}) \in \{0, 1\}$, and γ is rational, we call (γ, \mathcal{C}) a regular real Ressayre's pair.

Remark 2.8 *When $U = T$ is abelian, a couple (γ, \mathcal{C}) is a real Ressayre's pair when the real Bialynicki-Birula's submanifold \mathcal{C}^- is open and dense in \mathcal{Z} .*

2.4 Main result

The main result of this article is the following theorem.

Theorem 2.9 *For $\xi \in \mathfrak{a}_+^*$, the following statements are equivalent:*

1. $\xi \in \Delta_{\mathfrak{p}}(\mathcal{Z})$.
2. $\xi \in \Delta_{\mathfrak{p}}(\mathcal{C})$.
3. *For any regular infinitesimal real Ressayre's pair (γ, \mathcal{C}) of \mathcal{Z} , we have $\langle \xi, \gamma \rangle \geq \langle \Phi_{\mathfrak{p}}(\mathcal{C}), \gamma \rangle$.*
4. *For any regular real Ressayre's pair (γ, \mathcal{C}) of \mathcal{Z} , we have $\langle \xi, \gamma \rangle \geq \langle \Phi_{\mathfrak{p}}(\mathcal{C}), \gamma \rangle$.*

Remark 2.10 *In the previous theorem, the result still holds if we drop the “regular” hypothesis on (γ, \mathcal{C}) .*

3 Kirwan-Ness stratifications : complex and real settings

Let us now choose a rational U -invariant inner product on $\mathfrak{u}_{\mathbb{C}}$ that is invariant under the involution σ (see the Appendix). By rational we mean that for the maximal torus $T \subset U$ with Lie algebra \mathfrak{t} , the inner product takes integral values on the lattice $\Lambda := \frac{1}{2\pi} \ker(\exp : \mathfrak{t} \rightarrow T)$. Let us denote by $\Lambda^* \subset \mathfrak{t}^*$ the dual lattice : $\Lambda^* = \text{hom}(\Lambda, \mathbb{Z})$. We associate to the lattices Λ and Λ^* the \mathbb{Q} -vector space $\mathfrak{t}_{\mathbb{Q}}$ and $\mathfrak{t}_{\mathbb{Q}}^*$ generated by them: the vectors belonging to $\mathfrak{t}_{\mathbb{Q}}$ and $\mathfrak{t}_{\mathbb{Q}}^*$ are designed as rational.

The invariant scalar product on \mathfrak{u} induces an identification $\mathfrak{u}^* \simeq \mathfrak{u}, \xi \mapsto \xi^{\flat}$ such as $\mathfrak{t}_{\mathbb{Q}} \simeq \mathfrak{t}_{\mathbb{Q}}^*$. To simplify our notation, we will not distinguish between ξ and ξ^{\flat} : for example we write M^{λ} for the submanifold fixed by the subgroup generated by λ^{\flat} .

We come back to the setting of a Kähler Hamiltonian U -manifold (M, Ω, \mathbb{J}) with proper moment map $\Phi_{\mathfrak{u}} : M \rightarrow \mathfrak{u}^*$ and an anti-holomorphic involution τ . We fix a connected component \mathcal{Z} of the real part $Z = M^{\tau}$.

3.1 Bialynicki-Birula's submanifolds

Let us consider an element $\lambda \in \mathfrak{u}$ and a connected component C of the complex submanifold $M^{\lambda} := \{m \in M, \gamma \cdot m = 0\}$. As in the introduction, we define the subset $C^{-} := \{m \in M, \lim_{t \rightarrow \infty} \exp(-it\lambda)m \in C\}$ and the projection $p_C : C^{-} \rightarrow C$ that sends $m \in C^{-}$ to $\lim_{t \rightarrow \infty} \exp(-it\lambda)m \in C$. We have the following well-know fact [2, 11].

Proposition 3.1 *C^{-} is a locally closed complex submanifold of M , and the projection $p_C : C^{-} \rightarrow C$ is an holomorphic map.*

Suppose now that $\lambda \in \mathfrak{u}^{-\sigma}$, so $\lambda = i\beta$ with $\beta \in \mathfrak{p}$. The complex submanifold $M^{\lambda} = M^{\beta}$ is then stable under the involution τ . Suppose that $\mathcal{Z}^{\beta} = M^{\beta} \cap \mathcal{Z}$ is non-empty and let \mathcal{C} be a connected component of \mathcal{Z}^{β} . There exists a unique connected component $C \subset M^{\lambda}$ containing \mathcal{C} : the complex submanifold C is then stable under the anti-holomorphic involution τ , and C is a connected component of C^{τ} . We notice that for any $t \in \mathbb{R}$, and any $m \in M$ we have $\tau(\exp(-it\lambda)m) = \tau(\exp(t\beta)m) = \exp(t\beta)\tau(m)$. It shows that the complex Bialynicki-Birula's submanifolds C^{-} is stable under τ , hence $(C^{-})^{\tau} = \{z \in M^{\tau}, \lim_{t \rightarrow \infty} \exp(t\beta)m \in C^{\tau}\}$ is a locally closed submanifold of M^{τ} .

Corollary 3.2 *$C^{-} = \{z \in \mathcal{Z}, \lim_{t \rightarrow \infty} \exp(t\beta)m \in \mathcal{C}\}$ is a locally closed submanifold of \mathcal{Z} , and the projection $p_C : C^{-} \rightarrow \mathcal{C}$ is a smooth map.*

3.2 Stratification in the complex setting

Let

$$f_{\mathfrak{u}} := \frac{1}{2}(\Phi_{\mathfrak{u}}, \Phi_{\mathfrak{u}}) : M \longrightarrow \mathbb{R}$$

denote the norm-square of the moment map. Notice that $f_{\mathfrak{u}}$ is a proper function on M .

Definition 3.3 *The Kirwan vector field κ_u is defined by the relation*

$$\kappa_u(m) = \Phi_u(m) \cdot m, \quad \forall m \in M.$$

We consider the gradient $\text{grad}(f_u)$ of the function f_u relatively to the Riemannian metric $\Omega(-, \mathbb{J}-)$. We recall the following well-known facts [23].

Proposition 3.4 *1. The gradient of f_u is $\text{grad}(f_u) = \mathbb{J}(\kappa_u)$.*

2. The set of critical points of the function f_u is $\text{crit}(f_u) = \{\kappa_u = 0\}$.

3. We have the decomposition $\Phi_u(\text{crit}(f_u)) = \bigcup_{\lambda \in \mathcal{B}_u} U\lambda$ where the set $\mathcal{B}_u \subset \mathfrak{t}_+^$ is discrete. \mathcal{B}_u is called the set of types of M .*

4. We have $\text{crit}(f_u) = \bigcup_{\lambda \in \mathcal{B}_u} \text{crit}_\lambda$ where $\text{crit}_\lambda = \text{crit}(f_u) \cap \Phi_u^{-1}(U\lambda)$ is equal to $U(M^\lambda \cap \Phi_u^{-1}(\lambda))$.

Let $\varphi_u^t : M \rightarrow M$ be the flow of $-\text{grad}(f_u)$; since f_u is proper, φ_u^t exists for all times $t \in [0, \infty[$, and according to a result of Duistermaat [13] we know that any trajectory of φ_u^t has a limit when $t \rightarrow \infty$. For any $m \in M$, let us denote $m_\infty := \lim_{t \rightarrow \infty} \varphi_u^t(m)$.

The construction of the Kirwan-Ness stratification goes as follows. For each $\lambda \in \mathcal{B}_u$, let M_λ denote the set of points of M flowing to crit_λ : $M_\lambda := \{m \in M; m_\infty \in \text{crit}_\lambda\}$. From its very definition, the set M_λ is contained in $\{m \in M, f_u(m) \geq \frac{1}{2}\|\lambda\|^2\}$. The Kirwan-Ness stratification is the decomposition [9], [16]:

$$M = \bigcup_{\lambda \in \mathcal{B}_u} M_\lambda.$$

When 0 belongs to the image of Φ_u , the strata M_0 corresponds to the dense open subset of analytical semi-stable points : $M_0 = \{m \in M; \overline{U_{\mathbb{C}} m} \cap \Phi_u^{-1}(0) \neq \emptyset\}$.

Let us now explain the geometry of M_λ for a non-zero type λ . Let C_λ be the union of the connected components of M^λ intersecting $\Phi_u^{-1}(\lambda)$. Then C_λ is a Kähler Hamiltonian U_λ -manifold³ with proper moment map $\Phi_\lambda := \Phi_u|_{C_\lambda} - \lambda$.

The Bialynicki-Birula's complex submanifold

$$C_\lambda^- := \{m \in M, \lim_{t \rightarrow \infty} \exp(-it\lambda) \cdot m \in C_\lambda\}$$

corresponds to the set of points of M flowing to C_λ under the flow of $-\text{grad}\langle \Phi_u, \lambda \rangle$, as $t \rightarrow \infty$. The limit of the flow defines a projection $C_\lambda^- \rightarrow C_\lambda$. Notice that C_λ^- is invariant under the action of the parabolic subgroup $P_\lambda^u \subset U_{\mathbb{C}}$:

$$(6) \quad P_\lambda^u = \{g \in U_{\mathbb{C}}, \lim_{t \rightarrow \infty} \exp(-it\lambda)g \exp(it\lambda) \text{ exists}\}.$$

³ U_λ is the stabilizer subgroup of λ .

Consider now the Kirwan-Ness stratification of the Kähler Hamiltonian U_λ -manifold C_λ . Let $C_{\lambda,0}$ be the open strata of C_λ corresponding to the 0-type:

$$C_{\lambda,0} = \{x \in C_\lambda; \overline{(U_\lambda)_\mathbb{C} x} \cap \Phi_\lambda^{-1}(0) \neq \emptyset\}.$$

Let $C_{\lambda,0}^-$ denote the inverse image of $C_{\lambda,0}$ in C_λ^- .

Theorem 3.5 (Kirwan [9]) *Let M be a Kähler Hamiltonian U -manifold with proper moment map $\Phi_{\mathfrak{u}} : M \rightarrow \mathfrak{u}^*$. For each non zero type λ , M_λ is a $U_\mathbb{C}$ -invariant complex submanifold, and $U_\mathbb{C} \times_{P_\lambda^{\mathfrak{u}}} C_{\lambda,0}^- \rightarrow M_\lambda$, $[g, z] \mapsto g \cdot z$ is an isomorphism of complex $U_\mathbb{C}$ -manifolds.*

Kirwan gave a proof when M is a compact Kähler Hamiltonian K -manifold. When M is non-compact but the moment map is proper, a proof is given in [7] (see also [23]).

The following standard facts are going to be very important in the real setting. Let λ_s be the orthogonal projection of 0 on the closed convex polytope $\Delta_{\mathfrak{u}}(M)$.

Proposition 3.6 a) λ_s is the unique element of $\mathcal{B}_{\mathfrak{u}}$ with minimal norm.

b) M_{λ_s} is a connected, open, dense and $U_\mathbb{C}$ -invariant subset of M .

c) If $\lambda \neq \lambda_s$, then the strata M_λ has an empty interior.

3.3 Stratification in the real setting : first step

In this section we suppose that our Kähler Hamiltonian U -manifold (M, Ω, \mathbb{J}) admit an anti-holomorphic involution τ compatible with the complex-conjugate involution σ on $U_\mathbb{C}$. The main purpose of this section is to show that the Kirwan-Ness stratification $M = \bigcup_{\lambda \in \mathcal{B}_{\mathfrak{u}}} M_\lambda$ induces a stratification on the real part Z of M . In this section we suppose that the manifold Z is *non-empty*: notice that the real dimension of any connected component of Z is equal to the complex dimension of M .

Let us denote by $\sigma_+ : \mathfrak{t}_+^* \rightarrow \mathfrak{t}_+^*$ the involution of the Weyl chamber that is defined by the relations : $-\sigma(U\xi) = -U\sigma(\xi) = U\sigma_+(\xi)$ for any $\xi \in \mathfrak{t}_+^*$.

Recall that $\varphi_{\mathfrak{u}}^t : M \rightarrow M$ denotes the flow of $-\text{grad}(f_{\mathfrak{u}})$.

Proposition 3.7 1. For any $(m, t) \in M \times \mathbb{R}_{\geq 0}$, we have $\tau(\varphi_{\mathfrak{u}}^t(m)) = \varphi_{\mathfrak{u}}^t(\tau(m))$.

2. If $m_\infty = \lim_{t \rightarrow \infty} \varphi_{\mathfrak{u}}^t(m)$, then $\tau(m_\infty) = (\tau(m))_\infty$ for all $m \in M$.

3. For any $\lambda \in \mathcal{B}_{\mathfrak{u}}$, we have

- $\tau(M_\lambda) = M_{\sigma_+(\lambda)}$,
- $M_\lambda \cap Z \neq \emptyset$ only if $\sigma(\lambda) = -\lambda$.

4. If $\lambda \neq \lambda_s$, then the locally closed submanifold $M_\lambda \cap Z$ has an empty interior in Z .

5. $\sigma(\lambda_s) = -\lambda_s$.

6. $M_{\lambda_s} \cap Z$ is a dense open subset of Z .

7. If \mathcal{Z} is a connected component of Z , then $M_{\lambda_s} \cap \mathcal{Z}$ is a dense open subset of \mathcal{Z} , and the critical set crit_{λ_s} intersects \mathcal{Z} .

Proof : The first point is a direct consequence of the fact that, for any $m \in M$, the tangent map $\mathbf{T}_m \tau : \mathbf{T}_m M \rightarrow \mathbf{T}_{\tau(m)} M$ sends $\mathbb{J}(\kappa_u(m))$ to $\mathbb{J}(\kappa_u(\tau(m)))$. The second point follows from the first one.

By definition $m \in M_\lambda \Leftrightarrow \Phi_u(m_\infty) \in U\lambda$. Then if $m \in M_\lambda$, we have

$$\Phi_u((\tau(m))_\infty) = \Phi_u(\tau(m_\infty)) = -\sigma(\Phi_u(m_\infty)) \in -U\sigma(\lambda) = U\sigma_+(\lambda).$$

Hence the identity $\tau(M_\lambda) = M_{\sigma_+(\lambda)}$ is proven. Let $m \in M_\lambda \cap Z$. Thanks to 2. we know that $m_\infty \in \Phi_u^{-1}(U\lambda) \cap Z$: it implies that $\Phi_u(m_\infty) \in U\lambda \cap (\mathbf{u}^{-\sigma})^*$ and then $\sigma(\lambda) = -\lambda$ (see §4.1). The point 3. is settled.

Let $\lambda \neq \lambda_s$, and consider the decomposition of the (locally closed) complex submanifold M_λ into connected components : $M_\lambda = \mathcal{Y}_1 \cup \dots \cup \mathcal{Y}_p$. If $M_\lambda \cap Z \neq \emptyset$, then M_λ is stable under the involution τ : there exists a permutation $\epsilon \in \mathfrak{S}_p$ of order two such that $\tau(\mathcal{Y}_k) = \mathcal{Y}_{\epsilon(k)}$, and

$$M_\lambda \cap Z = \bigcup_{\epsilon(k)=k} \mathcal{Y}_k^r.$$

Since $\dim_{\mathbb{C}}(\mathcal{Y}_k) < \dim_{\mathbb{C}}(M)$, $\forall k$ (see point *c*) in Proposition 3.6), we see that $\dim_{\mathbb{R}}(\mathcal{Y}_k^r) < \dim_{\mathbb{R}}(Z)$ for all k such that $\epsilon(k) = k$. We can then conclude that the locally closed submanifold $M_\lambda \cap Z$ has an empty interior in Z . If \mathcal{Z} is a connected component of Z , we prove similarly that $M_\lambda \cap \mathcal{Z}$ has an empty interior in \mathcal{Z} .

Consider the decomposition $Z = (M_{\lambda_s} \cap Z) \cup \bigcup_{\lambda \neq \lambda_s} (M_\lambda \cap Z)$, and the proper map $f_u : M \rightarrow \mathbb{R}_{\geq 0}$ that we restrict to Z . If we denote $Z_{<R} = Z \cap \{f_u < R\}$, we have

$$Z_{<R} = M_{\lambda_s} \cap Z_{<R} \cup \bigcup_{\lambda \neq \lambda_s, \|\lambda\|^2 \leq 2R} M_\lambda \cap Z_{<R}.$$

Since $M_\lambda \cap Z_{<R}$ has an empty interior in $Z_{<R}$, we see that $M_{\lambda_s} \cap Z_{<R}$ is dense in $Z_{<R}$ for any $R \geq 0$. Then $M_{\lambda_s} \cap Z$ is dense in Z , and $M_{\lambda_s} \cap Z$ is open in Z because M_{λ_s} is open in M . In particular $M_{\lambda_s} \cap Z \neq \emptyset$: hence $\sigma(\lambda_s) = -\lambda_s$. We have proven the points 5. and 6..

If \mathcal{Z} is a connected component of Z , the decomposition $\mathcal{Z} = (M_{\lambda_s} \cap \mathcal{Z}) \cup \bigcup_{\lambda \neq \lambda_s} (M_\lambda \cap \mathcal{Z})$ shows similarly that $M_{\lambda_s} \cap \mathcal{Z}$ is a dense open subset of \mathcal{Z} . If $x \in M_{\lambda_s} \cap \mathcal{Z}$ then $x_\infty \in \text{crit}_{\lambda_s} \cap \mathcal{Z}$. We have proven point 7.. \square

3.4 Stratification in the real setting : second step

We have a stratification $\mathcal{Z} = (M_{\lambda_s} \cap \mathcal{Z}) \cup \bigcup_{\substack{\lambda \neq \lambda_s \\ \sigma(\lambda) = -\lambda}} (M_\lambda \cap \mathcal{Z})$ into locally closed submanifolds, that we are going to interpret in terms of the G -action on \mathcal{Z} and the gradient map $\Phi_{\mathfrak{p}}|_{\mathcal{Z}} : \mathcal{Z} \rightarrow \mathfrak{p}^*$. Let $f_{\mathfrak{p}} := \frac{1}{2}(\Phi_{\mathfrak{p}}, \Phi_{\mathfrak{p}}) : M \rightarrow \mathbb{R}$.

Definition 3.8 • If $\beta \in j^*(\mathcal{B}_u \cap (\mathfrak{u}^{-\sigma})^*)$, we define $\mathcal{Z}_\beta = M_{(j^*)^{-1}(\beta)} \cap \mathcal{Z}$.

- Let $\mathcal{B}_{\mathfrak{p}} = \{\beta \in j^*(\mathcal{B}_u \cap (\mathfrak{u}^{-\sigma})^*), \mathcal{Z}_\beta \neq \emptyset\} \subset \mathfrak{a}_+^*$.
- Let us denote $j^*(\lambda_s)$ by β_s .

At this stage we have a stratification $\mathcal{Z} = \bigcup_{\beta \in \mathcal{B}_{\mathfrak{p}}} \mathcal{Z}_\beta$ into submanifolds. First let us check that $\mathcal{B}_{\mathfrak{p}}$ parametrizes the set of critical points of the function $f_{\mathfrak{p}}|_{\mathcal{Z}}$. Recall that point 7. in Proposition 3.7 tell us that $0 \in \mathcal{B}_{\mathfrak{p}} \Leftrightarrow (\Phi_{\mathfrak{p}}|_{\mathcal{Z}})^{-1}(0) \neq \emptyset \Leftrightarrow \Phi_{\mathfrak{u}}^{-1}(0) \neq \emptyset$

Lemma 3.9 *The set of critical points of the function $f_{\mathfrak{p}}|_{\mathcal{Z}}$ admits the decomposition*

$$\text{crit}(f_{\mathfrak{p}}|_{\mathcal{Z}}) = \bigcup_{\beta \in \mathcal{B}_{\mathfrak{p}}} K(\mathcal{Z}^\beta \cap \Phi_{\mathfrak{p}}^{-1}(\beta)).$$

Proof : We have $f_{\mathfrak{u}} = \frac{1}{2}(\Phi_{\mathfrak{u}^\sigma}, \Phi_{\mathfrak{u}^\sigma}) + f_{\mathfrak{p}}$, and $\Phi_{\mathfrak{u}^\sigma} : M \rightarrow (\mathfrak{u}^\sigma)^*$ vanishes on \mathcal{Z} . Hence $\text{crit}(f_{\mathfrak{p}}|_{\mathcal{Z}}) = \text{crit}(f_{\mathfrak{u}}) \cap \mathcal{Z}$. Since $\text{crit}(f_{\mathfrak{u}}) = \bigcup_{\lambda \in \mathcal{B}_u} U(M^\lambda \cap \Phi_{\mathfrak{u}}^{-1}(\lambda))$, we obtain $\text{crit}(f_{\mathfrak{p}}|_{\mathcal{Z}}) = \bigcup_{\lambda \in \mathcal{B}_u} \mathcal{D}_\lambda$ with $\mathcal{D}_\lambda = U(M^\lambda \cap \Phi_{\mathfrak{u}}^{-1}(\lambda)) \cap \mathcal{Z}$. We see that $\mathcal{D}_\lambda = \emptyset$ if $\lambda \notin \mathcal{B}_u \cap (\mathfrak{u}^{-\sigma})^*$, otherwise $\mathcal{D}_\lambda = K(\mathcal{Z}^{j^*(\lambda)} \cap \Phi_{\mathfrak{p}}^{-1}(j^*(\lambda)))$ if $\lambda \in \mathcal{B}_u \cap (\mathfrak{u}^{-\sigma})^*$. \square

Consider the case where $0 \in \mathcal{B}_{\mathfrak{p}}$.

Lemma 3.10 *If $0 \in \mathcal{B}_{\mathfrak{p}}$ then $\mathcal{Z}_0 = \{z \in \mathcal{Z}, \overline{Gz} \cap \Phi_{\mathfrak{p}}^{-1}(0) \neq \emptyset\}$.*

Proof : By definition, $\mathcal{Z}_0 = M_0 \cap \mathcal{Z} = \{z \in \mathcal{Z}, z_\infty \in \Phi_{\mathfrak{u}}^{-1}(0)\}$. Here $m_\infty = \lim_{t \rightarrow \infty} \varphi_{\mathfrak{u}}^t(m)$, where $\varphi_{\mathfrak{u}}^t : M \rightarrow M$ denotes the flow of $-\text{grad}(f_{\mathfrak{u}})$. When $z \in \mathcal{Z}$, we know that $\varphi_{\mathfrak{u}}^t(z) \in \mathcal{Z}, \forall t \geq 0$, and moreover the tangent vector

$$\frac{d}{dt} \varphi_{\mathfrak{u}}^t(z) = -\text{grad}(f_{\mathfrak{u}})(\varphi_{\mathfrak{u}}^t(z)) = -\text{grad}(f_{\mathfrak{p}})(\varphi_{\mathfrak{u}}^t(z))$$

belongs to the subspace $\mathfrak{g} \cdot \varphi_{\mathfrak{u}}^t(z) = \{X \cdot \varphi_{\mathfrak{u}}^t(z), X \in \mathfrak{g}\}$. We see then that $\varphi_{\mathfrak{u}}^t(z) \in Gz$ for all $t \geq 0$. Since $\mathcal{Z} \cap \Phi_{\mathfrak{u}}^{-1}(0) = \mathcal{Z} \cap \Phi_{\mathfrak{p}}^{-1}(0)$, we have proved that $\mathcal{Z}_0 \subset \{z \in \mathcal{Z}, \overline{Gz} \cap \Phi_{\mathfrak{p}}^{-1}(0) \neq \emptyset\}$. On the other hand, for any $z \in \mathcal{Z}$,

$$\overline{Gz} \cap \Phi_{\mathfrak{p}}^{-1}(0) \neq \emptyset \implies \overline{U_{\mathbb{C}} z} \cap \Phi_{\mathfrak{u}}^{-1}(0) \neq \emptyset \implies z \in M_0.$$

The proof is complete. \square

Let $\beta \in \mathcal{B}_p$ be a non-zero element and let $\lambda = j^*(\beta) \in \mathcal{B}_u$. The strata M_λ is isomorphic to $U_{\mathbb{C}} \times_{P_\lambda^u} C_{\lambda,0}^-$. We see that $\mathcal{C}_\beta := C_\lambda \cap \mathcal{Z}$ is the union of the connected components of \mathcal{Z}^β intersecting $\Phi_p^{-1}(\beta)$. Similarly, $\mathcal{C}_\beta^- := C_\lambda^- \cap \mathcal{Z}$ corresponds to *real* Bialynicki-Birula's submanifold $\{z \in \mathcal{Z}, \lim_{t \rightarrow \infty} e^{t\beta} z \in \mathcal{C}_\beta\}$. In the same way, $\mathcal{C}_{\beta,0} := C_{\lambda,0} \cap \mathcal{Z}$ is the open and dense subset of \mathcal{C}_β formed by the elements $z \in \mathcal{C}_\beta$ such that $\overline{G_\beta z} \cap \Phi_p^{-1}(\beta) \neq \emptyset$ (see Lemma 3.10). Finally, $\mathcal{C}_{\beta,0}^- := C_{\lambda,0}^- \cap \mathcal{Z}$ is equal to the pullback of the open subset $\mathcal{C}_{\beta,0}$ through the projection $\mathcal{C}_\beta^- \rightarrow \mathcal{C}_\beta$.

The complex parabolic subgroup $P_\lambda^u \subset U_{\mathbb{C}}$ is stable under the complex-conjugate involution σ and the fixed point subgroup $(P_\lambda^u)^\sigma$ is equal to the real parabolic subgroup P_β of G defined by (5). We can now conclude that the strata $\mathcal{Z}_\beta := M_\lambda \cap \mathcal{Z}$ admits the following description.

Proposition 3.11 *Let $\beta \in \mathcal{B}_p$ be a non-zero element. The map $[g, z] \mapsto gz$ induces a diffeomorphism*

$$(7) \quad G \times_{P_\beta} \mathcal{C}_{\beta,0}^- \xrightarrow{\sim} \mathcal{Z}_\beta.$$

The existence of a stratification $\mathcal{Z} = \bigcup_{\beta \in \mathcal{B}_p} \mathcal{Z}_\beta$ where each strata is described through the isomorphism (7) was already obtained by P. Heinzner, G.W. Schwarz and H. Stötzel in a much more general setting [7]. Nevertheless we obtain in this particular framework a crucial information for the continuation: only one stratum has a non-empty interior, the open stratum \mathcal{Z}_{β_s} attached to the element $\beta_s \in \mathcal{B}_p$ of minimal norm.

3.5 Proof of the refined O'Shea-Sjamaar's theorem (in the Kähler case)

Suppose that $Z \neq \emptyset$. Point (6) of Proposition 3.7 shows that $0 \in \Delta_u(M)$ if and only if $0 \in \Delta_p(Z)$. Let us explain how O'Shea-Sjamaar's identify $\Delta_u(M) \cap (u^{-\sigma})^* \simeq \Delta_p(Z)$ follows directly from this observation thanks to the shifting trick.

The fact that $j^*(\Delta_u(M) \cap (u^{-\sigma})^*)$ contains $\Delta_p(Z)$ is immediate.

Take now $\lambda \in \Delta_u(M) \cap (u^{-\sigma})^*$, and consider the Kähler Hamiltonian U -manifold $N = M \times U(-\lambda)$. As $\sigma(\lambda) = -\lambda$, the map $(-\sigma)$ leaves $U(-\lambda)$ invariant and defines an anti-holomorphic involution $\tau_\lambda : U(-\lambda) \rightarrow U(-\lambda)$ by the relations $\tau_\lambda(-g\lambda) = -\sigma(g)\lambda$. By definition $Z_N = Z \times K(-\lambda)$ is the real part of N and Point (6) of Proposition 3.7 shows that $0 \in \Delta_p(Z_N)$ because $0 \in \Delta_u(N)$. We have proven that $j^*(\lambda) \in \Phi_p(Z) \cap \mathfrak{a}_+^*$. The proof of the identity $\Delta_u(M) \cap (u^{-\sigma})^* \simeq \Delta_p(Z)$ is complete.

The refinement of O'Shea-Sjamaar's theorem is a consequence Point (7) of Proposition 3.7 which says that if $0 \in \Delta_u(M)$ then the open subset M_0 of analytical semi-stable points intersects any connected component \mathcal{Z} of Z . But if $x \in \mathcal{Z} \cap M_0$, then $x_\infty \in \mathcal{Z} \cap \Phi_p^{-1}(0)$. In other words, $0 \in \Delta_u(M)$ if and only if $0 \in \Delta_p(\mathcal{Z})$. Like before, the shifting trick shows that $j^*(\Delta_u(M) \cap (u^{-\sigma})^*) = \Delta_p(\mathcal{Z})$.

4 Geometric properties

4.1 Symmetric coadjoint orbits

Let $\tilde{\lambda} \in \mathfrak{t}^*$. The coadjoint orbit $U\tilde{\lambda}$ admits a complex structure compatible with its symplectic structure which can be visualized through the isomorphisms $U\tilde{\lambda} \simeq U/U_{\tilde{\lambda}} \simeq U_{\mathbb{C}}/P_{\tilde{\lambda}}^{\mathfrak{u}}$. Here $P_{\tilde{\lambda}}^{\mathfrak{u}} \subset U_{\mathbb{C}}$ is the complex parabolic subgroup defined by (6). Notice that the Lie algebra of $P_{\tilde{\lambda}}^{\mathfrak{u}}$ is

$$\mathrm{Lie}(P_{\tilde{\lambda}}^{\mathfrak{u}}) = \mathfrak{t}_{\mathbb{C}} \oplus \sum_{\substack{\alpha \in \mathfrak{R}(\mathfrak{u}, \mathfrak{t}) \\ (\alpha, \tilde{\lambda}) \leq 0}} (\mathfrak{u}_{\mathbb{C}})_{\alpha}.$$

The conjugate linear involution $\sigma : \mathfrak{u}_{\mathbb{C}} \rightarrow \mathfrak{u}_{\mathbb{C}}$ leaves $\mathfrak{t}_{\mathbb{C}}$ invariant and sends the weight space $(\mathfrak{u}_{\mathbb{C}})_{\alpha}$ to $(\mathfrak{u}_{\mathbb{C}})_{-\sigma(\alpha)}$. That permits to see that the image of the parabolic subgroup $P_{\tilde{\lambda}}^{\mathfrak{u}}$ through σ is the parabolic subgroup $P_{-\sigma(\tilde{\lambda})}^{\mathfrak{u}}$.

Suppose now that coadjoint orbit $U\tilde{\lambda}$ intersects $(\mathfrak{u}^{-\sigma})^*$. If $\tilde{\lambda}$ is taken in the Weyl chamber, it is possible only if $\sigma(\tilde{\lambda}) = -\tilde{\lambda}$, and in this case we have $U\tilde{\lambda} \cap (\mathfrak{u}^{-\sigma})^* = K\tilde{\lambda}$. The map $\xi \in \mathfrak{u}^* \mapsto -\sigma(\xi) \in \mathfrak{u}^*$ leaves $U\tilde{\lambda}$ invariant and it defines an anti-holomorphic involution $\tau : U\tilde{\lambda} \rightarrow U\tilde{\lambda}$. The parabolic subgroup $P_{\tilde{\lambda}}^{\mathfrak{u}}$ is stable under σ , so that the involution $\tau : U_{\mathbb{C}}/P_{\tilde{\lambda}}^{\mathfrak{u}} \rightarrow U_{\mathbb{C}}/P_{\tilde{\lambda}}^{\mathfrak{u}}$ can also be defined by $\tau([g]) = ([\sigma(g)])$.

The submanifold fixed by the involution $\tau : U\tilde{\lambda} \rightarrow U\tilde{\lambda}$ is $K\tilde{\lambda}$, and the isomorphism $U\tilde{\lambda} \simeq U_{\mathbb{C}}/P_{\tilde{\lambda}}^{\mathfrak{u}}$ descends to an isomorphism $K\tilde{\lambda} \simeq G/G \cap P_{\tilde{\lambda}}^{\mathfrak{u}}$.

Let $j^* : \mathfrak{p}^* \rightarrow (\mathfrak{u}^{-\sigma})^*$ be the K -equivariant isomorphism. Let $\lambda \in \mathfrak{a}^*$ such that $j^*(\lambda) = \tilde{\lambda}$. Then $K\tilde{\lambda} \simeq K\lambda \simeq G/P_{\lambda}$ where $P_{\lambda} = G \cap P_{\tilde{\lambda}}^{\mathfrak{u}}$ is the real parabolic subgroup of G defined by (5).

Let ξ be an element belonging to the interior of the Weyl chamber \mathfrak{a}_+^* : we have $(\alpha, \xi) > 0$ for any restricted root $\alpha \in \Sigma^+$. In this case the orbit $K(-\xi)$ admits a natural identification with the quotient G/Q since the (minimal) parabolic subgroup Q coincides with $P_{-\xi}$ (see Definition 2.4).

4.2 Construction of real Ressayre's pairs

Let ξ be an element in the interior of the Weyl chamber \mathfrak{a}_+^* that does not belongs to the moment polytope $\Delta_{\mathfrak{p}}(\mathcal{Z})$. Let ξ' be the orthogonal projection of ξ on $\Delta_{\mathfrak{p}}(\mathcal{Z})$ and let $\gamma_{\xi} = \xi' - \xi \in \mathfrak{a}^* \simeq \mathfrak{a}$.

Lemma 4.1 *The set $(\Phi_{\mathfrak{p}}|_{\mathcal{Z}})^{-1}(\xi')$ is contained in the submanifold $\mathcal{Z}^{\gamma_{\xi}}$.*

Proof : Let us check that $\|\xi' - \xi\|^2$ is the minimal value of the function $\|\Phi_{\mathfrak{p}} - \xi\|^2 : \mathcal{Z} \rightarrow \mathbb{R}$. If $z \in \mathcal{Z}$ then $\Phi_{\mathfrak{p}}(z) = k\eta$, with $k \in K$ and $\eta \in \Delta_{\mathfrak{p}}(\mathcal{Z})$: $\|\Phi_{\mathfrak{p}}(z) - \xi\|^2 = \|\eta\|^2 + \|\xi\|^2 - 2(k\eta, \xi)$. Since $(k\eta, \xi) \leq (\eta, \xi), \forall k \in K$, we get

$$\|\Phi_{\mathfrak{p}}(z) - \xi\|^2 \geq \|\eta\|^2 + \|\xi\|^2 - 2(\eta, \xi) = \|\eta - \xi\|^2 \geq \|\xi' - \xi\|^2.$$

Then if $z \in \Phi_p^{-1}(\xi')$, the differential of $\|\Phi_p - \xi\|^2$ vanishes at z . But

$$d\|\Phi_p - \xi\|^2|_z = 2d\langle \Phi_p, \gamma_\xi \rangle|_z = 2(\gamma_\xi \cdot z, -).$$

Thus $\gamma_\xi \cdot z = 0$. \square

Let \mathcal{C}_{γ_ξ} be the union of the connected components of \mathcal{Z}^{γ_ξ} intersecting $(\Phi_p|_{\mathcal{Z}})^{-1}(\xi')$. The next result is the main tool to exhibit real Ressayre's pairs on the G -manifold \mathcal{Z} .

Theorem 4.2 *The data $(\gamma_\xi, \mathcal{C}_{\gamma_\xi})$ is a real Ressayre's pair on \mathcal{Z} .*

The rest of this section is dedicated to the proof of Theorem 4.2.

Let $\tilde{\xi}$ be the element of $\mathfrak{t}_+^* \cap (\mathfrak{u}^{-\sigma})^*$ such that $\xi = j^*(\tilde{\xi})$. Since $\xi \notin \Delta_p(\mathcal{Z})$, we know that $\tilde{\xi} \notin \Delta_u(M)$. Let $\tilde{\xi}'$ be the orthogonal projection of $\tilde{\xi}$ on $\Delta_u(M)$. We work with the proper Kähler Hamiltonian U -manifold

$$N := M \times U(-\tilde{\xi}) \simeq M \times U_{\mathbb{C}}/P_{-\tilde{\xi}}^u.$$

The submanifold $\mathcal{Z}_N := \mathcal{Z} \times K(-\tilde{\xi}) \simeq \mathcal{Z} \times G/Q$ is a connected component of its real part. Let $\Phi_u^N : N \rightarrow \mathfrak{u}^*$ be the moment map relative to the action of U on N .

We following result precises Lemma 4.1.

Lemma 4.3 • $\|g\tilde{\xi}' - \tilde{\xi}\| \geq \|\tilde{\xi}' - \tilde{\xi}\|, \forall g \in U$, and the equality holds if and only if $g\tilde{\xi}' \in U_{\tilde{\xi}} \cdot \tilde{\xi}'$.

- The function $\|\Phi_u^N\| : N \rightarrow \mathbb{R}$ reaches its minimum on $U(\Phi_u^{-1}(\tilde{\xi}') \times \{-\tilde{\xi}\})$.
- We have $\sigma(\tilde{\xi}') = -\sigma(\tilde{\xi})$. In other words, we have $\xi' = j^*(\tilde{\xi}')$ where ξ' is the orthogonal projection of ξ on $\Delta_p(\mathcal{Z})$.

Proof : The first point is a classical result of Hamiltonian geometry: we briefly recall the arguments. We have $\|g\tilde{\xi}' - \tilde{\xi}\| = \|\tilde{\xi}'\|^2 + \|\tilde{\xi}\|^2 - 2\phi(g\tilde{\xi}')$ where ϕ is the $\tilde{\xi}$ -th component of the moment map on $U\tilde{\xi}'$. The function ϕ has a unique local maximum on the coadjoint orbit $U\tilde{\xi}'$ which is reached on an orbit of the stabilizer subgroup $U_{\tilde{\xi}}$ (see [1, 5]). Finally, it is not hard to check that the point $\tilde{\xi}'$ belongs to this orbit.

If $n = (m, k\tilde{\xi}) \in N$, we write $\Phi_u(m) = g\eta$ with $\eta \in \Delta_u(M)$. Then

$$\|\Phi_u^N(n)\| = \|g\eta - k\tilde{\xi}\| \geq \|\eta - \tilde{\xi}\| \geq \|\tilde{\xi}' - \tilde{\xi}\|$$

and the equality $\|\Phi_u^N(n)\| = \|\tilde{\xi}' - \tilde{\xi}\|$ holds if and only if $\eta = \tilde{\xi}'$ and $k^{-1}g \in U_{\tilde{\xi}}$. It follows that the critical set $\text{crit}_{\lambda_s} = \{n, \|\Phi_u^N(n)\| = \|\tilde{\xi}' - \tilde{\xi}\|\}$ is $U(\Phi_u^{-1}(\tilde{\xi}') \times \{-\tilde{\xi}\})$. \square

Thanks to point 8) of Proposition 3.7, we know that $\text{crit}_{\lambda_s} \cap \mathcal{Z}_N$ is non-empty. If $g(m, -\tilde{\xi}) \in \text{crit}_{\lambda_s} \cap \mathcal{Z}_N$ then $gm \in \mathcal{Z}$ and $\Phi_u(m) = \tilde{\xi}'$ which implies that $\Phi_u(gm)$ belongs to $U\tilde{\xi}' \cap (\mathfrak{u}^{-\sigma})^*$. It follows that $\sigma(\tilde{\xi}') = -\sigma(\tilde{\xi})$ (see §4.1). \square

Thanks to the previous lemma we know that the minimal type of $N = M \times U_{\mathbb{C}}/P_{-\tilde{\xi}}^u$ is the non-zero element $\tilde{\gamma}_{\xi} := \tilde{\xi}' - \tilde{\xi} \in (\mathfrak{t}^{-\sigma})^*$: we denote by $\gamma_{\xi} := j^*(\tilde{\gamma}_{\xi}) = \xi' - \xi$ the corresponding element of $\mathfrak{a}^* \simeq \mathfrak{a}$. We will now use Proposition 3.11 in order to describe the open and dense strata $(\mathcal{Z}_N)_{\gamma_{\xi}}$ of \mathcal{Z}_N .

The critical set associated to the minimal type $\tilde{\gamma}_{\xi}$ is $(\Phi_u^N)^{-1}(U\tilde{\gamma}_{\xi}) = U(\Phi_u^{-1}(\tilde{\xi}') \times \{-\tilde{\xi}\})$. Let $C_{N, \tilde{\gamma}_{\xi}}$ be the connected component of $N^{\tilde{\gamma}_{\xi}}$ containing $(\Phi_u^N)^{-1}(U\tilde{\gamma}_{\xi})$: we have

$$C_{N, \tilde{\gamma}_{\xi}} = C_{\tilde{\gamma}_{\xi}} \times (U_{\tilde{\gamma}_{\xi}})_{\mathbb{C}} / ((U_{\tilde{\gamma}_{\xi}})_{\mathbb{C}} \cap P_{-\tilde{\xi}}^u),$$

where $C_{\tilde{\gamma}_{\xi}}$ is the connected component of $M^{\tilde{\gamma}_{\xi}}$ containing $\Phi_u^{-1}(\tilde{\xi}')$. The intersection $C_{N, \tilde{\gamma}_{\xi}} \cap \mathcal{Z}_N$ decomposes as follow

$$C_{N, \tilde{\gamma}_{\xi}} \cap \mathcal{Z}_N = \mathcal{C}_{\gamma_{\xi}} \times G_{\gamma_{\xi}} / G_{\gamma_{\xi}} \cap Q,$$

where $\mathcal{C}_{\gamma_{\xi}}$ is the union of the connected components of $\mathcal{Z}^{\gamma_{\xi}}$ intersecting $\Phi_p^{-1}(\xi')$, and $G_{\gamma_{\xi}} \subset G$ is the subgroup that stabilizes γ_{ξ} . The real Bialynicki-Birula's submanifold is then $(C_{N, \tilde{\gamma}_{\xi}} \cap \mathcal{Z}_N)^- = \mathcal{C}_{\gamma_{\xi}}^- \times P_{\gamma_{\xi}} / P_{\gamma_{\xi}} \cap Q$.

Let us consider the map

$$(8) \quad p_{\gamma_{\xi}} : G \times_{P_{\gamma_{\xi}}} \left(\mathcal{C}_{\gamma_{\xi}}^- \times P_{\gamma_{\xi}} / P_{\gamma_{\xi}} \cap Q \right) \longrightarrow \mathcal{Z} \times G/Q.$$

defined by $p_{\gamma}([g; z, [p]]) = (gz, [gp])$. Proposition 3.11 tells us that the image of $p_{\gamma_{\xi}}$ is a dense G -invariant subset of $\mathcal{Z} \times G/Q$ and that there exists a dense $P_{\gamma_{\xi}}$ -invariant subset \mathcal{U} of $\mathcal{C}_{\gamma_{\xi}}^- \times P_{\gamma_{\xi}} / P_{\gamma_{\xi}} \cap Q$, intersecting $\mathcal{C}_{\gamma_{\xi}} \times G_{\gamma_{\xi}} / G_{\gamma_{\xi}} \cap Q$, and such that $p_{\gamma_{\xi}}$ induces a diffeomorphism $G \times_{P_{\gamma_{\xi}}} \mathcal{U} \simeq G\mathcal{U}$.

In order finish the proof of Theorem 4.2, we have to compare the map $p_{\gamma_{\xi}}$ defined by (8) and the map $q_{\gamma_{\xi}} : Q \times_{Q \cap P_{\gamma_{\xi}}} \mathcal{C}_{\gamma_{\xi}}^- \rightarrow \mathcal{Z}$.

Consider the canonical isomorphism $\mathcal{C}_{\gamma_{\xi}}^- \times P_{\gamma_{\xi}} / P_{\gamma_{\xi}} \cap Q \simeq P_{\gamma_{\xi}} \times_{P_{\gamma_{\xi}} \cap Q} \mathcal{C}_{\gamma_{\xi}}^-$. More generally, for any $P_{\gamma_{\xi}}$ -invariant open subset $\mathcal{U} \subset \mathcal{C}_{\gamma_{\xi}}^- \times P_{\gamma_{\xi}} / P_{\gamma_{\xi}} \cap Q$ we have an isomorphism $\mathcal{U} \simeq P_{\gamma_{\xi}} \times_{P_{\gamma_{\xi}} \cap Q} U$ where U is the open $P_{\gamma} \cap Q$ -invariant subset of $\mathcal{C}_{\gamma_{\xi}}^-$ defined by the relation $U := \{x \in \mathcal{C}_{\gamma_{\xi}}^-; (x, [e]) \in \mathcal{U}\}$. Note that \mathcal{U} intersects $\mathcal{C}_{\gamma_{\xi}} \times G_{\gamma_{\xi}} / G_{\gamma_{\xi}} \cap Q$ if and only if U intersects $\mathcal{C}_{\gamma_{\xi}}$.

Now, we notice that $q_{\gamma_{\xi}}$ defines a diffeomorphism $Q \times_{Q \cap P_{\gamma_{\xi}}} U \simeq QU$ if and only if $p_{\gamma_{\xi}}$ defines a diffeomorphism $G \times_{P_{\gamma_{\xi}}} \mathcal{U} \simeq G\mathcal{U}$. That can be seen easily through the commutative diagram

$$\begin{array}{ccc} G \times_{P_{\gamma_{\xi}}} \mathcal{U} & \xrightarrow{\sim} & G \times_Q (Q \times_{Q \cap P_{\gamma_{\xi}}} U) \\ \downarrow p_{\gamma_{\xi}} & & \downarrow 1 \times q_{\gamma_{\xi}} \\ \mathcal{Z} \times G/Q & \xrightarrow{\sim} & G \times_Q \mathcal{Z}. \end{array}$$

The previous diagram shows also that $p_{\gamma_{\xi}}$ has a dense image if and only if $q_{\gamma_{\xi}}$ has a dense image. The proof of Theorem 4.2 is then complete. \square

4.3 Principal-cross-section Theorem

We propose here a principal-cross-section Theorem in the spirit of [14] that holds on the K -manifold \mathcal{Z} . Since $\Delta_{\mathfrak{p}}(\mathcal{Z})$ is a closed convex polyhedral subset of \mathfrak{a}_+^* , there exists a unique open face \mathfrak{s} of \mathfrak{a}_+^* such that

- $\Delta_{\mathfrak{p}}(\mathcal{Z}) \cap \mathfrak{s} \neq \emptyset$,
- $\Delta_{\mathfrak{p}}(\mathcal{Z}) \subset \bar{\mathfrak{s}}$.

Notice that $\Delta_{\mathfrak{p}}(\mathcal{Z}) \cap \mathfrak{s}$ is dense in $\Delta_{\mathfrak{p}}(\mathcal{Z})$. Recall that the linear isomorphism $j^* : (\mathfrak{t}^*)^{-\sigma} \rightarrow \mathfrak{a}^*$ induces a bijection $\mathfrak{t}_+^* \cap (\mathfrak{t}^{-\sigma})^* \simeq \mathfrak{a}_+^*$.

Lemma 4.4 1. *There exists a unique open face $\tilde{\mathfrak{s}}$ of the Weyl chamber \mathfrak{t}_+^* such that $\tilde{\mathfrak{s}} \cap (\mathfrak{t}^{-\sigma})^* \simeq_{j^*} \mathfrak{s}$.*

2. *The stabilizer subgroup K_{ξ} does not depend of $\xi \in \mathfrak{s}$: it is denoted by $K_{\mathfrak{s}}$.*

3. *The stabilizer subalgebra \mathfrak{g}_{ξ} does not depend $\xi \in \mathfrak{s}$: it is denoted by $\mathfrak{g}_{\mathfrak{s}}$. We have the decomposition : $\mathfrak{g}_{\mathfrak{s}} = \mathfrak{k}_{\mathfrak{s}} \oplus \mathfrak{p}_{\mathfrak{s}}$ where $\mathfrak{k}_{\mathfrak{s}}$ is the Lie algebra of $K_{\mathfrak{s}}$.*

Proof : Any $\xi_o \in \mathfrak{t}_+^*$ belongs to the open face $\tau(\xi_o) \subset \mathfrak{t}_+^*$ defined as follows : $\xi \in \tau(\xi_o)$ if and only if $\xi \in \mathfrak{t}_+^*$ and $(\alpha, \xi) = 0 \iff (\alpha, \xi_o) = 0$ for any $\alpha \in \mathfrak{R}(\mathfrak{u}, \mathfrak{t})$. Now, it is an easy matter to check that the face $\tau(\xi_o)$ does not depend on $\xi_o \in (j^*)^{-1}(\mathfrak{s}) \subset \mathfrak{t}_+^* \cap (\mathfrak{t}^{-\sigma})^*$: this face, denoted by $\tilde{\mathfrak{s}}$, satisfies the relation $\tilde{\mathfrak{s}} \cap (\mathfrak{t}^{-\sigma})^* \simeq_{j^*} \mathfrak{s}$.

All points in the open face $\tilde{\mathfrak{s}}$ have the same connected stabilizer $U_{\tilde{\mathfrak{s}}}$. If we take $\tilde{\xi} \in \tilde{\mathfrak{s}} \cap (\mathfrak{t}^{-\sigma})^*$, we see that $U_{\tilde{\xi}} = U_{\tilde{\mathfrak{s}}}$ is invariant under σ and that $K \cap U_{\tilde{\mathfrak{s}}} = K \cap U_{\tilde{\xi}}$ is equal to K_{ξ} where $\xi = j^*(\tilde{\xi}) \in \mathfrak{s}$. The second point is settled and the third point is leaved to the readers. \square

The following slice

$$\mathcal{T}_{\mathfrak{s}} = \{z \in \mathcal{Z}, \Phi_{\mathfrak{p}}(z) \in \mathfrak{s}\}.$$

is the key object of our principal-cross-section Theorem.

Theorem 4.5 1. *$\mathcal{T}_{\mathfrak{s}}$ is a $K_{\mathfrak{s}}$ -invariant submanifold of \mathcal{Z} .*

2. *The map $K \times_{K_{\mathfrak{s}}} \mathcal{T}_{\mathfrak{s}} \rightarrow \mathcal{Z}$, $[k, y] \mapsto ky$ is a diffeomorphism onto a K -invariant open and dense subset of \mathcal{Z} .*

3. *For any $x \in \mathcal{T}_{\mathfrak{s}}$, we have $[\mathfrak{k}_{\mathfrak{s}}, \mathfrak{p}_{\mathfrak{s}}] \subset \mathfrak{p}_x \subset \mathfrak{p}_{\mathfrak{s}}$.*

Proof : We consider the following open subset of $\mathfrak{u}_{\mathfrak{s}}^*$:

$$\mathcal{V}_{\tilde{\mathfrak{s}}} := U_{\tilde{\mathfrak{s}}} \left\{ \xi \in \mathfrak{t}_+^*, U_{\xi} \subset U_{\tilde{\mathfrak{s}}} \right\}.$$

The pull-back $Y_{\tilde{\mathfrak{s}}} = \Phi_{\mathfrak{u}}^{-1}(\mathcal{V}_{\tilde{\mathfrak{s}}})$ is the symplectic cross-section at $\tilde{\mathfrak{s}}$. It is an $U_{\tilde{\mathfrak{s}}}$ -invariant symplectic manifold of M such that the map $U \times_{U_{\tilde{\mathfrak{s}}}} Y_{\tilde{\mathfrak{s}}} \rightarrow M$, $[g, y] \mapsto gy$ defines a diffeomorphism onto the open and dense subset $UY_{\tilde{\mathfrak{s}}}$.

Lemma 4.6 1. $\mathcal{V}_{\tilde{\mathfrak{s}}} \subset \mathfrak{u}_{\tilde{\mathfrak{s}}}^*$ is invariant under the map $-\sigma$.

2. The submanifold $Y_{\tilde{\mathfrak{s}}}$ is stable under the involution τ .

3. The intersection $Y_{\tilde{\mathfrak{s}}} \cap \mathcal{Z}$ is equal to $\mathcal{T}_{\tilde{\mathfrak{s}}}$.

4. The intersection $UY_{\tilde{\mathfrak{s}}} \cap \mathcal{Z}$ is equal to $K\mathcal{T}_{\tilde{\mathfrak{s}}}$.

5. $K\mathcal{T}_{\tilde{\mathfrak{s}}}$ is open and dense in \mathcal{Z} .

Proof of the lemma : Recall that K' is the connected component of the centraliser subgroup $Z_K(\mathfrak{a})$. Let w'_0 be the longest element in the Weyl group $W' = N_{K'}(T)/T$. Then the linear map $\sigma_+(\xi) = -w'_0\sigma(\xi)$ of \mathfrak{t}^* preserves the Weyl chamber \mathfrak{t}_+^* , and for any $\xi \in \mathfrak{t}_+^*$, we have $-\sigma(U\xi) = U(\sigma_+(\xi))$. Let $k' \in K'$ be a representant of w'_0 .

Notice that the subgroup $U_{\tilde{\mathfrak{s}}}$ is stable under σ since $-\sigma$ fixes the elements of $\tilde{\mathfrak{s}} \cap (\mathfrak{t}^{-\sigma})^*$. Take $\eta \in \mathcal{V}_{\tilde{\mathfrak{s}}}$: so $\eta = g\xi$ where $g \in U_{\tilde{\mathfrak{s}}}$ and $\xi \in \mathfrak{t}_+^*$ satisfies $U_\xi \subset U_{\tilde{\mathfrak{s}}}$. We have $-\sigma(\eta) = \sigma(g)k'\sigma_+(\xi)$, where $\sigma(g)k' \in U_{\tilde{\mathfrak{s}}}$ because $K' \subset U_{\tilde{\mathfrak{s}}}$. Now we see that the stabilizer subgroup $U_{\sigma_+(\xi)}$ is equal to $Ad(k') \circ \sigma(U_\xi)$: as $U_\xi \subset U_{\tilde{\mathfrak{s}}}$, we obtain $U_{\sigma_+(\xi)} \subset U_{\tilde{\mathfrak{s}}}$. The first point is proved and the second point follows directly from the first.

The inclusion $\mathcal{T}_{\tilde{\mathfrak{s}}} \subset Y_{\tilde{\mathfrak{s}}} \cap \mathcal{Z}$ is immediate. Let us check the reverse inclusion. Let $z \in Y_{\tilde{\mathfrak{s}}} \cap \mathcal{Z}$ and $\eta = \Phi_u(z) \in \mathcal{V}_{\tilde{\mathfrak{s}}} \cap (\mathfrak{u}^{-\sigma})^*$. Taking the decomposition $\eta = g\xi$ as before, we see that $g\xi \in U_\xi \cap (\mathfrak{u}^{-\sigma})^* = K\xi$: there exists $k \in K$ such that $g\xi = k\xi$ or in other words $g^{-1}k \in U_\xi \subset U_{\tilde{\mathfrak{s}}}$. But $g \in U_{\tilde{\mathfrak{s}}}$ and so $k \in U_{\tilde{\mathfrak{s}}} \cap K = K_{\tilde{\mathfrak{s}}}$. At this stage we know that $\Phi_p(k^{-1}z) = j^*(\xi) \in \mathfrak{a}_+^*$. On one hand we know that $\xi \in \tilde{\mathfrak{s}}$ because $j^*(\xi) \in \Delta_p(\mathcal{Z}) \subset \tilde{\mathfrak{s}}$. On the other hand we know $U_\xi \subset U_{\tilde{\mathfrak{s}}}$. It shows that ξ belongs to $\tilde{\mathfrak{s}}$, and hence $k^{-1}z \in \mathcal{T}_{\tilde{\mathfrak{s}}}$. Since $k \in K_{\tilde{\mathfrak{s}}}$ we can conclude that z belongs to $\mathcal{T}_{\tilde{\mathfrak{s}}}$. The third point is settled.

Thanks to the third point we know that the intersection $UY_{\tilde{\mathfrak{s}}} \cap \mathcal{Z}$ contains $K\mathcal{T}_{\tilde{\mathfrak{s}}}$. Let us prove that $UY_{\tilde{\mathfrak{s}}} \cap \mathcal{Z} \subset K\mathcal{T}_{\tilde{\mathfrak{s}}}$. Let $(z, y, u) \in \mathcal{Z} \times Y_{\tilde{\mathfrak{s}}} \times U$ such that $z = uy$. We write $\Phi_u(y) = g\xi$, where $g \in U_{\tilde{\mathfrak{s}}}$ and $\xi \in \mathfrak{t}_+^*$ satisfies $U_\xi \subset U_{\tilde{\mathfrak{s}}}$. We see then that $\Phi_u(z) = ug\xi$ belongs to $U_\xi \cap (\mathfrak{u}^{-\sigma})^* = K\xi$: there exists $k \in K$ such that $ug\xi = k\xi$, so $k^{-1}ug \in U_{\tilde{\mathfrak{s}}}$. We have proved that there exists $g' \in U_{\tilde{\mathfrak{s}}}$ such that $u = kg'$. The identity $z = kg'y$ shows then that $g'y = k^{-1}z \in Y_{\tilde{\mathfrak{s}}} \cap \mathcal{Z} = \mathcal{T}_{\tilde{\mathfrak{s}}}$. We have proved that $z \in K\mathcal{T}_{\tilde{\mathfrak{s}}}$.

$K\mathcal{T}_{\tilde{\mathfrak{s}}}$ is open in \mathcal{Z} since $UY_{\tilde{\mathfrak{s}}}$ is open in M . Let $z \in \mathcal{Z}$, and assume that $\lambda = \Phi_p(z)$ belongs to the Weyl chamber \mathfrak{a}_+^* . The *local convexity theorem* of Sjamaar (see [21][Theorem 6.5] and [17][Theorem 8.2]) tells us that for any K -invariant neighborhood $U \subset \mathcal{Z}$ of z , the local moment polytope $\Delta_p(U) = \Phi_p(U) \cap \mathfrak{a}_+^*$ is a neighborhood of λ in $\Delta_p(\mathcal{Z})$. It implies that $U \cap \mathcal{T}_{\tilde{\mathfrak{s}}} \neq \emptyset$ for any K -invariant neighborhood $U \subset \mathcal{Z}$ of z . The density of $K\mathcal{T}_{\tilde{\mathfrak{s}}}$ in \mathcal{Z} is demonstrated. \square

We can now complete the proof of Theorem 4.5.

The identity $Y_{\tilde{\mathfrak{s}}} \cap \mathcal{Z} = \mathcal{T}_{\tilde{\mathfrak{s}}}$ shows that $\mathcal{T}_{\tilde{\mathfrak{s}}}$ corresponds to the union of the connected components of the submanifold $(Y_{\tilde{\mathfrak{s}}})^\tau$ contained in \mathcal{Z} . Hence $\mathcal{T}_{\tilde{\mathfrak{s}}}$ is a submanifold of \mathcal{Z} .

The last point of the Lemma shows that the diffeomorphism $U \times_{U_{\tilde{\mathfrak{s}}}} Y_{\tilde{\mathfrak{s}}} \xrightarrow{\sim} UY_{\tilde{\mathfrak{s}}}$ induces the diffeomorphism $K \times_{K_{\tilde{\mathfrak{s}}}} \mathcal{T}_{\tilde{\mathfrak{s}}} \xrightarrow{\sim} K\mathcal{T}_{\tilde{\mathfrak{s}}}$.

Let us check the last point of Theorem 4.5. For any $\beta \in \mathfrak{p}$, the vector field $z \in \mathcal{Z} \mapsto \beta \cdot z$ is the gradient vector field of the function $\langle \Phi_{\mathfrak{p}}, \beta \rangle : \mathcal{Z} \rightarrow \mathbb{R}$. Let $x \in \mathcal{T}_{\mathfrak{s}}$: then $\beta \in \mathfrak{p}_x$ if and only if the differential $d\langle \Phi_{\mathfrak{p}}, \beta \rangle|_x : \mathbf{T}_x \mathcal{Z} \rightarrow \mathbb{R}$ vanishes. Thanks to the second point of Theorem 4.5, we know that $\mathbf{T}_x \mathcal{Z} = \mathbf{T}_x \mathcal{T}_{\mathfrak{s}} + \mathfrak{k} \cdot x$. For any $X \in \mathfrak{k}$, we have

$$d\langle \Phi_{\mathfrak{p}}, \beta \rangle|_x(X \cdot x) = \langle \Phi_{\mathfrak{p}}(x), [\beta, X] \rangle$$

with $\Phi_{\mathfrak{p}}(x) \in \mathfrak{s}$. Thus $d\langle \Phi_{\mathfrak{p}}, \beta \rangle|_x$ vanishes on $\mathfrak{k} \cdot x$ if and only if $\beta \in \mathfrak{p}_{\mathfrak{s}}$. At this stage we have that $\mathfrak{p}_x \subset \mathfrak{p}_{\mathfrak{s}}$. The function $\Phi_{\mathfrak{p}}$, when restricted to the submanifold $\mathcal{T}_{\mathfrak{s}}$, takes value in \mathfrak{s} . If we take $\beta \in [\mathfrak{k}_{\mathfrak{s}}, \mathfrak{p}_{\mathfrak{s}}] \subset \mathfrak{p}_{\mathfrak{s}}$, the function $\langle \Phi_{\mathfrak{p}}, \beta \rangle$ is constant equal to zero on $\mathcal{T}_{\mathfrak{s}}$, thus $d\langle \Phi_{\mathfrak{p}}, \beta \rangle|_x$ vanishes on $\mathbf{T}_x \mathcal{T}_{\mathfrak{s}}$. We have checked that $[\mathfrak{k}_{\mathfrak{s}}, \mathfrak{p}_{\mathfrak{s}}] \subset \mathfrak{p}_x$. \square

5 Proof of Theorem 2.9

Let $\Delta_{\mathfrak{p}}(\mathcal{Z}) \subset \mathfrak{a}_+^*$ be the moment polytope of a connected component \mathcal{Z} of the real part Z (supposed non-empty) of a proper Kähler Hamiltonian U -manifold (M, Ω, \mathbb{J}) .

We define the following convex subsets of the chamber \mathfrak{a}_+^* .

- $\Delta_{\text{inf-RP}}$ is the set of points $\xi \in \mathfrak{a}_+^*$ satisfying the inequalities $\langle \xi, \gamma \rangle \geq \langle \Phi_{\mathfrak{p}}(\mathcal{C}), \gamma \rangle$, for any **infinitesimal real Ressayre's pair** (γ, \mathcal{C}) of \mathcal{Z} .
- $\Delta_{\text{inf-RP}}^{\text{reg}}$ is the set of points $\xi \in \mathfrak{a}_+^*$ satisfying the inequalities $\langle \xi, \gamma \rangle \geq \langle \Phi_{\mathfrak{p}}(\mathcal{C}), \gamma \rangle$, for any **regular infinitesimal real Ressayre's pair** (γ, \mathcal{C}) of \mathcal{Z} .
- Δ_{RP} is the set of points $\xi \in \mathfrak{a}_+^*$ satisfying the inequalities $\langle \xi, \gamma \rangle \geq \langle \Phi_{\mathfrak{p}}(\mathcal{C}), \gamma \rangle$, for any **real Ressayre's pair** (γ, \mathcal{C}) of \mathcal{Z} .
- $\Delta_{\text{RP}}^{\text{reg}}$ is the set of points $\xi \in \mathfrak{a}_+^*$ satisfying the inequalities $\langle \xi, \gamma \rangle \geq \langle \Phi_{\mathfrak{p}}(\mathcal{C}), \gamma \rangle$, for any **regular real Ressayre's pair** (γ, \mathcal{C}) of \mathcal{Z} .

By definition, we have the commutative diagram, where all the maps are inclusions:

$$\begin{array}{ccc} \Delta_{\text{inf-RP}} & \hookrightarrow & \Delta_{\text{inf-RP}}^{\text{reg}} \\ \downarrow & & \downarrow \\ \Delta_{\text{RP}} & \hookrightarrow & \Delta_{\text{RP}}^{\text{reg}} \end{array}$$

In §5.1 and §5.2, we prove the inclusions $\Delta_{\text{RP}} \subset \Delta_{\mathfrak{p}}(\mathcal{Z}) \subset \Delta_{\text{inf-RP}}$. It follows then that $\Delta_{\mathfrak{p}}(\mathcal{Z}) = \Delta_{\text{inf-RP}} = \Delta_{\text{RP}}$.

In §5.3, we prove the inclusion $\Delta_{\text{RP}}^{\text{reg}} \subset \Delta_{\mathfrak{p}}(\mathcal{Z})$, and since $\Delta_{\mathfrak{p}}(\mathcal{Z}) = \Delta_{\text{inf-RP}} \subset \Delta_{\text{inf-RP}}^{\text{reg}} \subset \Delta_{\text{RP}}^{\text{reg}}$, we get finally that $\Delta_{\mathfrak{p}}(\mathcal{Z}) = \Delta_{\text{RP}}^{\text{reg}} = \Delta_{\text{inf-RP}}^{\text{reg}}$. The proof of Theorem 2.9 is complete.

5.1 $\Delta_{\text{RP}} \subset \Delta_{\mathfrak{p}}(\mathcal{Z})$

Let $\xi_o \in \mathfrak{a}_+^*$ that does not belong to $\Delta_{\mathfrak{p}}(\mathcal{Z})$. The aim of this section is to prove that $\xi_o \notin \Delta_{\text{RP}}$. In other words, we will show the existence of a real Ressayre's pair (γ, \mathcal{C}) of \mathcal{Z} such that $\langle \xi_o, \gamma \rangle < \langle \Phi_{\mathfrak{p}}(\mathcal{C}), \gamma \rangle$.

Let $r > 0$ be the distance between ξ_o and $\Delta_{\mathfrak{p}}(\mathcal{Z})$, and let ξ be an element in the interior of the Weyl chamber \mathfrak{a}_+^* such as $\|\xi - \xi_o\| < \frac{r}{2}$: so the distance between ξ and $\Delta_{\mathfrak{p}}(\mathcal{Z})$ is strictly larger than $\frac{r}{2}$.

Let ξ' be the orthogonal projection of ξ on $\Delta_{\mathfrak{p}}(\mathcal{Z})$ and let $\gamma = \xi' - \xi \in \mathfrak{a}^* \simeq \mathfrak{a}$. Let \mathcal{C} be the union of the connected components of \mathcal{Z}^γ intersecting $\Phi_{\mathfrak{p}}^{-1}(\xi')$. Thanks to Theorem 4.2, we know that (γ, \mathcal{C}) is a real Ressayre's pair on \mathcal{Z} .

Using the fact that $\gamma = \xi' - \xi$, we compute

$$\begin{aligned} \langle \xi_o, \gamma \rangle - \langle \Phi_{\mathfrak{p}}(\mathcal{C}), \gamma \rangle &= \langle \xi_o, \gamma \rangle - \langle \xi', \gamma \rangle \\ &= \langle \xi_o - \xi, \gamma \rangle - \|\gamma\|^2 \\ &\leq -\|\gamma\|(\|\gamma\| - \|\xi_o - \xi\|) \\ &< 0. \end{aligned}$$

The last inequality comes from the fact that $\|\xi_o - \xi\| < \frac{r}{2}$ and that $\|\gamma\| > \frac{r}{2}$ since $\|\gamma\|$ represents the distance between ξ and $\Delta_{\mathfrak{p}}(\mathcal{Z})$.

5.2 $\Delta_{\mathfrak{p}}(\mathcal{Z}) \subset \Delta_{\text{inf-RP}}$

Let $\xi \in \mathfrak{a}_+^*$ belonging to $\Delta_{\mathfrak{p}}(\mathcal{Z})$. The aim of this section is to prove that $\xi \in \Delta_{\text{inf-RP}}$. Let $\tilde{\xi}$ be the corresponding element in $\mathfrak{t}_+^* \cap (\mathfrak{u}^{-\sigma})^*$: in other words $\xi = j^*(\tilde{\xi})$. Consider the Kähler Hamiltonian U -manifold $N := M \times U(-\tilde{\xi})$ and the connected component $\mathcal{Z}_N := \mathcal{Z} \times K(-\tilde{\xi})$ of its real part.

Let $\Phi_{\mathfrak{p}}^N : \mathcal{Z}_N \rightarrow \mathfrak{p}^*$ be the gradient map. We know that the strata

$$(\mathcal{Z}_N)_0 = \{n \in \mathcal{Z}_N, \overline{Gn} \cap (\Phi_{\mathfrak{p}}^N)^{-1}(0) \neq \emptyset\}$$

is a G -invariant dense open subset of \mathcal{Z}_N .

Let (γ, \mathcal{C}) be an infinitesimal real Ressayre's pair on \mathcal{Z} , and let $\mathcal{C}_N := \mathcal{C} \times (G_\gamma / P_{-\xi} \cap G_\gamma)$ be the corresponding connected component of \mathcal{Z}_N^γ . Notice that \mathcal{C}_N is invariant under the action of the stabilizer subgroup G_γ .

Let $\mathcal{C}_N^- := \{n \in \mathcal{Z}_N, \lim_{t \rightarrow \infty} e^{t\gamma} n \in \mathcal{C}_N\}$ be the real Bialynicki-Birula's submanifold.

Lemma 5.1 1. *The set $G\mathcal{C}_N^-$ has a non-empty interior in \mathcal{Z}_N .*

2. $\mathcal{C}_N^- \cap (\mathcal{Z}_N)_0 \neq \emptyset$.

Proof: The second point follows from the first one since $(\mathcal{Z}_N)_0$ is a dense G -invariant subset of \mathcal{Z}_N .

Let $x \in \mathcal{C}$ so that $\mathfrak{n}^{\gamma>0} \simeq (\mathbf{T}_x \mathcal{Z})^{\gamma>0}$. The point $n = (x, [e]) \in \mathcal{C}_N$ belongs to the interior of $G\mathcal{C}_N^-$ if we show that $\mathfrak{g} \cdot n + \mathbf{T}_n \mathcal{C}_N^- = \mathbf{T}_n \mathcal{Z}_N$. Since $\mathbf{T}_n \mathcal{C}_N^- = (\mathbf{T}_n \mathcal{Z}_N)^{\gamma \leq 0}$ it is sufficient to check that $(\mathbf{T}_n \mathcal{Z}_N)^{\gamma > 0} \subset \mathfrak{g} \cdot n$. We have the decomposition $(\mathbf{T}_n \mathcal{Z}_N)^{\gamma > 0} = (\mathbf{T}_x \mathcal{Z})^{\gamma > 0} \oplus \mathfrak{g}^{\gamma > 0} \cdot [e]$. So for any $v \in (\mathbf{T}_n \mathcal{Z}_N)^{\gamma > 0}$, there exists $X \in \mathfrak{g}^{\gamma > 0}$ so that $v - X \cdot (x, [e]) \in (\mathbf{T}_x \mathcal{Z}_N)^{\gamma > 0}$. But $\mathfrak{n}^{\gamma > 0} \simeq (\mathbf{T}_x \mathcal{Z})^{\gamma > 0}$, so there exists $Y \in \mathfrak{n}^{\gamma > 0}$ such as $v - X \cdot (x, [e]) = Y \cdot x$. The Lie algebra \mathfrak{n} is contained in the Lie algebra of the parabolic subgroup $P_{-\xi}$: hence $Y \cdot [e] = 0$. Finally we have proved that $v = (X + Y) \cdot (x, [e]) \in \mathfrak{g} \cdot n$. \square

The proof of the inclusion $\Delta_{\mathfrak{p}}(\mathcal{Z}) \subset \Delta_{\text{int-RP}}$ follows from the next result.

Lemma 5.2 *Let (γ, \mathcal{C}) be an infinitesimal real Ressayre's pair on \mathcal{Z} . Then, the following inequality holds $\langle \xi, \gamma \rangle \geq \langle \Phi_{\mathfrak{p}}(\mathcal{C}), \gamma \rangle$ for any $\xi \in \mathfrak{a}_+^*$ belonging to $\Delta_{\mathfrak{p}}(\mathcal{Z})$.*

Proof : Let $n \in \mathcal{C}_N^- \cap (\mathcal{Z}_N)_0$, and let $n_\gamma \in \mathcal{C}_N$ be the limit $\lim_{t \rightarrow \infty} e^{t\gamma} n$.

Let $P_\gamma \subset G$ be the parabolic subgroup associated to γ (see (6)). Since $G = KP_\gamma$, the fact that $n \in (\mathcal{Z}_N)_0$ means that $\overline{P_\gamma n} \cap (\Phi_{\mathfrak{p}}^N)^{-1}(0) \neq \emptyset$. In other words, $\min_{z \in P_\gamma n} \|\Phi_{\mathfrak{p}}^N(z)\| = 0$ which implies $0 \geq \min_{z \in P_\gamma n} \langle \Phi_{\mathfrak{p}}^N(z), \gamma \rangle$.

Consider now the function $t \geq 0 \mapsto \langle \Phi_{\mathfrak{p}}^N(e^{t\gamma} z), \gamma \rangle$ attached to $z \in P_\gamma n$. Since $\frac{d}{dt} \langle \Phi_{\mathfrak{p}}^N(e^{t\gamma} z), \gamma \rangle = -\|\gamma_{\mathcal{Z}_N}\|^2(e^{t\gamma} z) \leq 0$, we have

$$(9) \quad \langle \Phi_{\mathfrak{p}}^N(z), \gamma \rangle \geq \langle \Phi_{\mathfrak{p}}^N(e^{t\gamma} z), \gamma \rangle, \quad \forall t \geq 0.$$

Let's take $p \in P_\gamma$ and $z = pn$. Then, the limit $\lim_{t \rightarrow \infty} e^{t\gamma} z$ is equal to $g_\gamma n_\gamma$ where $g_\gamma = \lim_{t \rightarrow \infty} e^{t\gamma} p e^{-t\gamma} \in G_\gamma$. If we take the limit in (9) as $t \rightarrow \infty$, we get

$$\langle \Phi_{\mathfrak{p}}^N(z), \gamma \rangle \geq \langle \Phi_{\mathfrak{p}}^N(g_\gamma n_\gamma), \gamma \rangle = \langle \Phi_{\mathfrak{p}}^N(\mathcal{C}_N), \gamma \rangle = \langle \Phi_{\mathfrak{p}}(\mathcal{C}), \gamma \rangle - \langle \xi, \gamma \rangle, \quad \forall z \in P_\gamma n.$$

We obtain finally $0 \geq \min_{z \in P_\gamma n} \langle \Phi_{\mathfrak{p}}^N(z), \gamma \rangle \geq \langle \Phi_{\mathfrak{p}}(\mathcal{C}), \gamma \rangle - \langle \xi, \gamma \rangle$. \square

5.3 $\Delta_{\text{RP}}^{\text{reg}} \subset \Delta_{\mathfrak{p}}(\mathcal{Z})$

The aim of this section is the proof of the following

Theorem 5.3 *Let $\xi \in \mathfrak{a}_+^*$ satisfying the inequalities $\langle \xi, \gamma \rangle \geq \langle \Phi_{\mathfrak{p}}(\mathcal{C}), \gamma \rangle$, for any regular real Ressayre's pair (γ, \mathcal{C}) of \mathcal{Z} . Then $\xi \in \Delta_{\mathfrak{p}}(\mathcal{Z})$.*

Our arguments go as follows: we will show that there exists a collection $(\gamma_i, \mathcal{C}_i)_{i \in I}$ of regular real Ressayre's pairs for which we have

$$\bigcap_{i \in I} \left\{ \xi \in \mathfrak{a}_+^*, \langle \xi, \gamma_i \rangle \geq \langle \Phi(\mathcal{C}_i), \gamma_i \rangle \right\} = \Delta_{\mathfrak{p}}(\mathcal{Z}).$$

The set I will be finite when \mathcal{Z} is compact.

Before starting the description of the collection $(\gamma_i, \mathcal{C}_i)_{i \in I}$, recall the following fact concerning admissible elements (see §4.2 in [18]).

Remark 5.4 *Let $\gamma \in \mathfrak{a}$ be a rational element such that $K \mathcal{Z}^\gamma = \mathcal{Z}$. We have $\dim_{\mathfrak{p}}(\mathcal{Z}^\gamma) = \dim_{\mathfrak{p}}(\mathcal{Z})$, thus γ is admissible.*

5.3.1 The real Ressayre's pair $(\gamma_{\mathfrak{s}}, \mathcal{C}_{\mathfrak{s}})$

In this section, we show that a real Ressayre's pair describes the fact that $\Delta_{\mathfrak{p}}(\mathcal{Z})$ is contained in $\bar{\mathfrak{s}}$. Let us denote by $h_{\alpha} \in \mathfrak{a}$ the coroot associated to a root $\alpha \in \Sigma$: h_{α} is the rational element of $[\mathfrak{g}_{\alpha}, \mathfrak{g}_{-\alpha}] \cap \mathfrak{a}$ satisfying $\langle \alpha, X \rangle = (h_{\alpha}, X)_b, \forall X \in \mathfrak{a}$ (see the Appendix).

Let $\Sigma^+ \subset \Sigma$ be the set of positive roots associated to the choice of the Weyl chamber \mathfrak{a}_+^* : $\xi \in \mathfrak{a}_+^*$ if and only if $\langle \xi, h_{\alpha} \rangle \geq 0$ for every $\alpha \in \Sigma^+$. Let $\Sigma_{\mathfrak{s}}^+ \subset \Sigma^+$ be the set of positive roots orthogonal to \mathfrak{s} : $\alpha \in \Sigma_{\mathfrak{s}}^+$ if $\langle \xi, h_{\alpha} \rangle = 0$ for every $\xi \in \mathfrak{s}$.

Definition 5.5 Consider the following rational vector

$$\gamma_{\mathfrak{s}} := - \sum_{\alpha \in \Sigma_{\mathfrak{s}}^+} h_{\alpha} \in \mathfrak{a}.$$

Lemma 5.6 The element $\gamma_{\mathfrak{s}}$ satisfies the following properties:

- $\langle \xi, \gamma_{\mathfrak{s}} \rangle \leq 0$ for any $\xi \in \mathfrak{a}_{\geq 0}^*$.
- For any $\xi \in \mathfrak{a}_{\geq 0}^*$, $\langle \xi, \gamma_{\mathfrak{s}} \rangle = 0$ if and only if $\xi \in \bar{\mathfrak{s}}$.
- $\gamma_{\mathfrak{s}}$ acts trivially on the principal-cross-section⁴ $\mathcal{T}_{\mathfrak{s}}$.
- $\langle \alpha, \gamma_{\mathfrak{s}} \rangle < 0$ for any $\alpha \in \Sigma_{\mathfrak{s}}^+$.

Proof: The first two points follow from the fact that $\langle \xi, h_{\alpha} \rangle \geq 0$ for any $\xi \in \mathfrak{a}_{\geq 0}^*$ and any positive roots α . The third point is due to the fact that $\gamma_{\mathfrak{s}} \in [\mathfrak{k}_{\mathfrak{s}}, \mathfrak{p}_{\mathfrak{s}}]$. For the last point, see the Appendix. \square

Let $\mathcal{C}_{\mathfrak{s}}$ be the union of the connected components of $\mathcal{Z}^{\gamma_{\mathfrak{s}}}$ intersecting $\mathcal{T}_{\mathfrak{s}}$. We start with the following basic result.

Lemma 5.7 • $\gamma_{\mathfrak{s}}$ is an admissible element.

- For any $\xi \in \mathfrak{a}_+^*$, the inequality $\langle \xi, \gamma_{\mathfrak{s}} \rangle \geq \langle \Phi_{\mathfrak{p}}(\mathcal{C}_{\mathfrak{s}}), \gamma_{\mathfrak{s}} \rangle$ is equivalent to $\xi \in \bar{\mathfrak{s}}$.

Proof: We know that $\mathcal{T}_{\mathfrak{s}} \subset \mathcal{C}_{\mathfrak{s}}$ and that $K\mathcal{T}_{\mathfrak{s}}$ is dense in \mathcal{Z} . It follows that $K\mathcal{C}_{\mathfrak{s}} = \mathcal{Z}$, hence $\gamma_{\mathfrak{s}}$ is an admissible element (see Remark 5.4). Now we consider the inequality $\langle \xi, \gamma_{\mathfrak{s}} \rangle \geq \langle \Phi_{\mathfrak{p}}(\mathcal{C}_{\mathfrak{s}}), \gamma_{\mathfrak{s}} \rangle$ for an element $\xi \in \mathfrak{a}_+^*$. First we notice that $\langle \Phi_{\mathfrak{p}}(\mathcal{C}_{\mathfrak{s}}), \gamma_{\mathfrak{s}} \rangle = 0$, and the first two points of the previous Lemma tell us that $\langle \xi, \gamma_{\mathfrak{s}} \rangle \geq 0$ is equivalent to $\xi \in \bar{\mathfrak{s}}$. \square

Proposition 5.8 $(\gamma_{\mathfrak{s}}, \mathcal{C}_{\mathfrak{s}})$ is a regular real Ressayre's pair on \mathcal{Z} .

Proof: Using the identification $\mathfrak{a} \simeq \mathfrak{a}^*$, we view $\gamma_{\mathfrak{s}}$ as a rational element of \mathfrak{s}^* orthogonal to \mathfrak{s} . Take $\xi' \in \Delta_{\mathfrak{p}}(\mathcal{Z}) \cap \mathfrak{s}$ contained in the image $\Phi_{\mathfrak{p}}(\mathcal{C}_{\mathfrak{s}})$ and consider the elements $\xi(n) := \xi' - \frac{1}{n}\gamma_{\mathfrak{s}}$ for $n \geq 1$. We notice that for n large enough

⁴See §4.3.

1. $\xi(n)$ is a regular element of the Weyl chamber \mathfrak{a}_+^* ,
2. $\xi(n) \notin \Delta_{\mathfrak{p}}(\mathcal{Z})$,
3. ξ' is the orthogonal projection of $\xi(n)$ on $\Delta_{\mathfrak{p}}(\mathcal{Z})$.

So we can exploit the results of §4.2 with the elements $\xi(n)$ for $n \gg 1$. Proposition 4.2 and Lemma 5.7 tell us that $(\gamma_{\mathfrak{s}}, \mathcal{C}_{\mathfrak{s}})$ is a regular real Ressayre's pair. \square

5.3.2 The real Ressayre's pairs $(\gamma_l^{\pm}, \mathcal{C}_l^{\pm})$

Let $\mathbb{R}\mathfrak{s} \subset \mathfrak{a}^*$ be the rational vector subspace generated by the face \mathfrak{s} . The closed convex polytope $\Delta_{\mathfrak{p}}(\mathcal{Z})$ generate an affine subspace Π of $\mathbb{R}\mathfrak{s}$. In this section we show that a finite family of real Ressayre's pairs describe the fact that $\Delta_{\mathfrak{p}}(\mathcal{Z})$, viewed as a subset of $\mathbb{R}\mathfrak{s}$, is contained in the affine subspace Π .

In this section we will use the identifications $\mathfrak{p}^* \simeq \mathfrak{p}$, and $\mathfrak{a} \simeq \mathfrak{a}^*$ given by the invariant scalar product $(-, -)_b$ (see the Appendix).

We start with the orthogonal decompositions $\mathfrak{p} = \mathfrak{a} \oplus \mathfrak{q}$ and $\mathfrak{a} = \mathbb{R}\mathfrak{s} \oplus \mathbb{R}\mathfrak{s}^{\perp}$. It follows that $\mathfrak{p}_{\mathfrak{s}} = \mathfrak{a} \oplus \mathfrak{q}_{\mathfrak{s}}$. Let $\vec{\Pi}^{\perp}$ be the orthogonal of $\vec{\Pi}$ in $\mathbb{R}\mathfrak{s}$.

For any $x \in \mathcal{Z}$, we define $\mathfrak{a}_x = \{X \in \mathfrak{a}, X \cdot x = 0\}$.

Lemma 5.9 1. For any $x \in \mathcal{T}_{\mathfrak{s}}$, we have $\mathfrak{p}_x = \mathfrak{a}_x \oplus \mathfrak{q}_{\mathfrak{s}}$ with $\vec{\Pi}^{\perp} \oplus \mathbb{R}\mathfrak{s}^{\perp} \subset \mathfrak{a}_x$.

2. The equality $\vec{\Pi}^{\perp} \oplus \mathbb{R}\mathfrak{s}^{\perp} = \mathfrak{a}_x$ holds on an open subset of $\mathcal{T}_{\mathfrak{s}}$, thus

$$(10) \quad \dim_{\mathfrak{p}}(\mathcal{Z}) = \dim(\vec{\Pi}^{\perp}) + \dim(\mathbb{R}\mathfrak{s}^{\perp}) + \dim(\mathfrak{q}_{\mathfrak{s}}).$$

3. The subspaces $\vec{\Pi}^{\perp}$ and $\vec{\Pi}$ are rational.

Proof : Let us come back to the arguments used in the proof of Theorem 4.5. We know that $\mathfrak{p}_x \subset \mathfrak{p}_{\mathfrak{s}} = \mathfrak{a} \oplus \mathfrak{q}_{\mathfrak{s}}$ for any $x \in \mathcal{T}_{\mathfrak{s}}$, thus $\mathfrak{p}_x = \mathfrak{a}_x \oplus \mathfrak{q}_{\mathfrak{s}}$. An element β belongs to \mathfrak{a}_x if and only if the differential $d\langle \Phi_{\mathfrak{p}}, \beta \rangle|_x : \mathbf{T}_x \mathcal{T}_{\mathfrak{s}} \rightarrow \mathbb{R}$ vanishes. We see that for any $\beta \in \vec{\Pi}^{\perp} \oplus \mathbb{R}\mathfrak{s}^{\perp}$ the map $\langle \Phi_{\mathfrak{p}}, \beta \rangle : \mathcal{T}_{\mathfrak{s}} \rightarrow \mathbb{R}$ is constant : we obtain then that $\vec{\Pi}^{\perp} \oplus \mathbb{R}\mathfrak{s}^{\perp} \subset \mathfrak{a}_x, \forall x \in \mathcal{T}_{\mathfrak{s}}$.

Since the image of the map $\Phi_{\mathfrak{p}} : \mathcal{T}_{\mathfrak{s}} \rightarrow \Pi$ is open in Π , there exists $x_o \in \mathcal{T}_{\mathfrak{s}}$ such that $d\Phi_{\mathfrak{p}}|_{x_o} : \mathbf{T}_{x_o} \mathcal{T}_{\mathfrak{s}} \rightarrow \vec{\Pi}$ is surjective. It follows that $\vec{\Pi}^{\perp} \oplus \mathbb{R}\mathfrak{s}^{\perp} = \mathfrak{a}_{x_o}$. The second point is proved.

The vector subspaces $\mathbb{R}\mathfrak{s} \subset \mathfrak{a}$ and $\mathbb{R}\mathfrak{s}^{\perp} \subset \mathfrak{a}$ are rational. Let $x_o \in \mathcal{T}_{\mathfrak{s}}$ such that $\mathfrak{a}_{x_o} = \vec{\Pi}^{\perp} \oplus \mathbb{R}\mathfrak{s}^{\perp}$. The Lie algebra \mathfrak{t}_{x_o} is a rational subspace of \mathfrak{t} , thus $\mathfrak{a}_{x_o} = i(\mathfrak{t}_{x_o} \cap \mathfrak{t}^{-\sigma})$ is a rational subspace of \mathfrak{a} . It follows that $\vec{\Pi}^{\perp} = \mathfrak{a}_{x_o} \cap \mathbb{R}\mathfrak{s}$ is a rational subspace of \mathfrak{a} . \square

Let $(\eta_l)_{l \in L}$ be a rational basis of $\vec{\Pi}^{\perp}$. We consider then the rational elements

$$\gamma_l^{\pm} := \pm \eta_l + \gamma_{\mathfrak{s}} \in \vec{\Pi}^{\perp} \oplus \mathbb{R}\mathfrak{s}^{\perp}, \quad l \in L.$$

Thanks to Lemma 5.9, we know that $\mathcal{T}_{\mathfrak{s}} \subset \mathcal{Z}^{\gamma_l^{\pm}}, \forall l \in L$. Let $\xi'_o \in \Delta_{\mathfrak{p}}(\mathcal{Z}) \cap \mathfrak{s}$. For any $l \in L$, we denote by \mathcal{C}_l^{\pm} the union of connected components of $\mathcal{Z}^{\gamma_l^{\pm}}$ intersecting $\Phi_{\mathfrak{p}}^{-1}(\xi'_o) \subset \mathcal{T}_{\mathfrak{s}}$.

Lemma 5.10 • Any γ_l^\pm is an admissible element.

- The set of elements $\xi \in \mathbb{R}\mathfrak{s}$ satisfying the inequalities

$$(11) \quad \langle \xi, \gamma_l^\pm \rangle \geq \langle \Phi_{\mathfrak{p}}(\mathcal{C}_l^\pm), \gamma_l^\pm \rangle, \quad \forall l \in L$$

corresponds to the affine subspace Π .

Proof : Like in the proof of Lemma 5.7, we see that $K \cdot \mathcal{Z}^{\gamma_l^\pm} = \mathcal{Z}$ since $\mathcal{T}_{\mathfrak{s}} \subset \mathcal{Z}^{\gamma_l^\pm}$ and $K\mathcal{T}_{\mathfrak{s}}$ is dense in \mathcal{Z} . It follows that $\gamma_{\mathfrak{s}}$ is an admissible element.

The element ξ'_o belongs to Π . Since $\langle \xi, \gamma_{\mathfrak{s}} \rangle = 0, \forall \xi \in \mathbb{R}\mathfrak{s}$, the inequalities (11) are equivalent to $\pm \langle \xi - \xi'_o, \eta_l \rangle \geq 0, \forall l \in L$: in other words $\xi - \xi'_o \in \overrightarrow{\Pi}$, so $\xi \in \Pi$. \square

Proposition 5.11 For any $l \in L$, $(\gamma_l^\pm, \mathcal{C}_l^\pm)$ is a regular real Ressayre's pair of \mathcal{Z} .

Proof : The proof follows the lines of the proof of Proposition 5.8. The element ξ'_o is contained in the image $\Phi_{\mathfrak{p}}(\mathcal{C}_l^\pm)$. We consider the elements $\xi_l^\pm(n) := \xi'_o - \frac{1}{n}\gamma_l^\pm$ for $n \geq 1$. We notice that for n large enough

1. $\xi_l^\pm(n)$ is a regular element of the chamber \mathfrak{a}_+^* ,
2. $\xi_l^\pm(n) \notin \Delta_{\mathfrak{p}}(\mathcal{Z})$,
3. ξ'_o is the orthogonal projection of $\xi_l^\pm(n)$ on $\Delta_{\mathfrak{p}}(\mathcal{Z})$.

So we can exploit the results of §4.2 with the elements $a_l^\pm(n)$ for $n \gg 1$. Proposition 4.2 and Lemma 5.10 tell us that $(\gamma_l^\pm, \mathcal{C}_l^\pm)$ is a regular real Ressayre's pair. \square

5.3.3 The real Ressayre's pairs $(\gamma_F, \mathcal{C}_F)$

In this section, we show that the polytope $\Delta_{\mathfrak{p}}(\mathcal{Z})$, viewed as a subset of the affine subspace Π , is the intersection of the cone $\Pi \cap \mathfrak{a}_+^*$, with a collection of half spaces of Π parametrized by a family of real Ressayre's pairs.

Definition 5.12 An open facet F of $\Delta_{\mathfrak{p}}(\mathcal{Z})$ is called non trivial if $F \subset \mathfrak{s}$. We denote by $\mathcal{F}(\mathcal{Z})$ the set of non trivial open facets of $\Delta_{\mathfrak{p}}(\mathcal{Z})$.

Let $F \in \mathcal{F}(\mathcal{Z})$. There exists⁵ $\eta_F \in \overrightarrow{\Pi}$ such as the affine space generated by F is $\Pi_F = \{\xi \in \Pi, \langle \xi, \eta_F \rangle = \langle \xi_F, \eta_F \rangle\}$ for any $\xi_F \in F$. The vector η_F is chosen so that $\Delta_{\mathfrak{p}}(\mathcal{Z})$ is contained in the half space $\{\xi \in \Pi, \langle \xi, \eta_F \rangle \geq \langle \xi_F, \eta_F \rangle\}$.

By definition of the set $\mathcal{F}(\mathcal{Z})$, we have the following description of the polytope $\Delta_{\mathfrak{p}}(\mathcal{Z})$:

$$(12) \quad \Delta_{\mathfrak{p}}(\mathcal{Z}) = \bigcap_{F \in \mathcal{F}(\mathcal{Z})} \left\{ \xi \in \Pi, \langle \xi, \eta_F \rangle \geq \langle \xi_F, \eta_F \rangle \right\} \cap \mathfrak{a}_+^*.$$

⁵Here we see $\overrightarrow{\Pi}$ as a subspace of \mathfrak{a} through the identification $\mathfrak{a}^* \simeq \mathfrak{a}$.

The function $\langle \Phi_{\mathfrak{p}}, \eta_F \rangle : \mathcal{T}_{\mathfrak{s}} \rightarrow \mathbb{R}$ is locally constant on $\mathcal{T}_{\mathfrak{s}}^{\eta_F}$ and it takes its minimal value on $\Phi_{\mathfrak{p}}^{-1}(F) \subset \mathcal{T}_{\mathfrak{s}}$, thus $\Phi_{\mathfrak{p}}^{-1}(F)$ is an open subset of the submanifold $\mathcal{T}_{\mathfrak{s}}^{\eta_F}$. The map $\Phi_{\mathfrak{p}} : \Phi_{\mathfrak{p}}^{-1}(F) \rightarrow F$ is surjective, so it admits a regular value $\xi'_F \in F$. For any $x \in \Phi_{\mathfrak{p}}^{-1}(\xi'_F)$, the tangent map $\mathbf{T}_x \Phi_{\mathfrak{p}} : \mathbf{T}_x \mathcal{T}_{\mathfrak{s}}^{\eta_F} \rightarrow \vec{F}$ is surjective, thus

$$(13) \quad \mathfrak{a}_x = \mathbb{R}\eta_F \oplus \vec{\Pi}^{\perp} \oplus \mathbb{R}\mathfrak{s}^{\perp}, \quad \forall x \in \Phi_{\mathfrak{p}}^{-1}(\xi'_F).$$

Since the vector subspaces \mathfrak{a}_x , $\vec{\Pi}^{\perp}$, $\mathbb{R}\mathfrak{s}^{\perp}$ are rational, the vector η_F can be taken rational. We consider now the rational elements

$$\gamma_F = \eta_F + \gamma_{\mathfrak{s}}, \quad F \in \mathcal{F}(\mathcal{Z}).$$

Let \mathcal{C}_F be the union of the connected components of \mathcal{Z}^{γ_F} intersecting $\Phi_{\mathfrak{p}}^{-1}(\xi'_F)$. The identities (10) and (13) show that $\dim_{\mathfrak{p}}(\mathcal{C}_F) = \dim_{\mathfrak{p}}(\mathcal{Z}) + 1$.

Lemma 5.13 • γ_F is an admissible element, for any $F \in \mathcal{F}(\mathcal{Z})$.

- The set of elements $\xi \in \Pi \cap \mathfrak{t}_{\geq 0}^*$ satisfying the inequalities

$$(14) \quad \langle \xi, \gamma_F \rangle \geq \langle \Phi_{\mathfrak{p}}(\mathcal{C}_F), \gamma_F \rangle, \quad \forall F \in \mathcal{F}(\mathcal{Z})$$

corresponds to $\Delta_{\mathfrak{p}}(\mathcal{Z})$.

Proof : The first point is due to the fact that η_F is rational and that $\dim_{\mathfrak{p}}(\mathcal{C}_F) = \dim_{\mathfrak{p}}(\mathcal{Z}) + 1$. The last assertion is a consequence of (12). Notice that the relations $\langle \xi, \gamma_F \rangle \geq \langle \Phi_{\mathfrak{p}}(\mathcal{C}_F), \gamma_F \rangle$ and $\langle \xi, \eta_F \rangle \geq \langle \Phi_{\mathfrak{p}}(\mathcal{C}_F), \eta_F \rangle$ are equivalent for any $\xi \in \Pi$. \square

Proposition 5.14 For any $F \in \mathcal{F}(\mathcal{Z})$, the couple $(\gamma_F, \mathcal{C}_F)$ is a real Ressayre's pair on \mathcal{Z} .

Proof : The proof follows the lines of the proof of Proposition 5.8. Consider the elements $\xi_F(n) := \xi'_F - \frac{1}{n}\gamma_F$ for $n \geq 1$. We notice that for n is large enough

1. $\xi_F(n)$ is a regular element of the Weyl chamber,
2. $\xi_F(n) \notin \Delta_{\mathfrak{p}}(\mathcal{Z})$,
3. ξ'_F is the orthogonal projection of $\xi_F(n)$ on $\Delta_{\mathfrak{p}}(\mathcal{Z})$.

So we can exploit the results of §4.2 with the elements $\xi_F(n)$ for $n \gg 1$. Proposition 4.2 and Lemma 5.13 tell us that $(\gamma_F, \mathcal{C}_F)$ is a regular real Ressayre's pair on \mathcal{Z} . \square

6 Appendix

In this paper, $U_{\mathbb{C}}$ denote a connected complex reductive Lie group with maximal compact subgroup U . Let σ be a complex-conjugate involution on $U_{\mathbb{C}}$ leaving the subgroup U invariant. Let T be a maximal torus of U that is stable under σ : we suppose that the vector subspace $\mathfrak{t}^{-\sigma} = \{X \in \mathfrak{t}, \sigma(X) = -X\}$ has a maximal dimension.

Take a faithful representation $\rho : U \rightarrow U(V)$ where V is an Hermitian vector space. It extends to a morphism $\rho : U_{\mathbb{C}} \rightarrow \mathrm{GL}(V)$, so that $\rho(U_{\mathbb{C}})$ is a closed subgroup stable under the Cartan involution $\Theta : \mathrm{GL}(V) \rightarrow \mathrm{GL}(V)$.

We consider the real bilinear map

$$(15) \quad b : \mathfrak{u}_{\mathbb{C}} \times \mathfrak{u}_{\mathbb{C}} \longrightarrow \mathbb{R}$$

defined by $b(X, Y) = \mathrm{Re}\left(\mathrm{Tr}(d\rho(X)d\rho(Y)) + \mathrm{Tr}(d\rho(\sigma(X))d\rho(\sigma(Y)))\right)$. We see then that

- $b(\sigma(X), \sigma(Y)) = b(X, Y), \forall X, Y \in \mathfrak{u}_{\mathbb{C}},$
- $b(uX, uY) = b(X, Y), \forall u \in U_{\mathbb{C}}, \forall X, Y \in \mathfrak{u}_{\mathbb{C}},$
- $b(iX, iY) = -b(X, Y), \text{ if } X, Y \in \mathfrak{u}_{\mathbb{C}},$
- $b(X, Y) \in \mathbb{Z}, \text{ if } X, Y \in \wedge := \frac{1}{2\pi} \ker(\exp : \mathfrak{t} \rightarrow T).$

In the article, we work with the following U -invariant scalar product on $\mathfrak{u}_{\mathbb{C}}$:

$$(X, Y)_b = -b(X, \Theta(Y))$$

If $V \subset \mathfrak{u}_{\mathbb{C}}$ is a real vector subspace, we have an isomorphism $\xi \in V^* = \mathrm{hom}(V, \mathbb{R}) \rightarrow \xi^b \in V$ defined by the relation

$$\langle \xi, X \rangle = (\xi^b, X)_b, \quad \forall (\xi, X) \in V^* \times V.$$

Let G be the real reductive group equal to the connected component of $(U_{\mathbb{C}})^{\sigma}$: its maximal compact subgroup is $K = G \cap U$. At the level of Lie algebras we have $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ where $\mathfrak{k} = \mathfrak{u}^{\sigma}$ and $\mathfrak{p} = i\mathfrak{u}^{-\sigma}$. The bilinear map (15) defines a G -invariant bilinear map $b : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{R}$ that is positive definite on \mathfrak{p} and negative definite on \mathfrak{k} . The commutativity of the following diagram is frequently used in the body of the present paper :

$$\begin{array}{ccc} (\mathfrak{u}^{-\sigma})^* & \xrightarrow{j^*} & \mathfrak{p}^* \\ \downarrow b & & \downarrow b \\ \mathfrak{u}^{-\sigma} & \xleftarrow{j} & \mathfrak{p}. \end{array}$$

We consider the restricted root system $\Sigma = \mathfrak{R}(\mathfrak{g}, \mathfrak{a})$ associated to a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{p}$ and a system of positive roots Σ^+ associated to a choice of Weyl chamber

\mathfrak{a}_+^* . Let \mathfrak{s} be a face of \mathfrak{a}_+^* : the set of positive roots orthogonal to \mathfrak{s} is denoted by $\Sigma_{\mathfrak{s}}^+$. We consider

$$\rho_{\mathfrak{s}} := \sum_{\alpha \in \Sigma_{\mathfrak{s}}^+} \alpha \in \mathfrak{a}^*$$

and the corresponding dual element $(\rho_{\mathfrak{s}})^{\flat} = -\gamma_{\mathfrak{s}} \in \mathfrak{a}$ (see §5.3.1 where the element $\gamma_{\mathfrak{s}}$ is used).

Lemma 6.1 *We have $\langle \alpha, \gamma_{\mathfrak{s}} \rangle < 0$ for any $\alpha \in \Sigma_{\mathfrak{s}}^+$.*

Proof : Let α_o be a simple root of $\Sigma_{\mathfrak{s}}^+$. Let $s_{\alpha_o} : \mathfrak{a}^* \rightarrow \mathfrak{a}^*$ be the associated orthogonal symmetry. For any $\beta \in \Sigma_{\mathfrak{s}}^+$, standard computations give that $s_{\alpha_o}(\beta) \in \Sigma_{\mathfrak{s}}^+$ if β is not proportional to α , and $s_{\alpha_o}(\beta) = -\beta$ if β is proportional to α . It shows that $s_{\alpha_o}(\rho_{\mathfrak{s}}) = \rho_{\mathfrak{s}} - 2N\alpha_o$ for some integers $N \geq 1$, thus $(\alpha_o, \rho_{\mathfrak{s}})_b > 0$. This implies that $(\alpha, \rho_{\mathfrak{s}})_b > 0$ for any $\alpha \in \Sigma_{\mathfrak{s}}^+$. \square

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