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Subgroup induction property for branch groups

Dominik Francoeur, Paul-Henry Leemann

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Abstract

Recently, the so-called *subgroup induction property* attracted the attention of mathematicians working with branch groups. It was for example used to prove that groups with this property are subgroup separable (locally extensively residually finite) or to describe their finitely generated subgroups as well as their weakly maximal subgroups. Alas, until now, there were only two known examples of groups with this property: the first Grigorchuk group and the Gupta-Sidki 3-group.

The aim of this article is twofold. First, we investigate various consequences of the subgroup induction property, such as being just infinite or having all maximal subgroups of finite index. Then, we show that every torsion GGS group has the subgroup induction property, hence providing infinitely many new examples.

1 Introduction

Since their definition in 1997 (at the Groups St Andrews conference in Bath), branch groups (see Section 2 for all the relevant definitions) have attracted a lot of attention. Indeed, they are a rich source of examples of groups with exotic properties. They also naturally appear in the classification of *just infinite groups* [7], that is infinite groups whose proper quotients are all finite.

The most well-known branch group is the so-called first Grigorchuk group \mathcal{G} , which was the first example of a group of intermediate growth [6]. Another well-studied family of branch groups are the Gupta-Sidka groups G_p (p prime) and their generalization, the GGS groups (see Section 4). The first Grigorchuk group and the torsion GGS groups have been intensively studied and share many properties: being just infinite, being finitely generated infinite torsion groups, having all maximal subgroups of finite index, and more.

In 2003, Grigorchuk and Wilson showed [8] that \mathcal{G} has a property now known as the *subgroup induction property*. Roughly speaking, a group with the subgroup induction property is a branch group such that any sufficiently nice property of the group is shared by all of its finitely generated subgroups (see Section 3 for a proper definition). They then used this fact to show that every finitely generated infinite subgroup of \mathcal{G} is commensurable with \mathcal{G} . Recall that two groups G_1 and G_2 are *commensurable* if there exists H_i of finite index in G_i

with $H_1 \cong H_2$. In the same article, they also showed that \mathcal{G} has the rather rare property of being subgroup separable, where G is *subgroup separable* (also called *locally extensively residually finite* or *LERF*) if all of its finitely generated subgroups are closed in the profinite topology. Among other things, this implies that \mathcal{G} has a solvable membership problem. That is, there exists an algorithm that, given $g \in G$ and a finitely generated subgroup $H \leq G$, decides if g belongs to H . Indeed, this follows from subgroup separability and having a finite L -presentation, which is the case of \mathcal{G} and of GGS groups, see [10].

Adapting the strategy of [8], Garrido showed [5] that the Gupta-Sidki 3-group G_3 also has the subgroup induction property, is subgroup separable and that any finitely generated infinite subgroup $H \leq G_3$ is commensurable with G_3 or with $G_3 \times G_3$.

In recent years, the subgroup induction property of a branch group G turned out to be a versatile tool. For example, it was used in [14] to compute the Cantor-Bendixon rank of \mathcal{G} and G_3 , in [11] to describe the weakly maximal subgroups (subgroups that are maximal for the property of being of infinite index) of G and in [9] to give a characterization of finitely generated subgroups of G , which was used to show that (if G has the congruence subgroup property and some other minor technical hypothesis) G is subgroup separable.

The aim of this article is twofold: first, to expand the list of properties implied by the subgroup induction property, and second, to expand the list of groups known to satisfy this property. With regard to the first goal, we prove the following.

Theorem A. Let G be a finitely generated weakly branch group with the subgroup induction property, then G is torsion. If G is branch, it is just infinite.

If, moreover, G is a branch p -group, then all of its maximal subgroups are of finite index.

In fact, the condition of G being a p -group can be relaxed, see Proposition 3.7, Lemma 3.5 and the discussion below it. We also obtain (see Theorem 3.12 for a more general statement)

Theorem B. Let $G \leq \text{Aut}(T_d)$ be a self-replicating branch group such that $\text{Stab}_G(v) = \text{Stab}_G(\mathcal{L}_1)$ for every vertex v on the first level. Suppose that G has the subgroup induction property and let H be an infinite finitely generated subgroup of G . Then H is commensurable with one of G, G^2, \dots, G^{d-1} .

If moreover G is strongly self-replicating, has the congruence subgroup property and is a p -group, then all maximal subgroups of H are of finite index.

In view of the above, it is natural to ask for examples of groups with the subgroup induction property. Until now, the only known examples were \mathcal{G} and G_3 . The second part of this article is devoted to providing infinitely many new examples of such groups.

Theorem C. All the torsion GGS groups have the subgroup induction property.

In fact, by Theorem A we have that a GGS group has the subgroup induction property if and only if it is torsion.

Organisation The small Section 2 quickly recalls the definitions of branch groups, self-similar groups and other related notions. The definition of the subgroup induction property is found in Section 3, which also contains the proofs of Theorems A and B. The final section is dedicated to the proof of Theorem C.

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2 Reminders on branch groups

Let d be an integer greater than 1 and let T be a d -regular rooted tree. That is, T is a tree with a special vertex, called the root, of degree d and with all other vertices of degree $d + 1$. The set of all vertices at distance n from the root is the n^{th} level of the tree and will be denoted by \mathcal{L}_n . There is a natural (partial) order on the vertices of T defined by $w \geq v$ if the unique simple path starting from the root and ending in w passes through v . The root is the smallest element for this order. We identify T with the free monoid $\{1, \dots, d\}^*$ with the lexicographical order, where the children of $x_1 \dots x_n$ are the $x_1 \dots x_n y$ for $y \in \{1, \dots, d\}$. The set of infinite rays starting at the root is denoted by ∂T and is isomorphic to the set of infinite words $\{1, \dots, d\}^\infty$. There is a natural metrizable topology on ∂T obtained by declaring that two rays are near if they share a long common prefix. With this topology, the space ∂T is compact.

A subset X of T is a *transversal* (sometimes called a *spanning leaf set*, a *cut set* or a *section*) if every ray of ∂T passes through exactly one element of X . It follows from the compactness of ∂T that a transversal is always finite.

For any vertex v of T , we denote by T_v the subtree of T consisting of all $w \geq v$. It is naturally isomorphic¹ to T .

Let $G \leq \text{Aut}(T)$ be a group of automorphisms of T and V be a subset of vertices of T . The subgroup $\text{Stab}_G(V) = \bigcap_{v \in V} \text{Stab}_G(v)$ is the pointwise stabilizer of V , while $\text{Stab}_G(\mathcal{L}_n)$ is the n^{th} level stabilizer. The *rigid stabilizer* $\text{Rist}_G(v)$ of a vertex v in G is the set of elements of G acting trivially outside T_v . That is, $\text{Rist}_G(v) = \text{Stab}_G(V)$ for $V = \{w \in T \mid w \notin T_v\}$. Finally, the *rigid stabilizer* of a level $\text{Rist}_G(\mathcal{L}_n)$ is the subgroup generated by the $\text{Rist}_G(v)$ for all vertices v of level n .

Definition 2.1. Let T be a d -regular rooted tree. A subgroup of $G \leq \text{Aut}(T)$ is *weakly branch* if G acts transitively on the levels and all the rigid stabilizers of vertices are infinite.

¹If we identify T to $\{1, \dots, d\}^*$, then the identification of T and T_v is done by deleting of the prefix corresponding to v .

If moreover all the $\text{Rist}_G(\mathcal{L}_n)$ have finite index in G , then G is said to be *branch*.

The class of branch groups naturally arises in the description of *just infinite groups*, that is infinite groups whose quotients are all finite. More precisely, a just infinite group is either a just infinite branch group, virtually a product of simple groups, or virtually a product of *hereditary just infinite groups* (recall that a group G is hereditary just infinite if all of its finite index subgroups are just infinite), as was shown in [7]. On the other hand, not all branch groups are just infinite. It is thus interesting to investigate which branch groups are just infinite.

Apart from branch groups, another interesting class of subgroups of $\text{Aut}(T)$ is the class of self-similar groups. While these groups are of interest in their own right, they are often studied in conjunction with branch groups. Indeed, the intersection of these two classes (that is groups that are branch and self-similar) contains a lot of interesting examples, including the first Grigorchuk group and the GGS groups. In order to introduce self-similar groups, we will need a little more notation.

Let T be a d -regular rooted tree. Recall that we identify the vertices of T with the free monoid $\{1, \dots, d\}^*$. Thus, we have a natural operation of concatenation on the vertices of T . Geometrically, given $v, w \in T$, the vertex vw is the vertex of T_v associated to w under the natural identification between T_v and T .

It follows from the definitions that for every automorphism $g \in \text{Aut}(T)$ and every $v \in T$, there exists a unique automorphism $g|_{T_v} \in \text{Aut}(T)$ such that

$$g \cdot (vw) = (g \cdot v)(g|_{T_v} \cdot w)$$

for all $w \in T$. This gives us, for every $v \in T$, a map

$$\begin{aligned} \varphi_v: \text{Aut}(T) &\rightarrow \text{Aut}(T) \\ g &\mapsto g|_{T_v}. \end{aligned}$$

This map restricts to a homomorphism between $\text{Stab}(v)$ and $\text{Aut}(T)$, and it restricts further to an isomorphism between $\text{Rist}(v)$ and $\text{Aut}(T)$, where we denote here by $\text{Stab}(v)$ and $\text{Rist}(v)$ the stabilizer and rigid stabilizer of v with respect to the whole group $\text{Aut}(T)$. The element $\varphi_v(g)$ is called the *section* of g at v .

Definition 2.2. For a d -regular rooted tree T , a group $G \leq \text{Aut}(T)$ is said to be *self-similar* if for any v , the image $\varphi_v(\text{Stab}_G(v))$ is a subgroup of G . If moreover $\varphi_v(\text{Stab}_G(v)) = G$ for any vertex v , then G is said to be *self-replicating* (or *fractal*). Finally, G is said to be *strongly self-replicating* if it is self-similar and for every n and every v of level n we have $\varphi_v(\text{Stab}_G(\mathcal{L}_n)) = G$.

A subgroup H of $\prod_{i \in I} G_i$ is a *subdirect product* if for every $i \in I$, the canonical projection $H \twoheadrightarrow G_i$ is surjective. For example, $G \leq \text{Aut}(T)$ is strongly self-replicating if and only if it is self-similar and for every n the group $\text{Stab}_G(\mathcal{L}_n)$

is a subdirect product of $G^{|\mathcal{L}_n|}$. The notion of subdirect product plays an important role in the description of finitely generated subgroups of a group with the subgroup induction property, see [9].

Another property that we will need, and that is often studied for branch groups, is the congruence subgroup property, which asserts that two topologies on G are equal. Indeed, it is possible to endow any subgroup $G \leq \text{Aut}(T)$ with both the profinite topology (generated by all the subgroups of finite index) and the $\text{Aut}(T)$ topology (generated by the $\text{Stab}_G(\mathcal{L}_n)$ and sometimes called the *congruence topology*). The group G is said to have the *congruence subgroup property* if these two topologies coincide, or equivalently if every subgroup of finite index contains $\text{Stab}_G(\mathcal{L}_n)$ for some n .

Finally, we will sometimes require our groups to have the property that for every vertex v , the subgroup $\text{Stab}_G(v)$ stabilizes all the siblings of v . For self-similar groups, this is equivalent to asking that $\text{Stab}_G(\mathcal{L}_1) = \text{Stab}_G(w)$ for any vertex $w \in \mathcal{L}_1$, and is also equivalent to the fact that the action of $G/\text{Stab}_G(\mathcal{L}_1)$ on \mathcal{L}_1 is free. In particular, this property is satisfied as soon as G is self-similar and such that $G/\text{Stab}_G(\mathcal{L}_1)$ acts transitively and cyclically on \mathcal{L}_1 . This last condition happens to be true for a lot of interesting branch groups (it is usually a straightforward consequence of the definition), such as the first Grigorchuk group and the GGS groups.

3 The subgroup induction property

The first appearance (without a name) of the subgroup induction property goes back to [8], where it was expressed in terms of inductive classes of subgroups.

Definition 3.1. Let $G \leq \text{Aut}(T)$ be a self-similar group. A class \mathcal{X} of subgroups of G is said to be *inductive* if

- (A) Both $\{1\}$ and G belong to \mathcal{X} ,
- (B) If $H \leq L$ are two subgroups of G with $[L : H]$ finite, then L is in \mathcal{X} if and only if H is in \mathcal{X} ,
- (C) If H is a finitely generated subgroup of $\text{Stab}_G(\mathcal{L}_1)$ and all first level sections of H are in \mathcal{X} , then $H \in \mathcal{X}$.

The group G has the *subgroup induction property* if for any inductive class of subgroups \mathcal{X} , every finitely generated subgroup of G is contained in \mathcal{X} .

A slight modification of the above definition leads to the, a priori stronger, notion of strong subgroup induction property. We say that a class \mathcal{X} of subgroups of G is *weakly inductive* if it satisfies **A**, **C** and

- (B') If $H \leq L$ are two subgroups of G with $[L : H]$ finite, then L is in \mathcal{X} if H is in \mathcal{X} .

The group G has the *strong subgroup induction property* if for any weakly inductive class of subgroups \mathcal{X} , every finitely generated subgroup of G is contained in \mathcal{X} . The first Grigorchuk group [8], the Gupta-Sidki 3-group [5] as well as all torsion GGS groups (Theorem 4.23) have the strong subgroup induction property. On the other hand, there is no known examples of groups with the subgroup induction property, but without the strong one. Nevertheless, the equivalences given in [9] hint that Definition 3.1 is probably the right one.

On one hand, Definition 3.1 is convenient to prove that some group has the subgroup induction property. On the other hand, it is sometimes easier to use another characterization of it in order to study structural properties of groups with the subgroup induction property. In [9], the second named author, together with R. Grigorchuk and T. Nagnibeda, proposed the following alternative definition and proved [9, Proposition 4.3] that for self-similar groups the two definitions are equivalent.

Definition 3.2. A group $G \leq \text{Aut}(T)$ has the *subgroup induction property* if for every finitely generated subgroup $H \leq G$, there exists a transversal X of T such that for each $v \in X$, the section $\varphi_v(\text{Stab}_H(X))$ is either trivial or has finite index in $\varphi_v(\text{Stab}_G(v))$.

We will now use Definition 3.2 to prove several consequences of the subgroup induction property.

First of all, a finitely generated branch group with this property must be torsion and just infinite, hence proving the first part of Theorem A.

Proposition 3.3. *Let $G \leq \text{Aut}(T)$ be a finitely generated branch group with the subgroup induction property. Then G is torsion and just infinite.*

Proof. By [7, Theorem 4], a finitely generated torsion branch group is always just infinite. Hence, the proposition directly follows from the next lemma. \square

Lemma 3.4. *Let $G \leq \text{Aut}(T)$ be a group with the subgroup induction property and such that $\text{Rist}_G(v)$ is infinite for every $v \in T$. Then, G is torsion.*

Proof. Let g be any element of G and let $H := \langle g \rangle$. Let X be a transversal of T such that for every $v \in X$, the section $\varphi_v(\text{Stab}_H(X))$ is either trivial or has finite index in $\varphi_v(\text{Stab}_G(v))$. If g is of infinite order, then at least one of these sections is infinite, which implies that for some $v \in X$, the subgroup $\varphi_v(\text{Stab}_H(X))$ is isomorphic to \mathbf{Z} and hence $\varphi_v(\text{Stab}_G(v))$ is virtually \mathbf{Z} . But then, the subgroup $\text{Rist}_G(v) \cong \varphi_v(\text{Rist}_G(v))$ is also virtually \mathbf{Z} . In particular, there exists a finite index subgroup Z of $\text{Rist}_G(v)$ which is isomorphic to \mathbf{Z} . Since Z has finite index in $\text{Rist}_G(v)$, for every $w \geq v$ the subgroup $\text{Rist}_Z(w) \leq Z$ is non-trivial and hence also isomorphic to \mathbf{Z} . But then, by looking at all children w_1, \dots, w_d of v we have that $Z \cong \mathbf{Z}$ contains a subgroup isomorphic to $Z^d \cong \mathbf{Z}^d$ which is absurd. \square

Recall that the *Fratini subgroup* $\Phi(G)$ of a group G is the intersection of all the maximal subgroups of G . In the following, we will make use of the bigger

subgroup $\Phi_F(G)$ consisting of the intersection of maximal subgroups of finite index of G . This is obviously a characteristic subgroup of G . Observe that $\Phi_F(G)$ has finite index in G as soon as G is a finitely generated p -group. More generally, we have

Lemma 3.5. *Let G be a group such that every finite quotient of G is nilpotent. Then $\Phi_F(G)$ contains the derived subgroup G' . In particular, $\Phi_F(G)$ has finite index in G if G' does.*

Proof. Let M be a maximal subgroup of finite index and N be its normal core. Since M is of finite index, so is N , and M/N is a maximal subgroup of the finite nilpotent group G/N . Since a finite group is nilpotent if and only if every maximal subgroup is normal, we conclude that M is normal and hence G/M is cyclic of prime order. In particular, G/M is abelian, which implies that M contains G' . \square

Observe that the condition that G' has finite index in G is satisfied for a broad class of groups. This is the case for example for finitely generated torsion groups, or for non-abelian just infinite groups. In particular, if G is a finitely generated branch group with the subgroup induction property, then it is non-abelian and just infinite and therefore G' has finite index in G . This implies that for such a G , the subgroup $\Phi_F(G)$ has finite index in G as soon as every finite quotient of G is nilpotent.

Lemma 3.6. *Let G be a finitely generated group such that $\Phi_F(G)$ has finite index in G . If H is a subgroup of G which is dense for the profinite topology, then it contains a finitely generated subgroup which is also dense in G for the profinite topology.*

Proof. Since H is dense for the profinite topology and $\Phi_F(G)$ is of finite index, we have $H \cdot \Phi_F(G) = G$. In particular, we have elements h_1 to h_n of H such that $\langle h_1, \dots, h_n \rangle \Phi_F(G) = G$. Now, let N be a normal subgroup of finite index of G . We claim that $\langle h_1, \dots, h_n \rangle N = G$. Indeed, if it wasn't the case, there would exist some maximal subgroup M of G containing $\langle h_1, \dots, h_n \rangle N$. This subgroup is necessarily of finite index and hence contains $\Phi_F(G)$. In particular, we would have $M \geq \langle h_1, \dots, h_n \rangle \Phi_F(G) = G$, which is absurd. We hence have

$$\overline{\langle h_1, \dots, h_n \rangle} = \bigcap_{N \trianglelefteq_{\text{f.i.}} G} \langle h_1, \dots, h_n \rangle N = \bigcap_{N \trianglelefteq_{\text{f.i.}} G} G = G$$

as desired. \square

Using the above lemma, we prove that, under some technical hypothesis, the subgroup induction property implies that all maximal subgroups are of finite index.

Proposition 3.7. *Let $G \leq \text{Aut}(T)$ be a finitely generated branch group with the subgroup induction property. If $\Phi_F(G)$ is not trivial, then every maximal subgroup of G has finite index in G .*

Proof. First of all, by Proposition 3.3 G is just infinite and hence the normal subgroup $\Phi_F(G) \neq \{1\}$ has finite index in G .

Suppose now for a contradiction that there exists a maximal subgroup M that has infinite index in G . Such a group is dense for the profinite topology. By Lemma 3.6, M contains a finitely generated proper dense subgroup $H < G$.

Since G has the subgroup induction property, there must exist a transversal X of T such that for every $v \in X$, the section $\varphi_v(\text{Stab}_H(X))$ is either trivial or has finite index in $\varphi_v(\text{Stab}_G(v))$. This implies that for every $v \in X$, the subgroup $\varphi_v(\text{Stab}_H(v))$ is either finite or of finite index in $\varphi_v(\text{Stab}_G(v))$.

On the other hand, it follows from Theorem 3.2 and Lemma 3.1 of [3] that for every $v \in X$, the subgroup $\varphi_v(\text{Stab}_H(v))$ is a proper and dense subgroup of $\varphi_v(\text{Stab}_G(v))$ for the profinite topology on $\varphi_v(\text{Stab}_G(v))$, since H is proper and dense for the profinite topology on the just infinite branch group G . This immediately leads to a contradiction. Indeed, since $\varphi_v(\text{Stab}_G(v))$ is a residually finite group, both finite subgroups and finite index subgroups are closed for the profinite topology, and hence cannot be both dense and proper. \square

Since $\Phi_F(G)$ has finite index in G as soon as G is a finitely generated p -group, the above Proposition finishes the proof of Theorem A.

In certain cases, it is even possible to generalize the above Proposition 3.7 to groups commensurable with a power of G .

Theorem 3.8. *Let $G \leq \text{Aut}(T)$ be a finitely generated strongly self-replicating branch group with both the subgroup induction property and the congruence subgroup property. Suppose that $\Phi_F(G)$ is not trivial.*

Let $\Gamma := G^{(\times k)}$ be the direct product of k copies of G for some $k \in \mathbf{N}$. If H is a group commensurable with Γ then all maximal subgroups of H have finite index in H .

Proof. Since G is branch, it is residually finite, and since it has the subgroup induction property, it is just infinite by Proposition 3.3. Finally, by Proposition 3.7, all its maximal subgroups have finite index in G . Hence the result follows from the forthcoming Proposition 3.11. \square

In order to finish the proof of Theorem 3.8, let us recall some results from [5] and [8].

Lemma 3.9 ([8, Lemma 1]). *Let G be an infinite finitely generated group and let H be a subgroup of finite index in G . If G has a maximal subgroup M of infinite index, then H has a maximal subgroup of infinite index containing $H \cap M$.*

Recall that a *chief factor* of a group G is a non-trivial minimal normal subgroup of a quotient of G . The chief factors of a residually finite just infinite group are always finite.

Lemma 3.10 ([8, Lemma 3]). *Let G_1, \dots, G_n be groups with the property that all chief factors are finite and that all maximal subgroups have finite index. If*

L is a subdirect product of $\prod G_i$, then all chief factors of L are finite and all maximal subgroups of L have finite index.

The following is an easy generalization of [8, Lemma 4] and [5, Theorem 5].

Proposition 3.11. *Let $G \leq \text{Aut}(T)$ be a finitely generated branch group such that all chief factors are finite and all maximal subgroups have finite index. Suppose moreover that G has the congruence subgroup property and is strongly self-replicating.*

Let $\Gamma := G^{(\times k)}$ be the direct product of k copies of G . If H is a group commensurable with Γ then all maximal subgroups of H have finite index in H .

Proof. By definition of commensurability, there exist $K \leq H$ of finite index and $J \leq \Gamma$ of finite index with K isomorphic to J . For $i = 1, \dots, k$, let G_i denote the i^{th} direct factor of Γ . Then, for every i , the subgroup $J_i := J \cap G_i$ has finite index in G_i . By assumption, for each i , there is some n_i such that $\{1\} \times \dots \times \text{Stab}(\mathcal{L}_{n_i}) \times \dots \times \{1\} \leq J_i$ and so $L := \text{Stab}(\mathcal{L}_{n_1}) \times \dots \times \text{Stab}(\mathcal{L}_{n_k})$ is a finite index subgroup of J .

Now, by assumption, all of the maximal subgroups of G have finite index. Furthermore, by strong self-replicatingness, the subgroup $\text{Stab}(\mathcal{L}_n)$ is a subdirect product of copies of G . Thus L satisfies the assumptions of Lemma 3.10 and all of its maximal subgroups are of finite index. Lemma 3.9 then implies that all maximal subgroups of J must have finite index in J . The result now follows by applying Lemma 3.9 to $K \cong J$ and H . \square

We also prove Theorem B, which generalizes the main result of [8] and [5].

Theorem 3.12. *Let $G \leq \text{Aut}(T_d)$ be a self-replicating branch group such that $\text{Stab}_G(v) = \text{Stab}_G(\mathcal{L}_1)$ for every vertex v on the first level. Suppose that G has the subgroup induction property and let H be an infinite finitely generated subgroup of G . Then H is commensurable with one of G, G^2, \dots, G^{d-1} .*

If moreover G is strongly self-replicating, has the congruence subgroup property and $\Phi_F(G)$ is not trivial, then all maximal subgroups of H are of finite index.

Proof. We begin by showing that G, G^d and $\text{Rist}_G(v)$ are commensurable for any vertex v . Indeed, since G is branch, $\text{Rist}_G(\mathcal{L}_n) = \prod_{v \in \mathcal{L}_n} \text{Rist}_G(v)$ has finite index in G . In particular, for any vertex v of the n^{th} level, the section $\varphi_v(\text{Rist}_G(\mathcal{L}_n)) \cong \text{Rist}_G(v)$ has finite index in $\varphi_v(\text{Stab}_G(v)) = G$. Hence G is commensurable with $\text{Rist}_G(v)$ and G^d is commensurable with $\text{Rist}_G(\mathcal{L}_1)$, which is itself commensurable with G .

Now, let H be a finitely generated subgroup of G . By [9, Theorem 2.9], there exists a finite index subgroup B of H that is isomorphic to $\prod_{i=1}^m D_i$ where D_i is a finite index subgroup of $\text{Rist}_G(v_i)$ for some vertex v_i . In particular, H is commensurable with G^m for some integer m , with $m \geq 1$ if and only if H is infinite.

The assertion on maximal subgroups of H is a direct application of Theorem 3.8. \square

As a last application of the subgroup induction property, we expand a result of Skipper and Wesolek. See [14] for more and details on the Cantor–Bendixson rank.

Proposition 3.13. *Let $G \leq \text{Aut}(T)$ be a finitely generated regular branch group that is strongly self-replicating and such that for every vertex v of the first level, we have $\text{Stab}_G(v) = \text{Stab}_G(\mathcal{L}_1)$. Suppose that G has the congruence subgroup property and the subgroup induction property. Then $\text{Sub}(G)$ has Cantor–Bendixson rank ω .*

Proof. The group G is just infinite by Proposition 3.3 and subgroup separable by [9, Theorem 2.12]. Since it is subgroup separable and has the congruence subgroup property, it has “well-approximated subgroups” [14, Lemma 2.12]. Moreover, the so-called “Grigorchuk–Nagnibeda alternative” [14, Definition 4.1] is a weak form of the subgroup induction property in the sense of Definition 3.2. Hence [14, Corollary 5.4] applies. \square

4 Subgroup induction property for GGS groups

Until now, only two groups were known to possess the subgroup induction property, namely the first Grigorchuk group acting on the binary rooted tree [8] and the Gupta–Sidki group acting on the ternary rooted tree [5]. In this section, we exhibit an infinite family of groups with this property, the torsion GGS groups.

4.1 General facts about GGS groups

Defined as a generalisation of both the second Grigorchuk group and the Gupta–Sidki groups, the Grigorchuk–Gupta–Sidki groups, or GGS groups, are a well-studied family of groups acting on rooted trees. Let us first define them, before listing a few of their properties that will be useful later on.

Definition 4.1. Let p be an odd prime number, let T the p -regular tree and let $\mathbf{e} = (e_0, \dots, e_{p-2})$ be a vector in $(\mathbf{F}_p)^{p-1} \setminus \{0\}$. The GGS group $G_{\mathbf{e}} = \langle a, b \rangle$ with defining vector \mathbf{e} is the subgroup of $\text{Aut}(T)$ generated by the two automorphisms

$$\begin{aligned} a &= \varepsilon \cdot (1, \dots, 1) \\ b &= (a^{e_0}, \dots, a^{e_{p-2}}, b) \end{aligned}$$

where ε is the cyclic permutation $(12 \dots p)$.

Remark 4.2. The definition of GGS groups naturally extends to n -regular rooted tree for n an integer greater than 1. For example, Vokivsky studied proved Proposition 4.5 in the context of $n = p^r$ a prime power. However, it is usual to restrict the study of GGS groups on the ones acting on p -regular rooted tree, p -prime, as some results (for example the classification of Lemma 4.12) depend on this fact. This is why in the following we will always assume that T is a p -regular rooted tree with p prime.

The *Gupta-Sidki p -group* is the GGS group acting with defining vector $(1, -1, 0, \dots, 0)$.

A GGS group G acting on the p -regular rooted tree T is always infinite (since $\mathbf{e} \neq 0$, [16]), residually-(finite p). By [2], a GGS group $G_{\mathbf{e}}$ is branch if and only if \mathbf{e} is not constant, which is equivalent by [1] to the fact that $G_{\mathbf{e}}$ has the congruence subgroup property. Even if the GGS groups with constant defining vector are not branch, they are still weakly branch [1].

If a GGS group is torsion, then it is strongly self-replicating [15]. It follows directly from the definition that for every vertex $v \in \mathcal{L}_1$ on the first level, we have $\text{Stab}_G(v) = \text{Stab}_G(\mathcal{L}_1)$, or in other words, that the action of $G/\text{Stab}_G(\mathcal{L}_1)$ on \mathcal{L}_1 is free. Furthermore, G does not contain maximal subgroups of infinite index, a fact that is due to Pervova [13] in the torsion case and to the first named author and Anitha Thillaisundaram in the non-torsion case [4]. This implies that every maximal subgroup of G is normal and of index p , and thus, by Lemma 3.5, that the derived subgroup G' is contained $\Phi_F(G) = \Phi(G)$ the Frattini subgroup (that is, the intersection of all maximal subgroups) of G .

In what follows, we will frequently make use, often without explicitly mentioning them, of several structural properties of GGS groups, which we summarize in the proposition below. A proof of these classical facts can be found, among other sources, in [2].

Proposition 4.3. *Let p be a prime number, let G be a GGS group acting on the p -regular rooted tree T , and let G' be its derived subgroup. Then,*

$$(i) \text{Stab}_G(\mathcal{L}_1) = \langle b \rangle^G = \langle b, aba^{-1} \dots, a^{p-1}ba^{-(p-1)} \rangle,$$

$$(ii) G = \langle a \rangle \rtimes \text{Stab}_G(\mathcal{L}_1),$$

$$(iii) G/G' = \langle aG', bG' \rangle \cong C_p \times C_p,$$

$$(iv) \text{Stab}_G(\mathcal{L}_2) \leq G' \leq \text{Stab}_G(\mathcal{L}_1).$$

We will also use without mentioning it the following well-known trivial result.

Lemma 4.4. *Let T be a d -regular rooted tree and $a = \varepsilon \cdot (1, \dots, 1)$ where ε is the cyclic permutation $(12 \dots d)$. For any element (x_1, x_2, \dots, x_p) of $\text{Stab}(\mathcal{L}_1)$ we have*

$$(x_1, x_2, \dots, x_p)a = a(x_2, x_3, \dots, x_p, x_1).$$

As it follows from Lemma 3.4 that non-torsion GGS groups cannot have the subgroup induction property, we will now focus exclusively on torsion GGS groups. Luckily, there is a simple criterion, due to Vovkivsky, to decide whether a GGS group is torsion or not.

Proposition 4.5 ([16, Theorem 1]). *Let p be a prime, let $\mathbf{e} = (e_0, \dots, e_{p-2})$ be a vector in $(\mathbf{F}_p)^{p-1} \setminus \{0\}$ and let $G_{\mathbf{e}}$ be the corresponding GGS group. Then, G is torsion if and only if*

$$\sum_{i=0}^{p-2} e_i = 0.$$

It follows from this criterion that torsion GGS groups have non-constant defining vectors and hence are branch and have the congruence subgroup property.

Subgroups of torsion GGS groups that do not fix the first level satisfy a dichotomy that will be crucial in establishing the subgroup induction property. This dichotomy was first obtained by Garrido for Gupta-Sidki groups, and then generalised to all torsion GGS groups by the second named author.

Lemma 4.6 ([5, 11]). *Let G be a torsion GGS group and H be a subgroup of G that is not contained in $\text{Stab}_G(\mathcal{L}_1)$. Then either all first-level sections of H are equal to G , or they are all contained in $\text{Stab}_G(\mathcal{L}_1)$, so that $\text{Stab}_H(\mathcal{L}_1) = \text{Stab}_H(\mathcal{L}_2)$.*

4.2 Lengths in GGS groups

A GGS group G is always generated by two elements a and b , as in Definition 4.1. However, for our purposes, the word length corresponding to this generating set is not the most convenient, and we will prefer to it the word length given by the generating set consisting of all powers of a and b . To avoid all possible confusion, we will call the word length with this specific set of generators the *total length*.

Definition 4.7. Let p be a prime number, let G be a GGS group and let $g \in G$ be any element. The *total length* of g , which we will denote by $\lambda(g)$, is the word length of g with respect to the generating set $\{a, a^2, \dots, a^{p-1}, b, b^2, \dots, b^{p-1}\}$. In other words, $\lambda(g)$ is the smallest $n \in \mathbf{N}$ such that

$$g = s_1 s_2 \dots s_n$$

with $s_1, s_2, \dots, s_n \in \{a, a^2, \dots, a^{p-1}, b, b^2, \dots, b^{p-1}\}$.

Since every GGS group G is generated by two elements a and b , every element of G can be expressed as an alternating product of powers of a and powers of b . To simplify our analysis, we will often want to ignore the powers of a , since they have no action beyond the first level, and focus our attention only on the powers of b . To this end, we introduce a different “length” from the total length, that we will call the *b-length*.

Definition 4.8. Let p be a prime number, G be a GGS group and $g \in G$ be any element. The *b-length* of g , denoted by $|g|$, is the smallest $n \in \mathbf{N}$ such that

$$g = a^{i_1} b^{j_1} a^{i_2} b^{j_2} \dots a^{i_n} b^{j_n} a^{i_{n+1}}$$

with $i_1, \dots, i_{n+1}, j_1, \dots, j_n \in \mathbf{Z}$.

Remark 4.9. Please note that despite what its name and notation might suggest, the *b-length* is not a norm in the usual sense, but merely a pseudo-norm, since non-trivial elements, namely the powers of a , can have *b-length* 0.

The total length and the b -length are related to each other, and there are relations between the length of an element and the length of its sections, as the next proposition shows. These inequalities are all fairly standard, but we give brief proofs here for completeness.

Proposition 4.10. *Let p be a prime number, G be a GGS group and $g \in G$ be any element of G , and let $v \in \mathcal{L}_1$ be any vertex on the first level. We have*

$$(i) \quad 2|g| - 1 \leq \lambda(g) \leq 2|g| + 1,$$

$$(ii) \quad \lambda(\varphi_v(g)) \leq |g|,$$

$$(iii) \quad \lambda(\varphi_v(g)) \leq \frac{\lambda(g)+1}{2},$$

$$(iv) \quad |\varphi_v(g)| \leq \frac{|g|+1}{2},$$

$$(v) \quad \sum_{w \in \mathcal{L}_1} |\varphi_w(g)| \leq |g|.$$

Proof. (i) Let us write $m = \lambda(g)$, and let $s_1, \dots, s_m \in \{a, \dots, a^{p-1}, b, \dots, b^{p-1}\}$ be such that $g = s_1 \dots s_m$. By the minimality of m , if s_i is a power of b , then s_{i+1} , if it exists, must be a power of a . Consequently, g can be written as an alternating product of powers of a and powers of b with at most $\frac{m+1}{2}$ powers of b . The first inequality thus holds.

For the second inequality, let us write $n = |g|$, and let $i_1, \dots, i_{n+1}, j_1, \dots, j_n \in \mathbf{Z}$ be such that

$$g = a^{i_1} b^{j_1} a^{i_2} b^{j_2} \dots a^{i_n} b^{j_n} a^{i_{n+1}}.$$

As g can be written as a product of at most $2n + 1$ elements belonging to the set $\{a, \dots, a^{p-1}, b, \dots, b^{p-1}\}$, we must have $\lambda(g) \leq 2n + 1$.

(ii) We will proceed by induction on $n = |g|$. If $n = 0$, then $g = a^j$ for some $j \in \mathbf{Z}$. In particular, for any $v \in \mathcal{L}_1$, we have $\varphi_v(g) = 1$, so the result holds in this case. Now, let us assume that it holds for a given $n \in \mathbf{N}$. Then, if $|g| = n + 1$, we have $g = g' b^i a^j$ for some $g' \in G$ with $|g'| = n$ and some $i, j \in \mathbf{Z}$. Since we have

$$\varphi_v(g) = \varphi_v(g' b^i a^j) = \varphi_{a^j \cdot v}(g') \varphi_{a^j \cdot v}(b^i)$$

for any $v \in \mathcal{L}_1$, the result follows from the induction hypothesis, the fact that $\lambda(\varphi_w(b^i)) \leq 1$ for all $w \in \mathcal{L}_1$ and subadditivity of the total length.

(iii) and (iv) follow directly from (i) and (ii).

(v) This follows by induction on $n = |g|$, using the fact that if $g = g_1 g_2$, then

$$\sum_{w \in \mathcal{L}_1} |\varphi_w(g)| \leq \sum_{w \in \mathcal{L}_1} |\varphi_w(g_1)| + \sum_{w \in \mathcal{L}_1} |\varphi_w(g_2)|,$$

and that we have $\sum_{w \in \mathcal{L}_1} |\varphi_w(a^i)| = 0$ and $\sum_{w \in \mathcal{L}_1} |\varphi_w(b^i)| = 1$ for all $i \in \{1, 2, \dots, p-1\}$. \square

4.3 Proof of the subgroup induction property for torsion GGS groups

We will now prove that all torsion GGS group possess the subgroup induction property. In fact, we will prove something a priori stronger, namely that they possess the *strong* subgroup induction property, see Definition 3.1 and the discussion below it.

Let us begin by fixing some notation. For the rest of this section, p will denote an odd prime and G will denote a torsion GGS group with defining vector $\mathbf{e} = (e_0, \dots, e_{p-2})$ acting on the p -regular rooted tree T .

Notation 4.11. For any $g \in G$, we will denote by α_g and β_g the unique elements of $\{0, 1, \dots, p-1\}$ such that $g \equiv_{G'} a^{\alpha_g} b^{\beta_g}$. Note that the existence of such elements follows from point (iii) of Proposition 4.3.

The subgroups of a torsion GGS group G can be classified into three broad classes, as the following lemma shows.

Lemma 4.12. *Let $H \leq G$ be a subgroup. Then, either*

- (I) $H = G$,
- (II) there exists $j_H \in \{0, 1\}$ such that $HG'/G' = \langle b^{j_H} \rangle G'/G'$,
- (III) there exists $j_H \in \{0, 1, \dots, p-1\}$ such that $HG'/G' = \langle ab^{j_H} \rangle G'/G'$.

Proof. Since all maximal subgroups of G are of finite index [13], the Frattini subgroup $\Phi(G)$ is equal to $\Phi_F(G)$, see the discussion before Lemma 3.5. On the other hand, since G is a p -group, all its finite quotients are finite p -groups and hence nilpotent. Therefore, by Lemma 3.5 the derived subgroup G' is contained in $\Phi_F(G) = \Phi(G)$. This implies that HG' is a proper subgroup of G if and only if H is proper. Therefore, HG'/G' is a proper subgroup of G/G' if and only if H is a proper subgroup of G . As G/G' is isomorphic to $C_p \times C_p$, its proper subgroups are all cyclic. Thus, either $H = G$, or HG'/G' is a cyclic subgroup of G/G' . Cases II and III together cover all the possible cyclic subgroups of G/G' and are mutually exclusive. \square

In what follows, we will frequently need to study the first level stabiliser of a given subgroup. The following lemma gives us a convenient generating set for this stabiliser when the subgroup is of type III.

Lemma 4.13. *Let $H \leq G$ be a subgroup of type III (according to the classification of Lemma 4.12) generated by a set $S \subseteq H$. Then, for any $x \in S$ with $\alpha_x \neq 0$, the set*

$$S' = \{x^{k_1} y x^{k_2} \mid y \in S, k_1, k_2 \in \{0, 1, \dots, p-1\}, k_1 \alpha_x + \alpha_y + k_2 \alpha_x \equiv 0 \pmod{p}\}$$

generates $\text{Stab}_H(\mathcal{L}_1)$.

Proof. Let $x \in S$ be any element of S such that $\alpha_x \neq 0$. Then, since H is of type **III**, the set $\{x^k \mid 0 \leq k \leq p-1\}$ is a Schreier transversal for $\text{Stab}_H(\mathcal{L}_1)$. Therefore, by the Reidemeister-Schreier method (see for example [12]), the set

$$S'' = \{x^{k_1}yx^{-k_2} \mid y \in S, k_1, k_2 \in \{0, \dots, p-1\}, k_1\alpha_x + \alpha_y - k_2\alpha_x \equiv 0 \pmod{p}\}$$

generates $\text{Stab}_H(\mathcal{L}_1)$. Taking $k_1 = p-1$ and $y = x$, which forces $k_2 = 0$, we see that $x^p \in S''$. Thus, multiplying elements of S'' on the right by x^p when k_2 is not zero, we can transform the generating set S'' into the set S' , which must consequently also generate $\text{Stab}_H(\mathcal{L}_1)$. \square

The main idea behind the proof of the subgroup induction property for torsion GGS groups is to use an argument of length reduction on the generators of finitely generated subgroups to reduce the question to subgroups generated by elements of length at most 1. We will establish this length reduction property in the next several lemmas. While they are unfortunately rather technical at times, they ultimately make the proof of the main result very short and simple.

Lemma 4.14. *Let $x, y \in G$ be two elements, and suppose that $\alpha_x \neq 0$. Let $k_1, k_2 \in \{0, 1, \dots, p-1\}$ be such that $k_1\alpha_x + \alpha_y + k_2\alpha_x \equiv 0 \pmod{p}$. Then, for every $v \in \mathcal{L}_1$, there exist $l_1 \in \{k_1, k_1-p\}$ and $l_2 \in \{k_2, k_2-p\}$ such that*

$$|\varphi_v(x^{l_1}yx^{l_2})| \leq \frac{|y|+1}{2} + |x|.$$

Furthermore, this inequality is strict if $|x|$ is odd.

Proof. It suffices to show that for every $k \in \{0, 1, \dots, p-1\}$ and for every $w \in \mathcal{L}_1$, we have $|\varphi_w(x^k)| + |\varphi_w(x^{k-p})| \leq |x|$. Indeed, we have

$$|\varphi_v(x^{l_1}yx^{l_2})| \leq |\varphi_{w_1}(x^{l_1})| + |\varphi_{w_2}(y)| + |\varphi_v(x^{l_2})|,$$

where $w_1 = yx^{l_2} \cdot v$ and $w_2 = x^{l_2} \cdot v$. Therefore, if $|\varphi_w(x^k)| + |\varphi_w(x^{k-p})| \leq |x|$, one can choose $l_1 \in \{k_1, k_1-p\}$ and $l_2 \in \{k_2, k_2-p\}$ such that $|\varphi_{w_1}(x^{l_1})|, |\varphi_v(x^{l_2})| \leq \frac{|x|}{2}$. Note that if $|x|$ is odd, we have $|\varphi_{w_1}(x^{l_1})|, |\varphi_v(x^{l_2})| < \frac{|x|}{2}$, since the b -length of elements must always be an integer. The result then immediately follows from the fact that $|\varphi_{w_2}(y)| \leq \frac{|y|+1}{2}$ for all $w_2 \in \mathcal{L}_1$ by Proposition 4.10 (iv).

To conclude the proof, it thus remains only to show that

$$|\varphi_w(x^k)| + |\varphi_w(x^{k-p})| \leq |x|$$

for all $k \in \{0, 1, \dots, p-1\}$ and for all $w \in \mathcal{L}_1$. Let us write $x = a^{\alpha_x}(x_1, x_2, \dots, x_p)$. By direct computation, we find that

$$\varphi_w(x^k) = x_{w+(k-1)\alpha_x} x_{w+(k-2)\alpha_x} \cdots x_{w+\alpha_x} x_w$$

(with the exceptional case of $\varphi_w(x^0) = 1$ and $|\varphi_w(x^0)| = 0$) and that

$$\varphi_w(x^{k-p}) = x_{w+k\alpha_x}^{-1} x_{w+(k+1)\alpha_x}^{-1} \cdots x_{w+(p-1)\alpha_x}^{-1}.$$

Therefore,

$$\begin{aligned}
|\varphi_w(x^k)| + |\varphi_w(x^{k-p})| &\leq \sum_{i=0}^{k-1} |x_{w+i\alpha_x}| + \sum_{i=k}^{p-1} |x_{w+i\alpha_x}^{-1}| \\
&= \sum_{i=1}^p |x_i| \\
&\leq |x|,
\end{aligned}$$

where we used Proposition 4.10 (v) and the easily established fact that $|g^{-1}| = |g|$ for all $g \in G$. This concludes the proof. \square

Lemma 4.15. *Let $H < G$ be a subgroup of type II (according to the classification of Lemma 4.12) generated by a finite set S , and let M be the maximal b -length of elements in S . Then for all $v \in \mathcal{L}_1$, the subgroup $\varphi_v(H)$ is generated by the set $S_v := \varphi_v(S)$, whose elements are of b -length at most $\frac{M+1}{2}$. In particular, if $M > 1$, the elements of S_v are all of length strictly less than M .*

Proof. Since subgroups of type II are contained in the stabiliser of the first level, $\varphi_v(H)$ is generated by $\varphi_v(S)$. The inequality on the b -length is then given by Proposition 4.10 (iv). \square

Lemma 4.16. *Let $H < G$ be a finitely generated subgroup of type III (according to the classification of Lemma 4.12). If H is generated by elements of b -length at most M , then for every $v \in \mathcal{L}_1$, the subgroup $\varphi_v(\text{Stab}_H(v))$ is generated by elements of b -length at most $\frac{3M}{2}$. Furthermore, if there exists $x \in H$ with $\alpha_x \neq 0$ and $|x| < M$, then $\varphi_v(\text{Stab}_H(v))$ is generated by elements of b -length at most $\frac{3M-1}{2}$.*

Proof. Since H is generated by elements of b -length at most M , the set S of all elements of H of b -length at most M is a finite generating set for H . Let $x \in S$ be an element such that $\alpha_x \neq 0$. Such an element must necessarily exist, since H is of type III. By Lemma 4.13, the set

$$S' = \{x^{k_1} y x^{k_2} \mid y \in S, k_1, k_2 \in \{0, 1, \dots, p-1\}, k_1 \alpha_x + \alpha_y + k_2 \alpha_x \equiv 0 \pmod{p}\}$$

generates $\text{Stab}_H(\mathcal{L}_1)$. Notice that by taking $k_1 = p-1$, $y = x$ and $k_2 = 0$, we find $x^p \in S'$.

Let us fix $v \in \mathcal{L}_1$. Since the action of $G/\text{Stab}_G(\mathcal{L}_1)$ on \mathcal{L}_1 is free, we have $\text{Stab}_H(v) = \text{Stab}_H(\mathcal{L}_1)$, and is thus generated by S' . Using Lemma 4.14, we can define a set S'' obtained from S' by replacing every element of the form $x^{k_1} y x^{k_2} \in S'$ with $y \neq x$ by an element $x^{l_1} y x^{l_2}$ such that

$$|\varphi_v(x^{l_1} y x^{l_2})| \leq \frac{|y|+1}{2} + |x|,$$

with $l_1 \in \{k_1, k_1 - p\}$ and $l_2 \in \{k_2, k_2 - p\}$. This corresponds to multiplying on the left and on the right elements of S' different from x^p by either x^p or the identity. Therefore, S'' is also a generating set of $\text{Stab}_H(v)$.

It follows that $\varphi_v(S'')$ is a generating set of $\varphi_v(\text{Stab}_H(v))$. Let M' be the maximal b -length of any element in this generating set. By construction, we have

$$M' \leq \max \left\{ \frac{M+1}{2} + |x|, |\varphi_v(x^p)| \right\}.$$

Let us first show that $|\varphi_v(x^p)| \leq |x|$. As in the proof of Lemma 4.14, writing $x = a^{\alpha_x}(x_1, x_2, \dots, x_p)$, we find by direct computation that

$$\varphi_v(x^p) = x_{v+(p-1)\alpha_x} x_{v+(p-2)\alpha_x} \dots x_{v+\alpha_x} x_v.$$

Therefore, by subadditivity and Proposition 4.10 (v), we have

$$|\varphi_v(x^p)| \leq \sum_{i=0}^{p-1} |x_{v+i\alpha_x}| = \sum_{i=0}^{p-1} |x_i| \leq |x|.$$

Thus, we have $M' \leq \max \left\{ \frac{M+1}{2} + |x|, |x| \right\} = \frac{M+1}{2} + |x|$. Furthermore, it follows from Lemma 4.14 that this inequality is strict if $|x|$ is odd.

If $|x| < M$, then we have $|x| \leq M-1$, since the length must be an integer. Therefore, $\varphi_w(\text{Stab}_H(w))$ is generated by elements of b -length at most $\frac{3M-1}{2}$. This proves the second part of the lemma. To finish the proof of the first part, it remains only to treat the case where $|x| = M$.

If $|x| = M$ and M is odd, then $\varphi_w(\text{Stab}_H(w))$ is generated by elements of b -length strictly smaller than $\frac{3M+1}{2}$. Again, since the length must be an integer, we can say that the generators of $\varphi_w(\text{Stab}_H(w))$ are of b -length at most $\frac{3M-1}{2}$.

Similarly, if $|x| = M$ and M is even, we can bound the b -length of the generators of $\varphi_w(\text{Stab}_H(w))$ by $\frac{3M}{2}$, since $\frac{3M+1}{2}$ is not an integer.

Thus, in all cases, $\varphi_v(\text{Stab}_H(v))$ can be generated by elements of b -length at most $\frac{3M}{2}$, which concludes the proof. \square

Lemma 4.17. *Let $H < G$ be a finitely generated subgroup of type III (according to the classification of Lemma 4.12). If H is generated by elements of b -length at most M , then for every $w \in \mathcal{L}_2$, the subgroup $\varphi_w(\text{Stab}_H(w))$ is generated by elements of b -length at most $\frac{3M+2}{4}$.*

Proof. If $M = 0$, then we must have either $H = \{1\}$ or $H = \langle a \rangle$. Since H is of type III, the first case is impossible, and in the second case, we have $\varphi_w(\text{Stab}_H(w)) = \{1\}$ for all $w \in \mathcal{L}_2$. As the trivial subgroup is generated by elements of b -length at most 0, the result holds in this case. Therefore, for the rest of the proof, we may suppose that $M \geq 1$.

By Lemma 4.6, either $\varphi_v(\text{Stab}_H(v)) = G$ for every $v \in \mathcal{L}_1$, or the subgroup $\varphi_v(\text{Stab}_H(v))$ is of type II, according to the classification of Lemma 4.12, for every $v \in \mathcal{L}_1$. In the first case, the result is true, since $\varphi_w(\text{Stab}_H(w)) = G$ for every $w \in \mathcal{L}_2$ and is thus generated by elements of b -length at most $1 \leq \frac{3M+2}{4}$, using the fact that $M \geq 1$. In the second case, by Lemma 4.15, it suffices to show that for every $v \in \mathcal{L}_1$, the subgroup $\varphi_v(\text{Stab}_H(v))$ is generated by elements of b -length at most $\frac{3M}{2}$, which is the case according to Lemma 4.16. \square

Lemma 4.18. *Let $H < G$ be a subgroup of G generated by elements of b -length at most 1. Then, for all vertices $v \in T$, the subgroup $\varphi_v(\text{Stab}_H(v))$ is generated by elements of b -length at most 1.*

Proof. By induction, it suffices to prove the result for $v \in \mathcal{L}_1$.

The proof depends on the type of H , according to the classification of Lemma 4.12. If H is of type **I**, or in other words, if $H = G$, then $\varphi_v(\text{Stab}_H(v)) = G$ for all $v \in \mathcal{L}_1$, so the result is trivial in this case. If H is of type **II**, the result immediately follows from Lemma 4.15. Lastly, if H is of type **III**, then by Lemma 4.16, $\varphi_v(\text{Stab}_H(v))$ is generated by elements of length at most $\frac{3}{2}$. As the length must be an integer, the result follows. \square

Lemma 4.19. *Let $H < G$ be subgroup of type **III**, according to the classification of Lemma 4.12, generated by a finite set S . Let us assume that every element of S is of b -length at most 2 and of total length at most 3. Then $\varphi_w(\text{Stab}_H(w))$ is generated by elements of b -length at most 1 for all $w \in \mathcal{L}_2$.*

Proof. Using Lemma 4.6 and similarly to the proof of Lemma 4.17, we can suppose that $\varphi_v(\text{Stab}_H(v))$ is of type **II**, according to the classification of Lemma 4.12, for every $v \in \mathcal{L}_1$.

Thanks to Lemma 4.15, it suffices to show that $\varphi_v(\text{Stab}_H(v))$ is generated by elements of b -length at most 2 for all $v \in \mathcal{L}_1$, since the b -length must be an integer.

Let us first suppose that there exists a non-trivial $x \in S$ such that $|x| < 2$. Then, we cannot have $x \in G'$, since non-trivial elements of G' must have b -length at least 2. Indeed, any element of b -length strictly less than 2 can be written as $a^i b^j a^k$ with $i, j, k \in \mathbf{Z}$, and for this element to be in G' , by Proposition 4.3 (iii), we must have $j \equiv 0 \pmod{p}$ and $i \equiv -k \pmod{p}$, which forces this element to be the identity.

Therefore, x does not belong to G' , and as H is a subgroup of type **III**, we conclude that $\alpha_x \neq 0$. It then follows from Lemma 4.16 that $\varphi_v(\text{Stab}_H(v))$ is generated by elements of b -length at most $\frac{5}{2}$, and thus at most 2, for all $v \in \mathcal{L}_1$.

Thus, it only remains to show that the result holds when every non-trivial element of S is of b -length 2. Let us choose some non-trivial $x \in S$. In light of Lemma 4.13, we need to show that for every $v \in \mathcal{L}_1$, for every $y \in S$ and for every $k_1, k_2 \in \{0, 1, \dots, p-1\}$ such that $k_1 \alpha_x + \alpha_y + k_2 \alpha_x \equiv 0 \pmod{p}$, we have

$$|\varphi_v(x^{k_1} y x^{k_2})| \leq 2.$$

Since x and y are of b -length 2 and of total length at most 3, we must have $x = b^{m_x} a^{\alpha_x} b^{n_x}$ and $y = b^{m_y} a^{\alpha_y} b^{n_y}$, with $m_x, n_x, m_y, n_y \in \{1, 2, \dots, p-1\}$. Therefore, we have

$$x^{k_1} y x^{k_2} = b_0^* b_{\alpha_x}^* b_{2\alpha_x}^* \cdots b_{k_1 \alpha_x}^* b_{k_1 \alpha_x + \alpha_y}^* b_{k_1 \alpha_x + \alpha_y + \alpha_x}^* \cdots b_{k_1 \alpha_x + \alpha_y + k_2 \alpha_x}^*,$$

where the $*$ represent unspecified powers (possibly zero) and $b_l = a^l b a^{-l}$. For every $v \in \mathcal{L}_1$, we have

$$|\varphi_v(b_l^*)| = \begin{cases} 1 & \text{if } l \equiv v \pmod{p} \\ 0 & \text{otherwise.} \end{cases}$$

As $\varphi_v(x^{k_1}yx^{k_2}) = \varphi_v(b_0^*)\varphi_v(b_{\alpha_x}^*) \cdots \varphi_v(b_{k_1\alpha_x + \alpha_y + k_2\alpha_x}^*)$, it suffices to show that in the sequence

$$0, \alpha_x, 2\alpha_x, \dots, k_1\alpha_x, k_1\alpha_x + \alpha_y, k_1\alpha_x + \alpha_y + \alpha_x, \dots, k_1\alpha_x + \alpha_y + k_2\alpha_x,$$

every number (modulo p) appears at most twice. Since $\alpha_x \neq 0$, using the fact that p is prime and that $k_1 \in \{0, 1, \dots, p-1\}$, we see that $0, \alpha_x, 2\alpha_x, \dots, k_1\alpha_x$ are all pairwise distinct, modulo p . Likewise, $k_1\alpha_x + \alpha_y, k_1\alpha_x + \alpha_y + \alpha_x, \dots, k_1\alpha_x + \alpha_y + k_2\alpha_x$ are also pairwise distinct modulo p . Consequently, a number (modulo p) can appear at most twice in the sequence

$$0, \alpha_x, 2\alpha_x, \dots, k_1\alpha_x, k_1\alpha_x + \alpha_y, k_1\alpha_x + \alpha_y + \alpha_x, \dots, k_1\alpha_x + \alpha_y + k_2\alpha_x,$$

since if it appeared three times, it would have to appear at least twice in either $0, \alpha_x, 2\alpha_x, \dots, k_1\alpha_x$ or in $k_1\alpha_x + \alpha_y, k_1\alpha_x + \alpha_y + \alpha_x, \dots, k_1\alpha_x + \alpha_y + k_2\alpha_x$, by the pigeonhole principle, which is impossible.

Thus, $\varphi_v(\text{Stab}_H(v))$ is generated by elements of b -length at most 2 for all $v \in \mathcal{L}_1$, which concludes the proof. \square

Lemma 4.20. *Let $H \leq G$ be a finitely generated subgroup of G . There exists some $n \in \mathbf{N}$ such that for every $v \in \mathcal{L}_n$, the subgroup $\varphi_v(\text{Stab}_H(v))$ is generated by elements of b -length at most 1.*

Proof. Let us fix a finite set S of generators of H , and let M be the maximal b -length of elements in S .

If H is a subgroup of type **I**, according to the classification of Lemma 4.12, then $H = G$. In this case, the result is obviously true by taking $n = 0$. If H is of type **II**, then by Lemma 4.15, if $M > 1$, the subgroups $\varphi_v(\text{Stab}_H(v))$ are generated by elements of b -length strictly smaller than M for every $v \in \mathcal{L}_1$. Lastly, if H is of type **III**, then by Lemma 4.17, if $M > 2$, the subgroups $\varphi_v(\text{Stab}_H(v)) = G$ are generated by elements of b -length strictly smaller than M for every $v \in \mathcal{L}_2$.

Thus, using induction and Lemma 4.18, it suffices to prove the result for H of type **III** and generated by elements of b -length at most 2. Let $x \in S$ be a generator of H such that $\alpha_x \neq 0$. Since we have $|x| \leq 2$, it follows from Lemma 4.16 that $\varphi_v(\text{Stab}_H(v))$ is generated by elements of b -length at most 3 for all $v \in \mathcal{L}_1$. If $\varphi_v(\text{Stab}_H(v)) = G$ for all $v \in \mathcal{L}_1$ we are done. Otherwise, by Lemma 4.6, Lemma 4.15 and Proposition 4.10 (ii), for every $w \in \mathcal{L}_2$, the subgroup $\varphi_w(\text{Stab}_H(w))$ is generated by elements of b -length at most 2 and of total length at most 3. If $\varphi_w(\text{Stab}_H(w))$ is of type **I** or **II**, then the result follows from the argument above, and if it is of type **III**, the result follows from Lemma 4.19. \square

We are now missing only one piece to prove that a torsion GGS group has the subgroup induction property, namely that a subgroup generated by elements of length at most 1 must belong to every inductive class. We prove this fact in the following two lemmas.

Lemma 4.21. *Let $H \leq G$ be a subgroup of a torsion GGS group G , and let \mathcal{X} be a weakly inductive class of subgroups G (see Definition 3.1 and the discussion below it). If there exists $n \in \mathbf{N}$ such that $\varphi_v(\text{Stab}_H(v)) \in \mathcal{X}$ for all $v \in \mathcal{L}_n$, then $H \in \mathcal{X}$.*

Proof. We proceed by induction on n . For $n = 0$, the result is trivially true. Let us suppose that the result is true for some $n \in \mathbf{N}$, and let $H \leq G$ be a subgroup such that $\varphi_w(\text{Stab}_H(w)) \in \mathcal{X}$ for all $w \in \mathcal{L}_{n+1}$. For each $v \in \mathcal{L}_n$, let us write $H_v = \text{Stab}_H(v)$. Our assumptions on H imply that for every $v \in \mathcal{L}_n$ and for every $u \in \mathcal{L}_1$, we must have $\varphi_u(\text{Stab}_{H_v}(u)) \in \mathcal{X}$. Using Property C of Definition 3.1 and the fact that $\text{Stab}_{H_v}(\mathcal{L}_1) = \text{Stab}_{H_v}(u)$, we conclude that $\text{Stab}_{H_v}(\mathcal{L}_1) \in \mathcal{X}$. Since $\text{Stab}_{H_v}(\mathcal{L}_1)$ is of finite index in H_v , we must also have $H_v \in \mathcal{X}$ by Property B' of Definition 3.1.

We have just shown that for every $v \in \mathcal{L}_n$, we have $\text{Stab}_H(v) \in \mathcal{X}$. Thus, by our induction hypothesis, we have $H \in \mathcal{X}$. \square

Lemma 4.22. *Let \mathcal{X} be a weakly inductive class, and let $H \leq G$ be a subgroup generated by elements of b -length at most 1. Then, $H \in \mathcal{X}$.*

Proof. We begin by observing that if H is generated by elements of total length 1, then $H \in \mathcal{X}$. Indeed, in this case, we have either $H = G$, $H = \langle a \rangle$ or $H = \langle b \rangle$. Since both $\langle a \rangle$ and $\langle b \rangle$ are finite subgroups, the result follows.

If H is not generated by elements of total length 1, then H must be of type II or III, according to the classification of Lemma 4.12. If H is of type II, then for every $v \in \mathcal{L}_1$, the subgroup $\varphi_v(\text{Stab}_H(v))$ is generated by elements of total length 1 by Proposition 4.10 (ii), and thus belongs to \mathcal{X} by the previous argument. It follows that $H \in \mathcal{X}$.

Lastly, if H is of type III, then by Lemma 4.16, $\varphi_v(\text{Stab}_H(v))$ is generated by elements of b -length at most $\frac{3}{2}$, and thus at most 1, for every $v \in \mathcal{L}_1$. As $\varphi_v(\text{Stab}_H(v))$ is either G or of type II by Lemma 4.6, the preceding argument shows that $\varphi_v(\text{Stab}_H(v)) \in \mathcal{X}$ for every $v \in \mathcal{L}_1$. Therefore, we must have $H \in \mathcal{X}$ by Lemma 4.21. \square

We are now finally ready to prove that every torsion GGS has the strong subgroup induction property, see the discussion after Definition 3.1.

Theorem 4.23. *Every torsion GGS group has the strong subgroup induction property.*

Proof. The proof follows the same general strategy as in [5].

Let G be a torsion GGS group, and let \mathcal{X} be a weakly inductive class of subgroups. We need to show that every finitely generated subgroup of G belongs to \mathcal{X} .

Let $H \leq G$ be a finitely generated subgroup. By Lemma 4.20, there exists $n \in \mathbf{N}$ such that $\varphi_v(\text{Stab}_H(v))$ is generated by elements of b -length at most 1 for all $v \in \mathcal{L}_n$. Lemma 4.22 then implies that $\varphi_v(\text{Stab}_H(v)) \in \mathcal{X}$ for all $v \in \mathcal{L}_n$. Therefore, $H \in \mathcal{X}$ by Lemma 4.21. \square

As a corollary, we get that all torsion GGS groups are subgroup separable.

Corollary 4.24. *Every torsion GGS group is subgroup separable.*

Proof. This is a direct application of Theorem 4.23 and [9, Theorem 2.12]. More precisely, in order to apply [9, Theorem 2.12], we use the fact that torsion GGS groups are finitely generated, self-similar, satisfy $\text{Stab}_G(\mathcal{L}_1) = \text{Stab}_G(v)$ for every first level vertex v , are branch, have the congruence subgroup property and have the subgroup induction property. \square

There are other consequences of the subgroup induction property that could be of interest for the study of torsion GGS groups. For instance, the structure of their finitely generated subgroups can be described using the notion of blocks, and one can classify their weakly maximal subgroups. We refer the interested reader to [9] and [11] for more information about these consequences.

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