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Abstract

The method of asymptotic partial decomposition of a domain (MAPDD) proposed and justified earlier for thin domains (rod structures, tube structures consisting of a set of thin cylinders) generates some special interface conditions between the three-dimensional and one-dimensional parts. In the case of fluid mechanics this method generates a variational formulation of the velocity with Poiseuille type or Womersley type flow in the tubes at a small distance of the ends. However, the pressure should be then reconstructed using the obtained velocity field. In the present paper the procedure of the reconstruction of pressure is given and justified by the estimates between the exact pressure of the full geometry problem and reconstructed one. Numerical examples apply the method and assess its accuracy depending on the flow regime.

Key words: Stokes and Navier–Stokes equations, thin structures, asymptotic partial decomposition, hybrid dimension model

1. Introduction

Tube structures are domains which are unions of thin cylinders (or thin rectangles in two-dimensional setting). The ratio of the diameters of cylinders to their heights (or ratio of the sides of rectangles) is a small parameter ε. These domains can be considered as geometrical models of a blood circulation system. The method of asymptotic partial decomposition of a domain (MAPDD) combines the full-dimensional description in some neighborhoods

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of bifurcations and a reduced-dimensional description out of these small subdomains and it prescribes some special junction conditions at the interface between these 3D and 1D submodels (see [8, 10, 14, 11]). So, it reduces the dimension in the main part of the domain and allows to reduce essentially the computer resources needed for the numerical solution of such problems. For the models of viscous flows the original MAPDD was proposed in the form of a weak formulation containing only the velocity field with divergence free test functions only. This formulation is not suitable for a FEM numerical implementation because of difficulties of construction of divergence free finite elements. That is why the MAPDD weak formulation with pressure for the Stokes and Navier–Stokes equations was proposed and numerically tested in [2] but it was justified only for a particular class of tube structures when the number of vertices and nodes is greater by one than the number of edges. In particular, such structures cannot have loops. In the present paper we generalize this result to an arbitrary tube structure and prove the existence and uniqueness up to one constant of the MAPDD pressure.

The structure of the paper is as follows. In section 2 we recall the definitions of a tube structure, the formulation of the steady Stokes and non-stationary Naviers-Stokes equations set in a tube structure. In section 3 we recall the structure of an asymptotic expansions of the solution constructed in [9, 11, 14, 15]. Section 4 recalls the method of asymptotic partial decomposition for the Stokes and Navier–Stokes equations. Section 5 is the central section of the paper, where we formulate the main problem of reconstruction of the MAPDD pressure. This problem is related to a special extension of a linear functional generated by the variational formulation from the space of divergence free vector valued functions to the space of arbitrary functions of $\mathbf{H}_0^1$ (although having the shape of Womersley flow within the tubes). In order to construct this extension we study the extended space of test functions and prove a lemma on the divergence equation for the solutions being functions of this extended space. Finally we prove the existence and uniqueness up to an arbitrary function of time for the MAPDD pressure. Section 6 concludes with numerical examples, comparing the theoretical with numerical results, and testing the MAPDD model for flow regimes beyond the theoretical assumptions. The proofs and the extensive derivations are included in the appendix A.

2. Formulation of the fluid flow problem in a tube structure

In this section we will introduce the full dimensional fluid flow problem in a tube structure.

2.1. Thin tube structure domain

Let us remind the definition of a thin tube structure [9, 11, 14].

Let $O_1, O_2, \ldots, O_N$ be N different points in $\mathbb{R}^n, n = 2, 3$, and $e_1, e_2, \ldots, e_M$ be M closed segments each connecting two of these points (i.e. each $e_j = \overline{O_{i_j} O_{k_j}}$, where $i_j, k_j \in \{1, \ldots, N\}, i_j \neq k_j$). All points O_i are supposed to be the ends of some segments e_j. The segments e_j are called edges of the graph. The points O_i are called nodes. Any two edges e_j and e_i, $i \neq j$, can intersect only at the common node. A node is called vertex if it is an end point of only one edge. Assume that the set of vertices is $O_{N_1+1}, O_{N_1+2}, \ldots, O_N$, where $N_1 < N$. The graph

$\mathcal{B} = \bigcup_{j=1}^{M} e_j$ is the union of edges, and assume that $\mathcal{B}$ is a connected set. Let e be some edge, $e = \overline{O_i O_j}$. Consider two Cartesian coordinate systems in $\mathbb{R}^n$. The first one has the origin in O_i and the axis $O_i x_1^{(e)}$ has the direction of the ray $[O_i O_j)$; the second one has the origin in O_j and the opposite direction, i.e. $O_j \tilde{x}_1^{(e)}$ is directed over the ray $[O_j O_i)$. With every edge e_j we associate a bounded domain $\sigma_j \subset \mathbb{R}^{n-1}$ having a C^2-smooth boundary $\partial\sigma^j, j = 1, \dots, M$. For every edge $e_j = e$ and associated $\sigma_j = \sigma^{(e)}$ we denote by $B_\varepsilon^{(e)}$ the cylinder

$$B_\varepsilon^{(e)} = \{x^{(e)} \in \mathbb{R}^n : x_1^{(e)} \in (0,\, |e|),\, \frac{x^{(e)\,\prime}}{\varepsilon} \in \sigma^{(e)}\},$$

where $x^{(e)\prime} = (x_2^{(e)}, \dots, x_n^{(e)})$, $|e|$ is the length of the edge e and $\varepsilon > 0$ is a small parameter. Notice that the edges e_j and Cartesian coordinates of nodes and vertices O_j, as well as the domains σ_j, do not depend on ε. Denoting $\sigma_\varepsilon^{(e)} = \{x^{(e)\prime} \in \mathbb{R}^{n-1} : \frac{x^{(e)\,\prime}}{\varepsilon} \in \sigma^{(e)}\}$ we can write $B_\varepsilon^{(e)} = (0, |e|) \times \sigma_\varepsilon^{(e)}$. Let $\omega^1, \dots, \omega^N$ be bounded independent of ε domains in $\mathbb{R}^n$ boundaries $\partial\omega^j$; introduce the nodal domains $\omega_\varepsilon^j = \{x \in \mathbb{R}^n : \frac{x - O_j}{\varepsilon} \in \omega^j\}$. Denote $d = \max_{1 \le j \le N} \operatorname{diam} \omega^j$. By a tube structure we call the following domain

$$B_\varepsilon = \Big(\bigcup_{j=1}^{M} B_\varepsilon^{(e_j)} \Big) \bigcup \Big(\bigcup_{j=1}^{N} \omega_\varepsilon^j \Big).$$

So, the tube structure B_ε is a union of all thin cylinders connected by small smoothing domains ω_ε^j in the neighbourhoods of the nodes. Their role is to avoid artificial corners in the boundary of intersecting cylinders, and we will assume that B_ε is a bounded domain (connected open set) with a C^2-smooth boundary. In the present paper for numerical computations we will consider as well more general tube structures where one edge of the graph may correspond to two parallel non-intersecting tubes (cylinders) at the distance of order ε one from other (see Fig. 1). The theory and estimates are naturally generalized for such tube structures.

2.2. The full dimension fluid flow problem

Consider the stationary Stokes and the non-stationary Navier–Stokes equations in B_ε with the no-slip conditions at the boundary ∂B_ε except for some parts γ_ε^j of the boundary where the velocity field is given as known inflows and outflows.

Let us define these parts of the boundary. Denote $\gamma_\varepsilon^j = \partial\omega_\varepsilon^j \cap \partial B_\varepsilon$, $\gamma^j = \partial\omega^j \cap \partial B_1^j$ where $B_1^j = \{y : y\varepsilon + O_j \in B_\varepsilon\}$ and $\gamma_\varepsilon = \cup_{j=N_1+1}^{N} \gamma_\varepsilon^j$.

We will consider first the stationary Stokes equation as a simplest model and then the non-stationary Navier–Stokes equations.

So, the stationary setting is as follows. To find a solution to the Stokes problem:

$$
\begin{aligned}
-\nu\Delta\mathbf{u}_\varepsilon + \nabla p_\varepsilon &= 0,\\
\operatorname{div}\mathbf{u}_\varepsilon &= 0,\\
\mathbf{u}_\varepsilon\big|_{\partial B_\varepsilon} &= \mathbf{g}_\varepsilon,
\end{aligned}
\tag{1}
$$

where $\mathbf{u}_\varepsilon$ is the unknown velocity vector, p_ε is the unknown pressure, $\mathbf{g}_\varepsilon$ is a given vector-valued function satisfying the following conditions: $\mathbf{g}_\varepsilon(x) = \mathbf{g}_j(\frac{x-O_j}{\varepsilon})$ if $x \in \gamma_\varepsilon^j, j = N_1+1, ..., N$, and equal to zero for the remaining part of the boundary $\partial B_\varepsilon \backslash \gamma_\varepsilon$. Here $\mathbf{g}_j : \gamma_j \to \mathbb{R}^n$ belonging to $\mathbf{H}_0^{3/2}(\gamma_j)$. Assume that the following compatibility condition for the solvability of problem (1)

$$
\int_{\partial B_\varepsilon} \mathbf{g}_\varepsilon \cdot \mathbf{n} ds = \sum_{j=N_1+1}^{N} \int_{\gamma_\varepsilon^j} \mathbf{g}_j(\frac{x-O_j}{\varepsilon}) \cdot \mathbf{n} ds = 0 \tag{2}
$$

holds. Note that in this case one can prove that $\mathbf{g}_\varepsilon$ has a divergence free extension $\tilde{\mathbf{g}}$ defined in $B_\varepsilon \times [0,T]$ which we denote by the same symbol $\mathbf{g}_\varepsilon$, $\mathbf{g}_\varepsilon \in \mathbf{H}^2(B_\varepsilon))$.

Let us give the variational (weak) formulation of problem (1). Introduce the space $\mathbf{H}^1_{div0(\partial B_\varepsilon \backslash \gamma_\varepsilon)}(B_\varepsilon)$ as the subspace of vector valued functions from $\mathbf{H}^1(B_\varepsilon)$ satisfying the conditions div $\mathbf{v} = 0$, $\mathbf{v}|_{\partial B_\varepsilon \backslash \gamma_\varepsilon} = 0$, i.e.,

$$
\mathbf{H}^1_{div0(\partial B_\varepsilon \backslash \gamma_\varepsilon)}(B_\varepsilon) = \left\{ \mathbf{v} \in \mathbf{H}^1(B_\varepsilon) |\ \operatorname{div}\mathbf{v} = 0;\ \mathbf{v}|_{\partial B_\varepsilon \backslash \gamma_\varepsilon} = 0 \right\}.
$$

We consider as well the smaller subspace $\mathbf{H}^1_{div0}(B_\varepsilon) = \mathbf{H}^1_{div0(\partial B_\varepsilon \backslash \gamma_\varepsilon)}(B_\varepsilon) \cap \mathbf{H}^1_0(B_\varepsilon)$ of divergence free vector-valued functions vanishing at the whole boundary.

By a weak solution of problem (1) we understand the vector-field $\mathbf{u}_\varepsilon \in \mathbf{H}^1_{div0(\partial B_\varepsilon \backslash \gamma_\varepsilon)}(B_\varepsilon)$ such that $\mathbf{u}_\varepsilon = \mathbf{g}_\varepsilon$ on γ_ε and $\mathbf{u}_\varepsilon$ satisfies the integral identity:

$$
\nu \int_{B_\varepsilon} \nabla\mathbf{u}_\varepsilon(x) : \nabla\mathbf{v}(x) dx = 0 \tag{3}
$$

for any test function $\mathbf{v} \in \mathbf{H}^1_{div0}(B_\varepsilon)$.

According to the well-known theorem (known also as De Rham lemma) there exists a pressure function. For various regularities of the boundary this result was proved by several authors (see [5, 16, 7]), for more information on the history of the question see [4]. By this result, the formulation (3) is equivalent to another one: By a weak solution we understand the couple of the vector-field $\mathbf{u}_\varepsilon \in \mathbf{H}^1_{div0(\partial B_\varepsilon \backslash \gamma_\varepsilon)}(B_\varepsilon)$ and a scalar function $p_\varepsilon \in L^2(B_\varepsilon)$ such that $\mathbf{u}_\varepsilon = \mathbf{g}_\varepsilon$ on ∂B_ε and $(\mathbf{u}_\varepsilon, p_\varepsilon)$ satisfy the integral identity:

$$
\nu \int_{B_\varepsilon} \nabla\mathbf{u}_\varepsilon(x) : \nabla\mathbf{v}(x) dx = \int_{B_\varepsilon} p_\varepsilon \operatorname{div}\mathbf{v} dx \tag{4}
$$

for any test function $\mathbf{v} \in \mathbf{H}^1_0(B_\varepsilon)$.

There exists a unique solution to this problem (see [5, 7]).

Let us introduce now the initial boundary value problem for the non-stationary Navier–Stokes equations

$$\begin{aligned} \frac{\partial \mathbf{u}_\varepsilon}{\partial t} - \nu\Delta\mathbf{u}_\varepsilon + (\mathbf{u}_\varepsilon\cdot\nabla)\mathbf{u}_\varepsilon + \nabla p_\varepsilon = 0, \\ \operatorname{div}\mathbf{u}_\varepsilon = 0, \\ \mathbf{u}_\varepsilon\big|_{\partial B_\varepsilon} = \mathbf{g}_\varepsilon, \quad \mathbf{u}_\varepsilon(x,0) = 0, \end{aligned} \tag{5}$$

where $\mathbf{u}_\varepsilon$ is the unknown velocity vector, p_ε is the unknown pressure, $\mathbf{g}_\varepsilon$ is a given vector-valued function satisfying the following conditions: $\mathbf{g}_\varepsilon(x,t) = \mathbf{g}_j(\frac{x-O_j}{\varepsilon}, t)$ if $x \in \gamma_\varepsilon^j, j = N_1+1, ..., N$, and equal to zero for the remaining part of the boundary $\partial B_\varepsilon \backslash \gamma_\varepsilon$. Here $\mathbf{g}_j : \gamma_j \times [0,+\infty) \to \mathbb{R}^n$ belongs to $C^\alpha([0,T]; \mathbf{H}^{3/2}(\gamma_j))$ with some sufficiently large integer α and T is a positive number. Assume that $\mathbf{g}_j|_{t=0} = 0$ and that the compatibility condition

$$\int_{\partial B_\varepsilon} \mathbf{g}_\varepsilon\cdot\mathbf{n}ds = \sum_{j=N_1+1}^{N} \int_{\gamma_\varepsilon^j} \mathbf{g}_j(\frac{x-O_j}{\varepsilon}, t)\cdot\mathbf{n}ds = 0. \tag{6}$$

holds.

Then one can prove that $\mathbf{g}_\varepsilon$ has a divergence free extension $\tilde{\mathbf{g}}$ defined in $B_\varepsilon \times [0,T]$ which we denote by the same symbol $\mathbf{g}_\varepsilon$,

$$\mathbf{g}_\varepsilon \in C^\alpha([0,T]; \mathbf{H}^2(B_\varepsilon)), \tag{7}$$

satisfying for all $t \in [0,T]$ the following asymptotic estimates

$$\begin{aligned} \|\mathbf{g}_\varepsilon\|_{\mathbf{L}^2(B_\varepsilon)} + \|\frac{\partial \mathbf{g}_\varepsilon}{\partial t}\|_{\mathbf{L}^2(B_\varepsilon)} + \|\frac{\partial^2 \mathbf{g}_\varepsilon}{\partial t^2}\|_{\mathbf{L}_2(B_\varepsilon)} \le c\varepsilon^{\frac{n-1}{2}}, \\ \|\nabla\mathbf{g}_\varepsilon\|_{\mathbf{L}^2(B_\varepsilon)} + \|\nabla\frac{\partial \mathbf{g}_\varepsilon}{\partial t}\|_{\mathbf{L}^2(B_\varepsilon)} \le c\varepsilon^{\frac{n-3}{2}}, \\ \|\Delta\mathbf{g}_\varepsilon\|_{\mathbf{L}^2(B_\varepsilon)} \le c\varepsilon^{\frac{n-5}{2}}, \quad n = 2, 3, \end{aligned} \tag{8}$$

where the constant c is independent of ε.

Let us give the variational formulation. By a weak solution of problem (5) we understand the vector-field $\mathbf{u}_\varepsilon \in L^2(0,T; \mathbf{H}^1_{div0(\partial B_\varepsilon\backslash\gamma_\varepsilon)}(B_\varepsilon))$ with $\frac{\partial \mathbf{u}_\varepsilon}{\partial t} \in L^2(0,T; \mathbf{L}^2(B_\varepsilon))$ such that $\mathbf{u}_\varepsilon(x,0) = 0$, $\mathbf{u}_\varepsilon = \mathbf{g}_\varepsilon$ on γ_ε and $\mathbf{u}_\varepsilon$ satisfies for all $t \in (0,T)$ the integral identity

$$\int_{B_\varepsilon} \Big(\frac{\partial \mathbf{u}_\varepsilon}{\partial t}\cdot\phi + \nu\nabla\mathbf{u}_\varepsilon : \nabla\phi + (\mathbf{u}_\varepsilon, \nabla\mathbf{u}_\varepsilon)\cdot\phi\Big)dx = 0. \tag{9}$$

for every vector-field $\phi \in \mathbf{H}^1_{div0}(B_\varepsilon)$.

For sufficiently small ε this problem admits a unique solution (see [14]).

This formulation is equivalent (see [7, 16]) to the following one with pressure. By a weak solution we understand the couple of the vector-field $\mathbf{u}_\varepsilon \in L^2(0,T;\mathbf{H}^1_{div0(\partial B_\varepsilon\setminus\gamma_\varepsilon)}(B_\varepsilon))$ with $\mathbf{u}_{\varepsilon t} \in L^2(0,T;\mathbf{L}^2(B_\varepsilon))$ and a scalar function $p_\varepsilon \in L^2(0,T;L^2(B_\varepsilon))$ such that $\mathbf{u}_\varepsilon(x,0)=0$, $\mathbf{u}_\varepsilon = \mathbf{g}_\varepsilon$ on γ_ε and $(\mathbf{u}_\varepsilon, p_\varepsilon)$ satisfy for all $t \in (0,T)$ the integral identity or every vector-field $\phi \in \mathbf{H}^1_0(B_\varepsilon)$ for all $t \in (0,T)$,

$$\int_{B_\varepsilon} \Big(\frac{\partial \mathbf{u}_\varepsilon}{\partial t} \cdot \phi + \nu \nabla \mathbf{u}_\varepsilon : \nabla \phi + (\mathbf{u}_\varepsilon, \nabla)\mathbf{u}_\varepsilon \cdot \phi \Big) dx = \int_{B_\varepsilon} p_\varepsilon \operatorname{div} \phi dx \tag{10}$$

for every vector-field $\phi \in \mathbf{H}^1_0(B_\varepsilon)$.

This solution is defined uniquely up to an additive function dependent on time in the pressure component.

Note, that due to the incompressibility of the asymptotic approximation $\mathbf{u}_\varepsilon$ the convective term $\int_{B_\varepsilon}(\mathbf{u}_\varepsilon, \nabla \mathbf{u}_\varepsilon)\cdot \phi dx$ can be presented in the following form (used in the most part of theoretical works): $-\int_{B_\varepsilon}(\mathbf{u}_\varepsilon, \nabla)\phi \cdot \mathbf{u}_\varepsilon dx$.

3. About an asymptotic expansion of the solution

A complete asymptotic expansion of the solution of the stationary Stokes equations was constructed in [3], for the stationary Navier–Stokes equations in [9] and for the non-stationary Navier–Stokes equations in [14, 15] and has a form of a truncated series in powers of ε with coefficients being regular functions at some distance of the nodes and having boundary layer correctors near the nodes. In particular, the pressure's expansion of order J' up to terms of order $\varepsilon^{J'}$ denoted $p_\varepsilon^{(J')}$ is a sum of a regular part (macroscopic pressure) multiplied by a cut-off function and the boundary layer. The regular part is an affine function of $x_1^{(e)}$ for each edge e, satisfying the equation on the graph $\mathcal{B}$ corresponding to the tube structure B_ε (for more details see [14, 15, 12]).

In the case of stationary Stokes equations the regular part of the velocity is the Poiseuille flow (see below) and it is proved in [3] that the following estimates hold:

$$\|\mathbf{u}_\varepsilon^{(J')} - \mathbf{u}_\varepsilon\|_{H^1(B_\varepsilon)} = O(\varepsilon^{J'}) \ . \tag{11}$$

$$\|p_\varepsilon - p_\varepsilon^{(J')}\|_{L^2(B_\varepsilon)} = O(\varepsilon^{J'}), \tag{12}$$

and it satisfies the relation (see [3])

$$\int_{B_\varepsilon} \nu \nabla \mathbf{u}_\varepsilon^{(J')} : \nabla \phi dx = \int_{B_\varepsilon} p_\varepsilon^{(J')} \operatorname{div} \phi dx + \int_{B_\varepsilon} \mathbf{r}_\varepsilon^{(J')} \cdot \phi dx, \forall \phi \in \mathbf{H}^1_0(B_\varepsilon), \tag{13}$$

where

$$\|\mathbf{r}_\varepsilon^{(J')}\|_{L^2(B_\varepsilon)} = O(\varepsilon^{J'-1}). \tag{14}$$

For the non-stationary Navier–Stokes equations Theorems 5.2, 5.3 in [14] and 6.1, 6.2 in [15] state that given integer J for sufficiently smooth functions $\mathbf{g}_j$, i.e. for sufficiently

large α (see (7)), there exists J' and a function p_ε, an exact pressure of the Stokes or the Navier–Stokes problem such that

$$\|p_\varepsilon - p_\varepsilon^{(J')}\|_{L^2(0,T;L^2(B_\varepsilon))} = O(\varepsilon^J). \tag{15}$$

Besides this

$$\sup_{t\in(0,T)} \|\mathbf{u}_\varepsilon^{(J')} - \mathbf{u}_\varepsilon\|_{L^2(B_\varepsilon)} + \|\nabla(\mathbf{u}_\varepsilon^{(J')} - \mathbf{u}_\varepsilon)\|_{L^2((0,T);L^2(B_\varepsilon))} = O(\varepsilon^J). \tag{16}$$

Let us address the relation between α, J' and J in the case of the Navier–Stokes equations. Following the results from [14, 15]: if $\alpha = [\frac{3J}{2}] + n + 1 - 13(n-3)$, then for such regularity of data we can construct $p_\varepsilon^{(J')}$ with $J' = 2\alpha - 3$. Note that an asymptotic approximation of order J', constructed in [14, 15] for the pair $(\mathbf{u}_\varepsilon^{(J')}, p_\varepsilon^{(J')})$, satisfies problem (5) with a residual in the right hand side of the first equation of order $O(\varepsilon^{J'-1})$ in the norm $L^2(0,T;L^2(B_\varepsilon))$. It means that the following integral identity holds:

$$\int_{B_\varepsilon} \left(\frac{\partial \mathbf{u}_\varepsilon^{(J')}}{\partial t} \cdot \phi + \nu \nabla \mathbf{u}_\varepsilon^{(J')} : \nabla\phi - (\mathbf{u}_\varepsilon^{(J')} \cdot \nabla)\phi \cdot \mathbf{u}_\varepsilon^{(J')} \right) dx$$
$$= \int_{B_\varepsilon} p_\varepsilon^{(J')} \operatorname{div} \phi dx + \int_{B_\varepsilon} \mathbf{r}_\varepsilon^{(J')} \cdot \phi dx \quad \forall \phi \in \mathbf{H}_0^1(B_\varepsilon), \tag{17}$$

where

$$\|\mathbf{r}_\varepsilon^{(J')}\|_{L^2(0,T;L^2(B_\varepsilon))} = O(\varepsilon^{J'-1}). \tag{18}$$

Moreover it was proved in [14] that the boundary layer correctors in the asymptotic expansion near each node or vertex O_i can be multiplied by a cut-off function depending on $\frac{x^{(e)}-O_i}{\delta}$ and vanishing for $|x^{(e)}-O_i| \geq \frac{2}{3}\delta$ so that out of the $\frac{2}{3}\delta-$neighborhood of the nodes and vertices O_i the asymptotic expansion coincides with its regular part equal to some Poiseuille type flow. There exists a constant C_0 independent of ε and J such that if $\delta = C_0 J\varepsilon|\ln\varepsilon|$ then for the modified in this way asymptotic expansion $(\mathbf{u}_\varepsilon^{(J')}, p_\varepsilon^{(J')})$ all estimates (12)–(18) remain true.

4. MAPDD: the reduced geometry

Let δ be a small positive number much greater than ε but much smaller than 1. For any edge $e = \overline{O_iO_j}$ of the graph introduce two hyperplanes orthogonal to this edge and crossing it at the distance δ from its ends.

Denote the cross-sections of the cylinder $B_\varepsilon^{(e)}$ by these two hyperplanes $S_{i,j}$ (the cross-section at the distance δ from O_i), and $S_{j,i}$ (the cross-section at the distance δ from O_j), and denote the part of the cylinder between these two cross-sections by $B_{ij}^{dec,\varepsilon}$. Denote $B_i^{\varepsilon,\delta}$ the connected, truncated by the cross-sections $S_{i,j}$, part of B_ε containing the vertex or the node O_i.

Let us define the subspace $\mathbf{H}^{1,\delta}_{div0}(B_\varepsilon)$ (resp., $\mathbf{H}^{1,\delta}_{div0(\partial B_\varepsilon\setminus\gamma_\varepsilon)}(B_\varepsilon)$) of the space $\mathbf{H}^1_{div0}(B_\varepsilon)$ (resp., $\mathbf{H}^1_{div0(\partial B_\varepsilon\setminus\gamma_\varepsilon)}(B_\varepsilon)$) such that on every truncated cylinder $B^{dec,\varepsilon}_{ij}$ its elements (vector-valued functions) described in local variables have vanishing transversal (tangential) components while the longitudinal (normal) component has vanishing longitudinal (normal) derivative. Namely, if the local variables $x^{(e)}$ for the edge e coincide with the global ones x then they have a form of the Womersley flow $\mathbf{W}^{(e)}(x) = (v_1(x'/\varepsilon), 0, ..., 0)^T$, $v_1 \in H^1_0(\sigma^{(e)})$.

If e has the cosines directors $k_{e1}, ..., k_{en}$ and the local variables $x^{(e)}$ are related to the global ones by equation $x^{(e)} = x^{(e)}(x)$ then they are

$$\mathbf{W}^{(e)}(x) = const\ (k_{e1}v_1((x^{(e)}(x))'/\varepsilon), ..., k_{en}v_1((x^{(e)}(x))'/\varepsilon))^T,\ x' = (x_2, ..., x_n).$$

The weak MAPDD formulations for the stationary Stokes equations and non-stationary Navier–Stokes problem were given in [3, 14, 2] (for various versions of the method). Here we follow the formulation of [2].

For the Stokes equations it is as follows: find $\mathbf{u}_{\varepsilon,\delta} \in \mathbf{H}^{1,\delta}_{div0(\partial B_\varepsilon\setminus\gamma_\varepsilon)}(B_\varepsilon)$, such that $\mathbf{u}_{\varepsilon,\delta} = \mathbf{g}_\varepsilon$ on γ_ε and satisfies the following integral identity

$$\nu \int_{B_\varepsilon} \nabla \mathbf{u}_{\varepsilon,\delta}(x) : \nabla\phi(x)dx = 0 \quad \forall\, \phi \in \mathbf{H}^{1,\delta}_{div0}(B_\varepsilon). \tag{19}$$

Applying the Lax-Milgram argument one can prove that there exists a unique solution $\mathbf{u}_{\varepsilon,\delta}$ of the partially decomposed problem.

For the non-stationary Navier–Stokes equations the variational formulation of the MAPDD problem is: find a vector-field $\mathbf{u}_{\varepsilon,\delta} \in L^\infty(0,T;\mathbf{H}^{1,\delta}_{div0(\partial B_\varepsilon\setminus\gamma_\varepsilon)}(B_\varepsilon))$ with $\frac{\partial \mathbf{u}_{\varepsilon,\delta}}{\partial t} \in L^2(0,T;\mathbf{L}^2(B_\varepsilon))$, such that $\mathbf{u}_{\varepsilon,\delta}(x,0) = 0$, $\mathbf{u}_{\varepsilon,\delta} = \mathbf{g}_\varepsilon$ on γ_ε and $\mathbf{u}_{\varepsilon,\delta}$ satisfies for all $t \in (0,T)$ the integral identity

$$\int_{B_\varepsilon} \Big(\frac{\partial \mathbf{u}_{\varepsilon,\delta}}{\partial t} \cdot \phi + \nu \nabla \mathbf{u}_{\varepsilon,\delta} : \nabla\phi + (\mathbf{u}_{\varepsilon,\delta}, \nabla \mathbf{u}_{\varepsilon,\delta}) \cdot \phi\Big) dx = 0 \ \ \forall\, \phi \in \mathbf{H}^{1,\delta}_{div0}(B_\varepsilon). \tag{20}$$

Existence and uniqueness of a solution for sufficiently small ε is proved as in [14] by Galerkin method.

It was proved that

1. given natural number J and $\mathbf{g}_j \in \mathbf{W}^{3/2,2}(\partial\omega_j)$ there exists a constant C_0 (independent of ε and J) such that if $\delta = C_0 J\varepsilon|\ln\varepsilon|$ then
$$\|\mathbf{u}_{\varepsilon,\delta} - \mathbf{u}_\varepsilon\|_{\mathbf{H}^1(B_\varepsilon)} = O(\varepsilon^J) \tag{21}$$
for the steady Stokes problem, see [11];
2. for the non-stationary Navier–Stokes equations given natural number J and $\mathbf{g}_j \in C^{[\frac{J+4}{2}]+1}([0,T];\mathbf{W}^{3/2,2}(\partial\omega_j))$ there exists a constant C_0 (independent of ε and J) such that if $\delta = C_0 J\varepsilon|\ln\varepsilon|$ then
$$\sup_{t\in(0,T)} \|\mathbf{u}_{\varepsilon,\delta} - \mathbf{u}_\varepsilon\|_{\mathbf{L}^2(B_\varepsilon)} + \|\nabla(\mathbf{u}_{\varepsilon,\delta} - \mathbf{u}_\varepsilon)\|_{L^2((0,T);\mathbf{L}^2(B_\varepsilon))} = O(\varepsilon^J) \tag{22}$$
for the non-stationary Navier–Stokes equations (see [14, 15]).

5. MAPDD. Reconstruction of the pressure

The weak formulations of the partially decomposed problems (19) and (20) is formulation "without pressure" and they are not ready for the numerical implementation. In order to simplify the implementation we have to give an equivalent formulation of the partially decomposed problem with pressure. The first result was obtained in [2], where the FEM weak formulation for problem (20) was proposed. However, the proposed method was justified only in the case when the number of nodes and vertices N is equal to the number of edges M plus 1: $N = M + 1$. If this condition was not satisfied, the method was justified only for a more restrictive than just $\mathbf{H}_0^1$ space of test functions and without uniqueness of the pressure up to one additive constant. Now these restrictions are removed.

The main result of the present paper is the justification of the following weak formulation (see (20)).

find the vector-field $\mathbf{u}_{\varepsilon,\delta}$ and the pressure $p_{\varepsilon,\delta}$ such that $\mathbf{u}_{\varepsilon,\delta}(x,0) = 0$, $\mathbf{u}_{\varepsilon,\delta} \in L^\infty(0,T;\mathbf{H}^1(B_i^{\varepsilon,\delta}))$, for all $i = 1,...,N$, $\frac{\partial \mathbf{u}_{\varepsilon,\delta}}{\partial t} \in L^2(0,T;\mathbf{L}^2(B_i^{\varepsilon,\delta}))$, $\mathbf{u}_{\varepsilon,\delta} = \mathbf{g}_\varepsilon$ at γ_ε, $\mathbf{u}_{\varepsilon,\delta} = \mathbf{0}$ at $(\partial B_i^{\varepsilon,\delta} \cap \partial B_\varepsilon)\backslash\gamma_\varepsilon$, $p_{\varepsilon,\delta} \in L^2(0,T;L^2(B_i^{\varepsilon,\delta}))$ for all $i = 1,...,N$, $\mathbf{u}_{\varepsilon,\delta}\cdot\mathbf{t} = 0$ on $S_{ij}\cup S_{ji}$, $\mathbf{u}_{\varepsilon,\delta}\cdot\mathbf{n}\big|_{S_{ij}} + \mathbf{u}_{\varepsilon,\delta}\cdot\mathbf{n}\big|_{S_{ji}} = 0$, where $\mathbf{t}$ is the unit tangent vector, and the couple $(\mathbf{u}_{\varepsilon,\delta}, p_{\varepsilon,\delta})$ satisfies for all $t \in (0,T)$ the integral identity for every vector-field $\phi \in \mathbf{H}^1(B_i^{\varepsilon,\delta}), q \in L^2(B_i^{\varepsilon,\delta})$, for all $i = 1,...,N$, such that $\phi = 0$ at $\partial B_i^{\varepsilon,\delta}\cap\partial B_\varepsilon$, and for all edges O_iO_j, $\phi\cdot\mathbf{t} = 0$ at $S_{ij}\cup S_{ji}$ and $\phi\cdot\mathbf{n}|_{S_{ij}} + \phi\cdot\mathbf{n}|_{S_{ji}} = 0$:

$$\sum_{i=1}^{N}\int_{B_i^{\varepsilon,\delta}}\Big(\frac{\partial \mathbf{u}_{\varepsilon,\delta}}{\partial t}\cdot\phi + \nu\nabla\mathbf{u}_{\varepsilon,\delta} : \nabla\phi + (\mathbf{u}_{\varepsilon,\delta},\nabla\mathbf{u}_{\varepsilon,\delta})\cdot\phi - p_{\varepsilon,\delta}\operatorname{div}\phi + q\operatorname{div}\mathbf{u}_{\varepsilon,\delta}\Big)dx$$
$$+\sum_{l=1}^{M} d_l\int_{\sigma_\varepsilon^{(e_l)}}\frac{\partial \mathbf{u}_{\varepsilon,\delta}}{\partial t}\cdot\phi + \nu\nabla_{x^{(e_l),\prime}}\mathbf{u}_{\varepsilon,\delta} : \nabla_{x^{(e_l),\prime}}\phi dx^{(e_l),\prime} = 0. \quad (23)$$

A similar weak formulation with pressure can be given for the stationary Stokes equations: find the vector-field $\mathbf{u}_{\varepsilon,\delta}$ and the pressure $p_{\varepsilon,\delta}$ such that $\mathbf{u}_{\varepsilon,\delta} \in \mathbf{H}^1(B_i^{\varepsilon,\delta})$, for all $i = 1,...,N$, $\mathbf{u}_{\varepsilon,\delta} = \mathbf{g}_\varepsilon$ at γ_ε, $\mathbf{u}_{\varepsilon,\delta} = \mathbf{0}$ at $(\partial B_i^{\varepsilon,\delta}\cap\partial B_\varepsilon)\backslash\gamma_\varepsilon$, $p_{\varepsilon,\delta} \in L^2(B_i^{\varepsilon,\delta})$ for all $i = 1,...,N$, $\mathbf{u}_{\varepsilon,\delta}\cdot\mathbf{t} = 0$ on $S_{ij}\cup S_{ji}$, $\mathbf{u}_{\varepsilon,\delta}\cdot\mathbf{n}\big|_{S_{ij}} + \mathbf{u}_{\varepsilon,\delta}\cdot\mathbf{n}\big|_{S_{ji}} = 0$, and the couple $(\mathbf{u}_{\varepsilon,\delta}, p_{\varepsilon,\delta})$ satisfies the integral identity for every vector-field $\phi \in \mathbf{H}^1(B_i^{\varepsilon,\delta}), q \in L^2(B_i^{\varepsilon,\delta})$, for all $i = 1,...,N$, such that $\phi = 0$ at $\partial B_i^{\varepsilon,\delta}\cap\partial B_\varepsilon$, and for all edges O_iO_j, $\phi\cdot\mathbf{t} = 0$ at $S_{ij}\cup S_{ji}$ and $\phi\cdot\mathbf{n}|_{S_{ij}} + \phi\cdot\mathbf{n}|_{S_{ji}} = 0$:

$$\sum_{i=1}^{N}\int_{B_i^{\varepsilon,\delta}}\Big(\nu\nabla\mathbf{u}_{\varepsilon,\delta} : \nabla\phi - p_{\varepsilon,\delta}\operatorname{div}\phi + q\operatorname{div}\mathbf{u}_{\varepsilon,\delta}\Big)dx$$
$$+\sum_{l=1}^{M} d_l\int_{\sigma_\varepsilon^{(e_l)}}\nu\nabla_{x^{(e_l),\prime}}\mathbf{u}_{\varepsilon,\delta} : \nabla_{x^{(e_l),\prime}}\phi dx^{(e_l),\prime} = 0. \quad (24)$$

We will prove in the Appendix that these weak formulations are equivalent to the formulations without pressure given in the previous section in the following sense: there exists a

unique couple $(\mathbf{u}_{\varepsilon,\delta}, p_{\varepsilon,\delta})$ satisfying this formulation (23) (respectively (24)) up to an additive function of t for the pressure in the non-stationary case and up to an additive constant in the stationary case. Function ($\mathbf{u}_{\varepsilon,\delta}$ is the same as in the formulation without pressure. This formulation can be easily implemented.

Note that this formulation gives the solution only in the "octopuses" $B_i^{\varepsilon,\delta}$. However it can be reconstructed in the cut-off cylinders B_{ij}^{ε}. Namely, the velocity is the same as at the surfaces S_{ij} and S_{ji} while the pressure within B_{ij}^{ε} is a linear function of the longitudinal variable, such that its values at the cross-sections S_{ij} are equal to the averages of the pressure $p_{\varepsilon,\delta}$) in some thin truncated part of of thickness ε of $B_i^{\varepsilon,\delta}$ near S_{ij}. The error estimate is similar to that of (21), (22). They are formulated in Theorems 3 and 3' in Appendix: *There exists a constant C_0 such that if $\delta = C_0 J\varepsilon|\ln\varepsilon|$, then for the function $p_{\varepsilon,\delta}$ solution of problem (23) (respectively (24)) we can find the function p_ε solution of (10) (respectively (4)) such that*

$$\|p_{\varepsilon,\delta} - p_\varepsilon\|_{L^2(0,T;L^2(B_\varepsilon))} = O(\varepsilon^J), \tag{25}$$

respectively

$$\|p_{\varepsilon,\delta} - p_\varepsilon\|_{L^2(B_\varepsilon)} = O(\varepsilon^J). \tag{26}$$

Note that this result can be generalized in the case when the graph of the tube structure may contain two coinciding edges generating two parallel non-intersecting cylinders between the nodes O_i and O_j, separated by a distance of order ε (see Fig. 1).

6. Numerical examples

The extended MAPDD formulation is applied to a numerical test case. In contrast to the previous work [2], the requirement that the number of junction domains must exceed the number of connecting tubes is eliminated, which is reflected in the chosen test geometry. In addition, our new theoretical results provide an error estimate for the pressure.

The convergence of the MAPDD model for a sequence of ε is studied in terms of the velocity and pressure error estimates (22) and (25). Furthermore, we assess the accuracy of the method for different Reynolds numbers.

6.1. Problem setup

Consider the two-dimensional geometry illustrated in Fig. 1. Two junction domains, ω_ε^1 and ω_ε^2, are connected by two straight tubes, $B_{1,2}^{dec,\varepsilon}$ and $\widetilde{B}_{1,2}^{dec,\varepsilon}$, and respective buffer domains $B_i^{\varepsilon,\delta}$ and $\widetilde{B}_i^{\varepsilon,\delta}$, $i = 1, 2$. Flow enters the first junction domain through boundary Γ_{in} and exits the second junction through Γ_{out}.

The tubes $B_{1,2}^{dec,\varepsilon}$, $\widetilde{B}_{1,2}^{dec,\varepsilon}$ are included in the full reference model and truncated when the reduced MAPDD model is used. In the latter case, MAPDD interface conditions are applied to the boundaries S_{12}, S_{21}, $\widetilde{S}_{12}$, $\widetilde{S}_{21}$.

The radius R of the tubes scales with ε (we set $R = \varepsilon$). For each value of ε, the junction domains are contracted homothetically by a factor of ε with respect to the center points marked with crosses in Fig. 1. The distance between these points, L, remains the same for

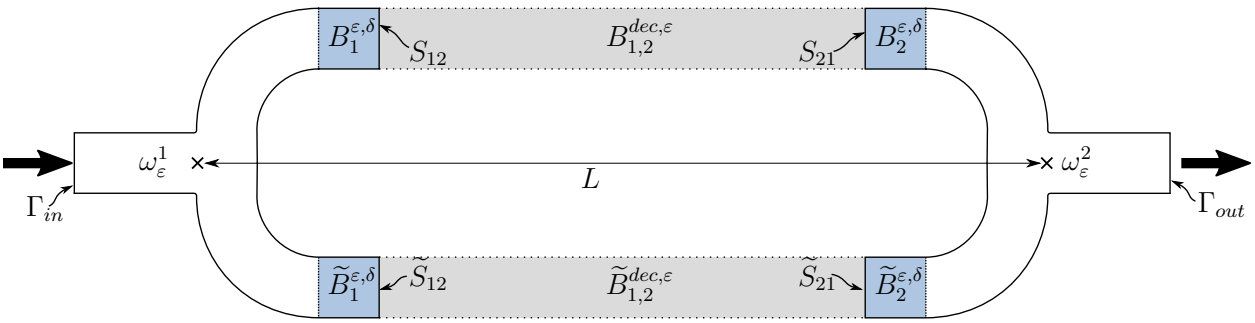


Figure 1: Illustration of the computational domain ($\varepsilon = 2^{-2}$).

all values of ε. $L = 7$ is selected arbitrarily for the experiments. Theorem 3 and estimate (22) require the associated distance, δ, from the centers of the junction domains to the interfaces, S_{ij}, $\widetilde{S}_{ij}$, to be defined as

$$\delta = C_0 J \varepsilon |\ln(\varepsilon)|, \tag{27}$$

where C_0 is some constant and J the a priori unknown convergence rates in (22) and (25). $\widetilde{C} = C_0 J$ is defined as a user parameter and for the selected geometry to be realizable has to satisfy $4\varepsilon < \delta < \frac{L}{2}$. Simulations are carried out for $\widetilde{C} = C/\ln(2)$, $C = \{3;\ 4;\ 5\}$. Pairs of full and truncated domains are created for a sequence of values $\varepsilon = 2^{-k}$, $k = 2, ..., 6$.

In both the full reference domain and the MAPDD formulation, the incompressible, time-dependent Navier–Stokes equations (10) (and the MAPDD version (23)) are considered for a time interval $t \in (0,\ T]$, $T = 0.4\,\mathrm{s}$. Pulsatile, parabolic velocity profiles are imposed on Γ_{in} and Γ_{out} via Dirichlet boundary conditions as $\mathbf{g}_\varepsilon = \left(U_0(1 - (x_2 - c_0)^2/\varepsilon^2)\sin(\pi t/T),\ 0\right)^T$, where c_0 is the x_2 coordinate of the center of the boundary. The initial velocity is $\mathbf{u}_\varepsilon(x, 0) = 0$. No-slip conditions hold on the remainder of the boundary, with exception of the interfaces of the truncated domain. In order to fix the undetermined pressure constant of the Dirichlet problem and to ensure that the resulting discretized system is invertible, a zero pressure is imposed on a single degree-of-freedom at the outlet.

Two different setups are considered. First, we verify numerically the error estimates (22) and (25) and estimate the rate of convergence in ε. Since the theory assumes that $\mathbf{g}_\varepsilon = \mathcal{O}(1)$, we set $U_0 = 1$. Furthermore, in accordance with the assumption of low Reynolds numbers, which we define as $Re(\varepsilon) = 4/3 U_0 \varepsilon/\nu$, the viscosity is chosen as $\nu = 0.01$. For the largest value of ε we get $Re \approx 33.3$.

Second, the dependency of the errors on the Reynolds number is analyzed. Fixed Reynolds numbers $1 \leq Re(\varepsilon) \leq 300$ are considered, with $\nu = 0.035$ and a variable inflow U_0 computed from Re. In addition, simulations are carried out in the convection-dominated regime using a Reynolds number of $Re = 1000$ and $C = 3$.

6.2. Discretization

The model equations are discretized with a monolithic, mixed finite element method using inf-sup stable Taylor–Hood basis functions (quadratic basis functions for the velocity,

linear ones for the pressure) on unstructured triangle meshes with a characteristic element size of $h = \varepsilon/10$ for $Re \leq 300$ and $h = \varepsilon/20$ for $Re = 1000$. An implicit, unconditionally stable Euler scheme discretizes the time-derivative with a time step size of $\Delta t = 0.01\,\mathrm{s}$. The convection term is written in skew-symmetric form and treated semi-implicitly.

The problem is implemented and solved using the FEniCS finite elements library [1].

6.3. Results

The error norms (22) and (25) are evaluated for the reference and MAPDD results obtained for the indicated sequence of ε and values of the constant $C \in \{3; 4; 5\}$, controlling the straight extension of the junction domain via (27). Fig. 2 displays the resulting errors.

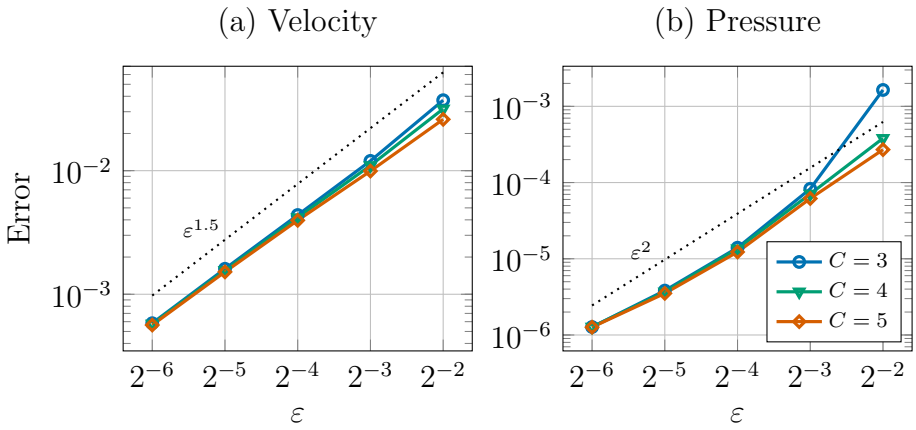


Figure 2: Errors of the MAPDD model in (a) the velocity (Eq. (22)) and (b) the pressure (Eq. (25)) for different values of C. The dotted lines indicate ε^J with (a) $J = 1.5$ and (b) $J = 2$ (cf. Tab. 1).

The velocity error shows a clear exponential decay for decreasing ε with a near constant rate. Increasing C leads to slightly smaller errors and slower convergence. The pressure error exhibits a similar behavior. For the largest ε, $C = 3$ leads to a pressure error larger than with $C = 4$ or 5 by almost one order of magnitude. At large values of ε, the rate of convergence appears to be faster than in the lower range of ε, where the slope seems to be similar to the velocity error. For reference, the figures include lines indicating the slopes of $J = 1.5$ (velocity) and $J = 2$ (pressure). Tab. 1 shows the empirically estimated average rates of convergence, J. As is visibile in the Figure, larger values of C slightly reduce the rate of convergence.

Relative errors are shown in Fig. 3 for the velocity and for the pressure. Here, the errors (22) and (25) are scaled by the full reference solutions measured in the same norms. The magnitude of the relative errors is very small. Even in the worst case ($Re = 300$), the error barely exceeds 10 % (velocity) or 1 % (pressure). For small Re, the relative errors reach 10^{-5} (velocity) and 10^{-8} (pressure), with the smallest considered value of ε. In the case of the velocity, the results of $Re = 1,\ 10,\ 100$ approximately coincide—only $Re = 100$ at $C = 3$ shows a significant increase in the error, which diminishes as ε decreases. Again, larger values of C lead to smaller errors in all cases. At $Re = 300$, the errors are significantly higher and

Table 1: Estimated rates of convergence J for different values of C.

	Rate of convergence J	
	velocity	pressure
$C = 3$	1.50	2.58
$C = 4$	1.44	2.06
$C = 5$	1.38	1.93

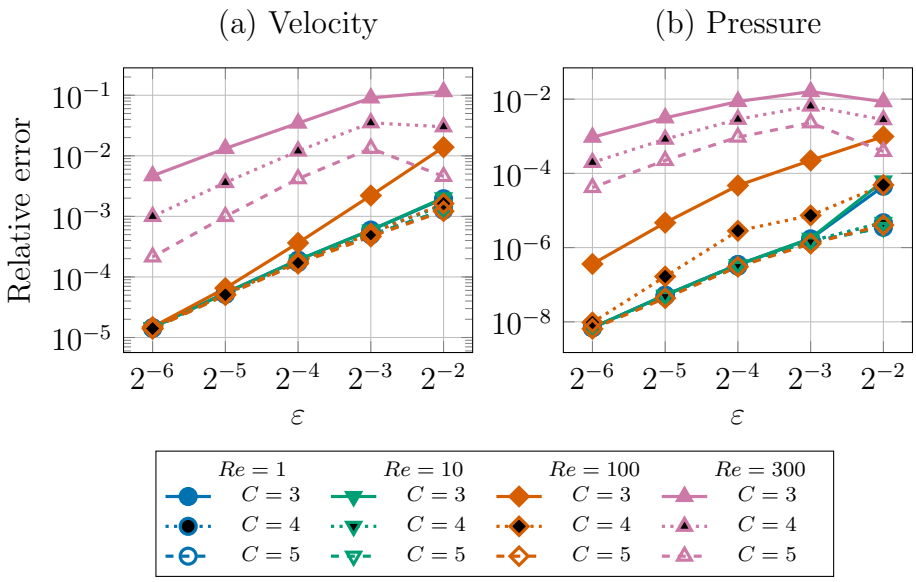
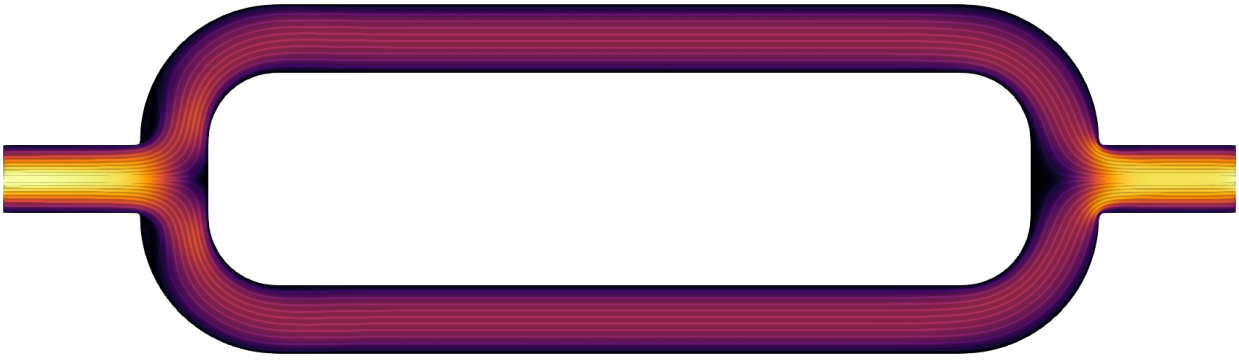


Figure 3: Relative velocity errors of the Navier–Stokes MAPDD model w.r.t. the full solution for different Reynolds numbers and C over ε.

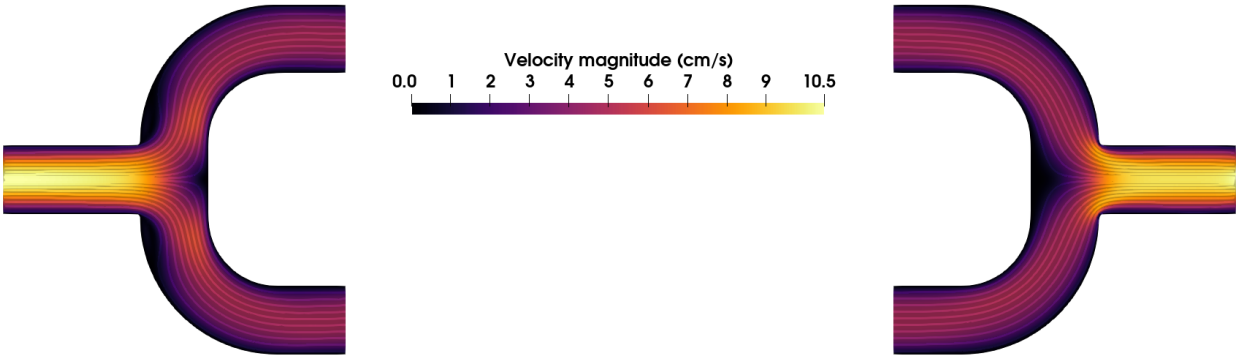
sensitive to C. In particular, the errors obtained with $C = 3,\ 4,\ 5$ are separated by roughly half an order of magnitude. For $\varepsilon \leq 2^{-3}$, the error can be seen to decrease exponentially with a rate comparable to the lower Reynolds numbers.

The relative pressure error shows almost identical behavior with $Re = 1$ and $Re = 10$. The error at $Re = 100$ is more sensitive to the selection of C than the velocity error. While with $C = 5$, the relative error coincides for $Re \in \{1; 10; 100\}$, with $C = 4$ and more so $C = 3$, the error is significantly higher. Under $Re = 300$, the pressure error replicates the features of the velocity error.

Velocity magnitude and pressure fields of both the full and the reduced MAPDD models are displayed in Figure 4–5 for the case of $Re = 100$, $\varepsilon = 0.25$, at the time of the peak inflow $t = T/2$. Inside the straight tube sections, the reference velocity streamlines are highly parallel, in accordance with the Womersley assumption of the MAPDD model. Likewise, the equidistantly spaced, vertical contour lines of the reference pressure (Fig. 5(a)) indicate a nearly linear pressure along the tubes. The velocity field obtained with the MAPDD model in the reduced domain is visually indiscernible from the reference solution. The MAPDD



(a) Velocity – Full order solution



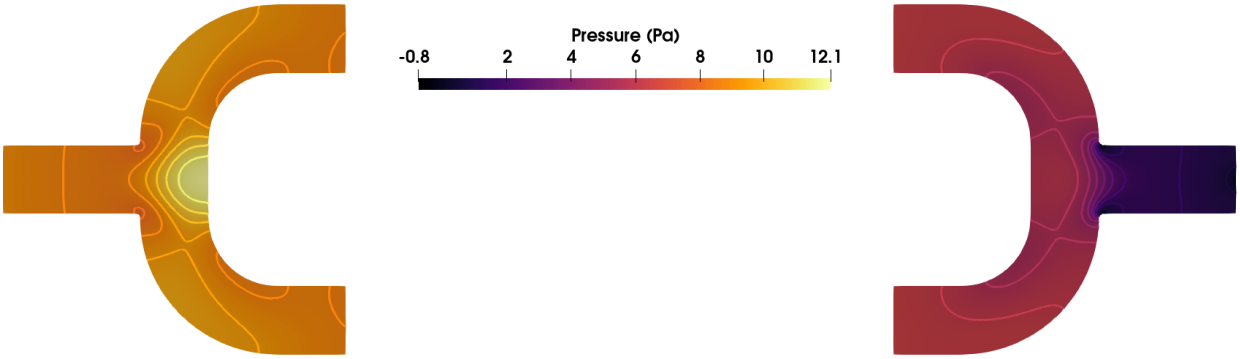
(b) Velocity – MAPDD solution

Figure 4: Velocity magnitude and streamlines for $\varepsilon = 0.25$, $Re = 100$, at peak time $t = T/2$, full reference model (a) versus MAPDD model (b).

pressure solution also matches the reference with high accuracy. Small differences in the pressure contour lines appear at the position of the upstream interfaces, where Womersley conditions are enforced by the MAPDD model.

In addition, Figures 6–7 show the velocity and pressure fields for an elevated Reynolds number of $Re = 1000$. Strong convective effects, flow separation and recirculation are present, mostly in the bifurcations of the upstream junction domains. Also this more complex flow is recovered with high fidelity by the MAPDD model. At the axial position of the upstream interfaces, the reference solution exhibits recirculation zones near the inner walls. In the MAPDD solution, the presence of the interfaces, imposing parallel-flow conditions, the recirculating flow is slightly modified with respect to the reference. In the junction domain on the downstream side, the full-order reference flow is recovered with excellent accuracy. Here, the interference of the interface conditions is minimal, since the straight tube sections have a parallelizing effect on the flow. Comparison of the pressure fields of the reference and the MAPDD solution, Figures 7(a) and (b), reveals similar features. The most pronounced characteristics of the pressure are the stagnation point in the upstream junction domain and

(a) Pressure – Full order solution



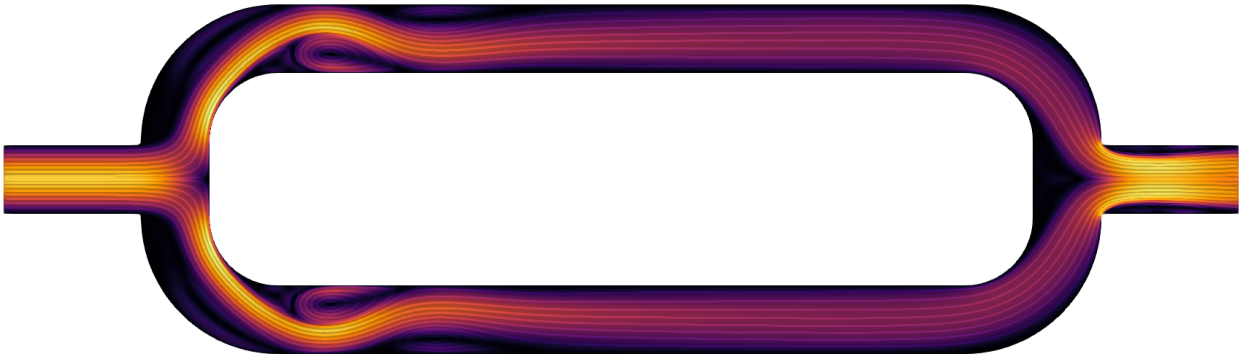
(b) Pressure – MAPDD solution

Figure 5: Pressure field and contours for $\varepsilon = 0.25$, $Re = 100$, at peak time $t = T/2$, full reference model (a) versus MAPDD model (b).

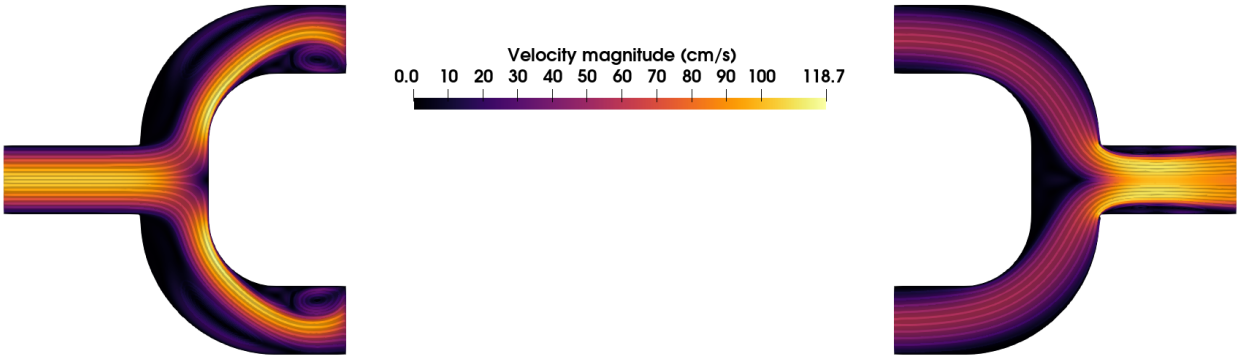
pressure minima close to the upstream interfaces, matching the positions of the recirculation zones, and a low pressure region in the outflow section of the downstream junction domain. All these features are accurately represented by the MAPDD solution. Again, only near the upstream interfaces, small differences can be observed, where the magnitude and extent of the MAPDD pressure minima is smaller than in the reference case. Elsewhere the pressure contours match almost exactly.

6.4. Discussion

The numerical experiments confirm the theoretical findings for a simple domain of two junction domains connected by two straight tubes, for small Reynolds numbers, when the Womersley flow assumption for the truncated tubes is accurate. The reduced geometry includes straight buffer zone extension of the junction domains at the interfaces of the truncated tubes. The length of these extensions is controlled by a user parameter, C. Longer extensions help to counteract the departure from the Womersley regime in the truncated tubes for elevated Reynolds numbers. The numerical results show that the velocity MAPDD



(a) Velocity – Full order solution



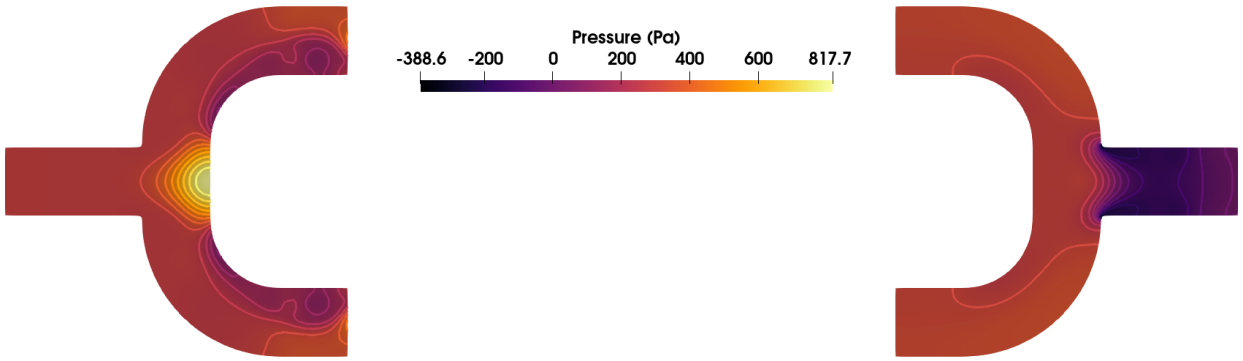
(b) Velocity – MAPDD solution

Figure 6: Velocity magnitude and streamlines for $\varepsilon = 0.25$, $Re = 1000$, at peak time $t = T/2$, full reference model (a) versus MAPDD model (b).

solution converges to the full solution with order ≈ 1.5 under ε, as indicated by the theoretical error bound. The pressure error appears to converge faster, but at non-constant rate, suggesting that for the selected parameters, the estimate is not sharp. Very small relative errors indicate that the MAPDD model is very accurate for low Reynolds numbers. In particular for larger Re, higher C could significantly improve the accuracy.

The MAPDD model delivered excellent results even for a large Reynolds number of $Re = 1000$, for which the theoretical error estimates do not hold. The MAPDD model results matched the reference velocity and pressure distributions with high accuracy and succeeded in recovering complex flow features. Only in proximity to the upstream interfaces, MAPDD the interface conditions slightly altered the flow in order to accommodate the model assumptions of a Womersley-like flow inside the truncated tubes. These new results are in agreement with our previous study [2].

(a) Pressure – Full order solution



(b) Pressure – MAPDD solution

Figure 7: Pressure field and contours for $\varepsilon = 0.25$, $Re = 1000$, at peak time $t = T/2$, full reference model (a) versus MAPDD model (b).

A. Proofs

In the present section we give the proof of the equivalence of the weak formulations of the problem with partially reduced dimension "without pressure" and "with the reconstructed pressure". The proof uses the main result of [2], proved for the domains with restriction $N = M + 1$ or with the restriction of the space of the test functions: $\int_{\partial B_i^{\varepsilon,\delta}} \phi \cdot \mathbf{n} = 0$, for all $i = 1, ..., N$. Below we will get rid of these restrictions and generalize it for the tube structures which may have loops.

Let us start with another (equivalent) form of the formulations of problems (23) and (24), using the spaces of test functions defined in the whole domain B_ε.

Introduce the following Sobolev space of test functions

$$\tilde{\mathbf{H}}_0^{1,\delta}(B_\varepsilon) = \{\phi \in \mathbf{H}_0^1(B_\varepsilon) | \phi(x) = \mathbf{W}^{(e)}(x), x \in B_{ij}^{dec,\varepsilon}, \forall e = \overline{O_i O_j}\}$$

and $\mathbf{H}_0^{1,\delta}(B_\varepsilon) = \{\phi \in \mathbf{H}_0^1(B_\varepsilon) | \phi(x) = \mathbf{W}^{(e)}(x), x \in B_{ij}^{dec,\varepsilon}, e = \overline{O_i O_j}, \int_{\partial B_i^{\varepsilon,\delta}} \phi \cdot \mathbf{n} = 0, i =$

$1, ..., N\big\}$. In comparison with $\tilde{\mathbf{H}}_0^{1,\delta}(B_\varepsilon)$, the space $\mathbf{H}_0^{1,\delta}(B_\varepsilon)$ has an additional restriction $\int_{\partial B_i^{\varepsilon,\delta}} \phi \cdot \mathbf{n} = 0$, for all $i = 1, ..., N$.

For the Stokes equations let us define $p_{\varepsilon,\delta}$ as a function belonging to the space $L^2(B_i^{\varepsilon,\delta})$ for all $i = 1, ..., N$ and satisfying the integral identity

$$\int_{B_\varepsilon} \nu \nabla \mathbf{u}_{\varepsilon,\delta} : \nabla \phi dx = \sum_{i=1}^{N} \int_{B_i^{\varepsilon,\delta}} p_{\varepsilon,\delta} \operatorname{div} \phi dx \quad \forall\, \phi \in \tilde{\mathbf{H}}_0^{1,\delta}(B_\varepsilon), \tag{28}$$

where $\mathbf{u}_{\varepsilon,\delta}$ is the solution of problem (19).

For the non-stationary Navier–Stokes equations $p_{\varepsilon,\delta} \in L^2(0, T; L^2(B_i^{\varepsilon,\delta}))$ for all $i = 1, ..., N$, and satisfies the integral identity for every vector-field $\phi \in \tilde{\mathbf{H}}_0^{1,\delta}(B_\varepsilon)$ for all $t \in (0, T)$:

$$\int_{B_\varepsilon} \Big(\frac{\partial \mathbf{u}_{\varepsilon,\delta}}{\partial t} \cdot \phi + \nu \nabla \mathbf{u}_{\varepsilon,\delta} : \nabla \phi + (\mathbf{u}_{\varepsilon,\delta}, \nabla \mathbf{u}_{\varepsilon,\delta}) \cdot \phi \Big) dx = \sum_{i=1}^{N} \int_{B_i^{\varepsilon,\delta}} p_{\varepsilon,\delta} \operatorname{div} \phi dx, \tag{29}$$

where $\mathbf{u}_{\varepsilon,\delta}$ is the solution of problem (20).

We will prove the existence and uniqueness up to an additive constant of so defined pressure (in the non-stationary case this constant depends on time), in such way that it is $O(\varepsilon^J)$-close in L^2-norm to some exact pressure. It can be extended linearly to all cylinders B_{ij}^{ε} and this extension is also $O(\varepsilon^J)$-close in L^2-norm to the exact pressure.

In what follows we will reconstruct the pressure and give all proofs for the non-stationary Navier–Stokes equations. For the stationary Stokes equations the proofs are similar and easier.

Let us start with the result for the Navier–Stokes equation obtained in [2], that states in the general case the existence of a function $p_{\varepsilon,\delta} \in L^2(0, T; L^2(B_i^{\varepsilon,\delta}))$ for all $i = 1, ..., N$, such that it satisfies identity (29) for all test functions from a smaller space $\mathbf{H}_0^{1,\delta}(B_\varepsilon)$. (In comparison with $\tilde{\mathbf{H}}_0^{1,\delta}(B_\varepsilon)$, the space $\mathbf{H}_0^{1,\delta}(B_\varepsilon)$ has an additional restriction $\int_{\partial B_i^{\varepsilon,\delta}} \phi \cdot \mathbf{n} = 0$, for all $i = 1, ..., N$). Note that so defined pressure exists (see the derivation of relation (4.5) and Lemma A4 in [2]) but it is not unique, it is defined up to arbitrary $\theta_i(t)$ in each subdomain $B_i^{\varepsilon,\delta}, i = 1, ..., N$. This follows immediately from the relation $\int_{\partial B_i^{\varepsilon,\delta}} \theta_i(t) \phi \cdot \mathbf{n} = 0$ for test functions from $\mathbf{H}_0^{1,\delta}(B_\varepsilon)$.

Let us denote for any function $q \in L^2(B_i^{\varepsilon,\delta})$, $\langle q \rangle_i = \int_{B_i^{\varepsilon,\delta}} q(x) dx / \operatorname{mes}(B_i^{\varepsilon,\delta})$ and $\tilde{q} = q - \langle q \rangle_i$. Then any pressure $p_{\varepsilon,\delta}$ satisfying (29) can be represended in the form $p_{\varepsilon,\delta}(x, t) = \tilde{p}_{\varepsilon,\delta}(x, t) + \theta_i(t)$. Recall that the pressure $\tilde{p}_{\varepsilon,\delta}$ is constructed in [2] in the following way. Let us denote by $\mathbf{U}_{ij}(x^{(e)\prime}, t)$ the trace of the solution $\mathbf{u}_{\varepsilon,\delta}$ to problem (20) at every cross-section S_{ij}. Then we get a standard Navier–Stokes problem (30) in each domain $B_i^{\varepsilon,\delta}$ with the known boundary value $\mathbf{U}_{ij}(x^{(e)\prime}, t)$ on S_{ij}, the no-slip boundary condition on $\partial B_i^{\varepsilon,\delta} \backslash \Sigma_i$ if $i = 1, ..., N_1$, or on $\partial B_i^{\varepsilon,\delta} \backslash (\Sigma_i \cup \gamma_i^{\varepsilon})$ if $i = N_1 + 1, ..., N$ and respectively with the condition

$\mathbf{u}_{\varepsilon,\delta} = \mathbf{g}_\varepsilon$ at γ_i^ε if $i = N_1 + 1, ..., N$; the initial condition is $\mathbf{u}_{\varepsilon,\delta}(x,0) = 0$:

$$\begin{aligned} \frac{\partial \mathbf{u}_{\varepsilon,\delta}}{\partial t} - \nu\Delta\mathbf{u}_{\varepsilon,\delta} + (\mathbf{u}_{\varepsilon,\delta},\nabla)\mathbf{u}_{\varepsilon,\delta} + \nabla p_{\varepsilon,\delta} &= 0, \qquad x \in B_i^{\varepsilon,\delta}, \\ \operatorname{div} \mathbf{u}_{\varepsilon,\delta} &= 0, \qquad x \in B_i^{\varepsilon,\delta}, \\ \mathbf{u}_{\varepsilon,\delta}\big|_{S_{ij}} &= \mathbf{U}_{ij}, \quad x \in S_{ij}, \\ \mathbf{u}_{\varepsilon,\delta}\big|_{\partial B_\varepsilon \cap \partial B_i^{\varepsilon,\delta}} &= \mathbf{g}_\varepsilon, \\ \mathbf{u}_{\varepsilon,\delta}(x,0) &= 0. \end{aligned} \tag{30}$$

Here Σ_i is a union $\cup_{j:\overline{O_iO_j}\in\{e_1,...,e_M\}} S_{ij}$ of all cross-sections S_{ij} belonging to the boundary of $B_i^{\varepsilon,\delta}$. This problem admits a unique solution-velocity (coinciding with $\mathbf{u}_{\varepsilon,\delta}$) and pressure $p_{\varepsilon,\delta}$ which is unique up to an additive function θ_i of t. Condition $\langle p_{\varepsilon,\delta}\rangle_i = 0$ selects a unique pressure satisfying (29) for test functions in $\mathbf{H}_0^{1,\delta}(B_\varepsilon)$. Let us prove now that there exist a unique set of functions $(\theta_i)_{i=1,...,N-1}$ with an arbitrary θ_N such that $p_{\varepsilon,\delta} = \tilde{p}_{\varepsilon,\delta} + \theta_i(t)$ satisfies (29) for test functions from the extended space $\tilde{\mathbf{H}}_0^{1,\delta}(B_\varepsilon)$.

Before we prove the existence and uniqueness up to an additive function of t of $p_{\varepsilon,\delta}$, let us prove several lemmas about the extended space $\tilde{\mathbf{H}}_0^{1,\delta}(B_\varepsilon)$ and linear functionals on it. Denote $\mathcal{Q}_i(\Phi) = \int_{\partial B_i^{\varepsilon,\delta}} \Phi \cdot \mathbf{n} ds$, where $\mathbf{n}$ is the outer normal vector with respect to $B_i^{\varepsilon,\delta}$, i.e. the flow rate (flux) through the boundary of $B_i^{\varepsilon,\delta}$. Evidently, $\sum_{i=0}^N \mathcal{Q}_i(\Phi) = 0$ because all fluxes through the cross sections S_{ij} and S_{ji} have the same absolute value but the opposite signes, and the common part of boundaries of the domains B_ε and $B_i^{\varepsilon,\delta}$ keeps the no-slip condition.

Lemma 1. *There exist $N-1$ vector-valued functions $\mathbf{U}_i \in \tilde{\mathbf{H}}_0^{1,\delta}(B_\varepsilon)$, $i = 1, ..., N-1$, such that $\mathcal{Q}_j(\mathbf{U}_i) = \delta_{ij}$, $j = 1, ..., N-1$ and*

$$\|\mathbf{U}_i\|^2_{\mathbf{L}^2(B_\varepsilon)} \le C_* \varepsilon^{-(n-1)}, \quad \|\mathbf{U}_i\|^2_{\mathbf{H}^1(B_\varepsilon)} \le C_* \varepsilon^{-(n+1)}, \tag{31}$$

with constant C_* independent of ε, δ. Consequently, $\mathcal{Q}_j(\mathbf{U}_N) = -1$, $j = 1, ..., N-1$.

Proof. Consider for any $i = 1, ..., N-1$ the following problem on the graph $\mathcal{B}$: Find a function $q_i \in H^1(\mathcal{B})$, affine at each edge e of the graph ($q_i(x) = -s_i^{(e)} x_1^{(e)} + a_i^{(e)}$) and satisfying conditions $q_i(O_N) = 0$ and

$$-\sum_{e:O_j\in e} L^{(e)} s_i^{(e)} = \delta_{ij}, \tag{32}$$

for each node or vertex O_j, $j = 1, ..., N-1$, where the local coordinate system has the origin O_j and $L^{(e)}$ is the operator relating the pressure slope and the flux: for any real $\mathcal{S}$, $L^{(e)}\mathcal{S} = \kappa_\varepsilon^{(e)}\mathcal{S}$, where $\kappa_\varepsilon^{(e)} = \int_{\sigma_\varepsilon^{(e)}} \mathcal{V}_\varepsilon(x^{(e)\prime}) dx^{(e)\prime}$, $\mathcal{V}_\varepsilon \in H_0^1(\sigma_\varepsilon^{(e)})$ is a solution of the problem

$$\begin{aligned} -\nu\Delta'_{x^{(e)\prime}} \mathcal{V}_\varepsilon(x^{(e)\prime}) &= 1, \quad x^{(e)\prime} \in \sigma_\varepsilon^{(e)}, \\ \mathcal{V}_\varepsilon(x^{(e)\prime})\big|_{\partial\sigma_\varepsilon^{(e)}} &= 0. \end{aligned} \tag{33}$$

Clearly, $\kappa_\varepsilon^{(e)} = \varepsilon^{n+1}\kappa^{(e)}$, where $\kappa^{(e)} = \int_{\sigma^{(e)}} \mathcal{V}(y^{(e)\prime})dy^{(e)\prime}$ and $\mathcal{V}$ is a unique solution of the problem

$$\begin{aligned} -\nu\Delta'_{y^{(e)\prime}}\mathcal{V}(y^{(e)\prime}) = 1, \quad y^{(e)\prime} \in \sigma^{(e)}, \\ \mathcal{V}(y^{(e)\prime})|_{\partial\sigma^{(e)}} = 0. \end{aligned} \tag{34}$$

Note that $\kappa^{(e)}$ and $\mathcal{V}$ do not depend on ε.

If we add the condition $L^{(e)}s_i^{(e)} = -1$ at the vertex O_N, then this problem is a particular case of problems on the graph considered in [11] and it admits a unique solution. Relation between $\kappa_\varepsilon^{(e)}$ and $\kappa^{(e)}$ yields: $s_i^{(e)} = \varepsilon^{-(n+1)}\tilde{s}_i^{(e)}$, where $\tilde{s}_i^{(e)}$ are the scaled slopes and they do not depend on ε. Let us construct now for every cylinder B_{ij}^ε, corresponding to an edge e, a function $\mathbf{U}_i$ as Poiseuille velocity equal to $c(\mathcal{V}_\varepsilon(x^{(e)\prime}), 0, ..., 0)$ in local variables $x^{(e)}$, choosing the constant c from the condition $c\kappa_\varepsilon^{(e)} = L^{(e)}s_i^{(e)}$, so that $c = s_i^{(e)}$ and

$$\begin{aligned} \|\mathbf{U}_i\|^2_{\mathbf{L}^2(B_{ij}^\varepsilon)} \le C_*\varepsilon^{-2(n+1)+(n-1)+4} = C_*\varepsilon^{-n+1}, \\ \|\mathbf{U}_i\|^2_{\mathbf{H}^1(B_{ij}^\varepsilon)} \le C_*\varepsilon^{-n-1}. \end{aligned}$$

(Here ε^{n-1} is the order of the measure of cross section and ε^4 is the square of the magnitude of the Poiseuille velocity in B_{ij}^ε).

Then we extend $\mathbf{U}_i$ inside the domains $B_k^{\varepsilon,\delta}$ as an arbitrary function from $\mathbf{H}^1(B_k^{\varepsilon,\delta})$ with the given boundary values. In particular, we can do it just by multiplication of the Poiseuille velocities by cut-off functions depending on $\frac{x_1^{(e)}-\delta}{\varepsilon}$ and we obtain the same estimates in $B_k^{\varepsilon,\delta}$, namely

$$\|\mathbf{U}_i\|^2_{\mathbf{L}^2(B_k^{\varepsilon,\delta})} \le C_*\varepsilon^{-n+1}, \quad \|\mathbf{U}_i\|^2_{\mathbf{H}^1(B_k^{\varepsilon,\delta})} \le C_*\varepsilon^{-(n+1)}.$$

In fact, the last estimates will contain an extra factor ε but it doesn't improve the overall result. Summing up all these estimates we get the assertion of the Lemma.

Lemma 2. *Let p be a function defined in $\cup_{i=1}^N B_i^{\varepsilon,\delta}$, belonging to $L^2(B_i^{\varepsilon,\delta})$ for all $i = 1, ..., N$ and such that*

$$\sum_{i=1}^N \int_{B_i^{\varepsilon,\delta}} p(x)dx = 0.$$

Then there exists a vector-valued function $\mathbf{U} \in \tilde{\mathbf{H}}_0^{1,\delta}(B_\varepsilon)$, such that

$$\operatorname{div}\mathbf{U}(x) = \begin{cases} p(x), \ x \in \cup_{i=1}^N B_i^{\varepsilon,\delta}, \\ 0, \ x \in B_\varepsilon \setminus \cup_{i=1}^N B_i^{\varepsilon,\delta}. \end{cases} \tag{35}$$

There holds the estimate

$$\|\mathbf{U}\|^2_{\mathbf{H}^1(B_\varepsilon)} \le C\varepsilon^{-2(n+1)}\delta\sum_{i=1}^N \|p\|^2_{L^2(B_i^{\varepsilon,\delta})} \tag{36}$$

with some positive constant C independent of ε and δ

Proof. Consider the sum

$$\mathbf{\Psi} = \sum_{i=1}^{N-1} \langle p \rangle_i \mathbf{U}_i,$$

where $\mathbf{U}_i$ are functions constructed in Lemma 1. For any domain $B_k^{\varepsilon,\delta}$ we construct a function $\mathbf{\Theta} \in \mathbf{H}_0^1(B_k^{\varepsilon,\delta})$ such that

$$\operatorname{div} \mathbf{\Theta} = -\operatorname{div} \mathbf{\Psi} + p. \tag{37}$$

Remind that $\mathcal{Q}_j(\mathbf{U}_i) = \delta_{ij}$. Therefore $\int_{B_k^{\varepsilon,\delta}} \operatorname{div} \mathbf{U}_k dx = 1$, and

$$\int_{B_k^{\varepsilon,\delta}} \operatorname{div} \mathbf{\Psi} = \langle p \rangle_k \int_{B_k^{\varepsilon,\delta}} \operatorname{div} \mathbf{U}_k dx = \langle p \rangle_k.$$

Thus,

$$\int_{B_k^{\varepsilon,\delta}} (p - \operatorname{div} \mathbf{\Psi}) dx = 0.$$

and, according to [13], there exists a solution $\mathbf{\Theta} \in H_0^1(B_k^{\varepsilon,\delta})$ of the divergence equation (37). After the dilation of the domain $1/\delta$-times and passing to the new small parameter ε/δ we get the estimate

$$\begin{aligned} \|\mathbf{\Theta}\|^2_{\mathbf{H}^1(B_k^{\varepsilon,\delta})} &\le c(\varepsilon/\delta)^{-2} \| - \operatorname{div} \mathbf{\Psi} + p\|^2_{L^2(B_k^{\varepsilon,\delta})} \\ &\le c(\varepsilon/\delta)^{-2} \left(\|\nabla \mathbf{\Psi}\|^2_{L^2(B_k^{\varepsilon,\delta})} + \|p\|^2_{L^2(B_k^{\varepsilon,\delta})} \right). \end{aligned} \tag{38}$$

Here and below c is a generic constant independent of small parameters. Let us evaluate the norm $\|\mathbf{\Psi}\|_{\mathbf{H}^1(B_k^{\varepsilon,\delta})}$. It is majorated by the sum

$$\sum_{i=1}^{N-1} |\langle p \rangle_i| \|\mathbf{U}_i\|_{\mathbf{H}^1(B_k^{\varepsilon,\delta})},$$

where

$$|\langle p \rangle_i| \le c\varepsilon^{-(n-1)}\delta^{-1} \left| \int_{B_k^{\varepsilon,\delta}} p dx \right| \le \varepsilon^{-(n-1)/2}\delta^{-1/2} \|p\|_{L^2(B_k^{\varepsilon,\delta})},$$

and

$$\|\mathbf{U}_i\|_{\mathbf{H}^1(B_k^{\varepsilon,\delta})} \le \|\mathbf{U}_i\|_{\mathbf{H}^1(B_\varepsilon)} \le C\varepsilon^{-(n+1)/2}$$

(see Lemma 1). So, finally,

$$\begin{aligned} \|\mathbf{\Psi}\|^2_{\mathbf{H}^1(B_k^{\varepsilon,\delta})} \le \|\mathbf{\Psi}\|^2_{\mathbf{H}^1(B_\varepsilon)} \le C\varepsilon^{-2n}\delta^{-1}\|p\|^2_{L^2(B_k^{\varepsilon,\delta})}, \\ \|\mathbf{\Theta}\|^2_{\mathbf{H}^1(B_k^{\varepsilon,\delta})} \le C\varepsilon^{-2(n+1)}\delta\|p\|^2_{L^2(B_k^{\varepsilon,\delta})}. \end{aligned} \tag{39}$$

Let us take $\mathbf{U} = \mathbf{\Psi} + \mathbf{\Theta}$, where $\mathbf{\Theta}$ is extended by zero to the cylinders B_{ij}^{ε}. Then $\mathbf{U} = \mathbf{\Psi}$ for the remaining part of the tube structure and using the estimate (39) we finalise the proof.

Now we are in position to study the main problem of the paper: find the function $p_{\varepsilon,\delta} \in L^2(0,T;L^2(B_i^{\varepsilon,\delta}))$ which satisfies for almost all $t \in (0,T)$ the integral identity

$$\int_{B_\varepsilon} \Big(\frac{\partial \mathbf{u}_{\varepsilon,\delta}}{\partial t} \cdot \phi + \nu \nabla \mathbf{u}_{\varepsilon,\delta} : \nabla \phi + (\mathbf{u}_{\varepsilon,\delta}, \nabla \mathbf{u}_{\varepsilon,\delta}) \cdot \phi \Big) dx = \sum_{i=1}^{N} \int_{B_i^{\varepsilon,\delta}} p_{\varepsilon,\delta} \operatorname{div} \phi dx. \quad (40)$$

for every vector-field $\phi \in \tilde{\mathbf{H}}^{1,\delta}_{div0}(B_\varepsilon)$. Here $\mathbf{u}_{\varepsilon,\delta}$ is the solution of problem (20).

Note that the existence of a couple $(\mathbf{u}_{\varepsilon,\delta}, p_{\varepsilon,\delta})$ (with non-unique pressure) satisfying (40) for test functions $\phi \in \mathbf{H}_0^{1,\delta}(B_\varepsilon)$ was proved in [2] and, in particular, $\mathbf{u}_{\varepsilon,\delta}$ is the unique solution of problem (20). Let us prove that there exists a function $p_{\varepsilon,\delta} \in L^2(0,T;L^2(B_i^{\varepsilon,\delta}))$ for all $i = 1, ..., N$, such that the integral identity (40) holds for test functions from the extended space $\tilde{\mathbf{H}}_0^{1,\delta}(B_\varepsilon)$. Moreover, we will prove that $p_{\varepsilon,\delta}(x,t) = \tilde{p}_{\varepsilon,\delta}(x,t) + \theta_i(t)$, where $\tilde{p}_{\varepsilon,\delta}$ is a unique solution of the problem (30) with the vanishing mean value in $B_i^{\varepsilon,\delta}$.

Theorem 1. *There exists a set of functions $\theta_1(t), ..., \theta_N(t)$ such that the couple $(\mathbf{u}_{\varepsilon,\delta}, \tilde{p}_{\varepsilon,\delta}(x,t) + \theta_i(t))$ is a solution to problem (40). The solution is unique up to an arbitrary function $\theta_N(t)$ in the pressure component.*

Proof. Consider the following linear functional defined on $\tilde{\mathbf{H}}_0^{1,\delta}(B_\varepsilon)$:

$$f(\phi) = -\int_{B_\varepsilon} \Big(\frac{\partial \mathbf{u}_{\varepsilon,\delta}}{\partial t} \cdot \phi + \nu \nabla \mathbf{u}_{\varepsilon,\delta} : \nabla \phi + (\mathbf{u}_{\varepsilon,\delta}, \nabla \mathbf{u}_{\varepsilon,\delta}) \cdot \phi \Big) dx + \sum_{i=1}^{N} \int_{B_i^{\varepsilon,\delta}} \tilde{p}_{\varepsilon,\delta} \operatorname{div} \phi dx.$$

Note that $f(\mathbf{U}_i)$ belong to the space $L^2(0,T)$ because all integrals in the definition of the functional belong to this space. Evidently, $f(\mathbf{H}_0^{1,\delta}(B_\varepsilon)) = 0$. Let us construct its extension. Define numbers $A_i = f(\mathbf{U}_i)$, $i = 1, ..., N-1$. For any function $\phi \in \tilde{\mathbf{H}}_0^{1,\delta}(B_\varepsilon)$ calculate $\mathcal{Q}_i(\phi)$, $i = 1, ..., N-1$. Then the function $\phi - \sum\limits_{i=1}^{N-1} \mathcal{Q}_i(\phi)\mathbf{U}_i \in \mathbf{H}_0^{1,\delta}(B_\varepsilon)$, because $\mathcal{Q}_j(\phi - \sum\limits_{i=1}^{N-1} \mathcal{Q}_i(\phi)\mathbf{U}_i) = 0$ for $j = 1, ..., N-1$ and consequently, $\mathcal{Q}_N(\phi - \sum\limits_{i=1}^{N-1} \mathcal{Q}_i(\phi)\mathbf{U}_i) = \sum\limits_{i=1}^{N} \mathcal{Q}_i(\phi) = 0$ (recall that $\mathcal{Q}_N(\mathbf{U}_j) = -1$ for all $j = 1, ..., N-1$). So, $f(\phi) = \sum\limits_{i=1}^{N-1} \mathcal{Q}_i(\phi) A_i$. Then for any

$\theta_i \in L^2(0,T)$,

$$\sum_{i=1}^{N} \int_{B_i^{\varepsilon,\delta}} (\tilde{p}_{\varepsilon,\delta} + \theta_i) \operatorname{div} \phi dx$$
$$- \int_{B_\varepsilon} \Big(\frac{\partial \mathbf{u}_{\varepsilon,\delta}}{\partial t} \cdot \phi + \nu \nabla \mathbf{u}_{\varepsilon,\delta} : \nabla \phi + (\mathbf{u}_{\varepsilon,\delta}, \nabla \mathbf{u}_{\varepsilon,\delta}) \cdot \phi \Big) dx$$
$$= f(\phi) + \sum_{i=1}^{N} \theta_i \mathcal{Q}_i(\phi) = \sum_{i=1}^{N-1} \mathcal{Q}_i(\phi) f(\mathbf{U}_i) + \sum_{i=1}^{N} \mathcal{Q}_i(\phi) \theta_i. \quad (41)$$

Let us take $\theta_i = -f(\mathbf{U}_i)$, $i = 1, ..., N-1$, $\theta_N = 0$. Then

$$\sum_{i=1}^{N} \int_{B_i^{\varepsilon,\delta}} (\tilde{p}_{\varepsilon,\delta} + \theta_i) \operatorname{div} \phi dx$$
$$- \int_{B_\varepsilon} \Big(\frac{\partial \mathbf{u}_{\varepsilon,\delta}}{\partial t} \cdot \phi + \nu \nabla \mathbf{u}_{\varepsilon,\delta} : \nabla \phi + (\mathbf{u}_{\varepsilon,\delta}, \nabla \mathbf{u}_{\varepsilon,\delta}) \cdot \phi \Big) dx$$
$$= \sum_{i=1}^{N-1} \mathcal{Q}_i(\phi) f(\mathbf{U}_i) + \sum_{i=1}^{N-1} \mathcal{Q}_i(\phi)(-f(\mathbf{U}_i)) = 0. \quad (42)$$

So, we have proved the existence of a set $\theta_1(t), ..., \theta_N(t)$ ($\theta_N(t)$ is an arbitrary function of t) such that the couple $(\mathbf{u}_{\varepsilon,\delta}, \tilde{p}_{\varepsilon,\delta}(x,t) + \theta_i(t))$ satisfies integral identity (40).

From relation (41) it follows that

$$\sum_{i=1}^{N} \int_{B_i^{\varepsilon,\delta}} (\tilde{p}_{\varepsilon,\delta} + \theta_i) \operatorname{div} \phi dx$$
$$- \int_{B_\varepsilon} \Big(\frac{\partial \mathbf{u}_{\varepsilon,\delta}}{\partial t} \cdot \phi + \nu \nabla \mathbf{u}_{\varepsilon,\delta} : \nabla \phi + (\mathbf{u}_{\varepsilon,\delta}, \nabla \mathbf{u}_{\varepsilon,\delta}) \cdot \phi \Big) dx = 0$$

if and only if $\sum\limits_{i=1}^{N-1} \mathcal{Q}_i(\phi) f(\mathbf{U}_i) + \sum\limits_{i=1}^{N} \mathcal{Q}_i(\phi) \theta_i = 0$ for all $\phi \in \tilde{\mathbf{H}}_0^{1,\delta}(B_\varepsilon)$. Recall that $\sum\limits_{i=1}^{N} \mathcal{Q}_i(\phi) f(\mathbf{U}_i) = 0$ because the total flux of any test function ϕ is equal to zero. Varying ϕ we get that for all $(x_1, ..., x_N) \in \mathbb{R}^N$, such that $x_1 + ... + x_N = 0$, θ_i are solutions of the equation

$$\sum_{i=1}^{N-1} x_i f(\mathbf{U}_i) + \sum_{i=1}^{N} x_i \theta_i = 0.$$

So, if we have two different sets $(\theta_1, ..., \theta_N)$ and $(\theta_1', ..., \theta_N')$ satisfying this equation, we obtain that the vector $(\theta_1 - \theta_1', ..., \theta_N - \theta_N')$ of $\mathbb{R}^N$ is orthogonal to the hyperplane $\{\mathbf{x} \in \mathbb{R}^N : x_1 + ... + x_N = 0\}$. It means that this vector is colinear to the vector $(1, ..., 1)$, so that

$$\theta_1 - \theta_1' = ... = \theta_N - \theta_N'.$$

The Theorem is proved.

Theorem 2. *There exist $p_{\varepsilon,\delta}$ a solution of (29) and p_ε a solution of (10) such that for all $k = 1, ..., N$,*

$$\|p_{\varepsilon,\delta} - p_\varepsilon\|_{L^2(0,T;L^2(B_k^{\varepsilon,\delta}))} = O(\varepsilon^{J+n/2}). \tag{43}$$

Proof: Consider the difference $p = p_\varepsilon^{(J')} - p_{\varepsilon,\delta}$ such that $\sum\limits_{i=1}^{N} \int_{B_i^{\varepsilon,\delta}} p(x,t)dx = 0$. Indeed, $p_\varepsilon^{(J')}$ as well as $p_{\varepsilon,\delta}$ are defined up to arbitrary constants. These constants can be chosen to satisfy this condition. Applying Lemma 2, we can construct a function $\mathbf{U} \in \tilde{\mathbf{H}}_0^{1,\delta}(B_\varepsilon)$, such that

$$\operatorname{div} \mathbf{U}(x) = \begin{cases} p(x), \ x \in \cup_{i=1}^N B_i^{\varepsilon,\delta} \ , \\ \\ 0, \ x \in B_\varepsilon \setminus \cup_{i=1}^N B_i^{\varepsilon,\delta} \end{cases} \tag{44}$$

and

$$\|\mathbf{U}\|^2_{\mathbf{H}^1(B_\varepsilon)} \le C\varepsilon^{-2(n+1)}\delta \sum_{i=1}^{N} \|p\|^2_{L^2(B_i^{\varepsilon,\delta})} \tag{45}$$

with some constant C independent of ε, and δ. Taking $\mathbf{U}$ as a test function in (17) and (29), consider the difference between identities (17) and (29). Denoting $\mathbf{u} = \mathbf{u}_\varepsilon^{(J')} - \mathbf{u}_{\varepsilon,\delta}$, we get

$$\int_{B_\varepsilon} \Big(\mathbf{u}_t \cdot \mathbf{U} + \nu \nabla \mathbf{u} : \nabla \mathbf{U} + (\mathbf{u}_\varepsilon^{(J')} \cdot \nabla)\mathbf{U} \cdot \mathbf{u}_\varepsilon^{(J')} - (\mathbf{u}_{\varepsilon,\delta} \cdot \nabla)\mathbf{U} \cdot \mathbf{u}_{\varepsilon,\delta} \Big) dx$$
$$= \sum_{i=1}^{N} \int_{B_i^{\varepsilon,\delta}} p \operatorname{div} \mathbf{U} dx + \int_{B_\varepsilon} \mathbf{r}_\varepsilon^{(J')} \cdot \mathbf{U} dx, \tag{46}$$

i.e.,

$$\int_{B_\varepsilon} \Big(\mathbf{u}_t \cdot \mathbf{U} + \nu \nabla \mathbf{u} : \nabla \mathbf{U} + (\mathbf{u} \cdot \nabla)\mathbf{U} \cdot \mathbf{u}_\varepsilon^{(J')} + (\mathbf{u}_{\varepsilon,\delta} \cdot \nabla)\mathbf{U} \cdot \mathbf{u} \Big) dx$$
$$= \sum_{i=1}^{N} \int_{B_i^{\varepsilon,\delta}} p \operatorname{div} \mathbf{U} dx + \int_{B_\varepsilon} \mathbf{r}_\varepsilon^{(J')} \cdot \mathbf{U} dx, \tag{47}$$

i.e.,

$$\int_{B_\varepsilon} \Big(\mathbf{u}_t \cdot \mathbf{U} + \nu \nabla \mathbf{u} : \nabla \mathbf{U} + (\mathbf{u} \cdot \nabla)\mathbf{U} \cdot \mathbf{u}_\varepsilon^{(J')} - (\mathbf{u} \cdot \nabla)\mathbf{U} \cdot \mathbf{u}$$
$$+ (\mathbf{u}_\varepsilon^{(J')} \cdot \nabla)\mathbf{U} \cdot \mathbf{u} \Big) dx = \sum_{i=1}^{N} \int_{B_i^{\varepsilon,\delta}} p \operatorname{div} \mathbf{U} dx + \int_{B_\varepsilon} \mathbf{r}_\varepsilon^{(J')} \cdot \mathbf{U} dx, \tag{48}$$

and so,

$$\sum_{i=1}^{N}\int_{B_i^{\varepsilon,\delta}} p^2dx \le c_1\Big(\|\mathbf{u}_t(\cdot,t)\|_{L^2(B_\varepsilon)} + \|\nabla\mathbf{u}(\cdot,t)\|_{\mathbf{L}^2(B_\varepsilon)} + \|\mathbf{u}(\cdot,t)\|_{\mathbf{L}^4(B_\varepsilon)}\|\mathbf{u}_\varepsilon^{(J')}(\cdot,t)\|_{\mathbf{L}^4(B_\varepsilon)} + \|\mathbf{u}(\cdot,t)\|^2_{\mathbf{L}^4(B_\varepsilon)} + \|\mathbf{r}_\varepsilon^{(J')}\|_{\mathbf{L}^2(B_\varepsilon)}\Big)\|\nabla\mathbf{U}\|_{\mathbf{L}^2(B_\varepsilon)}. \tag{49}$$

Applying (36) from Lemma 2 we get

$$\Big(\sum_{i=1}^{N}\int_{B_i^{\varepsilon,\delta}} p^2dx\Big)^{\frac{1}{2}} \le c_2\varepsilon^{-(n+1)}\sqrt{\delta}\Big(\|\mathbf{u}_t(\cdot,t)\|_{\mathbf{L}^2(B_\varepsilon)} + \|\nabla\mathbf{u}(\cdot,t)\|_{\mathbf{L}^2(B_\varepsilon)} + \|\mathbf{u}(\cdot,t)\|_{\mathbf{L}^4(B_\varepsilon)}\|\mathbf{u}_\varepsilon^{(J')}(\cdot,t)\|_{\mathbf{L}^4(B_\varepsilon)} + \|\mathbf{u}(\cdot,t)\|^2_{\mathbf{L}^4(B_\varepsilon)} + \|\mathbf{r}_\varepsilon^{(J')}\|_{\mathbf{L}^2(B_\varepsilon)}\Big). \tag{50}$$

Integration with respect to t yields

$$\int_0^T\sum_{i=1}^{N}\int_{B_i^{\varepsilon,\delta}} p^2dxdt \le c_3\varepsilon^{-2(n+1)}\delta\Bigg(\int_0^T\Big(\|\mathbf{u}_t(\cdot,t)\|^2_{\mathbf{L}^2(B_\varepsilon)} + \|\nabla\mathbf{u}(\cdot,t)\|^2_{\mathbf{L}^2(B_\varepsilon)}\Big)dt + \sup_{t\in[0,T]}\|\nabla\mathbf{u}(\cdot,t)\|^2_{\mathbf{L}^2(B_\varepsilon)}\int_0^T\Big(\|\nabla\mathbf{u}_\varepsilon^{(J')}(\cdot,t)\|^2_{\mathbf{L}^2(B_\varepsilon)} + \|\nabla\mathbf{u}(\cdot,t)\|^2_{\mathbf{L}^2(B_\varepsilon)}\Big)dt + \int_0^T\|\mathbf{r}_\varepsilon^{(J')}\|^2_{\mathbf{L}^2(B_\varepsilon)}dt\Bigg). \tag{51}$$

Applying now estimate (16) (which follows from Theorems 6.1, 6.2 [15] and Theorem 6.2 [14]) and constructing an asymptotic expansion $\mathbf{u}_\varepsilon^{(J')}$ with sufficiently large J', one can derive that there exists a positive independent of small parameters constant c_4 such that if $\delta = c_4 J\varepsilon|\ln\varepsilon|$ then

$$\int_0^T\|p(x,t)\|^2_{L^2(B_i^{\varepsilon,\delta})}dt = O(\varepsilon^{2J+2n}). \tag{52}$$

Applying then the estimate (15) for $p_\varepsilon - p_\varepsilon^{(J')}$, and then the triangle inequality we get the assertion of the Theorem.

Let us extend now $p_{\varepsilon,\delta}$ to the cylinders B_{ij}^{ε} as an affine function by the following formula:

$$p_{\varepsilon,\delta}(x_1^{(e)},t) = \langle p_{\varepsilon,\delta}\rangle_{S_{ij}} + \frac{\langle p_{\varepsilon,\delta}\rangle_{S_{ji}} - \langle p_{\varepsilon,\delta}\rangle_{S_{ij}}}{|e| - 2\delta + \varepsilon}\left(x_1^{(e)} - \delta + \frac{\varepsilon}{2}\right), \tag{53}$$

where for any function $p \in L^2(B_\varepsilon)$ we denote by $\langle p \rangle_{S_{ij}}$ the mean value of p in the cylinder $C_{ij} = \{x_1^{(e)} \in (\delta - \varepsilon, \delta)\} \times S_{ij}$, having one of the bases at the cross section S_{ij} and the height of the length ε. Note that for sufficiently large J', the asymptotic approximation $p_\varepsilon^{(J')}$ satisfies the relation analogous to (53):

$$p_\varepsilon^{(J')}(x_1^{(e)}, t) = \langle p_\varepsilon^{(J')} \rangle_{S_{ij}} + \frac{\langle p_\varepsilon^{(J')} \rangle_{S_{ji}} - \langle p_\varepsilon^{(J')} \rangle_{S_{ij}}}{|e| - 2\delta + \varepsilon} \left(x_1^{(e)} - \delta + \frac{\varepsilon}{2} \right), \tag{54}$$

because it is affine function of $x_1^{(e)}$ within the cylinder $\{x_1^{(e)} \in (\delta - \varepsilon, |e| - \delta + \varepsilon)\} \times \sigma^{(e)}$. On the other hand

$$|\langle p_{\varepsilon,\delta} \rangle_{S_{ij}} - \langle p_\varepsilon^{(J')} \rangle_{S_{ij}}|^2 \le c_5 (\operatorname{mes} C_{ij})^{-1} \|p_{\varepsilon,\delta} - p_\varepsilon^{(J')}\|^2_{L^2(B_i^{\varepsilon,\delta})}. \tag{55}$$

Applying the estimate (43) of Theorem 2, we get:

$$\int_0^T |\langle p_{\varepsilon,\delta} \rangle_{S_{ij}} - \langle p_\varepsilon^{(J')} \rangle_{S_{ij}}|^2 dt = O(\varepsilon^{2J}). \tag{56}$$

So, we get finally that

$$\|p_{\varepsilon,\delta} - p_\varepsilon^{(J')}\|_{L^2(0,T;L^2(B_\varepsilon))} = O(\varepsilon^J). \tag{57}$$

Applying as in the proof of Theorem 2 estimate (15) for $p_\varepsilon - p_\varepsilon^{(J')}$, and then the triangle inequality we get the following assertion.

Theorem 3. *There exists a constant C_0 such that if $\delta = C_0 J \varepsilon |\ln \varepsilon|$, then for the function $p_{\varepsilon,\delta}$ solution of problem (29) extended by formula (53), we can find the function p_ε solution of (10) such that*

$$\|p_{\varepsilon,\delta} - p_\varepsilon\|_{L^2(0,T;L^2(B_\varepsilon))} = O(\varepsilon^J). \tag{58}$$

In a similar way using the expansions from [3] we can prove for the steady Stokes equations

Theorem 3'. *There exists a constant c_1' such that if $\delta = c_1' J \varepsilon |\ln \varepsilon|$, then there is a pressure p_ε, solution of the problem (1), such that it is $O(\varepsilon^J)-$close in the norm $L^2(B_\varepsilon)$ to $p_{\varepsilon,\delta}$ solution of problem (28) extended to B_ε as the affine function*

$$\|p_{\varepsilon,\delta} - p_\varepsilon\|_{L^2(B_\varepsilon)} = O(\varepsilon^J). \tag{59}$$

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