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# The Neighbour Sum Distinguishing Relaxed Edge Colouring

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## Abstract

A  $d$ -relaxed  $k$ -edge colouring is an edge colouring using colours from the set  $\{1, \dots, k\}$  such that each monochromatic set of edges induces a subgraph with maximum degree at most  $d$ . A neighbour sum distinguishing  $d$ -relaxed  $k$ -edge colouring of  $G$  is a  $d$ -relaxed  $k$ -edge colouring such that for each edge  $uv \in E(G)$ , the sum of colours taken on the edges incident to  $u$  is different from the sum of colours taken on the edges incident to  $v$ . By  $\chi_{\Sigma}^d(G)$  we denote the smallest value  $k$  in such a colouring of  $G$ .

In this paper, we prove that  $\chi_{\Sigma}^d(G) \leq 4$  for every connected subcubic graph with at least three vertices. For complete graphs with at least three vertices, we show that  $\chi_{\Sigma}^d(K_n) \leq 4$  if  $d \in \{[(n-1)/2], \dots, n-1\}$  and we also determine the exact value of  $\chi_{\Sigma}^2(K_n)$ . Finally, we determine the value of  $\chi_{\Sigma}^d(T)$  for any tree  $T$  with at least three vertices.

**Keywords:** neighbour sum distinguish edge colouring, relaxed edge colouring, subcubic graphs

**Mathematics Subject Classification:** 05C15

# 1 Introduction

We consider undirected simple graphs and denote by  $V(G)$  and  $E(G)$  the sets of vertices and edges of a graph  $G$ , respectively. For a vertex  $v$  of a graph  $G$ ,  $N_G(v)$  denotes the set of vertices which are adjacent to  $v$ ,  $d_G(v)$  denotes the degree of the vertex  $v$  in  $G$ , or simply  $N(v), d(v)$  whenever the graph  $G$  is clear from the context. For undefined notations and terminology, we refer the reader to [2].

A  $k$ -edge colouring of  $G$  is a mapping  $\omega : E(G) \rightarrow \{1, \dots, k\}$ . The edge colouring naturally induces a vertex colouring  $\sigma_\omega : V(G) \rightarrow \mathbb{N}$  given by

$$\sigma_\omega(v) = \sum_{u \in N_G(v)} \omega(vu)$$

for every  $v \in V(G)$ . We say that the edge colouring  $\omega$  *distinguishes* vertices  $v, w \in V(G)$  if  $\sigma_\omega(v) \neq \sigma_\omega(w)$ . The edge colouring (vertex colouring) is *proper* if adjacent edges (vertices) receive different colours.

The edge colouring which induces a proper colouring of vertices gained a lot of attention, especially 1-2-3 Conjecture, addressed in 2004 by Karoński, Łuczak and Thomason [8]. More precisely, by  $\chi_\Sigma^e(G)$ , we denote the smallest value  $k$  for which there exists a  $k$ -edge colouring  $\omega$  of  $G$  (not necessarily proper) such that  $\sigma_\omega(v) \neq \sigma_\omega(u)$  for every edge  $uv \in E(G)$ .

**Conjecture 1.** [8](1-2-3 Conjecture) *If  $G$  is a connected graph on at least 3 vertices, then  $\chi_\Sigma^e(G) \leq 3$ .*

The best known upper bound on  $\chi_\Sigma^e$  is 5 and has been proved by Kalkowski, Karoński and Pfender [7]. Recently Przybyło [11] proved that  $\chi_\Sigma^e(G) \leq 4$  for every  $d$ -regular graph with  $d \geq 2$  and 1-2-3 Conjecture is true for  $d$ -regular graphs with  $d \geq 10^8$ . 1-2-3 Conjecture inspires a lot of studies on the original conjecture and variants of it. For more information on that topic, we refer to the survey [14].

The version of the edge colouring which distinguishes vertices and in which the edge colouring is proper has been introduced by Flandrin et al. [5]. If the  $k$ -edge colouring  $\omega$  is proper and satisfies that  $\sigma_\omega(v) \neq \sigma_\omega(u)$  for every edge  $uv \in E(G)$ , then we call such colouring the *neighbour sum distinguishing  $k$ -edge colouring*. By  $\chi'_\Sigma(G)$ , we denote the smallest value  $k$  for which  $G$  has a neighbour sum distinguishing  $k$ -edge proper colouring and we call it the *neighbour sum distinguishing index*.

Flandrin et al. [5] completely determined the neighbour sum distinguishing index for paths, trees, complete graphs and complete bipartite graphs. Based on these examples, they proposed the following conjecture.

**Conjecture 2.** [5] *If  $G$  is a connected graph on at least 3 vertices and  $G \neq C_5$ , then  $\chi'_\Sigma(G) \leq \Delta(G) + 2$ .*

The objective of our work is to generalize both problems, by introducing a sum distinguishing edge colouring which lies between them. More precisely, we will allow a vertex to be incident to edges having the same colour, in a limited way. Such a colouring will be called a  *$d$ -relaxed  $k$ -edge colouring*. Namely, a  *$d$ -relaxed  $k$ -edge colouring* is a  $k$ -edge colouring such that each monochromatic set of edges induces a subgraph with maximum degree at most  $d$ . If the  $d$ -relaxed  $k$ -edge colouring  $\omega$  satisfies that  $\sigma_\omega(v) \neq \sigma_\omega(u)$  for every edge  $uv \in E(G)$ , then we call such colouring the *neighbour sum distinguishing  $d$ -relaxed  $k$ -edge colouring*. By  $\chi_\Sigma^{d'}(G)$ , we denote the smallest value  $k$  for which there is a neighbour sum distinguishing  $d$ -relaxed  $k$ -edge colouring of  $G$ . Hence when  $d = 1$ , we have that  $\chi_\Sigma^{d'}(G) = \chi'_\Sigma(G)$  and when  $d = \Delta(G)$ , we have  $\chi_\Sigma^{d'}(G) = \chi_\Sigma^e(G)$ . In addition, for  $1 \leq d \leq \Delta(G)$ , the following inequality holds for the three parameters:

$$\chi_\Sigma^e(G) \leq \chi_\Sigma^{d'}(G) \leq \chi'_\Sigma(G)$$

Also, a natural lower bound for  $\chi_\Sigma^{d'}(G)$  is given by the definition of a  $d$ -relaxed edge colouring:

$$\chi_\Sigma^{d'}(G) \geq \left\lceil \frac{\Delta(G)}{d} \right\rceil$$

We will see further that this bound may be tight in some cases.

This paper is organized as follows. In Section 2 we determine the value of  $\chi_\Sigma^{d'}(T)$  for any tree  $T$  of order at least three and  $1 \leq d \leq \Delta(T)$ . In Section 3 we consider complete graphs with at least three vertices, we prove that  $\chi_\Sigma^{d'}(K_n) \leq 4$  if  $d \in \{[(n-1)/2], \dots, n-1\}$  and compute the exact value of the parameter when  $d = 2$ . In Section 4 we prove that  $\chi_\Sigma^{d'}(G) \leq 4$  for every connected subcubic graph of order at least three. It is worth noting that the best known upper bound on the neighbour sum distinguishing index of connected subcubic graphs of order at least three is 6 [6] (*i.e.*  $\chi'_\Sigma(G) \leq 6$ ). In every figure throughout the paper, we will use the following notations: the colours of edges are the numbers next to the edges, the colours of the vertices induced by the edge colouring are boxed next to the vertices.

## 2 Trees

It is known that  $\chi_{\Sigma}^e(T) \leq 2$  for any tree  $T$ , according to the following result of Chang et al. [3]:

**Theorem 3.** [3] *If  $G$  is a connected bipartite graph with at least three edges and  $\delta(G) = 1$ , then  $\chi_{\Sigma}^e(G) \leq 2$ .*

Furthermore, Flandrin et al. [5] obtain the following result for the neighbour sum distinguishing index of trees:

**Theorem 4.** [5] *Let  $T$  be a tree of order  $n \geq 3$  and maximum degree  $\Delta$ . Then  $\chi'_{\Sigma}(T) = \Delta + 1$  if there are two adjacent vertices of degree  $\Delta$ , and  $\chi'_{\Sigma}(T) = \Delta$  otherwise.*

In the next result we give the exact value of  $\chi_{\Sigma}^d(T)$  for a tree  $T$  and for any  $d$ .

**Theorem 5.** *Let  $T$  be a tree of order  $n \geq 3$  with the maximum degree  $\Delta$  and  $1 \leq d \leq \Delta$ . We have*

$$\chi_{\Sigma}^d(T) = \begin{cases} \frac{\Delta}{d} + 1, & \text{if } \Delta = 0 \pmod{d} \text{ and there are two adjacent vertices of} \\ & \text{degree } \Delta \\ \lceil \frac{\Delta}{d} \rceil, & \text{otherwise} \end{cases} .$$

*Proof.* First recall that  $\chi_{\Sigma}^d(T) \geq \lceil \Delta/d \rceil$ . Furthermore, if  $\Delta = 0 \pmod{d}$ , then the edges incident to two adjacent vertices of maximum degree cannot be coloured with  $\Delta/d$  colours, so if  $\Delta = 0 \pmod{d}$  and there are two adjacent vertices of degree  $\Delta$ , then  $\chi_{\Sigma}^d(T) \geq \Delta/d + 1$ . We prove by induction on  $n$  that  $(\chi_{\Sigma}^d(T) \leq \Delta/d + 1$  if  $\Delta = 0 \pmod{d}$  and there are two adjacent vertices of degree  $\Delta$ ) and  $\chi_{\Sigma}^d(T) \leq \lceil \Delta/d \rceil$ , otherwise.

Observe that the theorem trivially holds if  $T$  is a star  $K_{1,n-1}$ , hence, in particular, for  $n = 3$ . Suppose the theorem is true for all trees of order at most  $n - 1$  and let  $T$  be a tree of order  $n$ . We may assume that  $T \neq K_{1,n-1}$ . Let  $P$  be a longest path in  $T$  and  $x$  be an endvertex of  $P$ . Since  $T$  is not a star  $|V(P)| \geq 4$ . Let  $x$  be chosen such that the only neighbour  $y$  of  $x$  in  $P$  is degree at most  $\Delta - 1$ , whenever  $T$  has only one vertex of degree  $\Delta$ . Let  $T' = T - x$ . By our choice of  $x$ ,  $\Delta(T') = \Delta(T) = \Delta$  and  $y$  has only one neighbour, say  $z$ , of degree  $\geq 2$ . Moreover, for every  $v \in V(T') \setminus \{y\}$  we have  $d_{T'}(v) = d_T(v)$  and  $d_{T'}(y) = d_T(y) - 1$ . Let  $k = \Delta/d + 1$  if  $\Delta = 0 \pmod{d}$  and in  $T$  there are two adjacent vertices of degree  $\Delta$  or  $k = \lceil \Delta/d \rceil$ , otherwise. Thus, by induction hypothesis, there is a neighbour sum

distinguishing  $d$ -relaxed  $k$ -edge colouring of  $T'$ . Let  $\omega$  be such an edge colouring. Let  $F = \{c \in \{1, \dots, k\} : \text{there are } d \text{ edges incident to } y \text{ that are coloured with } c\}$ .

If in  $T'$  there is a neighbour  $v$  of  $y$  different from  $z$ , then  $\sigma_\omega(v) < \sigma_\omega(y)$ , so in order to extend the colouring  $\omega$  to a neighbour sum distinguishing  $d$ -relaxed  $k$ -edge colouring  $\omega'$  of  $T$  it is sufficient to choose a colour  $c$  for  $yx$  such that  $c \in \{1, \dots, k\} \setminus F$  and  $\sigma_{\omega'}(z) \neq \sigma_{\omega'}(y)$ . To prove that such a colour exists we consider two cases.

**Case 1.**  $\Delta = 0 \pmod d$

If in  $T$  there are two adjacent vertices of degree  $\Delta$ , then in  $\{1, \dots, k\} \setminus F$  there are two colours, hence, one of them is proper for the edge  $yx$ , *i.e.* if we put this colour on  $yx$ , then we obtain the colouring  $\omega'$  such that  $\sigma_{\omega'}(z) \neq \sigma_{\omega'}(y)$ . Suppose, now, that in  $T$  there are no two adjacent vertices of degree  $\Delta$ , in this case  $k = \Delta/d$ . Suppose that  $d_T(y) = \Delta$ . Then in  $\{1, \dots, k\} \setminus F$  there is only one colour. We colour  $yx$  with this colour, let  $\omega'$  be the resultant colouring. By our assumption  $d_T(z) < \Delta$  what implies that  $\sigma_{\omega'}(z) < \sigma_{\omega'}(y) = (1 + \dots + k)d$ . Suppose that  $d_T(y) < \Delta$ . If in  $\{1, \dots, k\} \setminus F$  there are at least two colours, then we can choose one for  $yz$  to obtain a neighbour sum distinguishing  $d$ -relaxed  $k$ -edge colouring of  $T$ . Suppose that in  $\{1, \dots, k\} \setminus F$  there is only one colour, say  $c$ . Thus for each colour  $c' \in \{1, \dots, k\} \setminus \{c\}$  there are  $d$  edges incident to  $y$  coloured with  $c'$  and there are at most  $d - 2$  edges incident to  $y$  coloured with  $c$ . In such a case it must be  $d \geq 2$ . Arguments  $d \geq 2$ ,  $k \geq 2$  and  $|\{1, \dots, k\} \setminus F| = 1$  imply that  $y$  has at least one pendant vertex  $w$  in  $T'$  such that  $\omega(yw) \neq c$ . First we colour  $yx$  with  $c$ . If  $\sigma_{\omega'}(z) \neq \sigma_{\omega'}(y)$ , then  $\omega'$  is a neighbour sum distinguishing  $d$ -relaxed  $k$ -edge colouring. If  $\sigma_{\omega'}(z) = \sigma_{\omega'}(y)$ , then we also recolour the edge  $yw$ , we put  $\omega'(yw) := c$ . The resultant edge colouring is neighbour sum distinguishing.

**Case 2.**  $\Delta \neq 0 \pmod d$

Thus  $k = \lceil \Delta/d \rceil$  and  $d \geq 2$ . If in  $\{1, \dots, k\} \setminus F$  there are two colours, then one of them is proper for the edge  $yx$  and we can extend the colouring  $\omega$  to a neighbour sum distinguishing  $d$ -relaxed  $k$ -edge colouring of  $T$ . Suppose that  $|\{1, \dots, k\} \setminus F| = 1$  and  $c \in \{1, \dots, k\} \setminus F$ . Similarly as in the case 1 we can observe that there are at most  $d - 2$  edges incident to  $y$  and coloured with  $c$  and  $y$  has at least one pendant vertex  $w$  in  $T'$  such that  $\omega(yw) \neq c$ . We colour  $yx$  with  $c$ . If  $\sigma_{\omega'}(z) \neq \sigma_{\omega'}(y)$ , then we are done. Otherwise, we also recolour the edge  $yw$ , we put  $\omega'(yw) := c$ . Finally, we obtain a neighbour sum distinguishing  $d$ -relaxed  $k$ -edge colouring  $\omega'$ .  $\square$

### 3 Complete graphs

In [3], it was proved that complete graphs verify the 1-2-3 Conjecture. More precisely, we have  $\chi_{\Sigma}^e(K_n) = 3$  for  $n \geq 3$ . Flandrin et al. [5] determined the neighbour sum distinguishing index of complete graphs:

**Proposition 6.** [5] *For every  $n \geq 3$*

$$\chi'_{\Sigma}(K_n) = \begin{cases} n; & \text{if } n \text{ is odd} \\ n + 1; & \text{if } n \text{ is even} \end{cases}$$

We now consider the neighbour sum distinguishing  $d$ -relaxed edge colouring of complete graphs for several cases when  $1 < d < \Delta(G)$ .

**Theorem 7.** *Let  $n \geq 4$  and  $d \in \{[(n-1)/2], \dots, n-1\}$  be two integers. We have  $\chi_{\Sigma}^d(K_n) \leq 4$ .*

*Proof.* We prove the result of the theorem for  $d = [(n-1)/2]$ , since having a higher value of  $d$  only gives more leeway. We will prove that there is a neighbour sum distinguishing  $d$ -relaxed 4-edge colouring of  $K_n$ . We use an inductive construction to get this colouring. The first case is  $K_4$ , and is depicted in Figure 1.

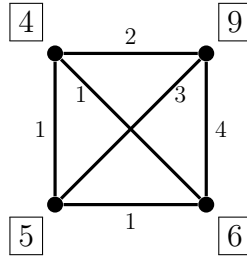


Figure 1: A neighbour sum distinguishing 2-relaxed 4-edge colouring for  $K_4$ .

Assume now that  $K_n$  has a neighbour sum distinguishing  $d$ -relaxed 4-edge colouring. We construct such a colouring of  $K_{n+1}$ . Let us call  $x$  the vertex we add to  $K_n$ .

First, assume that  $n$  is even. We order all the vertices of  $K_n$  by increasing colours. The first  $n/2$  vertices of  $K_n$  (those with the smaller colours) are linked to  $x$  with an edge coloured with 3. The other vertices are linked to  $x$  with an edge coloured with 4.

Now, assume that  $n$  is odd. We order all the vertices of  $K_n$  by increasing colours. The first  $[n/2]$  vertices of  $K_n$  (those with the smaller colours) are linked to  $x$  with

an edge coloured with 1. The other vertices are linked to  $x$  with an edge coloured with 2.

Those two constructions are depicted for  $K_5$  and  $K_6$  in Figure 2.

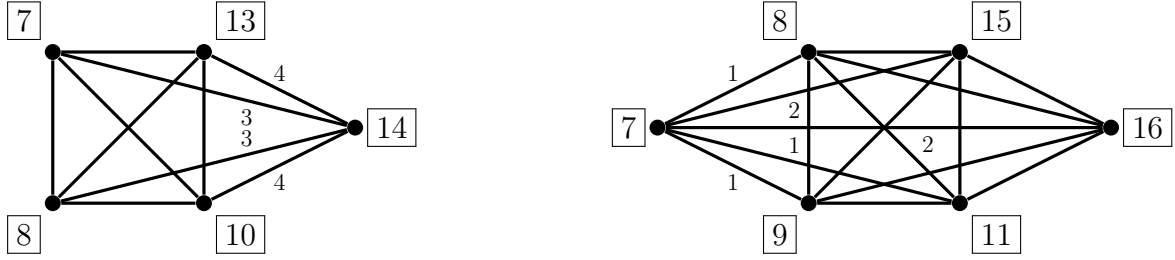


Figure 2: Neighbour sum distinguishing  $d$ -relaxed 4-edge colourings for  $K_5$  and  $K_6$  obtained by using our construction (for  $K_5$ , we have  $d = 2$ ; for  $K_6$ , we have  $d = 3$ ).

First, note that by alternating between those two constructions, the edge colouring will always be  $d$ -relaxed, since a vertex will gain an edge coloured with a certain colour once every two steps, and the construction starts from an even  $n$ .

Now, we need to verify that the colouring is neighbour sum distinguishing. The two following properties hold:

1. If  $n$  is even, then the highest colour is  $2 + 3\frac{n-2}{2} + 4\frac{n-2}{2}$  and the smallest colour is  $\frac{n}{2} + 2\frac{n-2}{2}$ .
2. If  $n$  is odd, then the highest colour is  $3\frac{n-1}{2} + 4\frac{n-1}{2}$  and the smallest colour is  $3 + \frac{n-1}{2} + 2\frac{n-3}{2}$ .

Indeed, when constructing the new colouring, if  $n$  is even then we add the vertex  $x$  with colour  $3n/2 + 4n/2$  which is the new highest colour in the graph, and we add 3 to the value of the smallest colour; and if  $n$  is odd then we add the vertex  $x$  with colour  $(n+1)/2 + 2(n-1)/2$  which is the new smallest colour in the graph, and we add 2 to the value of the highest colour.

Those two properties remaining true during our construction, the vertex  $x$  that we add is always distinguished from all the vertices that were already in the graph. Furthermore, the vertices that were in the graph are still neighbour sum distinguished, since we added the smallest value to the smallest colours. Thus, two vertices that had different colours cannot have the same colour in the new colouring, which concludes the proof.  $\square$

However, we also prove that this bound of 4 is not necessarily tight:



**Observation 8.** For  $n \in \{3, \dots, 7\}$  and  $d = \lceil (n-1)/2 \rceil$ , we have  $\chi_{\Sigma}^d(K_n) = 3$ .

*Proof.* First, note that a  $d$ -relaxed 2-edge colouring would not give us enough possible labels to be neighbour sum distinguishing, so we have  $\chi_{\Sigma}^d(K_n) \geq 3$ . Now, to prove the statement, we construct a neighbour sum distinguishing  $d$ -relaxed 3-edge colouring of  $K_3, \dots, K_7$ . This is shown in Figure 3 (the caption indicates how to read the constructions).  $\square$

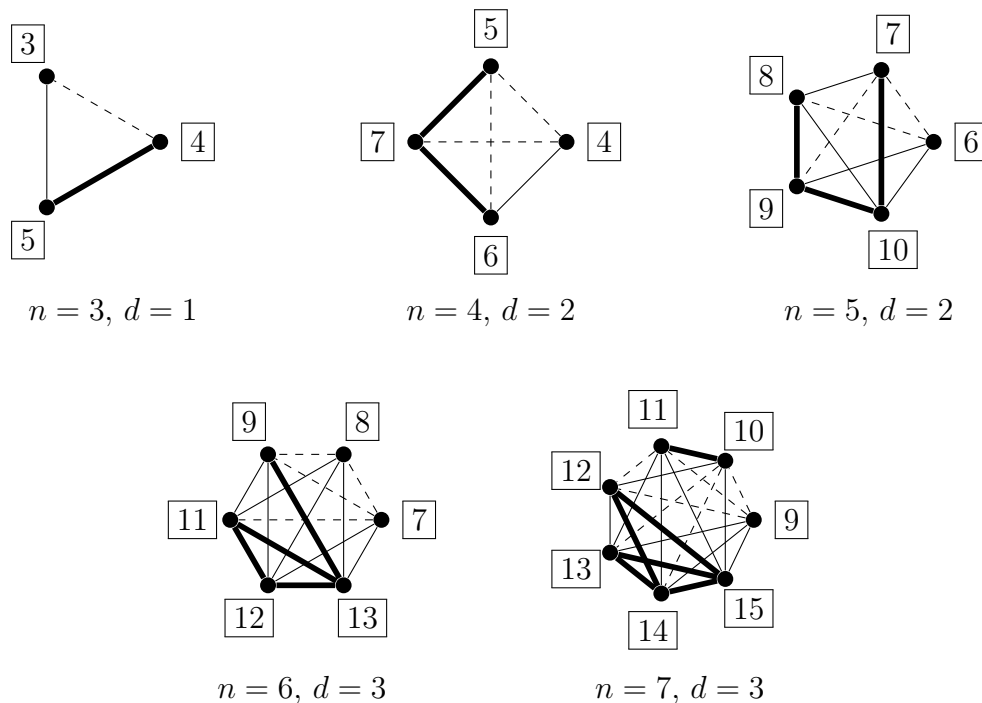


Figure 3: A neighbour sum distinguishing  $d$ -relaxed 3-edge colouring of  $K_3, \dots, K_7$ . Dashed lines indicate that the edges are coloured with 1, normal lines indicate that the edges are coloured with 2, and thick lines indicate that the edges are coloured with 3.

However, note that the constructions shown in Figure 3 were obtained by hand, and there does not seem to be a simple way to construct a neighbour sum distinguishing  $d$ -relaxed 3-edge colouring of  $K_{n+1}$  from the colouring of  $K_n$ . Thus, the exact value of  $\chi_{\Sigma}^d(K_n)$  remains open for  $n \geq 8$  and  $d \in \{\lceil (n-1)/2 \rceil, \dots, n-1\}$ .

We now compute the exact value of the parameter for complete graphs when  $d = 2$ .

**Theorem 9.** *Let  $n \geq 4$ . We have  $\chi_{\Sigma}^2(K_n) = \lceil \frac{n-1}{2} \rceil + 1$  if  $n \not\equiv 3 \pmod{4}$  and  $\chi_{\Sigma}^2(K_n) = \lceil \frac{n-1}{2} \rceil + 2$  otherwise.*

*Proof.* We first show that for all  $n \geq 4$ , there exist 2-relaxed distinguishing colourings having this number of colours. There are four cases according to  $n \pmod{4}$ , that all share a common basis that we present now. We denote and order the vertices of  $K_n$  by  $\{x_{-1}, \dots, x_{-\lceil n/2 \rceil}, x_1, \dots, x_{\lfloor n/2 \rfloor}\}$ . Given a vertex  $x_i$  of  $K_n$ , the vertex  $x_i + 1$  denotes its successor in the above ordering. In addition, we will consider this ordering in a cyclic way, *i.e.*  $x_{-\lceil n/2 \rceil} + 1 = x_1$  and  $x_{\lfloor n/2 \rfloor} + 1 = x_{-1}$ .

The following algorithm labels all the edges of  $K_n$  to provide a 2-relaxed colouring (but not distinguishing yet):

1. Label the edge  $x_{-\lceil n/2 \rceil} - k, x_1 + k$  with colour 1 for all  $k \in \{0, \dots, \lfloor n/2 \rfloor - 1\}$ .
2. Label the edge  $x_{-\lceil n/2 \rceil} - k, x_2 + k$  with colour 1 for all  $k \in \{0, \dots, \lfloor n/2 \rfloor - 2\}$ .
3. For each edge  $x_i x_j$  coloured 1, label the edge  $x_i + c - 1, x_j + c - 1$  with colour  $c$  for all  $c \in \{2, \dots, \lfloor n/2 \rfloor\}$ .
4. If  $n$  is odd, label the edge  $x_{\lfloor n/2 \rfloor} - k, x_{-1} + k$  with colour  $\lfloor n/2 \rfloor$  for all  $k \in \{0, \dots, \lfloor n/2 \rfloor - 1\}$ .

Figure 4 depicts the edges coloured 1 after steps 1 and 2 are done. Two cases are given,  $K_8$  and  $K_9$ . Figure 5 illustrates the edges coloured 2 after execution of step 3. They correspond to a rotation of the edges coloured 1. For the colour  $\lfloor n/2 \rfloor$ , it is concerned by step 3 when  $n$  is even and step 4 when  $n$  is odd. In the latter case, only the edges of step 1 are rotated, as depicted by Figure 6.

First note that the algorithm above labels exactly  $n(n-1)/2$  edges, *i.e.* the number of edges of  $K_n$ . Indeed, the first three steps label  $(\lfloor n/2 \rfloor + \lceil n/2 \rceil - 1) \times (\lfloor n/2 \rfloor - 1)$  edges, which equals  $n(n-1)/2$  if  $n$  is even and  $(n-1)^2/2$  if  $n$  is odd. In the latter case, the missing edges are given by step 4 that labels  $(n-1)/2$  edges.

We now prove that no edge is labeled twice. For that purpose, imagine the vertices of  $K_n$  as the vertices of a regular  $n$ -gon and edges are straight segments, as depicted by the figures above. Thus, all the edges coloured 1 by step 1 are parallel to  $x_{-\lceil n/2 \rceil}, x_1$ . There is no edge coloured twice according to the restriction given on  $k$ . For the same reason, all the edges coloured 1 by step 2 are parallel to  $x_{-\lceil n/2 \rceil}, x_2$  and pairwise distinct. Since the two directions  $x_{-\lceil n/2 \rceil}, x_1$  and  $x_{-\lceil n/2 \rceil}, x_2$  are not parallel,

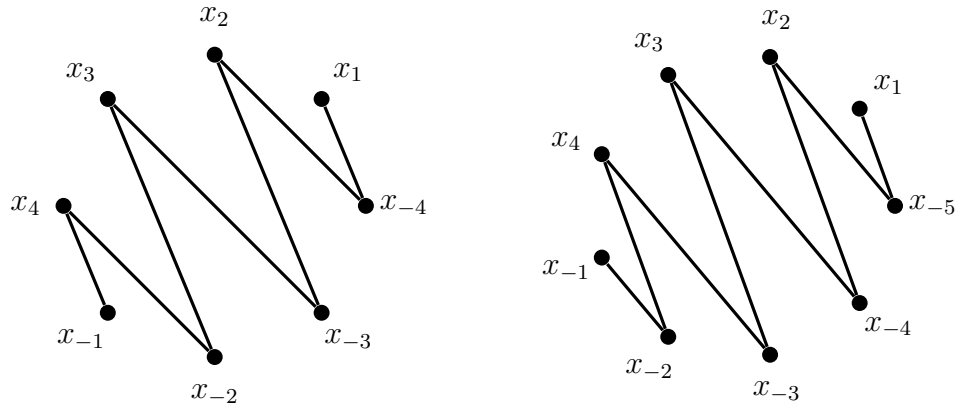


Figure 4: Edges of  $K_8$  and  $K_9$  labeled 1 after steps 1 and 2.

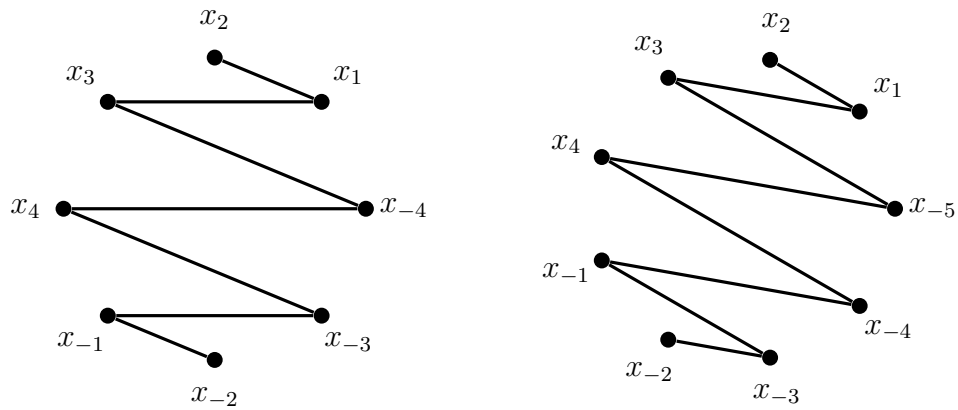


Figure 5: Edges of  $K_8$  and  $K_9$  labeled 2 after step 3.

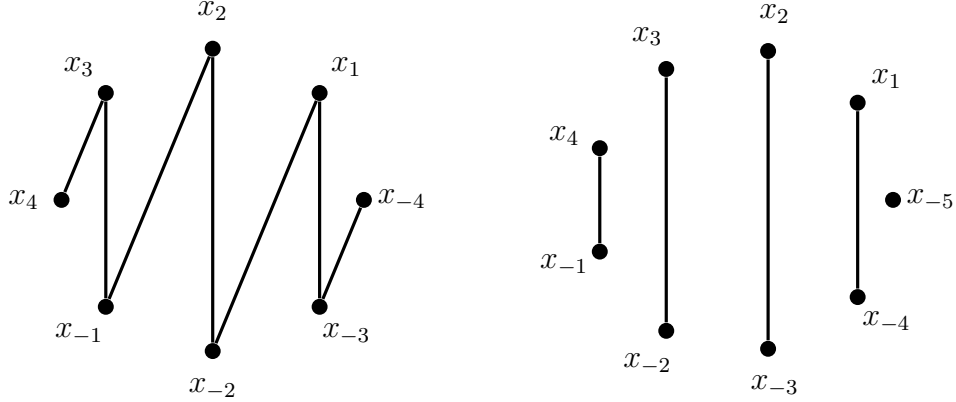


Figure 6: Edges of  $K_8$  and  $K_9$  labeled  $\lceil n/2 \rceil$  after step 3 or step 4.

all the edges coloured by steps 1 and 2 are distinct. Figure 4 illustrates this property. For steps 3 and 4, each edge coloured with  $c > 1$  is obtained by rotating the edges of colour 1 (see Figure 5 and 6). Consequently, all the edges of a given colour  $c$  and chosen by the algorithm are pairwise distinct. In addition, edges of colour  $c$  are parallel to  $x_{c-1}, x_c$  and  $x_{c-1}, x_{c+1}$ . Since  $c \leq \lfloor n/2 \rfloor$ , one can remark that on the  $n$ -gon, all the directions  $x_{c-1}, x_c$  and  $x_{c-1}, x_{c+1}$  are pairwise distinct. Thus, two edges of different colours necessarily have different directions. This ensures that the above algorithm labels all the edges exactly once.

Finally, each vertex is incident to at most twice the same colour. This is true for colour 1 as each vertex is concerned at most once by step 1 and at most once by step 2. Consequently, this is also true for the other colours, as step 3 (and step 4 when  $n$  is odd) consist in a rotation of the edges labeled 1.

If the above algorithm provides a 2-relaxed edge colouring, it is not distinguishing as the same sum may appear on two vertices. More precisely, we can compute the sum on each vertex. We define  $\Sigma_2$  as the quantity

$$\Sigma_2 = 2 \times \sum_{k=1}^{\lfloor n/2 \rfloor} k$$

One can remark that after steps 1 and 2, each vertex  $x_i$  with  $|i| \neq 1$  is incident to two edges with label 1, and vertices  $x_1$  and  $x_{-1}$  are incident to exactly one edge with label 1 (see Figure 4). Since step 3 consists in "rotating" the edges labeled with colour

1, we have that each vertex  $x_i$  is incident twice to every colour  $c \in \{1, \dots, \lfloor n/2 \rfloor\}$  for  $c \neq |i|$  and once to colour  $|i|$ . When  $n$  is odd, step 4 ensures that each vertex is incident to exactly one edge with colour  $\lceil n/2 \rceil$ , except  $x_{-\lceil n/2 \rceil}$  that is not incident to this colour (see Figure 6). Consequently, the above algorithm yields for each vertex  $x_i$  of  $K_n$ :

$$\sigma(x_i) = \begin{cases} \Sigma_2 - |i| & \text{for each vertex } x_i \text{ of } K_n \text{ when } n \text{ is even} \\ \Sigma_2 - |i| - \lceil n/2 \rceil & \text{for each vertex } x_i \text{ of } K_n \text{ when } n \text{ is odd} \end{cases}$$

In other words, the edge colouring is not distinguishing because vertices  $x_i$  and  $x_{-i}$  have the same sum. We will now break the ties by replacing some edges of colour 1 with an additional colour. Four cases are considered:

**Case  $n \equiv 0 \pmod{4}$**  : recolour the edges  $x_{-n/2} - k, x_1 + k$  with colour  $n/2 + 1$  for all  $k \in \{0, \dots, n/4 - 1\}$ . Hence, these edges form a perfect matching of  $K_{n/2}$ . Let  $V_{change}$  be the set of vertices of  $K_n$  incident to this new colour. We have  $V_{change} = \{x_{-n/4-1}, \dots, x_{-n/2}, x_1, \dots, x_{n/4}\}$ . Since each vertex of  $V_{change}$  is incident to exactly one recoloured edge and the colour  $n/2 + 1$  has not been used yet, it remains a 2-relaxed colouring. In addition, the new values of  $\sigma$  are the following:

$$\sigma(x_i) = \begin{cases} \Sigma_2 - |i| + n/2 & \text{if } x_i \text{ is in } V_{change} \\ \Sigma_2 - |i| & \text{otherwise} \end{cases}$$

Since for all  $i$  in  $\{1, \dots, n/2\}$ , exactly one vertex among  $\{x_i, x_{-i}\}$  is in  $V_{change}$ , all the sums are now different and cover all the values of the interval  $[\Sigma_2 - n/2, \Sigma_2 - 1 + n/2]$ .

**Case  $n \equiv 2 \pmod{4}$**  : recolour the edges  $x_{-n/2} - k, x_2 + k$  with colour  $n/2 + 1$  for all  $k \in \{0, \dots, \lfloor n/4 \rfloor - 1\}$ . Recolour also the edge  $x_{-n/2}, x_1$  with the same colour. Hence the two edges of colour 1 incident to  $x_{-n/2}$  have been recoloured, so that  $\sigma(x_{-n/2}) = \Sigma_2 - n/2 - 2 + 2(n/2 + 1) = \Sigma_2 + n/2$ . For the rest, the set  $V_{change}$  can be defined as in the previous case, and it has the same properties, *i.e.* half the vertices have their sum changed. Consequently:

$$\sigma(x_i) = \begin{cases} \Sigma_2 - |i| + n/2 & \text{if } x_i \text{ is in } V_{change} \text{ and } x_i \neq x_{-n/2} \\ \Sigma_2 + n/2 & \text{if } x_i = x_{-n/2} \\ \Sigma_2 - |i| & \text{otherwise} \end{cases}$$

and the values  $\sigma(x_i)$  cover  $[\Sigma_2 - n/2, \Sigma_2 - 1] \cup [\Sigma_2 + 1, \Sigma_2 + n/2]$ , *i.e.* the labeling is distinguishing.

**Case  $n \equiv 1 \pmod{4}$**  : recolour the edges  $x_{-1} - k, x_{-2} + k$  (initially coloured 1) with colour  $\lfloor n/2 \rfloor + 1$  for all  $k \in \{0, \dots, \lfloor n/4 \rfloor - 1\}$ . Since  $n$  is odd, the colour  $\lfloor n/2 \rfloor + 1$

used to replace the colour 1 is already used in step 4 of the algorithm. However, this colouring remains 2-relaxed as in step 4, each vertex is incident to at most one vertex of this colour. After this recolouring, by defining  $V_{change}$  as previously:

$$\sigma(x_i) = \begin{cases} \Sigma_2 - |i| - 1 & \text{if } x_i \text{ is in } V_{change} \\ \Sigma_2 - |i| - \lceil n/2 \rceil & \text{otherwise} \end{cases}$$

One can easily check that it covers all the  $n$  values in  $[\Sigma_2 - n - 1, \Sigma_2 - 2]$ .

**Case  $n = 3 \pmod 4$**  : recolour the edges  $x_{\lfloor n/2 \rfloor - k}, x_{-2} + k$  (initially coloured 1) with colour  $\lfloor n/2 \rfloor + 2$  for all  $k \in \{0, \dots, \lfloor n/4 \rfloor - 1\}$ . Recolour also the edge  $x_{-2}, x_{-1}$  with the same colour. Unlike the previous case, we here add a new colour instead of using the colour of step 4. Consequently, the colouring remains 2-relaxed. One can also easily check that it is distinguishing since half the vertices have their sum changed (and exactly one in each pair  $(x_i, x_{-i})$ ).

To sum up, the above algorithm uses  $\lfloor n/2 \rfloor + 1$  colours when  $n$  is even,  $\lfloor n/2 \rfloor$  colours when  $n = 1 \pmod 4$ , and  $\lceil n/2 \rceil + 1$  colours when  $n = 3 \pmod 4$ . This corresponds to the values given in the statement of the theorem.

We now prove that these values are lower bounds. When  $n$  is even, imagine there exists a 2-relaxed distinguishing colouring with  $\lfloor n/2 \rfloor$  colours. In that case, since the degree of each vertex is  $(n - 1)$ , there is only one colour in  $\{1, \dots, \lfloor n/2 \rfloor\}$  missing in the incident edges of any vertex. Thus, there will be two vertices not distinguished. When  $n = 1 \pmod 4$ , this argument remains true as there is no 2-relaxed colouring having at most  $\lfloor n/2 \rfloor$  colours. When  $n = 3 \pmod 4$ , it is less obvious to show that  $\lceil n/2 \rceil$  colours are not sufficient. By way of contradiction, imagine that there exists a 2-relaxed distinguishing colouring with  $\lceil n/2 \rceil$  colours. Since the degree of each vertex is  $(n - 1)$ , there are exactly two colours in  $\{1, \dots, \lceil n/2 \rceil\}$  missing in the incident edges of any vertex (possibly twice the same colour). Then the value  $\sigma(x_i)$  of any vertex  $x_i$  ranges in  $[\Sigma_2 - 2\lceil n/2 \rceil, \Sigma_2 - 2] = [\Sigma_2 - (n + 1), \Sigma_2 - 2]$ . As this interval is of size  $n$ , for each value  $s$  in  $[\Sigma_2 - (n + 1), \Sigma_2 - 2]$ , there exists  $x_i$  such that  $\sigma(x_i) = s$ . Now consider the quantity

$$Q = \sum_{x_i \in V} \sigma(x_i)$$

The above remark says it satisfies:

$$Q = \sum_{i=2}^{n+1} (\Sigma_2 - i) = n\Sigma_2 - \frac{(n+1)(n+2)}{2} - 1$$

Since  $\Sigma_2$  is even and  $n = 3 \pmod 4$ , it ensures that  $Q$  is odd.

Now consider the values  $\sigma(x_i)$  and for all  $i$ , let  $c_i$  and  $c'_i$  be the two values in  $\{1, \dots, \lceil n/2 \rceil\}$  such that  $\sigma(x_i) = \Sigma_2 - c_i - c'_i$ . For each value  $s$  in  $\{1, \dots, \lceil n/2 \rceil\}$ , let  $G_s$  be the subgraph of  $K_n$  induced by the edges of colour  $s$ . Since the colouring is 2-relaxed,  $G_s$  has maximum degree 2. Hence there is an even number of vertices with degree 1. By translating this property in our original context, a vertex  $x_i$  of degree 2 in  $G_s$  satisfies  $c_i \neq s$  and  $c'_i \neq s$ , a vertex of degree 0 in  $G_s$  satisfies  $c_i = c'_i = s$ , and a vertex of degree 1 has either  $c_i$  or  $c'_i$  equal to  $s$  (but not both). Consequently, there is an even total number of  $c_i$  and  $c'_i$  (added together) of value  $s$ . We denote this number  $val(s)$ . We now rewrite  $Q$  as follows:

$$Q = \sum_{x_i \in V} \sigma(x_i) = \sum_{x_i \in V} (\Sigma_2 - c_i - c'_i) = n\Sigma_2 - \sum_{x_i \in V} (c_i + c'_i) = n\Sigma_2 - \sum_{s=1}^{\lceil n/2 \rceil} val(s)$$

Since  $\Sigma_2$  is even, the above remark ensures that  $Q$  is even, a contradiction. □

Table 1 summarizes all the known results about the value of  $\chi_{\Sigma}^d(K_n)$ .

Value of $d$	Value of $\chi_{\Sigma}^d(K_n)$
1	$n$ if $n$ is odd $n + 1$ if $n$ is even
2	$\lceil (n-1)/2 \rceil + 1$ if $n \not\equiv 3 \pmod 4$ $\lceil (n-1)/2 \rceil + 2$ if $n \equiv 3 \pmod 4$
$2 < d < \lceil (n-1)/2 \rceil$	open
$\lceil (n-1)/2 \rceil < d \leq n-1$	$3$ if $n \in \{3, \dots, 7\}$ $\leq 4$ if $n > 7$

Table 1: Values of  $\chi_{\Sigma}^d(K_n)$

## 4 Subcubic graphs

Karoński et al. in [8] proved that every subcubic graph with at least three vertices admits a neighbour sum distinguishing 3-relaxed 3-edge colouring. On the other hand the best known upper bound for the neighbour sum distinguishing index of

subcubic graphs was proved by Huo et al. [6] and is equal to 6 (i.e. every subcubic graph of order at least three has a neighbour sum distinguishing 1-relaxed 6-edge colouring). In this section we prove that every connected subcubic graph with at least three vertices has a neighbour sum distinguishing 2-relaxed 4-edge colouring.

In the proof of this result, we need the following proposition observed by Flandrin et al. in [5].

**Proposition 10.** [5]  $\chi'_{\Sigma}(C_5) = 5$ ,  $\chi'_{\Sigma}(C_m) = 3$  if  $m = 0 \pmod{3}$  and  $\chi'_{\Sigma}(C_m) = 4$ , otherwise.

Some proofs in this section are based on the following theorem of Alon [1].

**Theorem 11** (Combinatorial Nullstellensatz [1]). *Let  $\mathbb{F}$  be an arbitrary field, and let  $P = P(x_1, \dots, x_n)$  be a polynomial in  $\mathbb{F}[x_1, \dots, x_n]$ . Suppose the degree  $\deg(P)$  of  $P$  equals  $\sum_{i=1}^n k_i$ , where each  $k_i$  is a nonnegative integer, and suppose the coefficient of  $x_1^{k_1} \dots x_n^{k_n}$  in  $P$  is nonzero. Then if  $S_1, \dots, S_n$  are subsets of  $\mathbb{F}$  with  $|S_i| > k_i$ , there are  $s_1 \in S_1, \dots, s_n \in S_n$  so that  $P(s_1, \dots, s_n) \neq 0$ .*

The main result of this section is that all connected subcubic graphs have a neighbour sum distinguishing 2-relaxed 4-edge colouring (Corollary). To prove this result, we prove by induction a stronger statement that is true for any subcubic graph that is neither  $K_2$  nor  $C_5$ .

**Theorem 12.** *Let  $G$  be a connected subcubic graph such that  $G \notin \{K_2, C_5\}$ . There is a neighbour sum distinguishing 2-relaxed 4-edge colouring of  $G$  such that all the vertices of degree 2 have their two adjacent edges of different colours.*

In order to prove this result, we need a first result that allows to simplify graphs having a pending  $C_5$  (i.e. an induced  $C_5$  connected to the rest of the graph by only one vertex).

**Lemma 13.** *Let  $G$  be a subcubic graph. Assume there exists in  $G$  an induced  $C_5$ ,  $C = \{u_0, u_1, u_2, u_3, u_4\}$ , such that only one vertex,  $u_0$ , is connected to the rest of  $G$ . Let  $G' = G \setminus \{u_1, u_2, u_3, u_4\}$ . If  $G'$  satisfies Theorem 12 then  $G$  also satisfies Theorem 12.*

*Proof.* Let  $\omega$  be a neighbour sum distinguishing 2-relaxed 4-edge colouring of  $G'$  such that all the vertices of degree 2 have their two adjacent edges of different colours.

Since  $G$  is subcubic,  $u_0$  has degree 1 in  $G'$ . Let  $v$  be its neighbour in  $G'$  and  $c_0$  the colour of the edge  $u_0v$ . Let  $c_1 \in \{1, 2, 3, 4\}$  such that  $c_1 \neq c_0$  and  $c_1 + 2c_0 \neq \sigma_{\omega}(v)$ . Let  $c_3 \in \{1, 2, 3, 4\}$  such that  $c_3 \neq c_0, c_1$ .



Then we can extend the colouring  $\omega$  by giving colour  $c_0$  to edges  $u_4u_0$  and  $u_1u_2$ , colour  $c_1$  to edges  $u_0u_1$  and  $u_3u_4$  and colour  $c_2$  to edge  $u_2u_3$ . Then all the adjacent vertices of  $C$  are distinguished as well as  $u_0$  and  $v$ . Moreover, vertices  $u_1$  to  $u_4$  are adjacent to edges of different colours, hence this colouring satisfies the conditions of Theorem 12. □

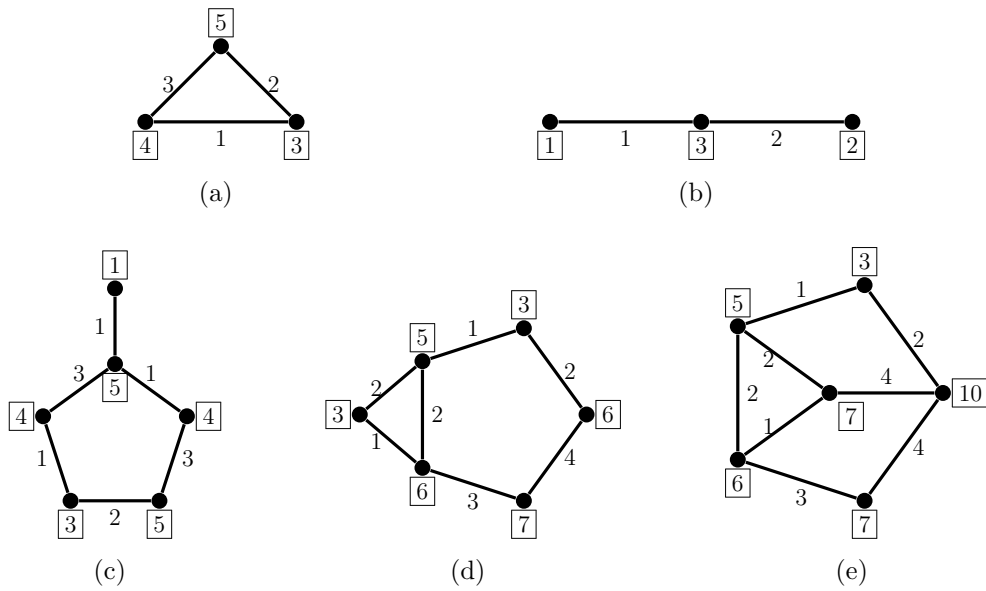


Figure 7: A neighbour sum distinguishing 2-relaxed 4-edge colouring for small graphs satisfying Theorem 12.

We are now ready to present the proof of Theorem 12.

*Proof of Theorem 12.* We prove Theorem 12 by induction on the number of edges of  $G$ . Figure 7a and Figure 7b show that the result is true for all connected graphs having three vertices. Now consider a connected subcubic graph  $G$  with at least four vertices and that is not  $C_5$ . If  $G$  is the graph of Figure 7c, then the result is true. Otherwise, from Lemma 13, one can assume that  $G$  has no pending  $C_5$ . The proof is now organized with four cases, according to the girth and the minimum degree of  $G$ .

**Case 1.**  $\delta(G) = 1$

Let  $u$  be a vertex of degree 1. Since  $G$  is connected, has at least four vertices and no pending  $C_5$ ,  $G-u$  satisfies the assumptions of the theorem. Hence the graph  $G-u$  has a colouring  $\omega$  that satisfies the theorem. We can extend  $\omega$  to  $G$  as follows. Let  $v$  be the neighbour of  $u$ . If  $v$  has degree 2 in  $G$ , then we put to the edge  $uv$  a colour different from the colour already adjacent to  $v$  and such that  $v$  is distinguished from its other neighbour. If  $v$  has degree 3 in  $G$ , let  $v_1$  and  $v_2$  be the neighbours of  $v$  in  $G-u$ . Then there are at most two forbidden colours for  $\omega(uv)$  to distinguish  $v$  from  $v_1$  and  $v_2$ . Thus there are two remaining possible colours for  $uv$  (since  $v$  has already two different colours in its neighbourhood). In both cases, we can extend the colouring  $\omega$ .

**Case 2.  $G$  has a triangle.**

**Subcase 2.1.  $G$  has a vertex of degree 2 in a triangle**

Let  $u$  be such a vertex and  $v, w$  be its (adjacent) neighbours. Let  $G' = G - u$ . Figure 7d shows that the result is true if  $G'$  is a  $C_5$ . Since  $G$  has at least four vertices,  $G'$  is not  $K_2$ . By induction  $G'$  satisfies the theorem so has a 2-relaxed 4-edge colouring  $\omega$  that satisfies the theorem.

Since  $G$  has at least four vertices, we can assume that  $w$  has degree 3 and let  $w_1$  be its other neighbour. If  $v$  has degree 2 in  $G$ , then the colour of  $uv$  must be different from the colour of  $vw$  and  $u$  must be distinguished from  $w$ , so there are at most two forbidden colours for  $uv$ . If  $v$  has degree 3 in  $G$ , then  $v$  must be distinguished from its neighbour in  $G'$  and  $u$  must be distinguished from  $w$ , so again there are at most two forbidden colours for  $uv$ . Let  $S_1$  be the set of colours that are not forbidden for  $uv$ . Similarly, there are at most two forbidden colours for  $uw$ . Let  $S_2$  be the set of colours that are not forbidden for  $uw$ . Thus, if we colour  $uv$  and  $uw$  with colours from  $S_1$  and  $S_2$  then we obtain a colouring that distinguishes vertices of  $G'$ , distinguishes  $u$  from  $w$ ,  $u$  from  $v$ , and guarantees that  $v$  is adjacent to edges coloured differently. Let  $x_1, x_2$  be colours attributed to edges  $uv, uw$ , respectively. To satisfy all conditions of the theorem for colours  $x_1, x_2$  it must verify:

- $x_1 \neq x_2$ , since the vertex  $u$  must be adjacent to edges coloured differently;
- $x_1 + \alpha \neq x_2 + \omega(wv_1)$  (where  $\alpha$  is a colour of the edge  $vv_1$ ,  $v_1 \in N(v) \setminus \{u, w\}$  if  $v_1$  exists and  $\alpha = 0$  otherwise), since  $v$  and  $w$  must be distinguished.

We construct a polynomial

$$P(x_1, x_2) = (x_1 - x_2)(x_1 + \alpha - x_2 - \omega(wv_1)).$$

The coefficient of the monomial  $x_1x_2$  is equal to  $-2$ , so is non-zero. Hence, by Theorem 11, there are  $x_1 \in S_1, x_2 \in S_2$  such that  $P(x_1, x_2) \neq 0$ , since  $|S_1| > 1, |S_2| > 1$ .

We put  $\omega(uv) = x_1, \omega(uw) = x_2$ , and the resulting colouring satisfies all the assumptions of the theorem.

**Subcase 2.2. All the vertices that are in a triangle have degree 3**

In this case, we can assume that there exists a vertex  $u$  such that there is exactly one edge in the graph induced by its neighbourhood. Indeed, take a vertex  $u$  in a triangle. Figure 1 shows that the result is true if  $G$  is  $K_4$ . Thus we can assume that  $G$  is not  $K_4$ , which means that there are at most two edges in  $G[N(u)]$ . Suppose there are exactly two edges, say  $vw$  and  $wy$ . Vertices  $v$  and  $y$  are in triangles so they must have degree 3. Consider now  $v$  instead of  $u$ . The third neighbour of  $v$ , says  $v_1$  is not  $y$  and thus is not adjacent to  $u$  and also not to  $w$  whose neighbours are  $v$ ,  $u$  and  $y$ .

Let  $u$  be such a vertex,  $v_1, v_2$  and  $v_3$  its neighbours with  $v_2$  and  $v_3$  adjacent. Let  $w_2$  and  $w_3$  be the other neighbours of  $v_2$  and  $v_3$  (we potentially have  $w_2 = w_3$ ).

Consider  $G' = G - u$ . If  $G'$  is connected, it cannot be  $K_2$ . If it is  $C_5$  then  $G$  is isomorphic to the graph of Figure 7e and the theorem is satisfied. If  $G'$  is not connected, it must have two components, say  $G_1$ , containing  $v_1$  and  $G_2$ , containing  $v_2$  and  $v_3$ . We can assume that the component  $G_1$  is not isomorphic to  $K_2$  (there would be a vertex of degree 1) nor  $C_5$  (it would be a pendant  $C_5$ ). Vertices  $v_2$  and  $v_3$  have degree 3 in  $G$  so  $G_2$  is not isomorphic to  $K_2$ . If  $G_2$  is isomorphic to  $C_5$  then one can replace  $u$  by  $u_2$  and then  $G - u$  is connected.

Thus, we can assume that all the connected components of  $G'$  are distinct from  $K_2$  and  $C_5$  and, by induction, there exists a colouring  $\omega$  of  $G'$  that satisfies the theorem (we colour independently the components if  $G'$  is not connected). We now extend the colouring  $\omega$  by colouring the three edges adjacent to  $u$ .

There are (at least) two possible colours for  $uv_1$  to distinguish  $v_1$  from its two other neighbours, if  $v_1$  has degree 3, or, if  $v_1$  has degree 2, to distinguish  $v_1$  from its neighbour and ensure that  $v_1$  is adjacent to two distinct colours.

Furthermore,  $v_2$  must be distinguished from  $w_2$  and  $v_3$  must be distinguished from  $w_3$ , so there at least three possible colours for  $uv_2$  and three possible colours for  $uv_3$  to ensure that. Let  $S_1, S_2, S_3$  be sets of possible colours for  $uv_1, uv_2, uv_3$ , respectively, so  $|S_2| = |S_3| = 3$  and  $|S_1| = 2$ . If we colour  $uv_1, uv_2, uv_3$  with colours from  $S_1, S_2, S_3$ , then all the vertices of  $N(u)$  have at least two distinct colours and are distinguished with their neighbours outside  $N[u]$ .

Let denote  $x_1, x_2, x_3$  be colours attributed to edges  $uv_1, uv_2, uv_3$ , respectively. To distinguish  $u$  from its neighbours and make sure that  $u$  is adjacent to edges coloured differently for colours  $x_1, x_2, x_3$ , it must hold:

- $x_2 \neq x_3$  or  $x_1 \neq x_2$  or  $x_1 \neq x_3$ , since the colouring must be 2-relaxed;

- $x_2 + \omega(v_2w_2) \neq x_3 + \omega(v_3w_3)$ , since  $v_2$  and  $v_3$  must be distinguished;
- $x_1 + x_3 \neq \omega(v_2w_2) + \omega(v_2v_3)$ ,  $x_1 + x_2 \neq \omega(v_3w_3) + \omega(v_2v_3)$ ,  $x_2 + x_3 \neq \sigma_\omega(v_1)$ , since  $u$  must be distinguished from its neighbours.

We construct a polynomial

$$P(x_1, x_2, x_3) = (x_2 - x_3)(x_2 - x_3 + \omega(v_2w_2) - \omega(v_3w_3))(x_1 + x_3 - \omega(v_2w_2) - \omega(v_2v_3)) \\ (x_1 + x_2 - \omega(v_3w_3) - \omega(v_2v_3))(x_2 + x_3 - \sigma_\omega(v_1)).$$

Consider the coefficient of the monomial  $x_1x_2^2x_3^2$ , observe that this coefficient in  $P$  is the same as in the following polynomial:

$$P(x_1, x_2, x_3) = (x_2 - x_3)(x_2 - x_3)(x_1 + x_3)(x_1 + x_2)(x_2 + x_3).$$

The coefficient of the monomial  $x_1x_2^2x_3^2$  is non-zero (is equal to  $-2$ ). Since  $|S_1| > 1$ ,  $|S_2| > 2$ ,  $|S_3| > 2$ , Theorem 11 implies that there are  $x_1 \in S_1, x_2 \in S_2, x_3 \in S_3$  such that  $P(x_1, x_2, x_3, x_4) \neq 0$ . Thus we can extend  $\omega$  to  $G$  by assigning  $\omega(uv_i) = x_i$  for  $i = 1, 2, 3$ , which proves the theorem.

### Case 3. $G$ has a $C_4$

Let  $u_1, u_2, u_3, u_4$  be the vertices of a 4-cycle  $C$  and let  $G'$  be obtained from  $G$  by removing the four edges of  $C$  (but not the vertices).

We can assume that  $G'$  has no component isomorphic to  $K_2$  (otherwise there would be a vertex of degree 1 or a triangle in  $G$ ). Furthermore, the only vertices that have changed their neighbourhood from  $G$  to  $G'$  are the vertices  $u_i$ , that have either degree 1 or degree 0 in  $G'$ . Since  $G$  has no component isomorphic to  $C_5$ , it is also the case for  $G'$  and we can apply induction. Thus, there is a colouring  $\omega$  of  $G'$  that satisfies the two conditions of the theorem. We now extend  $\omega$  by colouring the four edges of the cycle to obtain a neighbour sum distinguishing 2-relaxed edge colouring of  $G$  with the vertices of degree 2 adjacent to different colours.

For  $i \in \{1, 2, 3, 4\}$ , let  $x_i$  be the colour that will be assigned to  $u_iu_{i+1}$  (indices are taken modulo 4). If  $u_i$  has a neighbour, let  $v_i$  be this neighbour,  $c_i = \omega(u_iv_i)$  and  $\alpha_i = \sigma_\omega(v_i) - c_i$ . Otherwise, let  $c_i = 0 = \alpha_i = 0$ .

If for each  $i \in \{1, 2, 3, 4\}$  the following conditions are satisfied, then putting the colour  $x_i$  to  $u_iu_{i+1}$  will extend  $\omega$  to a colouring satisfying the theorem (indices are again taken modulo 4):

- $x_{i-1} + c_i \neq x_{i+1} + c_{i+1}$ , to distinguish  $u_i$  and  $u_{i+1}$ ;

- $x_i + x_{i-1} \neq \alpha_i$ , to distinguish  $u_i$  and  $v_i$  (if  $v_i$  exists, otherwise we keep the condition for simplicity reasons);
- $x_i \neq x_{i-1}$ , to have at least two colours adjacent to  $u_i$ .

We construct a polynomial

$$P(x_1, x_2, x_3, x_4) = \prod_{i=1}^4 ((x_{i-1} - x_{i+1} + c_i - c_{i+1})(x_i + x_{i-1} - \alpha_i)(x_i - x_{i-1})).$$

Consider the coefficient of the monomial  $x_1^3 x_2^3 x_3^3 x_4^3$ , observe that this coefficient in  $P$  is the same as in the following polynomial:

$$P'(x_1, x_2, x_3, x_4) = \prod_{i=1}^4 ((x_{i-1} - x_{i+1})(x_i^2 - x_{i-1}^2)).$$

The coefficient of the monomial  $x_1^3 x_2^3 x_3^3 x_4^3$  is non-zero (is equal to 8). Since there are initially four possible values for each  $x_i$ , Theorem 11 implies that there are  $x_i \in \{1, 2, 3, 4\}$  for  $i = 1, 2, 3, 4$  such that  $P(x_1, x_2, x_3, x_4) \neq 0$ . Thus we can extend  $\omega$  to  $G$  by assigning  $\omega(u_i u_{i+1}) = x_i$  for  $i \in \{1, \dots, 4\}$ , which proves the theorem.

**Case 4.  $G$  has girth at least 5.** We consider two subcases according to whether the graph is cubic or not.

**Subcase 4.1.**  $\delta(G) = 2$

By Proposition 10, the theorem is true for all cycles except  $C_5$ . Thus we may assume that  $G$  has a vertex of degree 3. Therefore, since  $G$  is connected, one can find a vertex of degree 2, say  $u$ , which has a neighbour of degree 3. Let  $v, w$  be the neighbours of  $u$  such that  $\deg(w) = 3$ . Let  $w_1$  and  $w_2$  be the neighbours of  $w$  different from  $u$ .

Since  $G$  has girth at least 5, the vertices  $v$  and  $w$  are not adjacent. This is also the case for the vertices  $\{v, w_1, w_2\}$  that form an independent set. Now observe that each component of  $G' = G \setminus \{u, w\}$  admits, by induction hypothesis, a colouring that satisfy the theorem. Indeed, since  $\delta(G) > 1$  and there is no pending  $C_5$ , there is no component isomorphic to  $K_2$  or  $C_5$  in  $G'$ , except if the graph is of the form of Figure 8. In that case, it suffices to consider  $y$  as the new vertex  $u$  of degree 2, and  $v$  its neighbour of degree 3.

Let  $\omega$  be such a colouring of  $G'$ .

The vertex  $v$  must be distinguished from its neighbours in  $G'$ . If  $v$  has two neighbours, then there are potentially two forbidden colours for  $vu$ . Thus there are two remaining possible colours for  $uv$ , since  $v$  has already two different colours in its

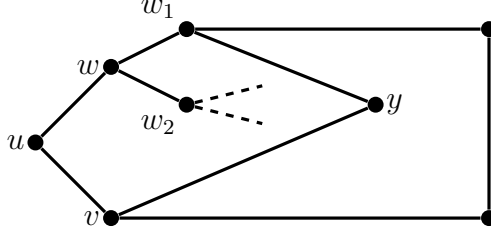


Figure 8: Subcase 4.1: if  $G$  is of this form, then consider  $G'$  as  $G \setminus \{y, v\}$ .

neighbourhood. If  $v$  has exactly one neighbour in  $G'$ , then  $v$  must be distinguished from its neighbour and additionally the colour of  $uv$  must be different from the colour of the other edge incident to  $v$ . Thus again we have at most two forbidden colours. Let  $S_1$  be a set colours that are not forbidden for  $uv$ . Vertices  $w_1$  and  $w_2$  must be distinguished from their neighbours in  $G'$  and if they have degree two in  $G$ , then they must be adjacent to edges coloured differently. Thus again there are potentially two forbidden colours for  $w_1w$  and  $w_2w$ . Let  $S_2$  and  $S_3$  be the sets of colours that are not forbidden for  $w_1w$  and  $w_2w$ , respectively. The vertex  $u$  must be distinguished from  $v$ , so there is at most one forbidden colour for  $wu$ . Let  $S_4$  be a set colours that are not forbidden for  $wu$ . Thus  $|S_1| = |S_2| = |S_3| = 2$  and  $|S_4| = 3$ . Summarize our reasoning, if for each edge  $uv, w_1w, w_2w, uw$  we choose colours from  $S_1, S_2, S_3, S_4$ , respectively, then we obtain a colouring that distinguishes all vertices of  $G'$ , distinguishes  $u$  from  $v$  and furthermore guarantees that  $w_1$  and  $w_2$  are adjacent to edges coloured differently. To obtain a colouring that satisfies all the conditions of the theorem we need to add some additional restrictions on colours that we choose for  $uv, w_1w, w_2w$  and  $wu$ . Let  $x_1, x_2, x_3, x_4$  be the colours attributed to edges  $uv, w_1w, w_2w, uw$ , respectively. Thus for colours  $x_1, x_2, x_3, x_4$  it must hold:

- $x_2 + x_3 \neq x_1$ , since  $u$  and  $w$  must be distinguished;
- $x_3 + x_4 \neq \sigma_1$  (where  $\sigma_1$  is the colour of  $w_1$  in  $G'$ ), since  $w$  and  $w_1$  must be distinguished;
- $x_2 + x_4 \neq \sigma_2$  (where  $\sigma_2$  is the colour of  $w_2$  in  $G'$ ), since  $w$  and  $w_2$  must be distinguished;
- $x_1 \neq x_4$ , since  $u$  must be adjacent to edges coloured differently;
- $x_3 \neq x_4$  or  $x_3 \neq x_2$  or  $x_2 \neq x_4$ , since  $w$  must be adjacent to edges coloured differently.

We construct a polynomial

$$P(x_1, x_2, x_3, x_4) = (x_2 + x_3 - x_1)(x_3 + x_4 - \sigma_1)(x_2 + x_4 - \sigma_2)(x_1 - x_4)(x_3 - x_4).$$

Observe that if there are  $x_i \in S_i$  ( $i \in \{1, 2, 3, 4\}$ ) such that  $P(x_1, x_2, x_3, x_4) \neq 0$ , then we put  $\omega(uv) = x_1, \omega(w_1w) = x_2, \omega(w_2w) = x_3, \omega(uw) = x_4$ , and the resulting colouring satisfies all the conditions of the theorem. To prove that there are  $x_i \in S_i$  ( $i \in \{1, 2, 3, 4\}$ ) such that  $P(x_1, x_2, x_3, x_4) \neq 0$ , we use the Combinatorial Nullstellensatz. The coefficient of the monomial  $x_1x_2x_3x_4^2$  is equal to  $-1$ , so is non-zero. Since  $|S_1| > 1, |S_2| > 1, |S_3| > 1, |S_4| > 2$ , Theorem 11 implies that there are  $x_i \in S_i$  ( $i \in \{1, 2, 3, 4\}$ ) such that  $P(x_1, x_2, x_3, x_4) \neq 0$ . Thus there is a colouring that satisfy all conditions of the theorem.

**Subcase 4.2.**  $\delta(G) = 3$

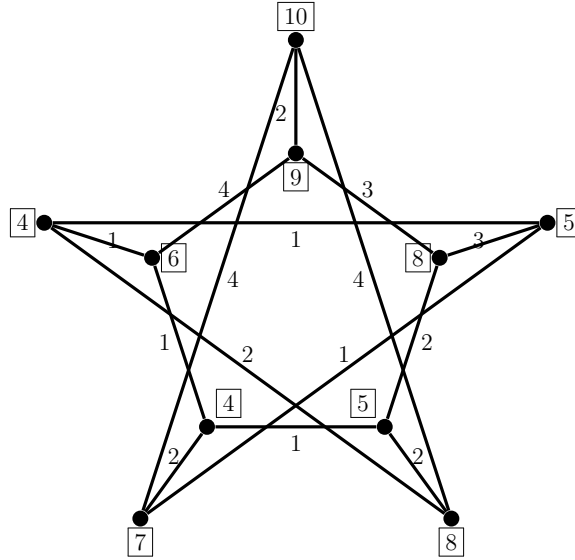


Figure 9: A neighbour sum distinguishing 2-relaxed 4-edge colouring for Petersen graph, appearing in Case 3 of Theorem 12.

Let  $C$  be a cycle of smallest size and  $u_0, \dots, u_{\ell-1}$  its vertices. For each vertex  $u_i$  of the cycle,  $u_i$  has a neighbour  $v_i$  outside  $C$  (since  $C$  is minimal).

We remove all the vertices  $u_i$  from  $G$  to obtain a graph  $G'$ .

If  $G' = C_5$ , then  $G$  is isomorphic to the Petersen graph and satisfies the theorem according to Figure 9. Thus we may assume that  $G$  is not isomorphic to the graph in Figure 9 and then  $G'$  has no component isomorphic to  $C_5$ . Since  $C$  is of smallest size and  $G$  is cubic, there is also no isolated  $K_2$  in  $G'$ . Thus by induction, there is a colouring  $\omega$  of  $G'$  that is distinguishing, 2-relaxed, and assigns two distinct colours

to the edges incident with vertices of degree 2.

Since  $C$  has size at least 5 and by minimality of  $C$ , all the vertices  $v_i$  are distinct (but might be adjacent). For each vertex  $v_i$ , we denote by  $\sigma_i$  the sum of the colours on the two edges adjacent to  $v_i$  that are already coloured (there are exactly two such edges). Note that to distinguish  $u_i$  and  $v_i$ , the colours of  $u_{i-1}u_i$  and  $u_iu_{i+1}$  must not sum to  $\sigma_i$ .

In order to keep  $v_i$  distinguished from its neighbours outside of  $C$ , at most two colours are forbidden for the edge  $u_iv_i$ , and thus at least two colours remain possible for the edge  $u_iv_i$ . We denote by  $L_i$  the list of possible colours for the edge  $u_iv_i$ .

Note that if  $v_i$  is adjacent to another vertex  $v_j$ , then, since there is no  $C_4$ , we have  $|j - i| > 1$  (subscripts are taken modulo  $\ell$ ). Furthermore, until either the edge  $v_ju_j$  or  $v_iu_i$  is coloured, the lists  $L_i$  and  $L_j$  have size at least three and at least two if one of the edges is coloured.

**Subcase 4.2.1**  $C$  has length 5 and there is a pair of adjacent vertices in  $\{v_0, \dots, v_4\}$ .

**Subcase 4.2.1.1.** There is  $\sigma_i \neq 5$  for  $i \in \{0, \dots, 4\}$

First we colour edges of the cycle  $C$  in such a way that every vertex  $u_i$  is adjacent to two differently coloured edges and the pairs  $(u_i, v_j)$  are distinguished for  $i \in \{0, \dots, 4\}$ . We claim using Theorem 11 that such a colouring exists. Without loss of generality we assume that  $\sigma_1 \neq 5$ . Observe that assumption on  $\sigma_1$  implies that there is a colour  $c_0 \in \{1, 2, 3, 4\}$  such that  $\sigma_1 - c_0 \notin \{1, 2, 3, 4\}$ . We colour  $u_0u_1$  with  $c_0$ . Thus if we colour  $u_1u_2$  with any colour from  $\{1, 2, 3, 4\}$ , then the pair  $(u_1, v_1)$  is distinguished. Since  $u_1$  must be adjacent to edges coloured differently, we assume that  $S_1 = \{1, 2, 3, 4\} \setminus \{c_0\}$  is the set of possible colours for  $u_1u_2$ . Let  $x_1, x_2, x_3, x_4$  be the colours attributed to edges  $u_1u_2, u_2u_3, u_3u_4, u_4u_0$ , respectively. Let  $S_i$  be the set of colours that are possible for  $x_i$ , so  $|S_1| = 3$  and  $|S_i| = 4$  for  $i \in \{2, 3, 4\}$ . To obtain the above described colouring we need that the colours additionally satisfy:

- $x_i + x_{i+1} \neq \sigma_{i+1}$  for  $i \in \{1, 2, 3\}$ , since  $(u_{i+1}, v_{i+1})$  must be distinguished;
- $x_i \neq x_{i+1}$  for  $i \in \{1, 2, 3\}$ , since  $u_{i+1}$  must be adjacent to edges with different colours;
- $x_4 + c_0 \neq \sigma_0$ , since  $(u_0, v_0)$  must be distinguished;
- $x_4 \neq c_0$ , since  $u_0$  must be adjacent to edges with different colours.

We construct a polynomial

$$P(x_1, x_2, x_3, x_4) = (x_1 + x_2 - \sigma_2)(x_1 - x_2)(x_2 + x_3 - \sigma_3)(x_2 - x_3) \\ (x_3 + x_4 - \sigma_4)(x_3 - x_4)(x_4 + c_0 - \sigma_0)(x_4 - c_0).$$



Consider the coefficient of the monomial  $x_1^2 x_2^2 x_3^2 x_4^2$ , observe that this coefficient in  $P$  is the same as in the following polynomial

$$P_1(x_1, x_2, x_3, x_4) = (x_1^2 - x_2^2)(x_2^2 - x_3^2)(x_3^2 - x_4^2)x_4^2.$$

The coefficient of the monomial  $x_1^2 x_2^2 x_3^2 x_4^2$  is 1, so since  $|S_1| > 2, |S_i| > 3$  for  $i \in \{2, 3, 4\}$ , Theorem 11 implies that there are  $x_i \in S_i$  ( $i \in \{1, 2, 3, 4\}$ ) such that  $P(x_1, x_2, x_3, x_4) \neq 0$  and equivalently there is a desired colouring of the edges of  $C$ . Let  $c_i \in S_i$  ( $i \in \{1, 2, 3, 4\}$ ) be colours such that  $P(c_1, c_2, c_3, c_4) \neq 0$ , we put  $\omega(u_1 u_2) = c_1, \omega(u_2 u_3) = c_2, \omega(u_3 u_4) = c_3, \omega(u_4 u_0) = c_4$ .

Now we colour the edges  $u_i v_i$  for  $i \in \{0, \dots, 4\}$ . The colours that we choose for these edges must distinguish adjacent vertices of  $C$  and adjacent vertices of  $\{v_0, \dots, v_4\}$ . Let  $t_i = |N(v_i) \cap \{v_0, v_1, v_2, v_3, v_4\}|$  for  $i \in \{0, \dots, 4\}$ .

**Claim 14.** *There are two consecutive vertices  $v_i, v_{i+1}$  such that  $t_i > t_{i+1}$ .*

*Proof.* Suppose that there is  $i$  such that  $t_i = 2$ . Since  $G$  is not isomorphic to the graph of Figure 9, there is  $t_j < 2$ . Thus we find two consecutive vertices  $v_i, v_{i+1}$  such that  $t_i > t_{i+1}$ . If  $t_i < 2$  for  $i \in \{0, \dots, 4\}$ , then there are at most two pairs of adjacent vertices in  $\{v_0, \dots, v_4\}$  and consequently there is  $t_i = 0$ . Since in  $\{v_0, \dots, v_4\}$  there are two adjacent vertices, there is  $t_j = 1$  and this implies that there are two vertices  $v_i, v_{i+1}$  such that  $t_i > t_{i+1}$ .  $\square$

Renaming vertices, if it is necessary, assume that  $t_0 > t_1$ . Note that this operation is possible, even though it has been assumed previously that  $\sigma_1 \neq 5$ . Indeed, we will not use anymore this property in the rest of the proof.

Observe that  $|L_i| \geq 2 + t_i$  for  $i \in \{0, \dots, 4\}$ . We choose the colour  $b_0 \in L_0$  for  $u_0 v_0$  such that  $|L_1 \setminus \{b_0 + c_4 - c_1\}| \geq 2 + t_1$ . Because  $t_0 > t_1$ , we can find such a colour. We colour  $u_0 v_0$  with  $b_0$ . Then we modify the list  $L_1$ , i.e. we delete the colour  $b_0 + c_4 - c_1$  from the list whenever such a colour is in  $L_1$ . Now each colour from  $L_1$  will distinguish  $u_0$  and  $u_1$  and still  $|L_1| \geq 2 + t_1$ . Moreover for every neighbour  $v_i$  of  $v_0$  in  $\{v_1, v_2, v_3, v_4\}$  we delete the colour  $\sigma_0 + b_0 - \sigma_i$  from the list  $L_i$ . After such a list modification, every colour in  $L_i$  will distinguish  $v_0$  and  $v_i$ .

Next we colour  $u_4 v_4$ , we choose the colour from  $L_4$  that distinguishes  $u_0$  and  $u_4$ . Because  $|L_4| \geq 2$  we can find the proper colour. Let  $b_4$  be such a colour, which we put on  $u_4 v_4$ . We modify the lists of neighbours of  $v_4$  in  $\{v_0, v_1, v_2, v_3\}$ : if  $v_i$  is the neighbour of  $v_4$ , then we delete the colour  $\sigma_4 + b_4 - \sigma_i$  from  $L_i$ . Then we repeat procedures of colouring and list modifications for the edges  $u_3 v_3, u_2 v_2$  and  $u_1 v_1$ . Observe that when we colour  $u_1 v_1$  we do not need to care about the vertex

$u_0$ , because the colour  $b_0$  on  $u_0v_0$  guarantees that every colour in  $L_1$  distinguishes  $u_0$  and  $u_1$ . Each time when we choose colour for the edge  $u_iv_i$  the list  $L_i$  contains at least two colours, because initially every list  $L_i$  had at least  $2 + t_i$  colours. Thus eventually we obtain the neighbour sum distinguishing 2-relaxed edge colouring of  $G$ .

**Subcase 4.2.1.2**  $\sigma_i = 5$  for  $i \in \{0, \dots, 4\}$

By our assumption there is a pair of adjacent vertices in  $\{v_0, \dots, v_4\}$ . Without loss of generality, we assume that  $v_0v_2 \in E(G)$ . Thus  $|L_0| \geq 3$ . Recall that  $|L_i| \geq 2 + t_i$ , where  $t_i = |N(v_i) \cap \{v_0, v_1, v_2, v_3, v_4\}|$ .

First we colour the edges of  $C$  in the following way  $\omega(u_0u_1) = 1, \omega(u_1u_2) = 3, \omega(u_2u_3) = 4, \omega(u_3u_4) = 3, \omega(u_4u_0) = 1$ . Such a colouring distinguishes the pairs  $(u_i, v_i)$  for  $i \in \{0, \dots, 4\}$ , furthermore, all vertices of  $C$ , except  $u_0$ , are adjacent to edges with different colours.

Now we colour edges  $u_iv_i$  ( $i \in \{1, \dots, 4\}$ ) in such a way that we distinguish all adjacent pairs of  $C$  and  $\{v_0, \dots, v_4\}$ . The vertex  $u_0$  must be adjacent to edges with different colours, so from  $L_0$  we delete the colour 1, whenever it is on the list  $L_0$ . Thus the edge  $u_0v_0$  has at least two possible colours.

Consider the pair  $(u_1, u_2)$ . If we give  $u_1v_1$  a colour  $b_1 \leq 3$ , then whatever the colour will be on  $u_2v_2$ , the pair  $(u_1, u_2)$  will be distinguished. Since  $u_1v_1$  has at least two possible colours, there is a colour  $b_1 \leq 3$ , which we can put on  $u_1v_1$ . After colouring  $u_1v_1$  with  $b_1$ , for every neighbour  $v_i$  of  $v_1$  in  $\{v_0, \dots, v_4\}$  we delete the colour  $\sigma_1 + b_1 - \sigma_i$  from  $L_i$ . After such a modification of  $L_i$  every colour in  $L_i$  will distinguish  $v_1$  and  $v_i$ .

Next, as in the case 3.1.1, we colour the edges  $u_0v_0, u_4v_4, u_3v_3, u_2v_2$ , one by one. We start with the edge  $u_0v_0$ , we choose the colour from  $L_0$  that distinguishes  $u_0$  and  $u_1$ , because  $|L_0| \geq 2$  we can choose the proper colour. After colouring  $u_0v_0$  we modify the lists of neighbours of  $v_0$  in  $\{v_0, \dots, v_4\}$  in such a way that every colour on the list of the neighbour will distinguish  $v_0$  from its neighbour, i.e. for every neighbour  $v_i$  of  $v_0$  we delete the colour  $\sigma_0 + b_0 - \sigma_i$  (where  $b_0$  is the colour of  $u_0v_0$ ) from  $L_i$ . Then we colour  $u_4v_4$  in such a way that the vertices  $u_0$  and  $u_4$  are distinguished, so if  $b_4$  is a colour of  $u_4v_4$ , then  $b_4 \in L_4$  and  $b_4 + 3 \neq b_0 + 1$ . We delete the colour  $\sigma_4 + b_4 - \sigma_i$  from  $L_i$  for every neighbour  $v_i$  of  $v_4$ . We do the same procedure for  $u_3v_3$  and  $u_2v_2$ . The colour  $b_1$ , which we chose for  $u_1v_1$  provide that every colour on  $L_2$  distinguishes vertices  $u_1$  and  $u_2$ , so if we colour  $u_2v_2$  we do not take care about the vertex  $v_1$ . Since initially every list  $L_i$  had at least  $2 + t_i$  colours, when we colour the edge  $u_iv_i$  the actual list has at least two colour and so we can find a proper colour. Eventually we obtain the neighbour sum distinguishing 2-relaxed edge colouring of  $G$ .

**Subcase 4.2.2**  $C$  has length 5 and the set  $\{v_0, \dots, v_4\}$  is independent or  $C$  has length at least 6.

We first consider that we are not in the case where all the lists  $L_i$  have size 2, are the same and all the  $\sigma_i$  sums to 5. In this case, we first colour some edges with particular conditions (given in Claim 15) and then extend this partial colouring to a complete colouring.

We say that a partial edge colouring is *good* if a vertex that has all its edges coloured has at least two colours in its neighbourhood and if two adjacent vertices  $x$  and  $y$  of degree 3 have their four edges distinct from  $xy$  coloured, then those vertices are distinguished (i.e. the sum in  $x$  and  $y$  are distinct). Observe that the partial edge colouring  $\omega$  of  $G$  is good.

**Claim 15.** *There is a partial good colouring that satisfies the following conditions :*

- the edges of  $C$  that are coloured are  $u_1u_2$  and  $u_2u_3$ ;
- the edges  $u_iv_i$  are coloured for  $0 \leq i \leq 2$ ;
- the pair  $(u_2, v_2)$  is distinguished;
- the pairs  $(u_0, u_1)$  and  $(u_1, u_2)$  are necessarily distinguished whatever will be the colours on  $u_{\ell-1}u_0$  and  $u_0u_1$ ;
- the vertices  $u_1$  and  $u_2$  are adjacent to edges of different colours.

*Proof of Claim 15.* Observe that our assumptions that either  $\{v_0, \dots, v_4\}$  is independent when  $C$  has length 5 or  $C$  has length at least 6 imply that if  $v_i$  is adjacent to another vertex  $v_j$ , then we have  $|j - i| > 2$ . Thus if we colour  $u_iv_i$ , then  $L_j$  does not change for  $|i - j| \leq 2$ . We first prove the claim is true in the following cases:

- (i) There are three consecutive lists  $L_i$  with 1 or 2 in the first one, 3 in the second and 1 in the last list.
- (ii) There are three consecutive lists  $L_i$  with 1 or 2 in the first one, 4 in the second and 1 in the last list.
- (iii) There are two consecutive lists  $L_i$  with 2 in the first one and 4 in the second.
- (iv) There are two consecutive lists  $L_i$  with 1 in the first one and 3 in the second.

(i-ii) *There are three consecutive lists  $L_i$  with 1 or 2 in the first one, 3 (resp. 4) in the second and 1 in the last list.*

Without loss of generality, we assume that  $1 \in L_0$  (or  $2 \in L_0$ ),  $3 \in L_1$  and  $1 \in L_2$ . We assign colour 1 (or 2) to the edge  $u_0v_0$ , colour 3 to the edge  $u_1v_1$  and 1 to the edge  $u_2v_2$ . Observe that by our assumption that  $C$  has length 5 and the set  $\{v_0, \dots, v_4\}$  is independent or  $C$  has length at least 6, the vertices  $v_0, v_1, v_2$  are independent, so

after colouring  $u_0v_0, u_1v_1$  and  $u_2v_2$  we still have a good colouring. Now we choose for the edges  $u_1u_2, u_2u_3$  colours 4, 1 if  $\sigma_2 \neq 5$  and 4, 2, otherwise (see Figure 10a and 10b). In this way we make sure that pairs of vertices  $(u_0, u_1)$ ,  $(u_1, u_2)$  and  $(u_2, v_2)$  will be distinguished. Indeed, for the last pair, we have  $\omega(u_1u_2) + \omega(u_2u_3) \neq \sigma_2$ . For the pair  $(u_1, u_2)$ , the sum in  $u_1$  will be at least 8 whereas the sum in  $u_2$  is 6 or 7. For the pair  $(u_0, u_1)$  the sum in  $u_1$  is  $\omega(u_0u_1) + 7$  whereas the sum in  $u_0$  will be at most  $\omega(u_0u_1) + 2 + \omega(u_{\ell-1}u_0)$  which is smaller, since  $7 > 2 + \omega(u_{\ell-1}u_0)$ . We also have that the vertices  $u_1$  and  $u_2$  are adjacent to edges of different colours.

Similarly we can show that if there exist three consecutive lists  $L_i$  with 1 or 2 in the first one, 4 in the second and 1 in the last list, then the claim holds (see Figure 10c and 10d).

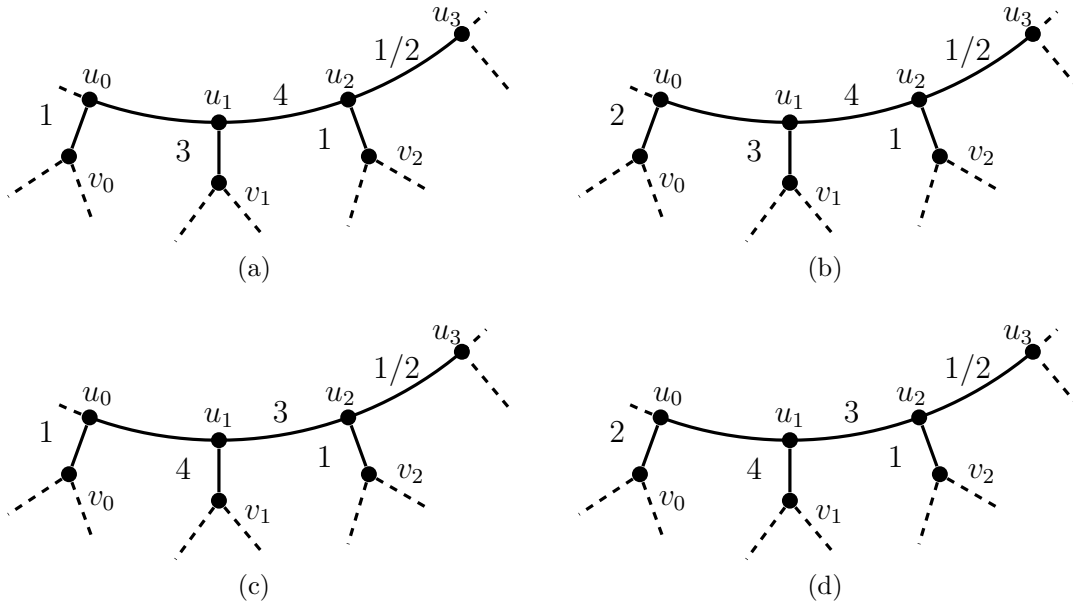


Figure 10: Pre-colouring of Claim 15 (i-ii)

(iii) *There are two consecutive lists  $L_i$  with 2 in the first one and 4 in the second.*

Without loss of generality, we assume that  $2 \in L_1$  and  $4 \in L_2$  and we assign colours 2 to the edge  $u_1v_1$  and 4 to the edge  $u_2v_2$ .

We now consider the list  $L_0$  and choose  $c_0 \in L_0$  such that  $c_0 \neq 2$  and assign this colour to  $u_0v_0$ . If possible, we choose  $c_0 > 2$ .

We first assume that we are not in the case where  $c_0 = 1$  and  $\sigma_2 = 7$ . Depending on  $c_0$  and the value of  $\sigma_2$ , we attribute colours to  $u_1u_2$  and  $u_2u_3$  in the following way:

	$\sigma_2$	$\omega(u_1u_2)$	$\omega(u_2u_3)$
$c_0 > 2$	$\neq 5$	1	4
$c_0 > 2$	5	1	3
$c_0 = 1$	$\neq 7$	4	3

We can see that we satisfy all the conditions of Claim 15.

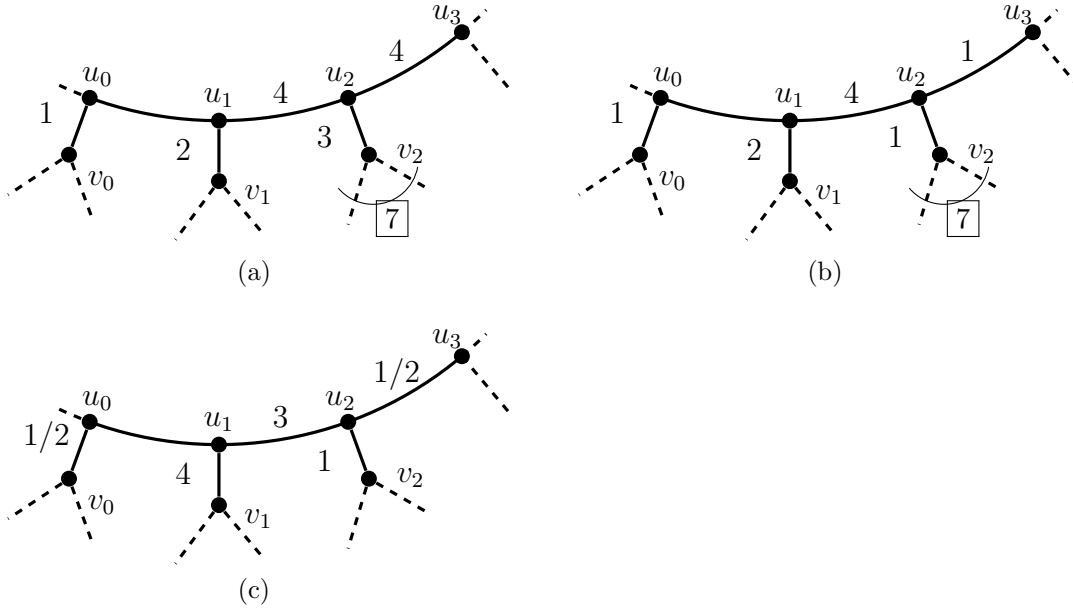


Figure 11: Pre-colouring of Claim 15 (iii)

Assume now that we are in the case  $c_0 = 1$  and  $\sigma_2 = 7$ . Then necessarily  $L_0 = \{1, 2\}$ . If  $3 \in L_2$ , we choose for the edges  $u_0v_0$ ,  $u_1v_1$ ,  $u_2v_2$  the colours 1, 2, 3 and for  $u_1u_2$  and  $u_2u_3$  the colour 4 (see Figure 11a). Then as before we satisfy all the conditions of Claim 15. If  $1 \in L_2$ , we choose for the edges  $u_0v_0$ ,  $u_1v_1$ ,  $u_2v_2$  the colours 1, 2, 1 and for  $u_1u_2$  and  $u_2u_3$  the colour 4 and 1 (see Figure 11b). Then as before we satisfy all the conditions of Claim 15. Hence we can assume that  $L_2 = \{2, 4\}$ . If  $3 \in L_1$  or  $4 \in L_1$  then we have three consecutive lists with 1 in the first one, 3 (or 4) in the second and 2 in the last list. Then as we show before Claim 15 holds. Hence we can assume that  $L_1 = \{1, 2\}$  and  $L_2 = \{2, 4\}$ .

Now, we reverse the role of  $u_1$  and  $u_2$ , assuming that  $L_1 = \{2, 4\}$  and  $L_2 = \{1, 2\}$ , and also the roles of  $u_0$  and  $u_3$ . We choose for  $u_1v_1$  colour 4 and for  $u_2v_2$  colour 1 and we put to  $u_1u_2$  colour 3 and to  $u_2u_3$  colour 1 or 2 to have a sum different to

the new  $\sigma_2$ . Then if 1 or 2 belongs to  $L_0$  we will satisfy Claim 15 (see Figure 11c). Thus we can assume that  $L_0 = \{3, 4\}$ . But then we are back to the first case by exchanging the role of 0 and 2. We have now  $L_1 = \{2, 4\}$  and  $L_2 = \{3, 4\}$  and thus can find colours to satisfy Claim 15.

Therefore, we have proved that we can satisfy Claim 15 whenever there are two consecutive lists with a 2 and a 4.

(iv) *There are two consecutive lists  $L_i$  with 1 in the first one and 3 in the second.*

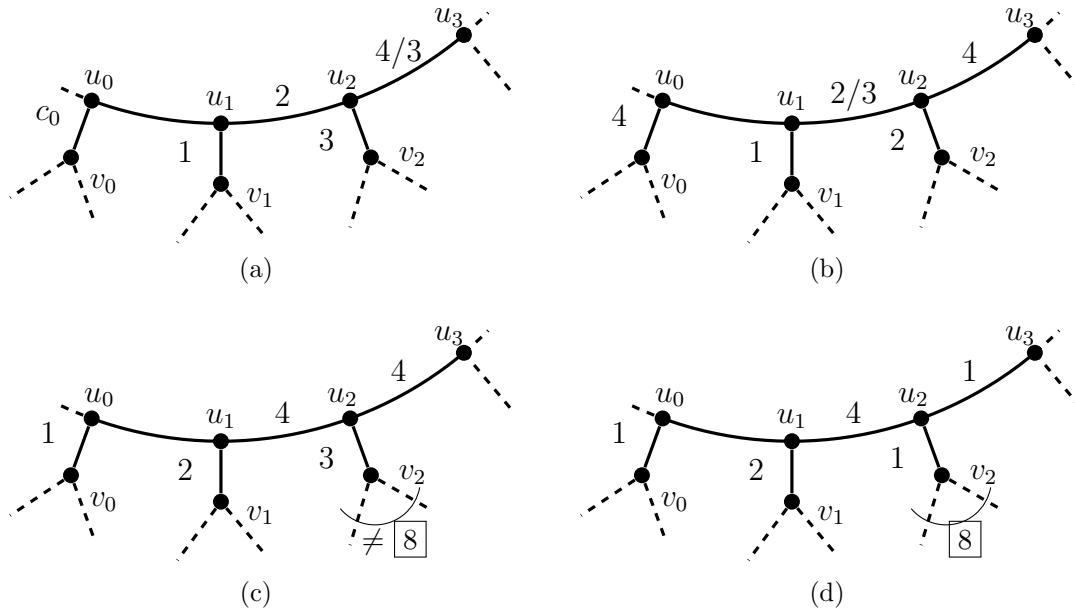


Figure 12: Pre-colouring of Claim 15 (iv)

Assume that  $1 \in L_1$  and  $3 \in L_2$  and we affect colour 1 to the edge  $u_1v_1$  and 3 to the edge  $u_2v_2$ .

Consider the list  $L_0$  and choose  $c_0 \in L_0$ . If  $c_0 > 2$ , then we assign colour 2 to the edge  $u_1u_2$  and we assign colour 4 to the edge  $u_2u_3$  whenever  $\sigma_2 \neq 6$  and colour 3 otherwise (see Figure 12a). Hence, we satisfy all the conditions of Claim 15.

Thus we may assume that  $L_0 = \{1, 2\}$ .

Since Claim 15 is true when there are two consecutive lists the first with 2 and the second with 4, we may assume that  $4 \notin L_1$ . If  $3 \in L_1$ , then  $1 \notin L_2$  and  $2 \notin L_2$ , since by our previous observation are no three consecutive lists with colours 1, 3, 1 or 1, 3, 2, respectively. Thus the argument  $3 \in L_1$  implies that  $L_2 = \{3, 4\}$ . In this case,

we reverse the role of  $u_0$  and  $u_2$  assuming that  $L_0 = \{3, 4\}$  and  $L_2 = \{1, 2\}$ . Then we may assign colours 4, 1, 2 to edges  $u_0v_0, u_1v_1, u_2v_2$  and assign to colours 2, 4 (if  $\sigma_0 \neq 6$ ) or 3, 4 (otherwise)  $u_1u_2, u_2u_3$  (see Figure 12b). Thus again Claim 15 holds.

Now assume that  $L_1 = \{1, 2\}$ . Observe that  $4 \notin L_2$  because, otherwise, there would be two consecutive lists the first with 2 and the second with 4 (which is case (iii)). Assume that  $1 \in L_2$ . If  $\sigma_2 \neq 8$ , then we assign colours 1, 2, 3 to edges  $u_0v_0, u_1v_1, u_2v_2$  and colours 4, 4 to edges  $u_1u_2, u_2u_3$  (see Figure 12c). If  $\sigma_2 = 8$ , then we assign colours 1, 2, 1 to edges  $u_0v_0, u_1v_1, u_2v_2$  and colours 4, 1 to edges  $u_1u_2, u_2u_3$  (see Figure 12d). Hence, we satisfy all the conditions of Claim 15.

We may assume that  $1 \notin L_2$  and  $4 \notin L_2$ , so  $L_2 = \{2, 3\}$ . Then we consider  $L_3$ . If  $4 \in L_3$  then we have two two consecutive lists the first one with 2 and the second with 4, so Claim 15 is true. Since  $|L_3| \geq 2$  we have that  $1 \in L_3$  or  $2 \in L_3$ . Thus we have three consecutive lists with colours 1, 3, 1 or 1, 3, 2 so by our previous observation Claim 15 is true.

Thus we may assume that there are no two consecutive lists  $L_i$  with 2 in the first one and 4 in the second and there are no two consecutive lists  $L_i$  with 1 in the first one and 3 in the second. If none of these two cases appear, but there are at least two different lists, it means that the lists are necessarily alternating  $\{1, 3\}$  with  $\{2, 4\}$ . But then three consecutive lists  $L_i$  with 1 in the first one, 3 in the second and 1 in the last list appear, so we are done.

Thus we can now assume that all the lists have size 2 and are the same, say  $L_i = \{a, b\}$  with  $a < b$ . By hypothesis, there exists  $i$  with  $\sigma_i \neq 5$  and without loss of generality, we can assume that  $i = 2$ . Assume first that  $a \neq 1$ . Then we assign the following colours:  $u_0v_0$  and  $u_2v_2$  get  $b$  whereas  $u_1v_1$  gets  $a$ ,  $u_1u_2$  gets 1 and  $u_2u_3$  gets 4 (see Figure 13a). If  $b \neq 4$ , we do the reverse:  $u_0v_0$  and  $u_2v_2$  get  $a$  whereas  $u_1v_1$  gets  $b$ ,  $u_1u_2$  gets 4 and  $u_2u_3$  gets 1. Finally, if  $\{a, b\} = \{1, 4\}$ , then  $u_0v_0$  and  $u_2v_2$  get 1 whereas  $u_1v_1$  gets 4,  $u_1u_2$  gets 3 and  $u_2u_3$  gets 2. In all these cases, Claim 15 is satisfied. □

We now extend the partial colouring of Claim 15 to a neighbour sum distinguishing 2-relaxed colouring in the following way.

We first colour the edge  $u_3v_3$  with a colour in  $L_3$  that is not the colour of  $\omega(u_2u_3)$  and we colour all the other edges  $u_iv_i$  with any colour in  $L_i$ .

Now we colour edge by edge the edges of  $C$  from  $u_3u_4$  to  $u_0u_1$ . To colour  $u_iu_{i+1}$  we ask for:

- $u_i$  and  $v_i$  to be distinguished, thus  $\omega(u_iu_{i+1}) + \omega(u_iu_{i-1}) \neq \sigma_i$

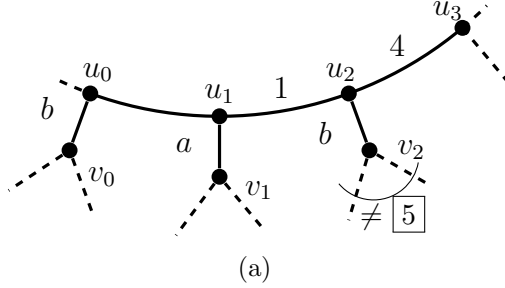


Figure 13: Pre-colouring of Claim 15

- $u_{i+1}$  to have their two adjacent edges of different colours:  $\omega(u_i u_{i+1})$  must be different from  $\omega(u_{i+1} v_{i+1})$
- $u_i$  and  $u_{i-1}$  to be distinguished.

To colour the last edge  $u_0 u_1$ , we replace the second condition (since  $u_1$  already has two colours) by a condition to distinguish  $u_1$  from  $v_1$ . There are three conditions giving three forbidden colours, thus at least one colour is always available.

This way, we colour all the edges of the cycle. At the end, all the adjacent vertices  $(u_i, v_i)$  are distinguished and all the adjacent vertices of the cycle are also distinguished (since the pairs  $u_0, u_1$  and  $u_1, u_2$  are sure to be distinguished by Claim 15). Moreover, there are at most twice the same colour on a vertex. Thus we obtain a neighbour sum distinguishing 2-relaxed colouring.

We now consider the final case where all the lists  $L_i$  are the same list  $\{a, b\}$  and all the  $\sigma_i$  are equal to 5. Then we colour all the edges  $u_i v_i$  with the colour  $a$ .

Let  $r = \ell \bmod 4$ . We colour the edges of the cycle following the pattern 1342, except for the last  $4 + r$  edges. For these ones, we use the same pattern but we double  $r$  colours that are not equal to  $a$ . For example, if  $r = 2$  and  $a = 3$ , we finish by 113442. This way, all the vertices of the cycle have at least two distinct colours. All the couples  $u_i, v_i$  are distinguished since there are no consecutive edges of the cycle that sum to 5. And finally all the pairs  $u_i, u_{i+1}$  are also distinguished since all the edges at distance 2 on the cycle are different, which is enough to distinguish the edge between them.

□

Since  $\chi_{\Sigma}^{\prime 2}(C_5) = 3$ , Theorem 12 implies the following corollary.

**Corollary 16.** *If  $G$  is a connected subcubic graph with at least three vertices, then  $\chi_{\Sigma}^{\prime 2}(G) \leq 4$ .*



## 5 Concluding remarks

In the paper we propose a new version of the distinguishing edge colouring of a graph, in which the monochromatic set of edges induces a subgraph with bounded maximum degree. If the maximum degree of the monochromatic subgraph is bounded either by maximum degree of the graph or by 1, then we obtain two well-know versions of distinguishing edge colourings related with Conjecture 1 or 2, respectively. According to these equivalences, the following conjecture can be considered as a generalization of both Conjectures 1 and 2.

**Conjecture 17.** *If  $G$  is a connected graph on at least 3 vertices and  $G \neq C_5$ , then  $\chi_{\Sigma}^d(G) \leq \left\lceil \frac{\Delta(G)}{d} \right\rceil + 2$ .*

As a support for Conjecture 17 we proved the validity of it for three families of graphs. In Section 3 we proved that for every value of  $d$ , it holds for trees. In Section 4 we proved that Conjecture 17 is true for complete graphs when  $d = 2$  or  $d \geq \lceil (V(G) - 1)/2 \rceil$ . In Section 4 we proved that Conjecture 17 is also true for subcubic graphs.

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