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# Lissajous-toric knots

Marc Soret and Marina Ville

## Abstract

A point in the  $(N, q)$ -torus knot in  $\mathbb{R}^3$  goes  $q$  times along a vertical circle while this circle rotates  $N$  times around the vertical axis. In the Lissajous-toric knot  $K(N, q, p)$ , the point goes along a vertical Lissajous curve (parametrized by  $t \mapsto (\sin(qt + \phi), \cos(pt + \psi))$ ) while this curve rotates  $N$  times around the vertical axis. Such a knot has a natural braid representation  $B_{N, q, p}$  which we investigate here. If  $\gcd(q, p) = 1$ ,  $K(N, q, p)$  is ribbon; if  $\gcd(q, p) = d > 1$ ,  $B_{N, q, p}$  is the  $d$ -th power of a braid which closes in a ribbon knot. We give an upper bound for the 4-genus of  $K(N, q, p)$  in the spirit of the genus of torus knots; we also give examples of  $K(N, q, p)$ 's which are trivial knots.

## 1 Introduction

We study a class of knots generalizing torus knots, which we call *Lissajous-toric*: a torus knot is generated by a circle rotating around an axis and a Lissajous-toric knot is generated by a Lissajous curve rotating around the axis. There are several ways of describing them.

### 1.1 Lissajous-toric knots: various points of view

#### 1.1.1 A description in $\mathbb{R}^3$

We recall the description of the  $(N, q)$ -torus knot in  $\mathbb{R}^3$  endowed with an orthonormal frame  $Oxyz$  (see for example [Cr] 1.5). If  $\Gamma$  is the circle of radius 1 centered at  $(0, 2, 0)$  in the  $yz$  plane, a point travelling along the knot goes  $q$  times around  $\Gamma$  while  $\Gamma$  is rotated  $N$  times around the axis  $Oz$ . In the case of the *Lissajous-toric knots*, we replace the vertical circle by a vertical Lissajous curve: we take three integers  $N, q, p$  with  $(N, q) = (N, p) =$

1 and a real number  $\phi$ , and we define a knot  $K(N, q, p, \phi)$  as follows. Consider the curve  $C_{q,p,\phi}$  given in a vertical plane by

$$t \in [0, 2\pi] \longrightarrow \mathbb{R}^3$$

$$t \mapsto \left( 2 + \sin (qt), 0, \cos (p(t + \phi)) \right)$$

and rotate  $C_{q,p,\phi}$   $N$  times around the axis generated by  $(0, 0, 1)$ . In Cartesian coordinates, we write the knot as

$$(*) \begin{cases} x = (2 + \sin(qt)) \cos(Nt) \\ y = (2 + \sin(qt)) \sin(Nt) \\ z = \cos(p(t + \phi)) \end{cases}$$

### 1.1.2 A description in the 3-dimensional cylinder

We write (\*) above in cylindrical coordinates:

$$\begin{cases} \theta = Nt \\ \rho = 2 + \sin(qt) \\ z = \cos(p(t + \phi)) \end{cases}$$

Thus  $K(N, q, p, \phi)$  is a closed  $N$ -braid which we can write in the 3-cylinder  $\mathbb{S}^1 \times \mathbb{R}^2$  as follows :

$$e^{it} \mapsto \left( e^{Nit}, \sin(qt), \cos(p(t + \phi)) \right) \quad (1)$$

Note the similarity with the  $(N, q)$ -torus knot which is written in the 3-sphere or the 3-cylinder as

$$e^{it} \mapsto (\frac{1}{\sqrt{2}}e^{Nit}, \frac{1}{\sqrt{2}}e^{qit}) \quad (2)$$

### 1.1.3 Billiard curve in a solid torus



Figure 1: Perspective front view of knots  $B(3, 4, 5)$  and  $K(3, 4, 5)$

Just as Billiard curves are equivalent to Lissajous knots (cf. [JP]) Lissajous toric knots are equivalent to billiard curves in a square solid torus, namely a cube where the top and bottom have been identified; C. Lamm introduced them in [La 1], see also the related [L-O]. Such billiard curves are parametrized similarly to Lissajous toric knots; the trigonometric functions are replaced by *saw-tooth functions* of the type  $g(t) := 2|t - [t] - \frac{1}{2}|$ . and  $h(t) := t - [t]$ .

$$C(N, p, q, \phi) : \left( \begin{array}{ccc} [0, 2\pi] & \longrightarrow & [-1, 1]^3 \\ t & \mapsto & (g(Nt), g(p(t + \phi)), h(qt)) \end{array} \right) \quad (3)$$

C. Lamm noticed that these billiard curves in a solid torus do not depend on the phase up to mirror transformation and stated that, if  $p$  and  $q$  are mutually prime, the knot  $K(N, q, p)$  is ribbon.

#### 1.1.4 Singularity knots of minimal surfaces

We first encountered the  $K(N, q, p, \phi)$ 's in [S-V] when we studied the singularities of minimal disks in  $\mathbb{R}^4$ ; having noticed that their knot types do not depend on the phase  $\phi$  up to mirror transformation, we dropped the  $\phi$  in the notation.

We consider a minimal, i.e. conformal harmonic, map  $F : \mathbb{D} \longrightarrow \mathbb{R}^4$  where  $\mathbb{D}$  is the unit disk in  $\mathbb{C}$ , with  $dF(0) = 0$ , i.e.  $F$  has a branch point at 0. If moreover  $F$  is a topological embedding, we can copy Milnor's construction of algebraic knots ([Mi]) and take the intersection of  $F(\mathbb{D})$  with a small sphere centered at  $F(0)$ : we obtain a *minimal knot*. Complex curves are a special case of minimal surfaces and the germ  $z \mapsto (z^N, z^q)$  yields the  $(N, q)$ -torus knot. In [S-V] the knots  $K(N, q, p, \phi)$ 's came from germs of singularities of the type

$$z \mapsto (Re(z^N), Im(z^N), Im(z^q), Re(e^{pi\phi} z^p)) \quad (4)$$

with

$$N < p, q \quad (5)$$

In [S-V] we called the  $K(N, q, p)$ 's *simple minimal knots*; in the present paper we drop the assumption (5) and study these knots *per se*; *Lissajous-toric* is a more appropriate name for the general case.

## 1.2 Contents of the paper

In [S-V] we defined a braid  $B_{N,q,p}$  naturally associated to the knot  $K(N, q, p)$ ; we describe it here in much greater detail. We view  $B_{N,q,p}$  as a collection of graphs of  $N$  functions from  $[\eta, 1 + \eta]$  to  $\mathbb{R}^2$ ; the purpose of the small positive number  $\eta$  is to avoid crossing points at the endpoints of the interval.

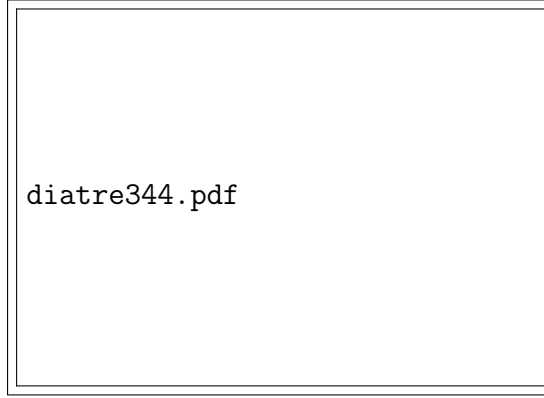


Figure 2: Braid shadow of  $B(5, q, p)$

We prove in §4.1 below

**Proposition 1.** *Let  $d = \gcd(p, q)$ ,  $\tilde{q} = \frac{q}{d}$ ,  $\tilde{p} = \frac{p}{d}$ ; then*

$$B_{N,q,p} = B_{N,\tilde{q},\tilde{p}}^d \quad (6)$$

Since  $\tilde{q}$  and  $\tilde{p}$  are mutually prime and since the knot type does not change if we interchange  $p$  and  $q$ , we make the

**Assumption 1.** *The numbers  $p$  and  $q$  are mutually prime and  $q$  is odd.*

In §2, we construct two braids  $\alpha_{N,q,p}$  and  $\beta_{N,q,p}$  of the form

$$\alpha_{N,q,p} = \prod_{2 \leq 2k \leq N-1} \sigma_{2k}^{\pm} \quad \beta_{N,q,p} = \prod_{1 \leq 2k+1 \leq N-1} \sigma_{2k+1}^{\pm} \quad (7)$$

where the  $\sigma_i$ 's are the standard generators of the braid group  $\mathbf{B}_N$  and the exponents  $\pm 1$  of the  $\sigma_i$ 's appearing in  $\alpha$  and  $\beta$  are given by simple formulae in  $N, q, p$ .

We will state below the Main Theorem which expresses the braid  $B_{N,q,p}$  as a product of the braids  $\alpha_{N,q,p}$ ,  $\alpha_{N,q,p}^{-1}$ ,  $\beta_{N,q,p}$  and  $\beta_{N,q,p}^{-1}$  as follows :

$$B_{N,q,p} = Q_{N,q,p} \alpha_{N,q,p} Q_{N,q,p}^{-1} \beta_{N,q,p} \quad (8)$$

where the  $N$ -braid  $Q_{N,q,p}$  is also a product of  $\alpha_{N,q,p}^{\pm 1}$ 's and  $\beta_{N,q,p}^{\pm 1}$ 's. We illustrate the Main Theorem in §3 by going through the examples we gave in [S-V] and we prove it in §4.

In the rest of the paper, we drop the Assumption 1 and study the topology of the knot. In §5.1, we reprove a theorem of Lamm:

**Theorem 1.** *If  $p$  and  $q$  are mutually prime, the knot  $K(N, q, p)$  is ribbon.*

**Corollary 1.** *If  $d = \gcd(p, q) > 1$ , the knot  $K(N, q, p)$  is periodic and its braid is the  $d$ -th power of a braid which closes in a ribbon knot.*

**Theorem 2.** *If  $d = \gcd(p, q)$ , the four-genus of  $K(N, q, p)$  verifies*

$$g_4(K(N, q, p)) \leq \frac{(N-1)(d-1)}{2}. \quad (9)$$

**Remark 1.** *The right-hand side of (9) is the genus of the  $K(N, d)$ -torus knot (cf. [K-M]).*

**Remark 2.** *The inequality (9) can be strict: for example the knot  $K(3, 5, 10)$  is  $10_{123}$  which is slice.*

There is one case where we know that (9) is an equality:

**Proposition 2.** *Let  $N, q, p$  be positive integers with  $(N, q) = (N, p) = 1$ ,  $d = \gcd(q, p)$  and let*

$$\tilde{p} = \frac{p}{d} \quad \tilde{q} = \frac{q}{d} \quad (10)$$

$$\text{If } \tilde{p} + \tilde{q} \equiv 0 \pmod{2N} \quad \text{or} \quad \tilde{p} - \tilde{q} \equiv 0 \pmod{2N} \quad (11)$$

*the knot  $K(N, q, p)$  is represented by a quasipositive braid and its 4-genus is*

$$g_4(K(N, q, p)) = \frac{(N-1)(d-1)}{2} \quad (12)$$

Finally, replacing  $t$  by  $t + \pi$  in the expression of  $K(N, q, p)$  given in §1.1.1 yields

**Proposition 3.** *If  $p$  and  $q$  have different parities (and thus  $N$  is odd), then  $K(N, q, p)$  is preserved by the involution*

$$(x, y, z) \mapsto (-x, -y, -z).$$

*Hence it is positive strongly amphicheiral.*

Some of the  $K(N, q, p)$ 's are actually trivial knots; in §6 show:

**Proposition 4.** *If  $N$  and  $q$  are mutually prime, the knots  $K(N, q, q + N)$ ,  $K(N, q, 1)$ ,  $K(N, q, 2Nq + 1)$  and  $K(N, q, 2Nq - 1)$  are trivial.*

Can we get all the trivial  $K(N, q, p)$ 's this way? We did computer simulations using the braid software from the Liverpool knot group ([br]) and KnotPlot ([KP]): they told us that in some cases (the  $K(4, 5, \cdot)$ 's for example) the answer is yes but in most cases the answer is no (see the lists of Jones polynomials at the end of the paper).

## Acknowledgment

We are grateful to Moshe Cohen whose stimulating conversation prompted us to embark on this work.

## 2 The structure of a simple minimal braid

### 2.1 Overview

Here is an informal description of the contents of the Main Theorem.

There are  $2q$  values of  $t$  in  $[\eta, 1 + \eta]$  (we call them *crossing values*) above which two or more of the  $N$  graphs forming  $B_{N,q,p}$  meet (at *crossing points*) and the data of all these crossing points make up the braid (see Figure 2); above each of the crossing values  $t$ 's, the generators of the braid group  $\mathbf{B}_N$  describing the corresponding crossing points are all even (i.e. of the form  $\sigma_{2k}^{\pm 1}$ ) or all odd (i.e. of the form  $\sigma_{2k+1}^{\pm 1}$ ).

The set of crossing points of  $B_{N,q,p}$  above a crossing value  $t$  can be represented by one of the braids:  $\alpha_{N,q,p}$ ,  $\alpha_{N,q,p}^{-1}$ ,  $\beta_{N,q,p}$  or  $\beta_{N,q,p}^{-1}$  which were introduced in formula (7); thus  $B_{N,q,p}$  is a product of the  $\alpha_{N,q,p}$ 's and  $\beta_{N,q,p}$ 's and of their inverses.

We order the  $2q$  crossing values  $t_1 < t_2 < \dots < t_q < \dots < t_{2q}$ . Going from  $t_k$  to  $t_{k+1}$  changes  $\alpha_{N,q,p}^{\pm 1}$  into  $\beta_{N,q,p}^{\pm 1}$  or vice-versa. A formula gives us the exponent  $+1$  or  $-1$  of the  $\alpha_{N,q,p}$  or  $\beta_{N,q,p}$  above a given crossing point  $t_k$  in terms only of  $N, q, p$  and  $k$ .

Finally we notice that, if we have an  $\alpha_{N,q,p}$  (resp.  $\alpha_{N,q,p}^{-1}$ ,  $\beta_{N,q,p}$ ,  $\beta_{N,q,p}^{-1}$ ) for  $t_k$  (with  $k \neq q$ ), we have a  $\alpha_{N,q,p}^{-1}$  (resp.  $\alpha_{N,q,p}$ ,  $\beta_{N,q,p}^{-1}$ ,  $\beta_{N,q,p}$ ) for  $t_{2q-k}$ : this explains the presence of  $Q_{N,q,p}$  and  $Q_{N,q,p}^{-1}$  in the product (8).

## 2.2 Statement of the structure theorem

**Main Theorem.** *Let  $N, p, q$  be three integers such that  $q$  is odd and  $(p, q) = (N, q) = (N, p) = 1$ ; and let  $A, B$  two integers such that*

$$2NA + Bq = 1 \quad (13)$$

For  $i \in \{1, \dots, N-1\}$ , we let

$$\epsilon_{N,q,p}(i) = (-1)^{[\frac{pBi}{N}]} \quad (14)$$

where  $[ ]$  denotes the integral part and we define

$$\alpha_{N,q,p} = \prod_{2 \leq 2i \leq N-1} \sigma_{2i}^{\epsilon_{N,q,p}(2i)} \quad \beta_{N,q,p} = \prod_{1 \leq 2i+1 \leq N-1} \sigma_{2i+1}^{\epsilon_{N,q,p}(2i+1)} \quad (15)$$

For  $k \in \{1, \dots, 2q\}$ ,  $k \neq q, k \neq 2q$ , we let

$$\lambda_{N,q,p}(k) = (-1)^{[\frac{2Apk}{q}]} \quad (16)$$

Up to mirror transformation, the knot  $K(N, q, p)$  is represented by the braid

$$B_{N,q,p} = \underbrace{\alpha_{N,q,p}^{\lambda(1)} \beta_{N,q,p}^{\lambda(2)} \dots \alpha_{N,q,p}^{\lambda(q-2)} \beta_{N,q,p}^{\lambda(q-1)}}_{Q_{N,q,p}} \alpha_{N,q,p} \underbrace{\beta_{N,q,p}^{-\lambda(q-1)} \alpha_{N,q,p}^{-\lambda(q-2)} \dots \beta_{N,q,p}^{-\lambda(2)} \alpha_{N,q,p}^{-\lambda(1)}}_{Q_{N,q,p}^{-1}} \beta_{N,q,p} \quad (17)$$

The  $k$ -th factor in this expression corresponds to the  $k$ -th crossing value  $t_k$ .

Notice that the arithmetic formulae (14) and (16) can be written in terms of the Conway sign:

**Definition 1.** ([Co]) *If  $m$  and  $n$  are two integers,  $n$  is said to be positive (resp. negative) modulo  $m$  if  $n$  is congruent to an integer inside  $(0, \frac{m}{2})$  (resp.  $(0, -\frac{m}{2})$ ).*

### 3 Illustrations and examples

In this section we go through the examples featured in [S-V] and we write their braid using the terminology of the Main Theorem.

We define three permutations of the crossing values  $t_k$ ,  $k \in \{1, \dots, 2q\}$  and their corresponding action on the blocks  $\alpha$  and  $\beta$  :

$$\begin{aligned} T(k) &= k + q, S(k) = 2q - k, R(k) = q - k \\ T : \alpha &\mapsto \beta, \beta \mapsto \alpha \\ S : \alpha &\mapsto \alpha^{-1}, \beta \mapsto \beta^{-1} \\ R : \alpha &\mapsto \beta^{-1}, \beta \mapsto \alpha^{-1} \end{aligned} \tag{18}$$

Since it is clear in each case of the following list what the  $N, q, p$  are, we dropped the indices  $N, q, p$ .

- $N = 3, q = 4, p = 5$ : square knot  $3_1 \# \bar{3}_1$

$$Q\sigma_2^{-1}Q^{-1}\sigma_1 \quad \text{where} \quad Q = \sigma_2\sigma_1^{-1}\sigma_2^{-1}\sigma_1$$

- $N = 3, q = 4, p = 7$ : trivial knot

$$Q\sigma_2Q^{-1}\sigma_1^{-1} \quad \text{where} \quad Q = (\sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-1})^2$$

- $N = 3, q = 4, p = 10$ : figure eight knot

$$B_{3,4,10} = B_{3,2,5}^2 = (Q\sigma_2Q^{-1}\sigma_1^{-1})^2 \quad \text{where} \quad Q = \sigma_2\sigma_1\sigma_2\sigma_1$$

- $N = 3, q = 5, p = 7$ :  $10_{155}$

$$Q\sigma_2^{-1}Q^{-1}\sigma_1^{-1} \quad \text{where} \quad Q = \sigma_2^{-1}\sigma_1\sigma_2^{-1}\sigma_1$$

Note that this knot verifies the assumptions of Theorem 2.

- $N = 3, q = 5, p = 10$ :  $10_{123}$

$$B_{3,5,10} = B_{3,1,2}^5 = (\sigma_2^{-1}\sigma_1)^5$$

- $N = 3, q = 7, p = 8$ :  $5_1\#\bar{5}_1$

$$Q\sigma_2^{-1}Q^{-1}\sigma_1 \quad \text{where} \quad Q = \sigma_2\sigma_1^{-1}\sigma_2\sigma_1\sigma_2^{-1}\sigma_1$$

- $N = 3, q = 7, p = 19$ :  $14N11995$

$$Q\sigma_2Q^{-1}\sigma_1 \quad \text{where} \quad Q = \sigma_2\sigma_1^{-1}\sigma_2^{-1}\sigma_1\sigma_2\sigma_1^{-1}$$

- $N = 4, q = 5, p = 7$ :  $5_2\#\bar{5}_2$

$$Q\alpha Q^{-1}\beta \quad \text{where} \quad \alpha = \sigma_2^{-1} \quad \beta = \sigma_1\sigma_3 \quad Q = \alpha^{-1}\beta^{-1}\alpha\beta$$

- $N = 4, q = 5, p = 13$ :  $9_{46}$

$$Q\alpha Q^{-1}\beta \quad \text{where} \quad \alpha = \sigma_2 \quad \beta = \sigma_1\sigma_3 \quad Q = \alpha\beta\alpha^{-1}\beta^{-1}$$

- $N = 5, q = 6, p = 22$ :  $7_7$

$$B_{5,6,22} = B_{5,3,11}^2 = (Q\alpha Q^{-1}\beta)^2 \quad \text{where} \quad \alpha = \sigma_2\sigma_4^{-1} \quad \beta = \sigma_1^{-1}\sigma_3 \quad Q = \alpha^{-1}\beta$$



Figure 3:  $B_{4,5,13}$ ,  $A = 4$ ,  $\alpha = \sigma_2, \beta = \sigma_1\sigma_3, \lambda(k) = (-1)^{[\frac{2k}{5}]}$



Figure 4:  $B_{5,6,22} = B_{5,3,11}^2, \alpha = \sigma_2\sigma_4^{-1}, \beta = \sigma_1^{-1}\sigma_3$

## 4 Proof of the Main Theorem on the simple minimal braid $B_{N,q,p}$

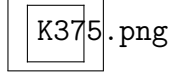


Figure 5: Graph of  $\{\psi_k\}_{k=1,2,3}$  of knot  $K(3, 7, 5)$

We recall some facts from [S-V]. We endow  $\mathbb{R}^3$  with coordinates  $(t, y, z)$ : the braid is the collections of the graphs in  $\mathbb{R}^3$  of the functions  $\psi_k$  for  $k = 1, \dots, N$ :

$$\begin{aligned} \psi_k &= (\psi_k^{(1)}, \psi_k^{(2)}) : [\eta, 1 + \eta] \longrightarrow \mathbb{R}^2 \\ t \mapsto (y, z) &= (\psi_k^{(1)}(t), \psi_k^{(2)}(t)) = \left( \sin \frac{2\pi q}{N}(t + k), \cos \frac{2\pi p}{N}(t + k + \phi) \right) \end{aligned} \quad (19)$$

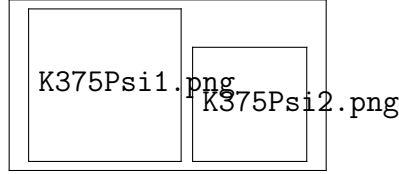


Figure 6: Graph of  $\{\psi_k^{(1)}\}_{k=1,2,3}$  and  $\{\psi_k^{(2)}\}_{k=1,2,3}$  of knot  $K(3, 7, 5)$

### 4.1 Periodic braids: proof of Proposition 1

We now prove Proposition 1 (stated in 1.2). We divide the interval  $[\eta, 1 + \eta]$  into  $d$  intervals

$$I_n = \left[ \frac{n}{d} + \eta, \frac{n+1}{d} + \eta \right], \quad n = 0, \dots, d-1.$$

After a change of variables  $t \mapsto s = dt$ , we see that the braid above an interval  $I_n$  consists in the collection of graphs of the functions  $[d\eta, 1 + d\eta] \longrightarrow \mathbb{R}^2$

$$s \mapsto \left( \sin \frac{2\pi \tilde{q}}{N}(s + dk), \cos \frac{2\pi \tilde{p}}{N}(s + dk + d\phi) \right) \quad (20)$$

Since  $(N, d) = 1$ , the map  $k \mapsto kd \pmod{N}$  induces a permutation of  $\{1, \dots, N-1\}$ ; hence the piece of  $B_{N,q,p}$  above  $I_n$  is the collection of graphs above  $[\eta, 1+\eta]$  of the functions

$$s \mapsto \left( \sin \frac{2\pi\tilde{q}}{N}(s+k), \cos \frac{2\pi\tilde{p}}{N}(s+k+d\phi) \right)$$

i.e. it is the braid  $B_{N,\tilde{q},\tilde{p}}$ , representing the knot  $K(N, \tilde{q}, \tilde{p}, d\phi)$ ; this proves Proposition 1.  $\square$

## 4.2 Crossing values and crossing points of the braid

The *braid shadow* is the projection of the braid onto the first two components  $(t, y)$  of  $\mathbb{R}^2$  i.e. the collection of the graphs of the  $\psi_k^{(1)}$ 's.

A *crossing point*  $P$  of the braid is the data of two different integers,  $k, l$  with  $0 \leq k, l \leq N-1$  and a number  $t \in [\eta, 1+\eta]$  called a *crossing value* such that

$$\psi_k^{(1)}(t) = \psi_l^{(1)}(t) \quad \text{i.e.} \quad \sin\left(\frac{2\pi}{N}q(t+k)\right) = \sin\left(\frac{2\pi}{N}q(t+l)\right).$$

There is a total of  $(N-1)q$  crossing points, as in the case of the  $(N, q)$  torus knot (where  $q = p$ ).

A straightforward computation (cf. [S-V]) shows that, for a crossing point  $P$  between the  $k$ -th and  $l$ -th strands of  $B_{N,q,p}$ , the corresponding crossing value  $t$  verifies for some integer  $m$

$$t = -\frac{k+l}{2} + \frac{N}{4q}(2m+1) \tag{21}$$

The sign  $\Sigma(P)$  of a crossing point  $P$  is

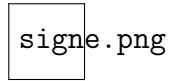


Figure 7: crossing numbers +1 and -1

$$\Sigma(P) = \text{sign of } (\psi_k^{(2)}(t) - \psi_l^{(2)}(t))(\psi_l^{(1)'}(t) - \psi_k^{(1)'}(t)) \tag{22}$$

In [S-V], we computed this  $\Sigma(P)$  as:

$$\Sigma(P) = (-1)^m (-1)^{[p\frac{m}{q} + \frac{p}{2q} + \frac{2p\phi}{N}]} (-1)^{[p\frac{k-l}{N}]} (-1)^{[q\frac{k-l}{N}]} \quad (23)$$

where  $[ \ ]$  denotes the integral part.

#### 4.2.1 Determination of the crossing points above a given crossing value

Let  $t$  be a crossing value of  $B_{N,q,p}$ . We look for the  $y$ 's such that  $(t, y)$  is a crossing point of the braid shadow.

We derive from (21) the existence of at least one ordered pair of integers  $(m, \mathfrak{s})$  such that

$$t = -\frac{\mathfrak{s}}{2} + \frac{N}{4q}(2m+1). \quad (24)$$

(with  $\mathfrak{s} = k + l$ ). There can be several  $(m, \mathfrak{s})$ 's verifying (24) for the same crossing value  $t$ ; however

$$1 \leq \mathfrak{s} = k + l \leq 2N - 3 \quad (25)$$

which implies that there are at most two possible  $(m, \mathfrak{s})$ 's (Lemma 1 below). We will see later that for  $t$  and  $(m, \mathfrak{s})$  given, a crossing point  $(t, y)$  of the braid shadow above  $t$  will be given by the data of  $\mathfrak{d} = k - l$ .

**Lemma 1.** *If  $t$  is a crossing value, one of the following two cases occurs:*

- **1st case.** *There is exactly one ordered pair  $(m, \mathfrak{s})$ ,  $1 \leq \mathfrak{s} \leq 2N - 3$  verifying (24); we denote it  $(m(t), \mathfrak{s}(t))$  and  $\mathfrak{s}(t)$  is either  $N - 2$ ,  $N - 1$  or  $N$ .*
- **2nd case.** *There exist exactly two  $(m, \mathfrak{s})$ 's satisfying (24) with  $1 \leq \mathfrak{s} \leq 2N - 3$ ; we denote them  $(m(t), \mathfrak{s}(t))$  and  $(m(t) + q, \mathfrak{s}(t) + N)$ .*

*Proof.* We let  $(m(t), \mathfrak{s}(t))$  be the ordered pair such that  $\mathfrak{s}(t)$  is the smallest  $\mathfrak{s}$  for the  $(m, \mathfrak{s})$ 's verifying (24) and (25). If  $(m_1, \mathfrak{s}_1)$  and  $(m_2, \mathfrak{s}_2)$  both verify (24) for the same  $t$ , we have

$$q(\mathfrak{s}_1 - \mathfrak{s}_2) = N(m_2 - m_1) \quad (26)$$

Since  $q$  and  $N$  are mutually prime, it follows that, for some integer  $a$ ,

$$\mathfrak{s}_2 = \mathfrak{s}_1 + aN \quad m_2 = m_1 + aq \quad (27)$$

Since  $\mathfrak{s}(t)$  is the smallest one, it verifies

$$\mathfrak{s}(t) \leq N. \quad (28)$$

If we are in the 1st case of the Lemma 1, i.e. a single  $(m, \mathfrak{s})$ , we derive from (27) that  $\mathfrak{s}(t) + N$  does not verify (25), i.e.

$$\mathfrak{s}(t) + N > 2N - 3 \quad (29)$$

Putting together (28) and (29), we get

$$N - 2 \leq \mathfrak{s}(t) \leq N$$

which concludes the proof of the 1st case.

The 2nd case is clear: since  $\mathfrak{s}(t) + 2N > 2N - 3$ ,  $\mathfrak{s}(t) + N$  is the only other integer  $\mathfrak{s}$  in  $[1, 2N - 3]$  which can appear in (24); the corresponding  $m$  is  $m(t) + q$ .  $\square$

Lemma 1 told us which  $\mathfrak{s}$ 's and  $m$  occur for crossing points  $(t, y)$  above a crossing value  $t$ : we now find the  $k, l$ 's such that  $\mathfrak{s} = k + l$  and derive the  $\sigma_i^\pm$ 's corresponding to the  $(t, y)$ 's.

**Definition 2.** Let  $P = (t, y)$  be a crossing point; we denote by  $i(P) \in \{1, \dots, N - 1\}$  the corresponding generator subscript, i.e.  $P$  is represented by  $\sigma_{i(P)}$  or  $\sigma_{i(P)}^{-1}$ .

**Lemma 2.** Let  $t$  be a crossing value of  $B_{N,q,p}$ .

1. The point  $P = (t, y)$  is a crossing point of the braid shadow if and only if

$$y = (-1)^{m(t)} \cos\left(\frac{q\mathfrak{d}}{N}\pi\right) \quad (30)$$

where  $\mathfrak{d}$  is any integer in  $[1, \dots, N - 1]$  of the same parity as  $\mathfrak{s}(t)$

2. To determine  $i(P)$ , we do the Euclidean division of  $q\mathfrak{d}$  by  $2N$

$$q\mathfrak{d} = 2Nn + w \quad (31)$$

with  $n \geq 0$ ,  $-N < w < N$ .

(a) If  $m(t)$  is even,

$$i(P) = i\left(t, \cos\left(\frac{q\mathfrak{d}}{N}\pi\right)\right) = |w| \quad (32)$$

(b) If  $m(t)$  is odd,

$$i(P) = i\left(t, -\cos\left(\frac{q\mathfrak{d}}{N}\pi\right)\right) = N - |w| \quad (33)$$

*Proof. Proof of 1:* we treat separately the two cases of Lemma 1.

1. **1st case:** a single ordered pair  $(m, \mathfrak{s})$ .

Let  $k, l$  such that  $k + l = \mathfrak{s}(t)$  and assume that  $l < k$ ; note that  $\mathfrak{d} = k - l$  has the parity of  $\mathfrak{s}(t)$ .

- (a) If  $\mathfrak{s}(t) = N - 1$ , the smallest possible value for  $l$  is 0 and  $k - l$  runs through all integers  $\mathfrak{d}$ ,  $1 \leq \mathfrak{d} \leq N - 1$  with the parity of  $\mathfrak{s}(t) = N - 1$ .
- (b) If  $\mathfrak{s}(t) = N - 2$  (resp.  $\mathfrak{s}(t) = N$ ), the smallest value for  $l$  is 0 (resp. 1) and  $\mathfrak{d}$  runs through the integers in  $[1, N - 2]$  with the parity of  $N - 2$  or  $N$ ; since  $N - 1$  has parity opposite to  $N$  and  $N - 2$ , we can actually assume  $\mathfrak{d}$  in  $[1, N - 1]$ .

To derive (30), we plug (24) into

$$y = \sin\left(\frac{2\pi}{N}q(t + k)\right) \quad (34)$$

2. **2nd case:** two ordered pairs:  $(m(t), \mathfrak{s}(t))$  and  $(m(t) + q, \mathfrak{s}(t) + N)$ .

- (a) We first consider the  $k, l$ 's such that  $l < k$  and  $k + l = \mathfrak{s}(t)$ . As above,  $k - l$  runs through the integers  $\mathfrak{d}$  with the parity of  $\mathfrak{s}(t)$  and such that

$$1 \leq \mathfrak{d} \leq \mathfrak{s}(t) \quad (35)$$

- (b) If  $k + l = \mathfrak{s}(t) + N$ , we look at  $\mathfrak{d} = k - l$ 's with  $l < k$ :

$$l = \mathfrak{s}(t) + N - k \geq \mathfrak{s}(t) + N - (N - 1) = \mathfrak{s}(t) + 1$$

hence  $\mathfrak{d} = \mathfrak{s}(t) + N - 2l \leq \mathfrak{s}(t) + N - 2\mathfrak{s}(t) - 2 = N - \mathfrak{s}(t) - 2$  and

$$1 \leq \mathfrak{d} \leq N - \mathfrak{s}(t) - 2 \quad (36)$$

Moreover every integer in  $\mathfrak{d} \in [1, N - \mathfrak{s}(t) - 2]$  with the parity of  $\mathfrak{s}(t) + N$  is a legitimate  $\mathfrak{d}$ , i.e. there exist  $k, l$  in  $\{1, \dots, N - 1\}$  with  $\mathfrak{d} = k - l$  and  $k + l = \mathfrak{s}(t) + N$ ; for example  $N - \mathfrak{s}(t) - 2 = (N - 1) - (\mathfrak{s}(t) + 1)$ .

Using (34), we derive the  $y$ -coordinate of the crossing point:

$$y = (-1)^{m(t)}(-1)^q \cos(\pi \frac{q\mathfrak{d}}{N}) = (-1)^{m(t)} \cos(\pi \frac{q\tilde{\mathfrak{d}}}{N}) \quad (37)$$

where  $\tilde{\mathfrak{d}} = N - \mathfrak{d}$ ; if  $\mathfrak{d}$  verifies (36), then

$$\mathfrak{s}(t) + 2 \leq \tilde{\mathfrak{d}} \leq N - 1 \quad (38)$$

Since  $\mathfrak{d}$  has the parity of  $\mathfrak{s}(t) + N$ ,  $\tilde{\mathfrak{d}}$  has the parity of  $\mathfrak{s}(t)$ .

Putting together the intervals (35) and (38) concludes the proof of the 2nd case.

**Proof of 2.** We derive from (31) that

$$\cos(\pi \frac{q\mathfrak{d}}{N}) = \cos(\pi \frac{w}{N}) = \cos(\pi \frac{|w|}{N})$$

so 2 (a) of Lemma 2 follows from the fact that the function  $\cos$  is decreasing on  $(0, \pi)$ :

$$\cos \frac{\pi}{N} > \cos \frac{2\pi}{N} > \dots > \cos \frac{(N-1)\pi}{N}.$$

and 2 (b) of Lemma 2 follows from

$$-\cos(\pi \frac{q\mathfrak{d}}{N}) = -\cos(\pi \frac{w}{N}) = -\cos(\pi \frac{|w|}{N}) = \cos(\pi \frac{N - |w|}{N}).$$

□

To see how many crossing points  $(t, y)$  occur above  $t$ , i.e. how many values (30) takes for a given  $t$ , we notice the following.

- If  $u, v \in \{1, \dots, N - 1\}$  and  $\cos(\pi \frac{qu}{N}) = \cos(\pi \frac{qv}{N})$ , then  $u = v$ .
- If  $u, v \in \{1, \dots, N - 1\}$  and  $\cos(\pi \frac{qu}{N}) = -\cos(\pi \frac{qv}{N})$ , then  $u = N - v$ .

So the sets  $\{\cos(\pi \frac{qu}{N})$  with  $1 \leq u \leq N-1\}$  and  $\{-\cos(\pi \frac{qu}{N})$  with  $1 \leq u \leq N-1\}$  are identical. We derive from Lemma 2

**Corollary 2.** *Let  $t$  be a crossing value. The indices  $i(P)$ 's for the crossing points  $P$ 's above  $t$  are all the  $i$ 's in  $\{1, \dots, N-1\}$  and with*

1. *the parity of  $\mathfrak{d}$  and  $\mathfrak{s}(t)$  if  $m(t)$  is even*
2. *the parity of  $\mathfrak{d} + N$  and  $\mathfrak{s}(t) + N$  if  $m(t)$  is odd.*

#### 4.2.2 The sign of the crossing points

We now compute the sign of the crossing points described in Lemma 2.

**Lemma 3.** *The sign of the crossing point corresponding to the  $(t, y)$  appearing in (30) is given by*

$$\Sigma(t, y) = (-1)^{m(t)} \sigma\left(p \frac{m(t)}{q} + \frac{p}{2q} + \frac{2p\phi}{N}\right) \sigma\left(\frac{q\mathfrak{d}}{N}\right) \sigma\left(\frac{p\mathfrak{d}}{N}\right) \quad (39)$$

where  $\sigma(r)$  is the parity of the integer part:

$$\sigma(r) = (-1)^{[r]} \quad (40)$$

*Proof.* We recall (Lemma 1) that  $k + l = \mathfrak{s}(t)$  or  $k + l = \mathfrak{s}(t) + N$ .

If  $k + l = \mathfrak{s}(t)$  and  $\mathfrak{d} = k - l$ , then (39) is just the formula (23) for the sign of a crossing point. So we assume that  $k + l = \mathfrak{s}(t) + N$ : we have seen above that  $y$  is given by (37) with  $k - l = N - \tilde{\mathfrak{d}}$  and we write (23) for the sign of the crossing point

$$\begin{aligned} & (-1)^{m(t)+q} \sigma\left(q\left(\frac{N-\tilde{\mathfrak{d}}}{N}\right)\right) \sigma\left(p\left(\frac{N-\tilde{\mathfrak{d}}}{N}\right)\right) \sigma\left(p \frac{m(t)+q}{q} + \frac{p}{2q} + \frac{2p\phi}{N}\right) \\ &= (-1)^{m(t)} \sigma\left(q \frac{\tilde{\mathfrak{d}}}{N}\right) \sigma\left(p \frac{\tilde{\mathfrak{d}}}{N}\right) \sigma\left(p \frac{m(t)}{q} + \frac{p}{2q} + \frac{2p\phi}{N}\right) \end{aligned}$$

□

We write the last two factors of (39) in terms of  $i(P)$ :

**Lemma 4.** *We let  $P = (t, (-1)^{m(t)} \cos \pi \frac{q\mathfrak{d}}{N})$  be a crossing point of  $B_{N,q,p}$  with  $m(t)$ ,  $\mathfrak{d}$  and  $w$  as in 2) of Lemma 2. Then*

$$\sigma\left(\frac{q\mathfrak{d}}{N}\right)\sigma\left(\frac{p\mathfrak{d}}{N}\right) = \begin{cases} \sigma\left(\frac{Bpi(P)}{N}\right) & \text{if } p \text{ is odd} \\ (-1)^{m(t)}\sigma\left(\frac{Bpi(P)}{N}\right) & \text{if } p \text{ is even} \end{cases} \quad (41)$$

*Proof.* It follows from (31) that

$$\sigma\left(\frac{q\mathfrak{d}}{N}\right) = \sigma\left(\frac{w}{N}\right). \quad (42)$$

We recall (13), namely  $2NA + Bq = 1$ , hence  $\mathfrak{d} = 2NA\mathfrak{d} + Bq\mathfrak{d}$ ; putting this together with the Euclidean division in (31), we have

$$\mathfrak{d} = 2NA\mathfrak{d} + 2NnB + Bw \quad (43)$$

Thus

$$\sigma\left(\frac{p\mathfrak{d}}{N}\right) = \sigma\left(\frac{pBw}{N}\right). \quad (44)$$

Now

$$\sigma\left(\frac{w}{N}\right)\sigma\left(\frac{pBw}{N}\right) = \sigma\left(\frac{|w|}{N}\right)\sigma\left(\frac{pB|w|}{N}\right) = \sigma\left(\frac{pB|w|}{N}\right) \quad (45)$$

- If  $m(t)$  is even, then  $i(P) = |w|$  and the Lemma is proved.
- If  $m(t)$  is odd, we use the fact that  $B$  is odd to write

$$\sigma\left(\frac{pB|w|}{N}\right) = \sigma\left(\frac{pB(N - i(P))|w|}{N}\right) = \begin{cases} \sigma\left(\frac{pBi(P)}{N}\right) & \text{if } p \text{ is odd} \\ -\sigma\left(\frac{pBi(P)}{N}\right) & \text{if } p \text{ is even} \end{cases}$$

□

It follows from Lemma 4 that the sign  $\Sigma(P)$  given in (39) of a crossing point  $P$  of crossing value  $t$  is ( $\epsilon$  has been defined in (14) above)

- $\sigma\left(p\frac{m(t)}{q} + \frac{p}{2q} + \frac{2p\phi}{N}\right)\epsilon(N, q, p)(i)$  if  $p$  is even
- $\sigma\left(p\frac{m(t)}{q} + \frac{p}{2q} + \frac{2p\phi}{N}\right)(-1)^{m(t)}\epsilon(N, q, p)(i)$  if  $p$  is odd.

We recall (see the formulae (15) in the Main Theorem) that the  $\pm 1$ -exponent of a  $\sigma_i$  in  $\alpha_{N,q,p}$  or  $\beta_{N,q,p}$  is  $\epsilon(N, q, p)(i)$ , hence

**Lemma 5.** *The  $\pm 1$ -exponent of the  $\alpha_{N,q,p}$  or  $\beta_{N,q,p}$  corresponding to a crossing value  $t$  in  $B_{N,q,p}$  is*

1.  $\sigma\left(p\frac{m(t)}{q} + \frac{p}{2q} + \frac{2p\phi}{N}\right)$  if  $p$  is even
2.  $(-1)^{m(k)}\sigma\left(p\frac{m(t)}{q} + \frac{p}{2q} + \frac{2p\phi}{N}\right)$  if  $p$  is odd.

The formulae in Lemma 5 depend on  $m(t)$ , where  $t$  goes through the  $2q$  crossing values. If  $t$  is the  $h$ -th crossing value, for  $h = 1, \dots, 2q$ , we want to have  $m(t)$  directly as an expression in  $h$  so we number the crossing values

$$t_1 < t_2 < \dots < t_q < \dots < t_{2q} \quad (46)$$

and for any integer  $k$ , with  $1 \leq k \leq 2q$ , we let

$$m(k) = m(t_k) \quad \mathfrak{s}(k) = \mathfrak{s}(t_k) \quad (47)$$

If  $t_k$  and  $t_{k+1}$  are two consecutive crossing values, we derive from (24)

$$t_{k+1} - t_k = \frac{1}{2q} \left[ q(\mathfrak{s}(k) - \mathfrak{s}(k+1)) + N(m(k+1) - m(k)) \right] \geq \frac{1}{2q} \quad (48)$$

Since there are  $2q$  crossing values in  $[\eta, 1 + \eta]$ , (48) is an equality and we have

$$q(\mathfrak{s}(k) - \mathfrak{s}(k+1)) + N(m(k+1) - m(k)) = 1 \quad (49)$$

We note in passing that, if we plug (49) into Proposition 2, we get the confirmation of the obvious fact

**Lemma 6.** *The crossing points above  $t_k$  and  $t_{k+1}$  are represented by  $\sigma_i^{\pm 1}$ 's with  $i$ 's of opposite parities.*

We now confront (49) with  $2NA + Bq = 1$  and derive the existence of an integer  $\nu_k$  such that

$$m(k+1) - m(k) = \nu_k q + 2A.$$

Thus, for any  $k$ , there exists an integer  $a_k$  such that

$$m(k) = m(1) + a_k q + 2(k-1)A \quad (50)$$

$$(-1)^{m(k)} = (-1)^{m(1)}(-1)^{a_k} \quad (51)$$

Define  $\phi_0$  by

$$p \frac{m(1)}{q} - \frac{2Ap}{q} + \frac{p}{2q} + \frac{2p\phi_0}{N} = 0 \quad (52)$$

We see (23) that  $\phi_0$  is a critical phase, i.e. a phase for which the knot  $K(N, p, q, \phi)$  is singular. So we pick a phase

$$\phi = \phi_0 + \xi \quad (53)$$

where  $\xi$  is a very small positive number. Using (50), we rewrite

$$\sigma\left(p \frac{m(t)}{q} + \frac{p}{2q} + \frac{2p\phi}{N}\right) = (-1)^{a_k p} (-1)^{\left[\frac{2Ap}{q}k + \xi\right]} \quad (54)$$

It follows from Lemma 5 and equations (50), (54) that the exponent of the  $\alpha_{N,q,p}$  or  $\beta_{N,q,p}$  at the  $k$ -th crossing value is

$$\begin{cases} (-1)^{m(1)}(-1)^{\left[\frac{2Ap}{q}k + \xi\right]} & \text{if } p \text{ is odd} \\ (-1)^{\left[\frac{2Ap}{q}k + \xi\right]} & \text{if } p \text{ is even} \end{cases}$$

Since we are working up to mirror transformation, we assume

$$(-1)^{m(1)} = 1.$$

We now conclude: the expression  $\left[\frac{2Ap}{q}k + \xi\right]$  is equal to 1 for  $k = q, 2q$  and equal to  $\lambda(k)$  for the other  $k$ 's. Going back to the statement of the Main Theorem, this gives us the exponent for the  $k$ -th crossing values with  $k \leq q$  or  $k = 2q$ . We settle the case of the  $k$ 's with  $q < k < 2q$  by noticing that

$$\lambda(2q - k) = -\lambda(k).$$

□

## 5 The four-genus

### 5.1 Ribbon knots

A *ribbon knot* in  $\mathbb{S}^3$  bounds a disk in  $\mathbb{S}^3$  with only ribbon singularities; equivalently it bounds an embedded disk in  $\mathbb{B}^4$  which does not have local maxima for the distance to the origin of  $\mathbb{B}^4$ . Thus a ribbon knot is slice, i.e. its 4-genus is zero; the long-standing Slice-Ribbon conjecture asks if the converse is true.

### 5.1.1 Proof of Theorem 1

Th. 1 has been proved by Lamm and also follows from his more general construction of ribbon symmetric unions ([La 2], [K-T]). His proof is fairly allusive so we felt it would be useful to give a more detailed proof.

We recall the well-known fact:

**Proposition 5.** *Let  $K$  be a knot in  $\mathbb{R}^3$  which is symmetric with respect to a plane  $P$  in  $\mathbb{R}^3$ ; then it is ribbon.*

*Proof.* We endow  $\mathbb{R}^3$  with the frame  $Oxyz$  and assume that  $P$  is defined by the equation  $x = 0$ . By genericity arguments, we assume

1.  $K$  meets  $P$  at a finite number of points
2. outside of  $P$ ,  $K$  is never tangent to the direction of  $Ox$ .

Since  $K$  has one component, it meets  $P$  at exactly two points.

We let  $K_+$  (resp.  $K_-$ ) be the intersection of  $K$  with the half-space of  $\mathbb{R}^3$  defined by  $z \geq 0$  (resp.  $z \leq 0$ ):  $K_+$  and  $K_-$  are both diffeomorphic to a closed interval.

Letting  $S$  be the symmetry in  $\mathbb{R}^3$  with respect to  $P$ , we let

$$\begin{aligned} \Phi &= [0, 1] \times K_+ \longrightarrow \mathbb{R}^3 \\ (t, X) &\mapsto tX + (1 - t)S(X) \end{aligned} \tag{55}$$

The self-intersections of  $\Phi$  are given by the data of  $t_1, t_2, X_1, X_2$  such that

$$t_1X_1 + (1 - t_1)S(X_1) = t_2X_2 + (1 - t_2)S(X_2) \tag{56}$$

We denote by  $(x_i, y_i, z_i)$ ,  $i = 1, 2$  the coordinates of  $X_i$ . Since  $S(x_i, y_i, z_i) = (-x_i, y_i, z_i)$ , (56) implies that

$$y_1 = y_2, z_1 = z_2.$$

Thus the line segments  $I_1 = X_1S(X_1)$  and  $I_2 = X_2S(X_2)$  are both included in the line which is defined by the equations  $y = y_1, z = z_1$ . Moreover, one of them is included in the other one and we have a ribbon singularity.  $\square$

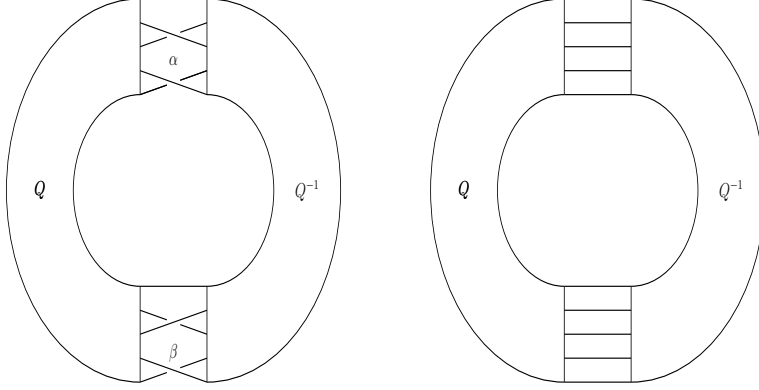


Figure 8:  $B(5, q, p)$  and the link  $L$

The Main Theorem tells us that, if  $p$  and  $q$  are mutually prime, with  $q$  odd,  $B_{N,q,p}$  is as in Fig. 8.

If  $N$  is the number of strands, there are  $N-1$  half-twist tangles connecting  $Q$  and  $Q^{-1}$ ; we replace them by  $N-1$  trivial tangles and get the  $N$ -component link  $L$  of Fig. 8 which is symmetric w.r.t. a plane.

Proposition 5 tells us that  $L$  bounds  $N$  ribbon disks  $D_1, \dots, D_N$ ; and the same arguments show us that two of these disks only have ribbon-type intersection.

We now connect each  $D_i$  to  $D_{i+1}$  by a half-twisted band bounded by the half-twist tangle of Fig. 8. The resulting surface is a topological disk with only ribbon singularities.  $\square$

## 5.2 General case: proof of Theorem 2

We use an idea by Brandenbursky and Kedra ([B-K]). If  $b$  is a  $N$ -braid, we denote by  $\hat{b}$  the link obtained by closing the braid  $b$ . If  $b_1$  and  $b_2$  are  $N$ -two braids, [B-K] constructed a cobordism of Euler characteristic  $-N$  between the closure of the product  $\widehat{b_1 b_2}$  and the disjoint union of the closures  $\hat{b}_1 \sqcup \hat{b}_2$ . Letting  $\tilde{q} = \frac{q}{N}$ ,  $\tilde{p} = \frac{p}{N}$ , we recall that

$$B_{N,q,p} = B_{N,\tilde{q},\tilde{p}}^d.$$

Applying [B-K]'s result  $d$  times, we derive a cobordism in  $\mathbb{B}^4$  of Euler characteristic  $-N(d-1)$  between  $\widehat{B_{N,q,p}}$  and  $\underbrace{\widehat{B_{N,\tilde{q},\tilde{p}}} \sqcup \widehat{B_{N,\tilde{q},\tilde{p}}} \sqcup \dots \widehat{B_{N,\tilde{q},\tilde{p}}}}_{d \text{ copies}}$ . Since  $\widehat{B_{N,\tilde{q},\tilde{p}}}$

is a ribbon knot (Theorem 1), it bounds an embedded disk in  $\mathbb{B}^4$ . Thus  $\widehat{B_{N,q,p}}$  bounds a surface of Euler characteristic

$$-d(N-1) + d = 1 - (d-1)(N-1).$$

We recover the formula (9) for the genus and Theorem 2 is proved.  $\square$

### 5.3 Quasipositive knots: proof of Proposition 2

Lee Rudolph (see [Ru] for details) defines a braid  $\gamma \in \mathbf{B}_N$  to be *quasipositive* if it is a product of conjugates  $w\sigma_i w^{-1}$  of positive braid generators, i.e.

$$\gamma = w_1 \sigma_{i_1} w_1^{-1} w_2 \sigma_{i_2} w_2^{-1} \dots w_k \sigma_{i_k} w_k^{-1} \quad (57)$$

**Theorem 3.** ([Ru]) *If  $\gamma$  is a quasipositive braid written as in (57) closing in a knot  $\hat{\gamma}$ , its four-genus verifies*

$$1 - 2g_4(\hat{\gamma}) = N - k.$$

It is easy to check that under the assumptions of Proposition 2, the exponents of all the  $\sigma_{2k}$ 's and  $\sigma_{2k+1}$ 's appearing respectively in  $\alpha_{N,q,p}$  and  $\beta_{N,q,p}$  are all of the same sign. Since we are working up to mirror symmetry, we can assume all these exponents to be equal to 1; thus

$$\alpha_{N,q,p} = \prod_{2 \leq 2k \leq N-1} \sigma_{2k} \quad \beta_{N,q,p} = \prod_{1 \leq 2k+1 \leq N-1} \sigma_{2k+1} \quad (58)$$

Hence  $B_{N,p,q} = (Q\alpha_{N,q,p}Q^{-1}\beta_{N,q,p})^d$  is a quasipositive braid; Theorem 3 tells us that  $1 - 2g_4(K(N, q, p)) = N - d(N-1)$  and Proposition 2 follows.

## 6 Trivial knots: proof of Proposition 4

### 6.1 The knot $K(N, q, q+N)$ is trivial

We set

$$A = \prod_{1 \leq 2k \leq N} \sigma_{2k} \quad B = \prod_{1 \leq 2k+1 \leq N} \sigma_{2k+1} \quad (59)$$

**Lemma 7.**

$$B_{N,q,q+N} = A(BA)^{\frac{q-1}{2}}(B^{-1}A^{-1})^{\frac{q-1}{2}}B^{-1} \quad (60)$$

The main thing to note about this formula is that all the positive generators are on one side and all the negative generators are on the other side.

*Proof.* We use the Main Theorem. It assumes that  $q$  is odd but in the present case, if  $q$  is even,  $N$  has to be odd, hence  $q + N$  is odd and we switch  $q$  and  $q + N$  to apply the theorem. We compute

$$\epsilon(i) = (-1)^i \quad \lambda(k) = (-1)^k \quad (61)$$

Thus  $\alpha_{N,q,q+N} = \prod_{1 \leq 2k \leq N} \sigma_{2k} = A$  and  $\beta_{N,q,q+N} = \prod_{1 \leq 2k+1 \leq N} \sigma_{2k+1}^{-1} = B^{-1}$ . We conclude by noticing that

$$(AB)^{\frac{q-1}{2}} A = A(BA)^{\frac{q-1}{2}}$$

□

We construct a trivial pure braid  $\mathcal{B}_N$ ; we will show that  $B_{N,N+q,q}$  is the product of a power of  $\mathcal{B}_N$  and of a piece of  $\mathcal{B}_N$ .

If  $N = 2k$  is even, we let

$$\mathcal{B}_N = (BA)^k (B^{-1}A^{-1})^k. \quad (62)$$

If  $N = 2k + 1$  is odd, we let

$$\mathcal{B}_N = (BA)^k BA^{-1} (B^{-1}A^{-1})^k. \quad (63)$$

In both cases, we check that the corresponding permutation between the endpoints of the braid is the identity, thus  $\mathcal{B}_N$  is a pure braid.

To prove that it is a trivial braid, we discuss when one strand of  $\mathcal{B}_N$  is above another one; so let us fix some terminology.

We number the strands of  $\mathcal{B}_N$ : the  $j$ -th strand,  $0 \leq j \leq N - 1$  is the strand starting at the  $(j + 1)$ -th point on the left (the points being counted from top to bottom).

We say that the  $j$ -th strand is above the  $k$ -th strand if, wherever there is a crossing point between these two strands, the  $j$ -th strand is above the  $k$ -th strand. As an example, in Fig. 9, the red strand is above all the other strands.



1.  $0 \leq x \leq N$  and the  $k$ -th (resp.  $j$ -th) strand is  $\searrow$  (resp.  $\nearrow$ ). Then the  $k$ -th strand is as 1. above and the  $j$ -th strand is as 3.
2.  $N \leq x \leq 2N$  and the  $k$ -th (resp.  $j$ -th) strand  $\nearrow$  (resp.  $\searrow$ ). Then the  $k$ -th strand is as 3. above and the  $j$ -th strand is as 5.

In both cases it is easy to check that  $k < j$ .

□

We conclude the proof of the proposition in the case when  $N$  is even; the odd case is similar. We derive from Lemma 8 that for an  $n > k$ ,

$$\begin{aligned} A(BA)^n(B^{-1}A^{-1})^nB &= A(BA)^{n-k}(BA)^k(B^{-1}A^{-1})^k(B^{-1}A^{-1})^{n-k}B \\ &= A(BA)^{n-k}(B^{-1}A^{-1})^{n-k}B. \end{aligned}$$

Thus, if  $b$  is the remainder of the division of  $\frac{q-1}{2}$  by  $k$ , we have

$$B_{N,q,q+N} = A(BA)^b(B^{-1}A^{-1})^bB \quad (64)$$

The braid (64) is a piece of the braid  $(BA)^k(B^{-1}A^{-1})^k$  where the  $i$ -th strand is above the  $j$ -th strands, for  $j > i$ . Thus the same is true for (64) which closes therefore in a trivial knot.

## 6.2 The knot $K(N, 1, p)$ is trivial

This follows from the Main Theorem. We can also prove it directly by computing the crossing points and their sign: we see that every  $\sigma_i^\pm$  appears once and only once in the braid  $B_{N,1,p}$  and so braid represents a trivial knot.

## 6.3 The other knots of Proposition 4

We have now seen two cases where  $K(N, q, p)$  is trivial. We know that  $K(N, q, p)$  and  $K(N, p, q)$  are isotopic; and  $K(N, q, k)$  and  $K(N, q, 2qN + k)$  (resp.  $K(N, q, 2qN - k)$ ) are isotopic (resp. mirror image of one another). Thus we can get more examples of trivial knots, e.g.  $K(3, 5, 29)$  and  $K(3, 5, 31)$ .

## 7 Lists of Jones polynomials

Jones polynomial for  $q = 11$  and

$$p = 1 : 1$$

$$p = 2 : 1$$

$$p = 4 : 3 - \frac{1}{t^3} + \frac{1}{t^2} - \frac{1}{t} - t + t^2 - t^3$$

$$p = 5 : 1$$

$$p = 7 : 21 + \frac{1}{t^8} - \frac{4}{t^7} + \frac{8}{t^6} - \frac{14}{t^5} + \frac{20}{t^4} - \frac{24}{t^3} + \frac{26}{t^2} - \frac{24}{t} - 14t + 8t^2 - 4t^3 + t^4$$

$$p = 8 : 1$$

$$p = 10 : 7 - \frac{1}{t^7} + \frac{1}{t^6} - \frac{2}{t^5} + \frac{3}{t^4} - \frac{4}{t^3} + \frac{5}{t^2} - \frac{5}{t} - 5t + 5t^2 - 4t^3 + 3t^4 - 2t^5 + t^6 - t^7$$

$$p = 11 : t^{10} + t^{12} - t^{22}$$

$$p = 13 : 32 + \frac{1}{t^{10}} - \frac{2}{t^9} + \frac{5}{t^8} - \frac{10}{t^7} + \frac{16}{t^6} - \frac{24}{t^5} + \frac{31}{t^4} - \frac{36}{t^3} + \frac{38}{t^2} - \frac{36}{t} - 24t + 16t^2 - 10t^3 + 5t^4 - 2t^5 + t^6$$

$$p = 14 : 1$$

$$p = 16 : 193 - \frac{1}{t^9} + \frac{5}{t^8} - \frac{14}{t^7} + \frac{31}{t^6} - \frac{56}{t^5} + \frac{89}{t^4} - \frac{126}{t^3} + \frac{159}{t^2} - \frac{183}{t} - 183t + 159t^2 - 126t^3 + 89t^4 - 56t^5 + 31t^6 - 14t^7 + 5t^8 - t^9$$

$$p = 17 : 1$$

$$p = 19 : 825 + \frac{1}{t^{12}} - \frac{8}{t^{11}} + \frac{30}{t^{10}} - \frac{78}{t^9} + \frac{165}{t^8} - \frac{296}{t^7} + \frac{465}{t^6} - \frac{652}{t^5} + \frac{824}{t^4} - \frac{946}{t^3} + \frac{990}{t^2} - \frac{946}{t} - 652t + 465t^2 - 296t^3 + 165t^4 - 78t^5 + 30t^6 - 8t^7 + t^8$$

$$p = 20 : 1$$

$$p = 22 :$$

$$4863 - \frac{1}{t^{11}} + \frac{11}{t^{10}} - \frac{55}{t^9} + \frac{176}{t^8} - \frac{429}{t^7} + \frac{869}{t^6} - \frac{1518}{t^5} + \frac{2343}{t^4} - \frac{3245}{t^3} + \frac{4070}{t^2} - \frac{4652}{t} - 4652t + 4070t^2 - 3245t^3 + 2343t^4 - 1518t^5 + 869t^6 - 429t^7 + 176t^8 - 55t^9 + 11t^{10} - t^{11}$$

$$p = 23 : 1$$

$$p = 25 : 825 + \frac{1}{t^{12}} - \frac{8}{t^{11}} + \frac{30}{t^{10}} - \frac{78}{t^9} + \frac{165}{t^8} - \frac{296}{t^7} + \frac{465}{t^6} - \frac{652}{t^5} + \frac{824}{t^4} - \frac{946}{t^3} + \frac{990}{t^2} - \frac{946}{t} - 652t + 465t^2 - 296t^3 + 165t^4 - 78t^5 + 30t^6 - 8t^7 + t^8$$

$$p = 26 : 3 - \frac{1}{t^3} + \frac{1}{t^2} - \frac{1}{t} - t + t^2 - t^3$$

$$p = 28 : 193 - \frac{1}{t^9} + \frac{5}{t^8} - \frac{14}{t^7} + \frac{31}{t^6} - \frac{56}{t^5} + \frac{89}{t^4} - \frac{126}{t^3} + \frac{159}{t^2} - \frac{183}{t} - 183t + 159t^2 - 126t^3 + 89t^4 - 56t^5 + 31t^6 - 14t^7 + 5t^8 - t^9$$

$$p = 29 : 21 + \frac{1}{t^4} - \frac{4}{t^3} + \frac{8}{t^2} - \frac{14}{t} - 24t + 26t^2 - 24t^3 + 20t^4 - 14t^5 + 8t^6 - 4t^7 + t^8$$

$$p = 31 : 32 + \frac{1}{t^{10}} - \frac{2}{t^9} + \frac{5}{t^8} - \frac{10}{t^7} + \frac{16}{t^6} - \frac{24}{t^5} + \frac{31}{t^4} - \frac{36}{t^3} + \frac{38}{t^2} - \frac{36}{t} - 24t + 16t^2 - 10t^3 + 5t^4 - 2t^5 + t^6$$

$$p = 32 : 7 - \frac{1}{t^7} + \frac{1}{t^6} - \frac{2}{t^5} + \frac{3}{t^4} - \frac{4}{t^3} + \frac{5}{t^2} - \frac{5}{t} - 5t + 5t^2 - 4t^3 + 3t^4 - 2t^5 + t^6 - t^7$$

Figure 10: List of Jones polynomials of knots  $K(3, 11, p)$

Jones polynomial for  $q = 11$  and

$p = 1 :$  1

$p = 3 :$   $2 - t + t^2 - 2t^3 + t^4 - t^5 + t^6$

$p = 5 :$   $3 - \frac{1}{t^9} + \frac{5}{t^8} - \frac{12}{t^7} + \frac{18}{t^6} - \frac{22}{t^5} + \frac{23}{t^4} - \frac{20}{t^3} + \frac{15}{t^2} - \frac{8}{t} + 2t - 3t^2 + t^3$

$p = 7 :$  1

$p = 9 :$

$$166 - \frac{1}{t^8} + \frac{6}{t^7} - \frac{23}{t^6} + \frac{50}{t^5} - \frac{74}{t^4} + \frac{72}{t^3} - \frac{27}{t^2} - \frac{58}{t} - 264t + 331t^2 - 346t^3 + 307t^4 - 228t^5 + 138t^6 - 66t^7 + 23t^8 - 6t^9 + t^{10}$$

$p = 11 :$   $t^{15} + t^{17} + t^{19} - t^{20} + t^{21} - t^{22} + t^{23} - t^{24} + t^{25} - t^{26} - t^{28}$

$p = 13 :$

$$-317 + \frac{1}{t^{14}} - \frac{7}{t^{13}} + \frac{31}{t^{12}} - \frac{102}{t^{11}} + \frac{272}{t^{10}} - \frac{586}{t^9} + \frac{1040}{t^8} - \frac{1546}{t^7} + \frac{1952}{t^6} - \frac{2108}{t^5} + \frac{1938}{t^4} - \frac{1474}{t^3} + \frac{832}{t^2} - \frac{182}{t} + 570t - 576t^2 + 426t^3 - 240t^4 + 102t^5 - 31t^6 + 7t^7 - t^8$$

$p = 15 :$  1

$p = 17 :$

$$924 + \frac{1}{t^{11}} - \frac{8}{t^{10}} + \frac{34}{t^9} - \frac{106}{t^8} + \frac{256}{t^7} - \frac{490}{t^6} + \frac{763}{t^5} - \frac{963}{t^4} + \frac{949}{t^3} - \frac{616}{t^2} - \frac{45}{t} - 1820t + 2513t^2 - 2839t^3 + 2744t^4 - 2305t^5 + 1687t^6 - 1071t^7 + 586t^8 - 272t^9 + 106t^{10} - 34t^{11} + 8t^{12} - t^{13}$$

$p = 19 :$   $2 - t + t^2 - 2t^3 + t^4 - t^5 + t^6$

$p = 21 :$

$$-1784 + \frac{1}{t^{16}} - \frac{9}{t^{15}} + \frac{42}{t^{14}} - \frac{141}{t^{13}} + \frac{385}{t^{12}} - \frac{896}{t^{11}} + \frac{1803}{t^{10}} - \frac{3156}{t^9} + \frac{4829}{t^8} - \frac{6473}{t^7} + \frac{7605}{t^6} - \frac{7792}{t^5} + \frac{6845}{t^4} - \frac{4916}{t^3} + \frac{2449}{t^2} - \frac{38}{t} + 2719t - 2779t^2 + 2244t^3 - 1491t^4 + 824t^5 - 377t^6 + 141t^7 - 42t^8 + 9t^9 - t^{10}$$

$p = 23 :$  1

$p = 25 :$

$$7200 + \frac{1}{t^{13}} - \frac{10}{t^{12}} + \frac{53}{t^{11}} - \frac{199}{t^{10}} + \frac{591}{t^9} - \frac{1450}{t^8} + \frac{2981}{t^7} - \frac{5164}{t^6} + \frac{7542}{t^5} - \frac{9180}{t^4} + \frac{8943}{t^3} - \frac{6006}{t^2} + \frac{324}{t} - 14944t + 21101t^2 - 24284t^3 + 23994t^4 - 20751t^5 + 15816t^6 - 10638t^7 + 6308t^8 - 3289t^9 + 1502t^{10} - 595t^{11} + 199t^{12} - 53t^{13} + 10t^{14} - t^{15}$$

$p = 27 :$   $3 + \frac{1}{t^3} - \frac{3}{t^2} + \frac{2}{t} - 8t + 15t^2 - 20t^3 + 23t^4 - 22t^5 + 18t^6 - 12t^7 + 5t^8 - t^9$

$p = 29 :$

$$-19300 + \frac{1}{t^{18}} - \frac{11}{t^{17}} + \frac{65}{t^{16}} - \frac{269}{t^{15}} + \frac{874}{t^{14}} - \frac{2373}{t^{13}} + \frac{5578}{t^{12}} - \frac{11548}{t^{11}} + \frac{21190}{t^{10}} - \frac{34536}{t^9} + \frac{50031}{t^8} - \frac{64383}{t^7} + \frac{73381}{t^6} - \frac{73449}{t^5} + \frac{63208}{t^4} - \frac{44242}{t^3} + \frac{20698}{t^2} + \frac{2071}{t} + 28329t - 29251t^2 + 24438t^3 - 17198t^4 + 10334t^5 - 5318t^6 + 2339t^7 - 872t^8 + 269t^9 - 65t^{10} + 11t^{11} - t^{12}$$

$p = 31 :$

$$166 + \frac{1}{t^{10}} - \frac{6}{t^9} + \frac{23}{t^8} - \frac{66}{t^7} + \frac{138}{t^6} - \frac{228}{t^5} + \frac{307}{t^4} - \frac{346}{t^3} + \frac{331}{t^2} - \frac{264}{t} - 58t - 27t^2 + 72t^3 - 74t^4 + 50t^5 - 23t^6 + 6t^7 - t^8$$

$p = 33 :$

$$-48510 - \frac{1}{t^6} + \frac{21}{t^5} - \frac{209}{t^4} + \frac{1287}{t^3} - \frac{5566}{t^2} + \frac{18315}{t} + 107635t - 205909t^2 + 347017t^3 - 523775t^4 + 717125t^5 - 899424t^6 + 1041140t^7 - 1118700t^8 + 1120834t^9 - 1051138t^{10} + 925980t^{11} - 768768t^{12} + 603174t^{13} - 447986t^{14} + 314941t^{15} - 209112t^{16} + 130558t^{17} - 76165t^{18} + 41174t^{19} - 20406t^{20} + 9142t^{21} - 3631t^{22} + 1244t^{23} - 353t^{24} + 78t^{25} - 12t^{26} + t^{27}$$

$p = 35 :$

$$-317 - \frac{1}{t^8} + \frac{7}{t^7} - \frac{31}{t^6} + \frac{102}{t^5} - \frac{240}{t^4} + \frac{426}{t^3} - \frac{576}{t^2} + \frac{570}{t} - 182t + 832t^2 - 1474t^3 + 1938t^4 - 2108t^5 + 1952t^6 - 1546t^7 + 1040t^8 - 586t^9 + 272t^{10} - 102t^{11} + 31t^{12} - 7t^{13} + t^{14}$$

$p = 37 :$

$$-19300 + \frac{1}{t^{18}} - \frac{11}{t^{17}} + \frac{65}{t^{16}} - \frac{269}{t^{15}} + \frac{874}{t^{14}} - \frac{2373}{t^{13}} + \frac{5578}{t^{12}} - \frac{11548}{t^{11}} + \frac{21190}{t^{10}} - \frac{34536}{t^9} + \frac{50031}{t^8} - \frac{64383}{t^7} + \frac{73381}{t^6} - \frac{73449}{t^5} + \frac{63208}{t^4} - \frac{44242}{t^3} + \frac{20698}{t^2} + \frac{2071}{t} + 28329t - 29251t^2 + 24438t^3 - 17198t^4 + 10334t^5 - 5318t^6 + 2339t^7 - 872t^8 + 269t^9 - 65t^{10} + 11t^{11} - t^{12}$$

$p = 39 :$

$$924 - \frac{1}{t^{13}} + \frac{8}{t^{12}} - \frac{34}{t^{11}} + \frac{106}{t^{10}} - \frac{272}{t^9} + \frac{586}{t^8} - \frac{1071}{t^7} + \frac{1687}{t^6} - \frac{2305}{t^5} + \frac{2744}{t^4} - \frac{2839}{t^3} + \frac{2513}{t^2} - \frac{1820}{t} - 45t - 616t^2 + 949t^3 - 963t^4 + 763t^5 - 490t^6 + 256t^7 - 106t^8 + 34t^9 - 8t^{10} + t^{11}$$

$p = 41 :$

$$7200 + \frac{1}{t^{13}} - \frac{10}{t^{12}} + \frac{53}{t^{11}} - \frac{199}{t^{10}} + \frac{591}{t^9} - \frac{1450}{t^8} + \frac{2981}{t^7} - \frac{5164}{t^6} + \frac{7542}{t^5} - \frac{9180}{t^4} + \frac{8943}{t^3} - \frac{6006}{t^2} + \frac{324}{t} - 14944t + 21101t^2 - 24284t^3 + 23994t^4 - 20751t^5 + 15816t^6 - 10638t^7 + 6308t^8 - 3289t^9 + 1502t^{10} - 595t^{11} + 199t^{12} - 53t^{13} + 10t^{14} - t^{15}$$

$p = 43 :$

$$-1784 - \frac{1}{t^{10}} + \frac{9}{t^9} - \frac{42}{t^8} + \frac{141}{t^7} - \frac{377}{t^6} + \frac{824}{t^5} - \frac{1491}{t^4} + \frac{2244}{t^3} - \frac{2779}{t^2} + \frac{2719}{t} - 38t + 2449t^2 - 4916t^3 + 6845t^4 - 7792t^5 + 7605t^6 - 6473t^7 + 4829t^8 - 3156t^9 + 1803t^{10} - 896t^{11} + 385t^{12} - 141t^{13} + 42t^{14} - 9t^{15} + t^{16}$$

Figure 11: List of Jones polynomials of knots  $K(4, 11, p)$

## 8 Appendix

We give a better proof of the following fact from [S-V]:

**Proposition 6.** *Let  $\phi_1$  and  $\phi_2$  two real numbers. The knots  $K(N, q, p, \phi_1)$  and  $K(N, q, p, \phi_2)$  defined in (1) are either isotopic or mirror image of one another.*

*Proof.* Without loss of generality, we assume  $\phi_1 < \phi_2$ .

If there is no critical phase (i.e. a phase for which the knot is singular) between  $\phi_1$  and  $\phi_2$ , the two knots are isotopic.

In [S-V] we showed that the difference between two critical phases is of the form

$$\frac{N}{2} \left( \frac{m}{p} + \frac{n}{q} \right) \quad (65)$$

for two integers  $m, n$ .

Thus it is enough to prove that, for a given  $\phi_3$ , and integers  $m$  and  $n$ , the knots  $K(N, q, p, \phi_3)$  and  $K(N, q, p, \phi_3 + \frac{N}{2}(\frac{m}{p} + \frac{n}{q}))$  are the same or mirror images of one another.

Consider the parametrization of  $K(N, q, p, \phi)$  given in (19); we change its variable by setting

$$s = t + \frac{Nn}{2q} \quad (66)$$

and we rewrite the expression in (19)

$$\begin{aligned} & \left( \sin \frac{2\pi q}{N} (t + k), \cos \frac{2\pi p}{N} \left( t + k + \phi_3 + \frac{N}{2} \left( \frac{m}{p} + \frac{n}{q} \right) \right) \right) \\ &= \left( (-1)^n \sin \frac{2\pi q}{N} (s + k), (-1)^m \cos \frac{2\pi p}{N} (s + k + \phi_3) \right) \end{aligned} \quad (67)$$

Thus, if  $m$  and  $n$  have the same (resp. opposite) parities, the two knots are isotopic (resp. mirror images of one another).  $\square$

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